OPTIMAL DIVIDENDS AND CAPITAL INJECTIONS
IN THE DUAL MODEL WITH DIFFUSION

BY

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ABSTRACT

The dual model with diffusion is appropriate for companies with continuous expenses that are offset by stochastic and irregular gains. Examples include research-based or commission-based companies. In this context, Avanzi and Gerber (2008) showed how to determine the expected present value of dividends, if a barrier strategy is followed. In this paper, we further include capital injections and allow for (proportional) transaction costs both on dividends and capital injections.

We determine the optimal dividend and (unconstrained) capital injection strategy (among all possible strategies) when jumps are hyperexponential. This strategy happens to be either a dividend barrier strategy without capital injections, or another dividend barrier strategy with forced injections when the surplus is null to prevent ruin. The latter is also shown to be the optimal dividend and capital injection strategy, if ruin is not allowed to occur. Both the choice to inject capital or not and the level of the optimal barrier depend on the parameters of the model.

In all cases, we determine the optimal dividend barrier and show its existence and uniqueness. We also provide closed form representations of the value functions when the optimal strategy is applied. Results are illustrated.

KEYWORDS

Dual model, diffusion, dividends, capital injections, HJB equation.

1. INTRODUCTION

1.1. The stability problem

What decisions should a company make in order to ensure 'stable' operations? Criteria that are used in the actuarial literature to address this 'stability problem' (see, for instance, Bühlmann, 1970) include the probability of ruin (see Asmussen and Albrecher, 2010, for an excellent broad reference) and the

Astin Bulletin 41(2), 611-644. doi: 10.2143/AST.41.2.2136990

expected present value of dividends (as introduced by de Finetti, 1957). More recently, some authors introduced capital injections and proposed to maximise the expected present value of the difference between dividends and capital injections.

The expected present value of dividends as an alternative to the probability of ruin was first proposed by de Finetti (1957). If a company makes decisions so that the probability of ruin is minimised, then it is implicit that it should let its surplus grow to the infinity. As this behaviour is arguably unrealistic, de Finetti (1957), in his model, allowed some surplus to be distributed. These leakages are likely to benefit the company's owners, hence explaining their qualification of 'dividends'. Usually, the way these are distributed (the 'dividend strategy') is determined such that the expected present value of dividends is maximised; see Albrecher and Thonhauser (2009) and Avanzi (2009) for reviews of the related literature.

The time value of money provides an incentive to distribute dividends earlier and more often. When these are maximised, ruin is usually certain. In some cases, it may be profitable (or required) to rescue the company by injecting some capital. Irrespective of ruin, injecting capital may have a positive net present value. This idea goes back to Borch (1974, Chapter 20) and Porteus (1977), and recent references on capital injections include Avram et al. (2007) for spectrally negative processes, Løkka and Zervos (2008) and He and Liang (2008) in the Brownian risk model, Yao et al. (2010) in the dual model, Dai et al. (2010) in the dual model with diffusion. In the case of the Cramér-Lundberg model without diffusion, Kulenko and Schmidli (2008) provide a proof of the optimality of a barrier strategy under general jump distributions when capital injections are forced (that is, when ruin is not allowed to occur).

It is worthwhile noting that the broader issue is relevant to other fields as well, such as corporate finance. In their excellent review of the literature on dividend payout policy, Allen and Michaely (2003, Chapter 7) state: "We believe that […] how payout policy interacts with capital-structure decisions (such as debt and equity issuance) are important questions and a promising field for further research."

In this paper, we are interested in determining the joint optimal dividend and capital injection strategy in the dual model with diffusion as described in the next section.

1.2. The dual model with diffusion

We consider the dual model with diffusion. In this model, the company surplus at time t is described as

$$U(t) = x - ct + S(t) + \sigma W(t),\ t \geq 0, \tag{1.1}$$

where $U(0-) = x \geq 0$ is the initial surplus, $c > 0$ is the expense rate per unit of time and where $\{S(t)\}$ is a compound Poisson process with intensity λ. The process $\{W(t)\}$ is a standard Brownian motion which is independent of $\{S(t)\}$, with volatility of σ per unit of time. Such a model is appropriate for companies with stochastic gains and deterministic expenses, such as research-based companies that develop inventions or patents. Such companies make discoveries at random times, and can crystallise the gain by selling the associated intellectual property to a buyer, or requiring patent licence fees from firms using the technology (see, for instance, Sharma and Clark, 2008). Other examples include commission-based firms such as real estate agents. The Brownian motion term reflects additional uncertainty in the firm's expenses and gains.

The dual risk model was first named so by Mazza and Rullière (2004) because of its duality to the Cramér-Lundberg model. Without diffusion, Avanzi et al. (2007) and Cheung and Drekic (2008) provide results when a dividend barrier strategy is applied, whereas Ng (2009) considers threshold strategies. Model (1.1) is dual to the Cramér-Lundberg model with diffusion as introduced by Dufresne and Gerber (1991). In this framework, results about dividends with a barrier strategy are derived in Avanzi and Gerber (2008).

We will assume that the distribution P of the jumps in $\{S(t)\}$ is a mixture of exponentials, namely:

$$\frac{dP(y)}{dy} = p(y) = \sum_{i=1}^{n} w_i \beta_i e^{-\beta_i y}, \text{ for } y > 0, \tag{1.2}$$

with

$$\sum_{i=1}^{n} w_i = 1, \; w_i > 0 \text{ for all } i, \text{ and } 0 < \beta_1 < \beta_2 < \ldots < \beta_n < \infty. \tag{1.3}$$

Mixtures of exponentials can be used to approximate certain long-tailed distributions such as the Pareto and Weibull. In the case of 'completely monotone' probability distribution functions, algorithms are readily available (see, for instance, Feldmann and Whitt, 1998). The broader class of combinations of exponentials (for which $w_i > 0$ is no more required) is also useful to approximate probability distributions (see, for instance, Dufresne, 2007). Although the optimality results of this paper do not extend to combinations, the closed form solutions for the value functions are still valid under mild assumptions (see also Remark 2.1).

Furthermore, note that (1.2) can be interpreted in the following way. If a research and development firm has n different departments, each with gains distribution being exponential with parameter β_i, expenses $w_i \cdot c$, and initial investment $w_i \cdot x$ ($i = 1, \ldots, n$), then (1.1) represents its global surplus (because of the properties of compound Poisson processes); see also Remark 4.3.

1.3. Formulation of the general optimal control problem

In this paper, we consider two types of controls: dividend payments (surplus outflows) and equity issuance (surplus inflows). We assume that a complete

https://doi.org/10.2143/AST.41.2.2136990 Published online by Cambridge University Press

filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is given, such that $\{U(t)\}$ is adapted. The controlled surplus process is

$$X_\pi(t) = U(t) - D_\pi(t) + E_\pi(t),\ t \geq 0. \tag{1.4}$$

Here, $\{D_\pi(t)\}$ represents the aggregate dividends distributed up until time t, according to strategy π. A dividend strategy is said to be admissible if $\{D_\pi(t)\}$ is a non-decreasing, $\{\mathcal{F}_t\}$-adapted process with $D_\pi(0-) = 0$. We assume that $\{D_\pi(t)\}$ has càdlàg sample paths. In addition, we restrict the possible control processes so that a firm cannot pay out an amount of dividends that is larger than the current surplus. That is,

$$\Delta D_\pi(t) \leq X_\pi(t-) \text{ for all } t, \tag{1.5}$$

where

$$\Delta D_\pi(t) = D_\pi(t) - D_\pi(t-) \tag{1.6}$$

represents the size of the dividend paid at time t. On the other hand, $\{E_\pi(t)\}$ represents the aggregate capital injected up until time t. We assume that $\{E_\pi(t)\}$ has càdlàg sample paths. A capital injection strategy is admissible if $\{E_\pi(t)\}$ is a non-decreasing, $\{\mathcal{F}_t\}$-adapted process with $E_\pi(0-) = 0$. An admissible joint control strategy is then denoted by $\pi = (D_\pi, E_\pi)$, and the set of admissible control strategies is denoted by Π so that $\pi \in \Pi$.

Our objective is to determine the optimal control strategy π that maximises the expected present value of dividends less capital injections until ruin, which we define to be

$$J(x;\pi) := \mathbb{E}^x\left[\limsup_{t\to\infty}\left(\eta\int_{0-}^{t\wedge\tau_\pi -} e^{-\delta s} dD_\pi(s) - \kappa\int_{0-}^{t\wedge\tau_\pi -} e^{-\delta s} dE_\pi(s)\right)\right], \tag{1.7}$$

where τ_π is the time of ruin, $a \wedge b$ denotes the minimum of a and b, and where $\mathbb{E}^x$ is the conditional expectation given the initial surplus x. We assume that dividends are paid out of the surplus to the same group of investors that inject capital into the surplus, and the force of interest $\delta > 0$ reflects the time preference of those investors. Proportional costs on dividend transactions are taken into account through the value of η, with $0 < \eta \leq 1$ representing the net proportion of leakages from the surplus received by investors after transaction costs have been paid. Proportional transaction costs on capital injections are taken into account through the value of κ, with $1 \leq \kappa < \infty$ representing the 'total costs' of injecting a single dollar of capital, where these are defined to be the amount of capital injected, plus any transaction costs required to inject this capital. Given initial capital $x \geq 0$, we define the value of the optimal strategy to be

$$V(x;\pi^*) := \sup_{\pi\in\Pi} J(x;\pi). \tag{1.8}$$

It follows from results in the discrete-time setting of Miyasawa (1962) and Takeuchi (1962) that the barrier strategy should be the optimal dividend strategy in the dual model, although it has yet to be formally proven. In the case where the dual model is perturbed by a diffusion term, Bayraktar and Egami (2008, without capital injections) and Dai et al. (2010) proved that the barrier strategy is optimal if the gains distribution is exponential and has a finite right endpoint, respectively.

Note that some papers force capital injections when the surplus is null to prevent ruin. Such a compulsion may be justified by strictly negative surplus at ruin (because of downwards jumps) or by regulation (in the case of insurance companies). These reasons are less relevant in the dual model, which gives us grounds for allowing any capital injection strategy as above.

1.4. Structure of the paper

In order to solve the general optimal control problem as described above, we need to consider two sub-problems first.

Section 2 restricts the problem to dividends only and shows that a barrier strategy is optimal, whether the drift of (1.1) is positive or not. Furthermore, a closed form representation of the value function is developed, which did not appear in Avanzi and Gerber (2008).

In Section 3, capital injections are forced when the surplus hits 0 to prevent ruin. Again, it is shown that a dividend barrier strategy is optimal irrespective of the drift of (1.1), and a closed form representation for the value function is given.

The optimal joint strategy π^* as well as a closed form for (1.8) are developed in Section 4. The solution of the problem is a combination of the two sub-problems above. Whereas the barrier strategy is always optimal for dividends, the decision whether capital should be injected or not and the level of the optimal barrier depend on the parameters of the model. This general solution is illustrated in Section 5.

2. Optimality of the barrier without capital injections

We first examine the optimal dividend problem without equity issuance, such that $E_{\pi_d}(t) \equiv 0$ for all t. This is a special case of (1.4), where

$$X_{\pi_d}(t) = U(t) - D_{\pi_d}(t), \quad t \geq 0. \tag{2.1}$$

An admissible control strategy is then denoted by $\pi_d = (D_{\pi_d}, E_{\pi_d})$, such that $\pi_d \in \Pi$. The time of ruin for such a strategy is defined as

$$\tau_{\pi_d} := \inf\{t : X_{\pi_d}(t-) = 0\}, \tag{2.2}$$

because of diffusion and because the surplus process is spectrally positive.

Our objective is to determine the optimal control strategy π_d that maximises the expected present value of dividends until ruin, which we define to be

$$J(x;\pi_d) := \mathbb{E}^x\Big[\eta\int_{0-}^{\tau_{\pi_d}-} e^{-\delta t}\,dD_{\pi_d}(t)\Big]. \tag{2.3}$$

Here the upper limit of the integral is $\tau_{\pi_d}-$ to reflect the fact that in general, $X(t) \neq X(t-)$ due to the possibility of a jump in the compound Poisson process. Given initial capital $x > 0$, we consider the expected present value of dividends under the optimal strategy, denoted by

$$V(x;\pi_d^*) := \sup_{\pi_d \in \Pi_d} J(x;\pi_d) \tag{2.4}$$

where the set of admissible strategies is $\Pi_d := \{\pi_d = (D_{\pi_d}, E_{\pi_d}) \in \Pi\}$. We will identify the form of the value function $V(x;\pi_d^*)$ and the optimal strategy π_d^*.

2.1. Hamilton-Jacobi-Bellman (HJB) equation

Suppose that for a given level of initial surplus $x \geq 0$, the value function under π_d^* is denoted by $G(x)$. According to the Hamilton-Jacobi-Bellman (HJB) equation for this problem, if the value function G is twice continuously differentiable then we expect it to satisfy

$$\max\{(\mathcal{A}-\delta)G(x),\ \eta - G'(x)\} = 0 \ \text{ with } \ G(0) = 0, \tag{2.5}$$

where the operator $\mathcal{A}$ is the infinitesimal generator

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2 f''(x) - cf'(x) - \lambda f(x) + \lambda\int_0^\infty f(x+y)\,dP(y). \tag{2.6}$$

The HJB (2.5) can be obtained from the following heuristic argument. Consider the small time interval $(0, dt)$. Suppose that on this time interval, we follow an arbitrary strategy whereby surplus is released at a rate $l \geq 0$ to cover dividend distribution plus transaction costs, and thereafter, an optimal strategy is applied. By conditioning on the number of jumps that occur, the size of the jump if it does occur, and the value of $W(dt)$, we see that the expected present value of dividends until ruin under this strategy is (by Taylor expansions)

$$\begin{aligned}
&l\eta dt + (1-\delta dt)\Big\{(1-\lambda dt)\mathbb{E}\big[G(x-(c+l)dt+\sigma W(dt))\big] \\
&+ \lambda dt\int_0^\infty \mathbb{E}\big[G(x+y-(c+l)dt+\sigma W(dt))\big]dP(y)\Big\} + o(dt) \qquad (2.7)\\
=\ & G(x) + \big\{l[\eta - G'(x)] \\
&+ \frac{1}{2}\sigma^2 G''(x) - cG'(x) - (\lambda+\delta)G(x) + \lambda\int_0^\infty G(x+y)\,dP(y)\}\,dt + o(dt). \qquad (2.8)
\end{aligned}$$

Since $G(x)$ is the optimal value, its value must be greater than or equal to the value of equation (2.8). Thus, it follows that the expression in braces must have maximal value of zero, suggesting

$$\max_{l\geq 0}\left\{l\left[\eta - G'(x)\right] + (\mathcal{A}-\delta)\,G(x)\right\} = 0. \tag{2.9}$$

Note that if $G'(x) < \eta$ we can make the first part of (2.9) unbounded by letting l tend to infinity, so we must restrict the first derivative to

$$G'(x) \geq \eta. \tag{2.10}$$

Conversely, when $G'(x) \geq \eta$, the first part of (2.9) is less than or equal to zero for any $l \geq 0$. Now since (2.9) holds when $l = 0$, we must have

$$(\mathcal{A}-\delta)\,G(x) \leq 0. \tag{2.11}$$

Since we allowed the initial surplus $x \geq 0$ to be arbitrary, (2.10) and (2.11) must hold for any $x \geq 0$. Thus, we can rewrite (2.9) by splitting it into two parts, as given in the HJB equation (2.5). The boundary condition $G(0) = 0$ holds because if the initial surplus is zero, then by definition the firm is immediately ruined.

2.2. Construction of a candidate solution

We conjecture that the barrier strategy is optimal. Let

$$G(x) = \mathbb{E}^x\left[\eta\int_{0-}^{\tau_{b_d}-} e^{-\delta t}\,dD_{b_d}(t)\right] \tag{2.12}$$

denote the expected present value of the dividends distributed until ruin using a barrier strategy with level b_d, given an initial surplus of x. It follows from the results in Avanzi and Gerber (2008) that $G(x)$ satisfies the integro-differential equation (IDE)

$$\frac{1}{2}\sigma^2 G''(x) - cG'(x) - (\lambda+\delta)\,G(x) + \lambda\int_0^\infty G(x+y)\,dP(y) = 0, \quad 0 \leq x \leq b_d, \tag{2.13}$$

leading to

$$G(x) := \begin{cases} G(x;b_d) & x \in [0, b_d];\ \text{and} \\ \eta(x-b_d) + G(b_d;b_d) & x \in (b_d, \infty); \end{cases} \tag{2.14}$$

where we define

$$G(x;b_d) := \sum_{k=0}^{n+1} C_k(b_d)e^{r_k x}, \ \text{for}\ \ x \geq 0, \tag{2.15}$$

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and where the r_k's are the roots of the characteristic equation

$$f(\xi) = \frac{1}{2}\sigma^2\xi^2 - c\xi - (\lambda+\delta) + \lambda\sum_{i=1}^{n} w_i \frac{\beta_i}{\beta_i - \xi} = 0. \tag{2.16}$$

It is easy to show that the r_k's satisfy the following 'interweaving root' condition:

$$r_0 < 0 < r_1 < \beta_1 < \ldots < r_n < \beta_n < r_{n+1}. \tag{2.17}$$

The optimal barrier b_d^* and the associated $n + 2$ coefficients $C_k(b_d^*)$ are the solution of the following $n + 3$ equations:

$$G(0; b_d^*) = \sum_{k=0}^{n+1} C_k(b_d^*) = 0, \tag{2.18}$$

$$G'(b_d^*-; b_d^*) = \sum_{k=0}^{n+1} r_k C_k(b_d^*) e^{r_k b_d^*} = \eta, \tag{2.19}$$

$$G''(b_d^*-; b_d^*) = \sum_{k=0}^{n+1} r_k^2 C_k(b_d^*) e^{r_k b_d^*} = 0, \text{ and} \tag{2.20}$$

$$\sum_{k=0}^{n+1} \frac{\beta_i r_k}{\beta_i - r_k} C_k(b_d^*) e^{r_k b_d^*} = \eta, \text{ for } i = 1, 2, \ldots, n. \tag{2.21}$$

Conditions (2.18) and (2.20) are equivalent to Conditions (3.6) and (5.3) of Avanzi and Gerber (2008), respectively, and Conditions (2.19) and (2.21) are analogous to Conditions (3.7) and (3.5) of Avanzi and Gerber (2008), respectively, with the incorporation of the transaction costs η. The latter are derived using a similar approach. Note that Conditions (2.18), (2.19) and (2.21) hold for *any* level of barrier b_d, whereas (2.20) is the condition for the optimal barrier b_d^* only.

Remark 2.1. *When the coefficients w_i in* (1.2) *are allowed to be negative, that is, when jumps are distributed according to a combination of exponentials, Condition* (2.17) *– crucial for optimality – does not necessarily hold any more. Nevertheless, closed form expressions for the value functions, throughout the paper, hold as long as all r_k are real and distinct.*

2.3. Explicit form of the value function

In this section, we focus on the optimal strategy b_d^* and first solve equations (2.19)-(2.21) to get a closed form representation for the $C_k(b_d^*)$'s. We then show that (2.18) leads to a unique optimal barrier $b_d^* > 0$, and that this one exists if and only if the drift of the process $\{U(t)\}$,

$$\mu := \mathbb{E}\left[U(t+1) - U(t)\right] = \lambda\sum_{i=1}^{n} \frac{w_i}{\beta_i} - c, \; t \geq 0, \tag{2.22}$$

is strictly positive. If $\mu \le 0$, the optimal barrier is null; this is discussed in Section 2.6.

2.3.1. Determining the $C_k(b_d^*)$ coefficients

We start by defining the rational function Q:

$$Q(\xi) := \sum_{k=0}^{n+1} \frac{r_k^2 C_k(b_d^*) e^{r_k b_d^*}}{\xi - r_k}. \tag{2.23}$$

The objective in this section is to find an equivalent representation of Q, and to use the fact that

$$\lim_{\xi \to r_k} (\xi - r_k) Q(\xi) = r_k^2 C_k(b_d^*) e^{r_k b_d^*} \text{ for } k = 1, 2, \ldots, n \tag{2.24}$$

to determine the $C_k(b_d^*)$ coefficients. We observe that Q satisfies the following properties:

(P1) By factorising the denominator of (2.23), we see that Q is a rational function with the denominator being a polynomial of degree $n+2$. The numerator is a polynomial of degree n since the coefficient of ξ^{n+1} is zero due to (2.20).

(P2) Its poles are $r_0, r_1, r_2, \ldots, r_n, r_{n+1}$;

(P3) $Q(0) = -\eta$ due to (2.19);

(P4) $Q(\beta_i) = 0$ for $i = 1, 2, \ldots, n$, by factorising the difference between (2.21) and (2.19).

The four points (P1)-(P4) uniquely determine Q. (P1) and (P2) give us the form of the denominator, and these can be combined with (P3) and (P4) to determine the form of the numerator. Hence, we can write

$$Q(\xi) = -\eta \frac{\prod_{j=0}^{n+1} r_j \prod_{i=1}^{n} \frac{\xi - \beta_i}{\beta_i}}{\prod_{j=0}^{n+1} (\xi - r_j)}. \tag{2.25}$$

Applying (2.24) we find

$$C_k(b_d^*) = -\eta \frac{e^{-r_k b_d^*}}{r_k} \prod_{\substack{j=0 \\ j \neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i} \text{ for } k = 0, 1, \ldots, n+1. \tag{2.26}$$

Because of (2.17), for all $b_d^* \geq 0$,

$$C_0(b_d^*) < 0, \lim_{b_d^* \to \infty} C_0(b_d^*) = -\infty, \text{ and} \tag{2.27}$$

$$C_k(b_d^*) > 0, \lim_{b_d^* \to \infty} C_k(b_d^*) = 0, \quad k = 1, 2, \ldots, n+1. \tag{2.28}$$

As a result, (2.15) – with the optimal barrier b_d^*, can now be explicitly written as

$$G(x; b_d^*) := -\eta \sum_{k=0}^{n+1} \frac{e^{-r_k b_d^*}}{r_k} \prod_{\substack{j=0 \\ j \neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i} e^{r_k x}, \ x \geq 0, \tag{2.29}$$

where b_d^* is determined by condition (2.18), which can now be rewritten as

$$-\eta \sum_{k=0}^{n+1} \frac{e^{-r_k b_d^*}}{r_k} \prod_{\substack{j=0 \\ j \neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i} = 0. \tag{2.30}$$

Substituting (2.14), (2.19) and (2.20) into the IDE (2.13) with $x = b_d^*$ yields

$$G(b_d^*; b_d^*) = \frac{\eta\mu}{\delta} \tag{2.31}$$

which is the present value of a perpetuity of $\eta\mu$ using force of interest δ.

Remark 2.2. *From* (2.29) *we can see that the inclusion of proportional transaction costs on the dividends through η simply scales the size of the value function. A heuristic argument for this property is as follows: suppose that there are no transaction costs and the optimal barrier is b_d^*. Then introduce proportional transaction costs on dividends. The introduction of the costs does not affect the surplus process, since whenever dividends are paid out, the same amount is removed from the surplus, but the investors simply receive less dividends. Thus, it is still optimal to use the same barrier b_d^*. However, since only η of each dollar is distributed as dividends, the value function is scaled by η.*

In light of this remark, we note that equation (2.31) *is an updated version of the analogous formulas from Gerber (1972), Avanzi et al. (2007) and Avanzi and Gerber (2008), who found that in the absence of transaction costs on dividends,*

$$G(b_d^*; b_d^*) = \frac{\mu}{\delta}$$

in the Brownian risk model, dual model and dual model with diffusion, respectively. Note that it can be shown that μ, δ, the r_k and the β_i satisfy the following elegant relationship,

$$\frac{\mu}{\delta} = \sum_{k=0}^{n+1} \frac{1}{r_k} - \sum_{i=1}^{n} \frac{1}{\beta_i}. \tag{2.32}$$

which does not seem to have any particular interpretation. It is remarkable that the weights w_i do not appear on the right-hand side.

Remark 2.3. *The approach of defining a rational function and finding an equivalent representation to determine the form of the C_k's was used in Section 6 of Dufresne and Gerber (1991) and Section 4 of Albrecher et al. (2010) to solve problems on ruin probabilities and the discounted penalty function respectively.*

Remark 2.4. *It should be noted that the $C_k(b_d^*)$ derived here is a general form which applies to other problems in the dual model with diffusion, provided that the gains distribution is a mixture of exponentials, $G'(b_d^*-; b_d^*) = \eta$ and $G''(b_d^*-; b_d^*) = 0$. This fact will be used in Section 3 (with capital injections), which uses a different boundary condition.*

2.3.2. Existence and uniqueness of b_d^*

Let us first define

$$\chi(b_d^*) := \sum_{k=0}^{n+1} C_k(b_d^*),\ b_d^* \geq 0, \tag{2.33}$$

such that (2.18) is equivalent to

$$\chi(b_d^*) = 0. \tag{2.34}$$

The problem is now to show that $\chi(b_d^*)$ has a unique root. We first note that

$$\chi'(b_d^*) = \sum_{k=0}^{n+1} -r_k C_k(b_d^*) < 0, \tag{2.35}$$

because r_k and $C_k(b_d^*)$ have the same sign for all k; see (2.17), (2.27) and (2.28). Hence, χ is a decreasing function in b_d^*. Since

$$\chi(0) = G(b_d^*; b_d^*) = \frac{\eta\mu}{\delta} \tag{2.36}$$

and

$$\lim_{b_d^* \to \infty} \chi(b_d^*) = -\infty \tag{2.37}$$

because of (2.27), (2.28) and the continuity of χ, it follows that (2.30) has a unique positive solution that exists if and only if $\mu > 0$. This also shows that the optimal barrier b_d^* is independent of the initial surplus x.

2.4. Verification of all the conditions of the HJB equation

By construction, our candidate solution satisfies $G'(x) = \eta$ for $x \in [b_d^*, \infty)$, $(\mathcal{A} - \delta)\,G(x) = 0$ for $x \in [0, b_d^*]$ and the boundary condition $G(0) = 0$. Furthermore,

$$\begin{aligned}(\mathcal{A}-\delta)\,G(x) &= \frac{1}{2}\sigma^2 G''(x) - cG'(x) - (\lambda+\delta)\,G(x) + \lambda\int_0^\infty G(x+y)\,dP(y)\\ &= 0 - c\eta - (\lambda+\delta)\big[\eta(x-b_d^*) + G(b_d^*;b_d^*)\big]\\ &\quad + \lambda\int_0^\infty \big[\eta(x+y-b_d^*) + G(b_d^*;b_d^*)\big]\,dP(y)\\ &= -\,\delta\eta(x-b_d^*) < 0,\ \ x > b_d^*.\end{aligned} \tag{2.38}$$

where we have used (2.31) to go from the second to the last line. Hence, it only remains to show that

$$G'(x) \geq \eta,\ \ 0 \leq x \leq b_d^*. \tag{2.39}$$

Because $G''(b_d^*-;\, b_d^*) = 0$ and

$$G'''(x;b_d^*) = \sum_{k=0}^{n+1} r_k^3 C_k(b_d^*)e^{r_k x} > 0,\ 0 \leq x \leq b_d^*, \tag{2.40}$$

$G''(x)$ is negative and $G'(x)$ decreasing when $0 \leq x \leq b_d^*$. It follows then from (2.19) that (2.39) holds.

2.5. Verification lemma

Lemma 2.1. *If non-negative function $G \in \mathbb{C}^1(\mathbb{R}_+)$ is also twice continuously differentiable except at countably many points and satisfies*

1. $(\mathcal{A} - \delta)\,G(x) \leq 0,\ x \geq 0,$
2. $G''(x) \leq 0,\ x \geq 0,$
3. $G'(x) \geq \eta,\ x \geq 0,$

then

$$G(x) \geq V(x;\pi_d^*),\ \ x \geq 0. \tag{2.41}$$

Moreover, if there exists a point $b_d^* \in \mathbb{R}_+$ such that $G \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{b_d^*\})$ with

4. $(\mathcal{A} - \delta)\,G(x) = 0,\ G'(x) \geq \eta$ *for* $x \in [0, b_d^*]$,
5. $(\mathcal{A} - \delta)\,G(x) < 0,\ G(x) = \eta(x - b_d^*) + G(b_d^*)$ *for* $x \in (b_d^*, \infty)$,

in which the integro-differential operator $\mathcal{A}$ *is defined by* (2.6)*, then*

$$G(x) = V(x;\pi_d^*),\ x \in \mathbb{R}_+,\ \textit{and}, \tag{2.42}$$

$$dD_{\pi_d}(t) = (X_{\pi_d}(t-) - b_d^*)\,\mathbb{1}_{\{X_{\pi_d}(t-) > b_d^*\}} + dL_{X_{\pi_d}}^{b_d^*}(t),\ t \geq 0, \tag{2.43}$$

is optimal, where

$$L_{X_{\pi_d}}^{b_d^*}(t) = \int_0^t \mathbb{1}_{\{X_{\pi_d}(s) = b_d^*\}}\, dL_{X_{\pi_d}}^{b_d^*}(s),\ t \geq 0, \tag{2.44}$$

is the local time of the process X at the barrier b_d^**, representing dividends due to oscillations of the Brownian Motion when the surplus is at the barrier, and*

$$(X_{\pi_d}(t-) - b_d^*)\,\mathbb{1}_{\{X_{\pi_d}(t-) > b_d^*\}} \tag{2.45}$$

represents the dividend distributed at time t if the surplus process jumps above the barrier.

A proof is discussed in Appendix A.

2.6. The case $\mu \leq 0$

In the previous sections, we found that there is a unique positive barrier b_d^* that maximises the value function $G(x)$ if and only if $\mu > 0$. We now consider the case when $\mu \leq 0$ and will show that $b_d^* = 0$ if and only if $\mu \leq 0$. This means that if the business is not profitable, the optimal strategy is to remove any surplus that is available as a final dividend and stop the business. This is not necessarily trivial when $\eta < 1$.

2.6.1. Case 1: $b_d^* = 0 \Rightarrow \mu \leq 0$

Suppose that $b_d^* = 0$. This means that the value function $G(x)$ is maximised when the barrier is at zero, and it is optimal to immediately release the entire surplus as dividends. In this case, it follows that

$$G(x) = \eta x. \tag{2.46}$$

However, we know from the HJB equation (2.5) that any optimal strategy should satisfy $(\mathcal{A} - \delta)\, G(x) \leq 0$ for all $x \geq 0$. Upon substitution with (2.46), this condition reduces to $\mu \leq 0$. Thus, we see that if the optimal barrier is $b_d^* = 0$, then the drift μ should satisfy $\mu \leq 0$. Going backwards, it follows that if $\mu \leq 0$, then $G(x) = \eta x$ satisfies the HJB equation (2.5).

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2.6.2. Case 2: $\mu \leq 0 \Rightarrow b_d^* = 0$

Consider an alternative strategy, say $\hat{\pi}_d$, whereby the surplus x is immediately paid as a dividend, so that ruin occurs immediately. The value under this strategy is $J(x; \hat{\pi}_d) = \eta x$. However, this strategy must have value less than the optimal strategy, so it follows that $J(x; \hat{\pi}_d) = \eta x \leq V(x; \pi_d^*)$.

Moreover, we showed that the function $G(x) = \eta x$ satisfies the HJB equation in the case when $\mu \leq 0$. Thus, it follows from Lemma 2.1 that $G(x) = \eta x \geq V(x; \pi_d^*)$.

Based on these two arguments, it follows that $V(x; \pi_d^*) = \eta x$, and so, that the optimal barrier is $b_d^* = 0$.

3. Dividend maximisation with forced capital injections to prevent ruin

In this section, as a stepping stone in solving the general optimal control problem, we first assume that the set of admissible control strategies π_e is determined such that the surplus X_{π_e} is never ruined. This can be achieved by injecting extra capital in order to keep the surplus above zero. The surplus process becomes

$$X_{\pi_e}(t) = U(t) - D_{\pi_e}(t) + E_{\pi_e}(t),\ t \geq 0, \tag{3.1}$$

where the set of admissible strategies is

$$\Pi_e := \{\pi_e = (D_{\pi_e}, E_{\pi_e}) \in \Pi \text{ such that } X_{\pi_e}(t-) \geq 0 \ \text{ for all } t \geq 0\}. \tag{3.2}$$

In this model, ruin does not occur. The objective function for this problem is

$$J(x; \pi_e) := \mathbb{E}^x\left[\limsup_{t\to\infty}\left(\eta\int_{0-}^{t} e^{-\delta s}\, dD_{\pi_e}(s) - \kappa\int_{0-}^{t} e^{-\delta s}\, dE_{\pi_e}(s)\right)\right]. \tag{3.3}$$

Given initial surplus $x > 0$, we consider the expected present value of dividends distributed less the total costs of equity issuance under the optimal strategy, denoted by

$$V(x; \pi_e^*) := \sup_{\pi_e \in \Pi_e} J(x; \pi_e). \tag{3.4}$$

We will identify the form of the value function $V(x; \pi_e^*)$ and the optimal strategy π_e^*.

3.1. HJB equation

Suppose that for a given level of initial surplus $x \geq 0$, the value function under the optimal joint dividend and capital injection strategy is denoted by $H(x)$.

According to the Hamilton-Jacobi-Bellman (HJB) equation for this problem, if the value function H is twice continuously differentiable then we expect it to satisfy

$$\max\{(\mathcal{A}-\delta)H(x),\ \eta - H'(x),\ H'(x)-\kappa\} = 0 \text{ with } H'(0) = \kappa. \tag{3.5}$$

Using the same techniques as described in Section 2.1 and allowing for capital injection $m\kappa dt$, the analogous result to (2.8) is

$$H(x) + \left\{l\left[\eta - H'(x)\right] + m\left[H'(x)-\kappa\right] + (\mathcal{A}-\delta)H(x)\right\}dt + o(dt). \tag{3.6}$$

Since $H(x)$ is the optimal value, it follows that the expression in braces must have maximal value of zero, suggesting

$$\max_{l\geq 0,\, m\geq 0}\left\{l\left[\eta - H'(x)\right] + m\left[H'(x)-\kappa\right] + (\mathcal{A}-\delta)H(x)\right\} = 0. \tag{3.7}$$

We restrict then the first derivative of the value function such that

$$\eta \leq H'(x) \leq \kappa, \tag{3.8}$$

otherwise we can make the first or second part of (3.7) unbounded, by letting l or m tend to infinity respectively. Now since (3.7) holds for $l = m = 0$, we require

$$(\mathcal{A}-\delta)H(x) \leq 0. \tag{3.9}$$

Since we allowed the initial surplus $x \geq 0$ to be arbitrary, equations (3.8) and (3.9) must hold for any $x \geq 0$, and we can rewrite (3.7) by splitting it into three parts, leading to (3.5).

The boundary condition can be explained by the following heuristic argument. Consider two sample paths of the surplus process: one starting at some small $\varepsilon > 0$, and another starting at zero. If the latter path moves down to $-\varepsilon$, and the former path moves parallel to this path, we must have

$$H(0) = H(\varepsilon) - \varepsilon\kappa. \tag{3.10}$$

Subtracting $H(0)$ from both sides, dividing by ε and letting ε tend to zero shows that $H'(0) = \kappa$. This is also supported by the following discussion.

Consider the representation of the expected present value of dividends less capital injections under the arbitrary strategy given in (3.6). The optimal value $H(x)$ is obtained when the value of the expression in braces is maximised. We now consider the value of m that will maximise this expression. Since $(\mathcal{A}-\delta)H(x)$ and $l[\eta - H'(x)]$ are independent of m, we wish to consider

$$\max_{m\geq 0}\left\{m\left[H'(x)-\kappa\right]\right\}. \tag{3.11}$$

It is clear that the value of m that maximises this expression will depend on the value of $H'(x)$. However, because our objective function (3.3) is penalised by capital injections, we will minimise m whenever possible. Together with $\eta \leq H'(x) \leq \kappa$ it follows that at any time $t > 0$, the appropriate value of m is determined by $H'(X_{\pi_e}(t))$ in the following way (with slight abuse of notation):

$$\text{If } H'(X_{\pi_e}(t)) \begin{cases} < \kappa & \text{then } m = 0; \\ = \kappa & \text{then } m \in [0, \infty]. \end{cases} \tag{3.12}$$

Since we wish to minimise m whenever possible (because of transaction costs), then ideally we would like to set $m = 0$ at all times. However, in the problem formulation outlined at the start of Section 3, we are required to inject capital to prevent ruin. With this being the case, the only time when it is possibly optimal to inject capital is when $H'(X_{\pi_e}(t)) = \kappa$, and this should only happen when the surplus is null. Intuitively, this is because discounting will unnecessarily penalise capital injections that are made before they are absolutely necessary, and these can be absolutely necessary only when the surplus is null (to avoid imminent ruin).

3.2. Construction of a candidate solution

We conjecture that the optimal dividend strategy is a barrier strategy b_e^*. Furthermore, due to the fact that our objective function (3.3) is penalised by capital injections, and these capital injections are discounted for time, we conjecture that the optimal capital injection strategy is to issue the minimum amount of capital, and to delay the injection of capital for as long as possible. We will then consider a strategy that only injects capital when the surplus process $\{X_{\pi_e}(t)\}$ hits the level of zero.

We construct our candidate solution to satisfy $\eta - H'(x) = 0$ above the barrier, and $(\mathcal{A} - \delta) H(x) = 0$ below the barrier, which yields

$$H(x) := \begin{cases} H(x; b_e^*) & x \in [0, b_e^*]; \text{ and} \\ \eta(x - b_e^*) + H(b_e^*; b_e^*) & x \in (b_e^*, \infty), \end{cases} \tag{3.13}$$

where we define

$$H(x; b_e^*) := \sum_{k=0}^{n+1} C_k(b_e^*)\, e^{r_k x}, \; x \geq 0. \tag{3.14}$$

Here, the r_k's remain the solutions of (2.16). The $C_k(b_e^*)$'s and the optimal barrier b_e^* have to satisfy the following conditions:

$$H'(0; b_e^*) = \sum_{k=0}^{n+1} r_k C_k(b_e^*) = \kappa \tag{3.15}$$

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$$H'(b_e^*-;\, b_e^*) \;=\; \sum_{k=0}^{n+1} r_k\, C_k(b_e^*)\, e^{r_k b_e^*} = \eta \tag{3.16}$$

$$H''(b_e^*-;\, b_e^*) \;=\; \sum_{k=0}^{n+1} r_k^2\, C_k(b_e^*)\, e^{r_k b_e^*} = 0 \tag{3.17}$$

$$\sum_{k=0}^{n+1} \frac{r_k \beta_i}{\beta_i - r_k}\, C_k(b_e^*)\, e^{r_k b_e^*} = \eta,\ \text{for}\ \ i = 1, 2, \ldots, n. \tag{3.18}$$

Condition (3.15) is the boundary condition of the HJB equation. Conditions (3.16)-(3.18) are obtained by analogous reasoning to Section 2.3.

As (3.16)-(3.18) are identical to (2.19)-(2.21), with b_e^* substituted for b_d^*, it follows that

$$C_k(b_e^*) \;=\; -\eta\, \frac{e^{-r_k b_e^*}}{r_k} \prod_{\substack{j=0\\ j\neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i} \tag{3.19}$$

for $k = 0, 1, \ldots, n+1$ and all $b_e^* > 0$. The optimal barrier b_e^* is then determined by (3.15), as explained in the following section. We have then

$$H(x; b_e^*) \;:=\; -\eta \sum_{k=0}^{n+1} \frac{e^{-r_k b_e^*}}{r_k} \prod_{\substack{j=0\\ j\neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i}\, e^{r_k x},\ \ x \geq 0, \tag{3.20}$$

where b_e^* is determined by (3.15), which can be rewritten as

$$-\sum_{k=0}^{n+1} e^{-r_k b_e^*} \prod_{\substack{j=0\\ j\neq k}}^{n+1} \frac{r_j}{r_k - r_j} \prod_{i=1}^{n} \frac{r_k - \beta_i}{\beta_i} \;=\; \frac{\kappa}{\eta}. \tag{3.21}$$

Remark 3.1. *Note that in this problem $H(0; b_e^*)$ is no longer zero because equity is issued to prevent ruin. Given the initial surplus of zero, if the present value of the total costs of injecting future capital outweighs the present value of the dividends distributed in the future then $H(0; b_e^*)$ will be negative. Since b_d^* is defined to be the unique positive solution to the equation $G(0; b_d^*) = 0$, and $G(\cdot;\, b_d^*)$ and $H(\cdot;\, b_e^*)$ have the same form, it follows that $H(0;\, b_e^*) = 0$ if and only if $b_d^* = b_e^*$.*

3.3. The optimal dividend barrier b_e^*

Now that we have determined the form of the $C_k(b_e^*)$, we show that there is a unique value of b_e^* that solves (3.15) in conjunction with (3.19). Using the function χ as defined in (2.33), we define the related function

$$\chi_1(b_e^*) \;:=\; -\chi'(b_e^*) \;=\; \sum_{k=0}^{n+1} r_k\, C_k(b_e^*). \tag{3.22}$$

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We want to show that there is a unique solution to (3.15), which is equivalent to

$$\chi_1(b_e^*) = \kappa. \tag{3.23}$$

We first note that

$$\chi_1'(0) = H''(b_e^*-;\, b_e^*) = 0 \tag{3.24}$$

because of (3.17), and that

$$\chi_1''(b_e^*) = \sum_{k=0}^{n+1} r_k^3 C_k(b_e^*) > 0, \tag{3.25}$$

because r_k and $C_k(\cdot)$ have the same sign for all k; see (2.17), (2.27) and (2.28). Hence, χ_1 is an (increasingly) increasing function in b_e^*. Since

$$\chi_1(0) = H'(b_e^*-;\, b_e^*) = \eta \tag{3.26}$$

and since

$$\lim_{b_e^* \to \infty} \chi_1(b_e^*) = \infty \tag{3.27}$$

it follows from (3.8) that there exists a unique non-negative solution to (3.15) that is independent of the initial surplus x. Furthermore, this holds for any μ (positive, null or negative).

Remark 3.2. *Note that $b_e^* = 0$ if and only if $\eta = \kappa = 1$, and that in this case the value function is $H(x) = x + \mu/\delta$. That is, if there are no proportional transaction costs on dividend distributions or capital injections, then the optimal strategy is to pay out all of the surplus as a dividend, and to offset all future surplus cash flows by dividends or capital injections (with present value μ/δ). As these are not penalised, there is no benefit in holding any surplus.*

Remark 3.3. *Equation* (3.21) *shows that the optimal barrier b_e^* is now dependent on η, which is not the case when only dividends are considered; see Remark 2.2. However, as the r_k's are independent of η and κ, only the ratio of κ to η matters.*

3.4. Verification of all the conditions of the HJB equation

By construction, our candidate solution satisfies $H'(x) = \eta$ for $x \in [b_e^*, \infty)$ and $(\mathcal{A} - \delta)\, H(x) = 0$ for $x \in [0, b_e^*]$. Hence, it only remains to show that

$$(\mathcal{A} - \delta)\, H(x) \le 0,\ x > b_e^*, \tag{3.28}$$

$$H'(x) \ge \eta,\ 0 \le x < b_e^*,\ \text{and} \tag{3.29}$$

$$H'(x) \le \kappa,\ x \ge 0. \tag{3.30}$$

The proof of (3.28) is similar to the one developed in Section 2.4.

Considering (3.13) with Conditions (3.15) and (3.16) implies that $H'(x)$ goes from κ to η as x goes from 0 to b_e^*, and then stays equal to η for $x \geq b_e^*$. Since $\kappa \geq \eta$, in order to show that (3.29) and (3.30) hold, it suffices to show that $H'(x)$ decreases monotonically over $0 \leq x < b_e^*$. This follows from $H''(b_e^*) = 0$ because of Condition (3.17) and from the observation that

$$H'''(x) = \sum_{k=0}^{n+1} r_k^3 C_k(b_e^*) e^{r_k x} > 0, \quad 0 \leq x < b_e^*.$$

Remark 3.4. *The observation that $H'''(x) > 0$ for $0 \leq x \leq b_e^*$ allows us to deduce the concavity of the value function. An alternative proof of the concavity for general jump distributions is also provided in Appendix B. Unfortunately, this proof does not hold when ruin is allowed, hence the need to explicitly determine the sign of $G''(x)$ in Sections 2 and 4.*

3.5. Verification lemma

We use the following verification lemma to prove that in the case when ruin is not allowed, the optimal joint dividend and capital injection strategy is to distribute dividends according to a barrier strategy, and to inject capital only when the surplus reaches the level of zero. This verification lemma extends the lemma from Section 2.5 by introducing capital injections.

Lemma 3.1. *If function $H \in \mathbb{C}^1(\mathbb{R}_+)$ is also twice continuously differentiable except at countably many points and satisfies*

1. $(\mathcal{A} - \delta) H(x) \leq 0$, $x \geq 0$,
2. $H''(x) \leq 0$, $x \geq 0$,
3. $\eta \leq H'(x) \leq \kappa$, $x \geq 0$,

then

$$H(x) \geq V(x; \pi_e^*), \; x \geq 0. \tag{3.31}$$

Moreover, if there exists a point $b_e^* \in \mathbb{R}_+$ such that $H \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{b_e^*\})$ with

4. $(\mathcal{A} - \delta) H(x) = 0$, $H'(x) \geq \eta$ *for* $x \in [0, b_e^*]$,
5. $(\mathcal{A} - \delta) H(x) < 0$, $H(x) = \eta(x - b_e^*) + H(b_e^*)$ *for* $x \in (b_e^*, \infty)$,

in which the integro-differential operator $\mathcal{A}$ is defined by (2.6), *and*

6. $H'(0) = \kappa$,

then

$$H(x) = V(x; \pi_e^*) \;\; x \in \mathbb{R}_+, \tag{3.32}$$

and the joint strategy

$$dD_{\pi_e}(t) = (X_{\pi_e}(t-) - b_e^*)\,\mathbb{1}_{\{X_{\pi_e}(t-) > b_e^*\}} + dL_{X_{\pi_e}}^{b_e^*}(t),\ t \geq 0, \tag{3.33}$$

and

$$E_{\pi_e}(t) = L_{X_{\pi_e}}^0(t),\ t \geq 0, \tag{3.34}$$

is optimal, where

$$L_{X_{\pi_e}}^{b_e^*}(t) = \int_0^t \mathbb{1}_{\{X_{\pi_e}(s) = b_e^*\}}\, dL_{X_{\pi_e}}^{b_e^*}(s),\ t \geq 0, \tag{3.35}$$

is the local time of the process X at the barrier b_e^*, representing dividends due to oscillations of the Brownian Motion when the surplus is at the barrier,

$$(X_{\pi_e}(t-) - b_e^*)\,\mathbb{1}_{\{X_{\pi_e}(t-) > b_e^*\}} \tag{3.36}$$

represents the dividend paid at time t if the surplus process jumps above the barrier, and

$$L_{X_{\pi_e}}^0(t) = \int_0^t \mathbb{1}_{\{X_{\pi_e}(s) = 0\}}\, dL_{X_{\pi_e}}^0(s),\ t \geq 0, \tag{3.37}$$

represents capital injected when the surplus is at the level of zero.

A proof is discussed in Appendix A.

4. The optimal joint dividend and capital injection strategy

In this section we consider the general optimal control problem as defined in Section 1.3. Since there are now no restrictions on capital injections, the surplus may become negative. The time of ruin for a given control strategy π is then defined as

$$\tau_\pi := \inf\{t : X_\pi(t-) < 0\}. \tag{4.1}$$

Note the strict inequality, which is required because of the capital injections. In fact, it is possible that $\tau_\pi = \infty$.

We consider the value function (1.8). Since $V(x; \pi^*)$ is the optimal strategy from the unrestricted set of admissible strategies Π, it follows that we must have

$$V(x; \pi^*) \geq \max\{V(x; \pi_d^*), V(x; \pi_e^*)\}, \tag{4.2}$$

where $V(x; \pi_d^*)$ and $V(x; \pi_e^*)$ are defined as in equations (2.4) and (3.4). In this section, we determine $V(x; \pi^*)$ and the optimal strategy π^*.

4.1. HJB equation and verification lemma

We first use the following verification lemma to prove the optimality of any concave solution of the HJB equation

$$\max\{(\mathcal{A}-\delta)V(x),\ \eta - V'(x), V'(x)-\kappa\} = 0 \text{ with } \max\{-V(0), V'(0)-\kappa\} = 0. \tag{4.3}$$

The boundary conditions are explained as follows. If $V'(0) > \kappa$ then capital is injected up to a level α such that $V'(\alpha) = \kappa$. This does not make sense because if capital is injected, ruin does not happen and then it is useless to keep the surplus at a higher level than 0. We restrict then $V'(0) \leq \kappa$. However, if $V'(0) = \kappa$ then capital is injected when the surplus is null to prevent ruin. This can only make sense if $V(0) \geq 0$. Otherwise, the expected present value of capital injections would be higher than that of the dividends, and the company would then never choose to inject capital, which leads to a contradiction.

Lemma 4.1. *If non-negative function $V \in \mathbb{C}^1(\mathbb{R}_+)$ is also twice continuously differentiable except at countably many points and satisfies*

1. $(\mathcal{A}-\delta)V(x) \leq 0,\ x \geq 0,$
2. $V''(x) \leq 0,\ x \geq 0,$
3. $\eta \leq V'(x) \leq \kappa,\ x \geq 0,$

then

$$V(x) \geq V(x;\pi^*),\ \ x \geq 0. \tag{4.4}$$

A proof is discussed in Appendix A.

4.2. Characterisation of the optimal strategy

In this section we characterise the optimal strategy to maximise $J(x;\pi)$ and show how it depends on the drift μ and the relationship between the barriers b_d^* and b_e^* determined in the previous sections.

Theorem 4.2. *Let $\{X_\pi(t)\}$, π, π^* and $V(x;\pi^*)$ be as defined in Section 1, and let μ be as in* (2.22)*. Furthermore, π_d^*, b_d^*, π_e^* and b_e^* are the optimal strategies and associated optimal dividend barriers as developed in Sections 2 and 3, respectively. The optimal joint dividend and capital injection strategy π^* is then characterised as follows:*

$$\pi^* = \pi_d^* \quad \text{if } \mu \leq 0, \tag{4.5}$$

$$\pi^* = \pi_d^* \quad \text{if } \mu > 0 \text{ and } b_e^* > b_d^*, \tag{4.6}$$

$$\pi^* = \pi_e^* \quad \text{if } \mu > 0 \text{ and } b_e^* < b_d^*, \text{ and} \tag{4.7}$$

$$\pi^* = \pi_d^* \text{ or } \pi_e^* \quad \text{if } \mu > 0 \text{ and } b_e^* = b_d^*. \tag{4.8}$$

In the next four sections, we provide a proof of Theorem 4.2 by showing (4.5)-(4.8) sequentially.

4.2.1. Proof of (4.5)

From Section 2.6 we see that $V(x;\pi_d^*) = \eta x$, and $V(x;\pi^*) \geq V(x;\pi_d^*)$ because of equation (4.2). If we can show that $V(x;\pi^*) \leq V(x;\pi_d^*)$, then we have proved that the optimal strategy is to use a barrier of zero. In order to do this, we need to verify that $V(x;\pi_d^*)$ satisfies the conditions of the HJB equation (4.3).

We have previously shown that

$$\max\left\{(\mathcal{A}-\delta)\,V(x;\pi_d^*),\ \eta - V'(x;\pi_d^*)\right\} = 0, \tag{4.9}$$

so it remains to show that

$$\max\{V'(x;\pi_d^*) - \kappa\} = 0 \text{ with } \max\{-V(0;\pi_d^*), V'(0;\pi_d^*) - \kappa\} = 0. \tag{4.10}$$

We have

$$V'(x;\pi_d^*) = \eta \leq \kappa,\ x \geq 0, \tag{4.11}$$

which also means that

$$V'(0;\pi_d^*) - \kappa = \eta - \kappa \leq 0. \tag{4.12}$$

In addition,

$$-V(0;\pi_d^*) = -\eta\cdot 0 = 0, \tag{4.13}$$

which completes the proof.

4.2.2. Proof of (4.6)

From Lemma 2.1 we see that $G(x) = V(x;\pi_d^*)$, and $V(x;\pi^*) \geq G(x)$ because of equation (4.2). If we can show that $V(x;\pi^*) \leq G(x)$ for $b_d^* \leq b_e^*$, then it follows that the optimal joint dividend and capital injection strategy is to use a barrier of b_d^* to distribute dividends, and to issue no capital.

In order to do this, we need to verify that $G(x)$ satisfies the conditions of the HJB equation (4.3). By construction, $G(0) = 0$; see (2.18). It remains thus to show that

$$G'(x) \leq \kappa,\ x \geq 0. \tag{4.14}$$

In Section 2.4, we showed that $G''(x) < 0$ for $x \in [0, b_d^*)$, and since G is linear on $[b_d^*, \infty)$, it follows that $G(x)$ is concave. Hence, (4.14) holds if and only if $G'(0) \leq \kappa$, which follows from

$$\chi_1(b_d^*) = \sum_{k=0}^{n+1} r_k C_k(b_d^*) = G'(0) \le \chi_1(b_e^*) = \sum_{k=0}^{n+1} r_k C_k(b_e^*) = H'(0) = \kappa \quad (4.15)$$

because $\chi_1(\xi)$ is an increasing function; see Section 3.3.

Due to this result and Lemma 2.1, $G(x)$ satisfies the conditions of Lemma 4.1. Hence $G(x) \ge V(x; \pi^*)$ so that $V(x; \pi^*) = V(x; \pi_d^*)$, which completes the proof.

4.2.3. Proof of (4.7)

From Lemma 3.1 we see that $H(x) = V(x; \pi_e^*)$ and $V(x; \pi^*) \ge H(x)$ due to equation (4.2), so it is sufficient to show that $V(x; \pi^*) \le H(x)$ if $b_e^* \le b_d^*$. As above, we wish to verify that $H(x)$ satisfies the boundary conditions in HJB equation (4.3), and proceed in a similar way. Due to Lemma 3.1, all conditions of the HJB equation (4.3) have been confirmed except for $H(0) \ge 0$. This follows from

$$\chi(b_e^*) = \sum_{k=0}^{n+1} C_k(b_e^*) = H(0) \ge \chi(b_d^*) = \sum_{k=0}^{n+1} C_k(b_d^*) = G(0) = 0 \quad (4.16)$$

because $\chi(\xi)$ is a decreasing function; see Section 2.3.2.

Due to this result and Lemma 3.1, $H(x)$ satisfies the conditions of Lemma 4.1. Hence, $H(x) \ge V(x; \pi^*)$ so that $V(x; \pi^*) = V(x; \pi_e^*)$, which completes the proof.

4.2.4. Proof of (4.8)

Because of equation (4.2), $V(x; \pi^*) \ge \max\{G(x), H(x)\}$. Furthermore, it follows from the proofs of (4.6) and (4.7) that

$$G(x) \ge V(x; \pi^*) \iff b_d^* \le b_e^*, \text{ and} \quad (4.17)$$

$$H(x) \ge V(x; \pi^*) \iff b_e^* \le b_d^*. \quad (4.18)$$

But

$$b_d^* = b_e^* \iff H(x) = G(x); \quad (4.19)$$

see Remark 3.1. Hence, $V(x; \pi^*) = V(x; \pi_d^*) = V(x; \pi_e^*)$, which completes the proof. Note that this means that when the surplus hits 0, management will be indifferent between injecting capital to rescue the business and stopping the business.

Remark 4.1. *There are two alternative representations to the conditions on b_e^* and b_d^* in Theorem 4.2. From* (4.15) *and* (4.16) *it follows that*

$$b_e^* < b_d^* \iff G'(0; b_e^*) > \kappa = H'(0; b_d^*) \iff H(0; b_e^*) > 0 = G(0; b_d^*), \quad (4.20)$$

and vice versa. This is interpreted using similar arguments to the ones developed to explain the conditions in (4.3). *If* (4.20) *holds, then capital injections are profitable for low levels of surplus (because* $G'(0) > \kappa$*), which results in* $H(0) > 0$ *and* $\pi^* = \pi_e^*$*. Conversely, if* $G'(0) < \kappa$ *then capital will never be injected and* $H(0) < 0$*, so that* $\pi^* = \pi_d^*$*.*

Remark 4.2. *Injecting capital can be considered as a real option (see, for instance, Dixit and Pindyck, 1994). This option has an aggregate positive value equal to* $H(x) - G(x)$ *when* (4.20) *is satisfied.*

Remark 4.3. *Gerber and Shiu (2006) consider the merger of two companies when their surplus is a pure diffusion. There, merger is considered as profitable when*

$$W(x_1 + x_2; b_m^*) > W(x_1; b_1^*) + W(x_2; b_2^*), \tag{4.21}$$

where $W(x; b)$ *is the expected present value of dividends until ruin when a barrier strategy* b *is applied, where* x_i *and* b_i^* *are the initial surplus and optimal barrier of company* i *(*$i = 1, 2$*), respectively, and where* b_m^* *is the optimal barrier of the merged surpluses. This work gives rise to two remarks.*

Firstly, this approach can easily be extended to the dual model with diffusion as the sum of two (independent) compound Poisson processes with mixture of exponential jumps is compound Poisson with mixture of exponential jumps again, as dependence can still be modeled between the two diffusion components. Numerical calculations indicate that capital injections are more likely to be optimal for lower levels of dependence.

Secondly, merger can be seen as a 'cheap' way of injecting capital, as the aggregation of the surpluses is not penalised by (proportional) transaction costs. However, the level of the barrier b_m^* *is likely to change, resulting in an indeterminate net profit. On the other hand, if one of the companies is comparatively small then its impact on the optimal barrier will be negligible, and the merger will be more profitable as the surplus is lower (since* $W'(x; b) > 1$ *and decreasing for* $x < b$*). Note also that in practice, a merger would attract transaction costs, but inclusion of these is trivial as they only need to be subtracted from* $x_1 + x_2$ *on the left-hand side of* (4.21).

5. Numerical illustrations

5.1. The choice between π_d and π_e

Let $c = 0.2$, $\delta = 0.08$, $\lambda = 1$, $\sigma = 5$ and

$$p(y) = 0.5\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) + 0.5\left(2e^{-2y}\right)$$

such that $\mu = 0.8 > 0$.

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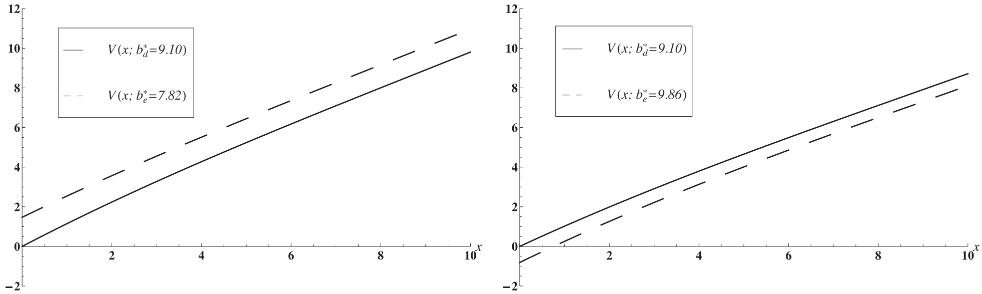

FIGURE 1: Value functions when $\eta = 0.9$ and $\kappa = 1.1$ on the left and when $\eta = 0.8$ and $\kappa = 1.1$ on the right.

We first consider $\eta = 0.9$ and $\kappa = 1.1$. In this case, $b_e^* = 7.8159 < b_d^* = 9.1045$, so it is optimal to inject capital and $V(x; b_e^*) > V(x; b_d^*)$ for all x. If we increase the transaction costs so $\eta = 0.8$ and $\kappa = 1.1$, then it is no longer optimal to inject capital, since $b_d^* = 9.1045 < b_e^* = 9.8606$. In this case, $V(x; b_d^*) > V(x; b_e^*)$ for all x. These two cases are shown in Figure 1. Note that this also illustrates Remark 3.1.

5.2. The effect of the drift

In this example, we consider the same parameters as in Section 5.1, using $\eta = 0.9$ and $\kappa = 1.1$, but vary the drift of the process by changing c in order to study its impact on the optimal strategy. Figure 2 shows two cases, when $\sigma = 0.5$ and $\sigma = 5$, respectively.

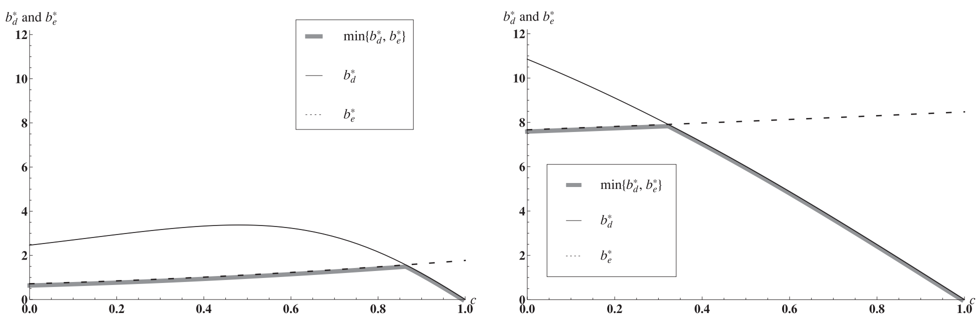

FIGURE 2: Optimal dividend barriers according to π_d^*, π_e^* and π^* when the drift changes, for $\sigma = 0.5$ and $\sigma = 5$.

The impact of the drift on b_e^* is monotone for all levels of volatility. As the drift decreases (c increases), this barrier increases slowly to try to avoid capital injections.

In contrast, μ has a mixed impact on the barrier b_d^*. There, two conflicting forces are at work. On one hand, a lower drift increases risk which calls for a higher barrier. On the other hand, when the drift gets closer to 0, it is better to distribute a greater proportion of the surplus that is available as a dividend because of bad prospects. In the limit $\mu = 0$ ($c = 1$), $b_d^* = 0$. In the case $\sigma = 5$, the second force dominates.

The optimal dividend barrier according to π^*, $\min\{b_e^*, b_d^*\}$, is shown in grey. We observe that injecting capital is in general better when the drift is high. As risk increases, the optimal strategy π^* switches from π_e^* to π_d^* for higher levels of drift.

5.3. The effect of the force of interest

We now consider the effects of a change in the force of interest. Increasing the force of interest decreases the value of dividends, but also decreases the cost of injecting capital. We plot the levels of the barriers for the mixture from Section 5.1 with parameters $\kappa = 1.1$, $\eta = 0.9$, $\lambda = 1$ and $c = 0.5$. We look at the cases when the Brownian motion volatility is $\sigma = 0.5$ and $\sigma = 5$ as the force of interest δ varies from 0 to 0.2.

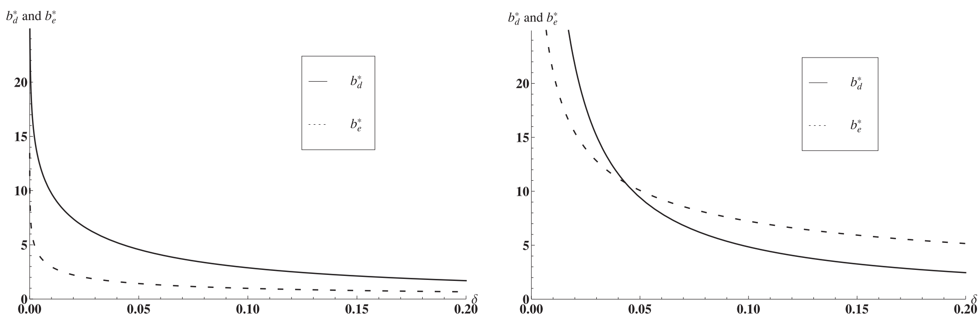

FIGURE 3: Sensitivity of the Optimal Barriers to changes in the Force of Interest δ, for $\sigma = 0.5$ and $\sigma = 5$.

The two graphs show that the relationship between b_d^* and b_e^* (as a function of δ) depends on the volatility of the surplus. If the volatility is 'low', then b_e^* seems to be always lower than b_d^* as δ changes. However, if the volatility is high, then as δ increases, the decreased value of dividends is not sufficient to justify further investments, particularly since the high volatility means that more capital will need to be injected, and the present value of the capital injections will far outweigh the present value of the dividends distributed.

ACKNOWLEDGMENTS

The authors acknowledge financial support of an Australian Actuarial Research Grant from the Institute of Actuaries of Australia. Jonathan Shen is indebted to Mr Edwin Blackadder for providing him with the EJ Blackadder Honours Scholarship. The authors are grateful to anonymous referees for helpful comments.

APPENDIX

A. Proofs of Lemmas 2.1, 3.1 and 4.1

This appendix details the proofs of Lemma 4.1 and the second section of Lemma 3.1. Similar approaches to the ones taken here can be used to prove the first sections of Lemmas 2.1 and 3.1, and the second section of 2.1, respectively.

We will first prove Lemma 4.1. The first sections of Lemma 2.1 and Lemma 3.1 can be proved by making the following modifications. For Lemma 2.1, set $\mathcal{E}_t$ as the empty set, and replace V with G. For Lemma 3.1, replace V with H and replace $t \wedge \tau_\pi$ with t.

Proof of Lemma 4.1. For a given strategy $\pi \in \Pi$, we define the following sets:

$$\mathcal{D}_t = \{s \leq t : D_\pi(s-) \neq D_\pi(s) \text{ and } S(s-) = S(s)\}; \tag{A.1}$$

$$\mathcal{E}_t = \{s \leq t : E_\pi(s-) \neq E_\pi(s) \text{ and } S(s-) = S(s)\}; \tag{A.2}$$

$$(\mathcal{D}\mathcal{E})_t = \mathcal{D}_t \cup \mathcal{E}_t. \tag{A.3}$$

That is, $\mathcal{D}_t$ is the set containing the jump times of the process $\{D_\pi(t)\}$ due to dividend distributions that do not occur at the same time as the jumps in the compound Poisson process, $\mathcal{E}_t$ is the set containing the jump times of the process $\{E_\pi(t)\}$ due to capital injections that do not occur at the same time as the jumps in the compound Poisson process, and $(\mathcal{D}\mathcal{E})_t$ is the set containing the times when the dividend and/or capital injection processes jump, but the compound Poisson process does not jump. Also, let $Z^{(c)}$ denote the continuous part of arbitrary process Z, defined as:

$$Z^{(c)}(t) := Z(t) - \sum_{s \leq t} [Z(s) - Z(s-)]. \tag{A.4}$$

By the Itô formula for jump-diffusion processes, we have

$$\begin{aligned} &e^{-\delta(t\wedge\tau_\pi-)} V(X_\pi(t\wedge\tau_\pi-)) \\ &\quad = V(x) - \int_{0-}^{t\wedge\tau_\pi-} \delta e^{-\delta s} V(X_\pi(s))\, ds + \int_{0-}^{t\wedge\tau_\pi-} e^{-\delta s} V'(X_\pi(s))\, dX_\pi^{(c)}(s) \\ &\qquad + \int_{0-}^{t\wedge\tau_\pi-} \frac{\sigma^2}{2} e^{-\delta s} V''(X_\pi(s))\, ds \\ &\qquad + \sum_{s \leq t\wedge\tau_\pi-}^{\Delta X_\pi(s)\neq 0} e^{-\delta s} [V(X_\pi(s-) + \Delta X_\pi(s)) - V(X_\pi(s-))]. \end{aligned} \tag{A.5}$$

The summation term in (A.5) represents changes due to jumps in the compound Poisson process $\{S(t)\}$, the aggregate dividend process $\{D_\pi(t)\}$ and the capital injection process $\{E_\pi(t)\}$. Using a similar approach as in Bayraktar and Egami (2008, with dividends only), we split these jumps into three categories:

1. Jumps in either or both the dividend process and the capital injection process, that do not occur at the same time as a jump in the compound Poisson process;
2. All jumps due to the compound Poisson process; and
3. The 'extra' jumps due to jumps in either or both the dividend process and capital injection process that occur at the same time as a jump in the compound Poisson process.

Thus, we can write the summation as

$$\begin{aligned}
&\sum_{s\le t\wedge\tau_\pi-}^{\Delta X_\pi(s)\neq 0} e^{-\delta s}[V(X_\pi(s-)+\Delta X_\pi(s)) - V(X_\pi(s-))] \\
&\quad = \sum_{s\in(\mathcal{DE})_{t\wedge\tau_\pi-}} e^{-\delta s}[V(X_\pi(s)) - V(X_\pi(s-))] \\
&\qquad + \int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s-)+y) - V(X_\pi(s-))]\,N(ds,dy) \\
&\qquad + \int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s)) - V(X_\pi(s-)+y)]\,N(ds,dy).
\end{aligned} \tag{A.6}$$

Noting that $X_\pi^{(c)}$, the continuous part of X_π, satisfies

$$dX_\pi^{(c)}(t) = -c dt + \sigma dW(t) - dD_\pi^{(c)}(t) + dE_\pi^{(c)}(t), \tag{A.7}$$

and expressing the first integral in (A.6) with the 'compensated' jump measure, we can write (A.5) as:

$$\begin{aligned}
&e^{-\delta(t\wedge\tau_\pi-)}V(X_\pi(t\wedge\tau_\pi-)) \\
&\quad = V(x) + \int_{0-}^{t\wedge\tau_\pi-} e^{-\delta s}(\mathcal{A}-\delta)V(X_\pi(s))\,ds + \int_{0-}^{t\wedge\tau_\pi-}\sigma e^{-\delta s}V'(X_\pi(s))\,dW(s) \\
&\qquad - \int_{0-}^{t} e^{-\delta s}V'(X_\pi(s))\,dD_\pi^{(c)}(s) + \int_{0-}^{t\wedge\tau_\pi-} e^{-\delta s}V'(X_\pi(s))\,dE_\pi^{(c)}(s) \\
&\qquad + \sum_{s\in(\mathcal{DE})_{t\wedge\tau_\pi-}} e^{-\delta s}[V(X_\pi(s)) - V(X_\pi(s-))] \\
&\qquad + \int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s-)+y) - V(X_\pi(s-))](N(ds,dy) - v(ds,dy)) \\
&\qquad + \int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s)) - V(X_\pi(s-)+y)]N(ds,dy).
\end{aligned} \tag{A.8}$$

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Rewriting $dD_\pi^{(c)}(s)$ and $dE_\pi^{(c)}(s)$ using the decomposition as in (A.4) yields

$$\begin{aligned}
&e^{-\delta(t\wedge\tau_\pi-)}V(X_\pi(t\wedge\tau_\pi-))\\
&\quad= V(x)+\int_{0-}^{t\wedge\tau_\pi-}e^{-\delta s}(\mathcal{A}-\delta)V(X_\pi(s))ds+\int_{0-}^{t\wedge\tau_\pi-}\sigma e^{-\delta s}V'(X_\pi(s))dW(s)\\
&\qquad-\int_{0-}^{t\wedge\tau_\pi-}\eta e^{-\delta s}dD_\pi(s)+\int_{0-}^{t\wedge\tau_\pi-}e^{-\delta s}[\eta-V'(X_\pi(s))]dD_\pi(s)\\
&\qquad+\int_{0-}^{t\wedge\tau_\pi-}\kappa e^{-\delta s}dE_\pi(s)+\int_{0-}^{t\wedge\tau_\pi-}e^{-\delta s}[V'(X_\pi(s))-\kappa]dE_\pi(s)\\
&\qquad+\sum_{s\in(\mathcal{DE})_{t\wedge\tau_\pi-}}e^{-\delta s}[V(X_\pi(s))-V(X_\pi(s-))-[X_\pi(s)-X_\pi(s-)]V'(X_\pi(s-))]\\
&\qquad+\int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s-)+y)-V(X_\pi(s-))](N(ds,dy)-v(ds,dy))\\
&\qquad+\int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[V(X_\pi(s))-V(X_\pi(s-)+y)]N(ds,dy)\\
&\qquad-\int_{0-}^{t\wedge\tau_\pi-}\int_0^\infty e^{-\delta s}[X_\pi(s)-X_\pi(s-)-y]V'(X_\pi(s-)+y)N(ds,dy).
\end{aligned}\tag{A.9}$$

We note that V is a concave function due to point 2 of the verification lemma, and in conjunction with point 3, we have $\eta\le V'(X_\pi(t))\le\kappa$ so that the stochastic integral with respect to the Brownian motion in (A.9) is a uniformly integrable martingale, because $\delta>0$ and $V'(x)$ is bounded. In addition, $(\mathcal{A}-\delta)V(x)\le 0$, $(\eta-V'(X_\pi(s))\le 0$ and $(V'(X_\pi(s)-\kappa)\le 0$ due to points 1 and 3 of the verification lemma, and combining $V''(x)\le 0$ from point 2 of the verification lemma with the Mean Value Theorem, we have

$$V(y)-V(x)-(y-x)V'(x)\le 0 \text{ for } y\ge x;\text{ and}$$
$$V(x)-V(y)-(x-y)V'(y)\le 0 \text{ for } y\ge x.$$

After applying all of these results to (A.9), taking expectations and rearranging, it follows that

$$\begin{aligned}
V(x)\ \ge\ &\mathbb{E}^x\Big[e^{-\delta(t\wedge\tau_\pi-)}V(X_\pi(t\wedge\tau_\pi-))\Big]+\mathbb{E}^x\Big[\int_{0-}^{t\wedge\tau_\pi-}\eta e^{-\delta s}dD_\pi(s)\Big]\\
&-\mathbb{E}^x\Big[\int_{0-}^{t\wedge\tau_\pi-}\kappa e^{-\delta s}dE_\pi(s)\Big].
\end{aligned}\tag{A.10}$$

Using the fact that $V(0)\ge 0$ and conditioning on the value of τ_π, we can then write

$$\begin{aligned}
&\liminf_{t\to\infty} e^{-\delta(t\wedge\tau_\pi-)} V(X_\pi(t\wedge\tau_\pi-)) \\
&\quad = \liminf_{t\to\infty} e^{-\delta(t\wedge\tau_\pi-)} V(X_\pi(t\wedge\tau_\pi-))\, \mathbb{1}_{\{\tau_\pi-<\infty\}} \\
&\qquad + \liminf_{t\to\infty} e^{-\delta(t\wedge\tau_\pi-)} V(X_\pi(t\wedge\tau_\pi-))\, \mathbb{1}_{\{\tau_\pi-=\infty\}} \\
&\quad = e^{-\delta(\tau_\pi-)} V(X_\pi(\tau_\pi-))\, \mathbb{1}_{\{\tau_\pi-<\infty\}} + \liminf_{t\to\infty} e^{-\delta t} V(X_\pi(t))\, \mathbb{1}_{\{\tau_\pi-=\infty\}} \\
&\quad \geq e^{-\delta(\tau_\pi-)} V(0)\, \mathbb{1}_{\{\tau_\pi-<\infty\}} \geq 0.
\end{aligned}$$

Finally, taking limits as $t\to\infty$ in (A.10), we find

$$V(x) \geq \mathbb{E}^x\left[\limsup_{t\to\infty}\left(\eta\int_{0-}^{t\wedge\tau_\pi-} e^{-\delta s}\, dD_\pi(s) - \kappa\int_{0-}^{t\wedge\tau_\pi-} e^{-\delta s}\, dE_\pi(s)\right)\right] = J(x;\pi). \tag{A.11}$$

Since the strategy π is arbitrary, it follows that

$$V(x) \geq V(x;\pi^*). \tag{A.12}$$

□

We now proceed by proving the second section of Lemma 3.1. A similar approach to the one taken here can be used to prove the second section of Lemma 2.1 by setting $\mathcal{E}_t$ as the empty set, replacing H with G, replacing π_e with π_d, replacing b_e^* with b_d^* and replacing t with $(t\wedge\tau_{\pi_d}-)$ in the upper limits of the integrals.

Proof. We know that $(\mathcal{A}-\delta)H(X_{\pi_e}(s))\equiv 0$ because $0\leq X_{\pi_e}(s)\leq b_e^*$. After taking expectations on both sides of (A.9), replacing V with H, replacing $t\wedge\tau_\pi$ with t and using points 4 and 5 of Lemma 3.1, we can write

$$\begin{aligned}
&\mathbb{E}^x\left[e^{-\delta t} H(X_{\pi_e}(t))\right] \\
&= H(x) - \mathbb{E}^x\left[\int_{0-}^{t} e^{-\delta s} H'(X_{\pi_e}(s))\, dD_{\pi_e}(s)\right] \\
&\quad + \mathbb{E}^x\left[\int_{0-}^{t} e^{-\delta s} H'(X_{\pi_e}(s))\, dE_{\pi_e}(s)\right] \\
&\quad + \mathbb{E}^x\left[\sum_{s\in(\mathcal{DE})_t} e^{-\delta s}\left[H(X_{\pi_e}(s)) - H(X_{\pi_e}(s-)) - (X_{\pi_e}(s) - X_{\pi_e}(s-))H'(X_{\pi_e}(s-))\right]\right] \\
&\quad + \mathbb{E}^x\left[\int_{0-}^{t}\int_0^\infty e^{-\delta s}[H(X_{\pi_e}(s)) - H(X_{\pi_e}(s-)+y)]\, N(ds,dy)\right] \\
&\quad - \mathbb{E}^x\left[\int_{0-}^{t}\int_0^\infty e^{-\delta s}[X_{\pi_e}(s) - X_{\pi_e}(s-) - y]\, H'(X_{\pi_e}(s-)+y)\, N(ds,dy)\right].
\end{aligned} \tag{A.13}$$

The last three terms are zero due to the definition of the proposed joint dividend and equity strategy. By construction there are no jumps in the equity process. Hence, the only jumps in the process $\{X_{\pi_e}(t)\}$ occur due to jumps in the dividend and/or compound Poisson process. The summation term in (A.13) is summing over all points in time when there is a jump in the dividend process, but no jump in the compound Poisson process. This can only possibly occur at time zero, if the initial surplus x is larger than the barrier b_e^*, so we have

$$X_{\pi_e}(0-) = x,\ X_{\pi_e}(0) = b_e^*,\ H'(X_{\pi_e}(0-)) = \eta,$$
$$H(X_{\pi_e}(0-)) = \eta(x - b_e^*) + H(b_e^*) \text{ and } H(X_{\pi_e}(0)) = H(b_e^*),$$

from which it follows that the summation term is zero. The last two integral terms in (A.13) apply to the points in time when there is a jump in both the dividend process and the compound Poisson process. At these points, the surplus rises above the barrier, and the value function is linear, so we have

$$X_{\pi_e}(s) = b_e^*,\ H(X_{\pi_e}(s)) = H(b_e^*),\ H'(X_{\pi_e}(s-) + y) = \eta, \text{ and}$$
$$H(X_{\pi_e}(s-) + y) = \eta(X_{\pi_e}(s-) + y - b_e^*) + H(b_e^*),$$

so the sum of the last two integral terms is zero.

It follows that (A.13) simplifies to

$$\begin{aligned}\mathbb{E}^x\left[e^{-\delta t}H(X_{\pi_e}(t))\right] = H(x) - \mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\,dD_{\pi_e}(s)\right] \\ + \mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\,dE_{\pi_e}(s)\right].\end{aligned} \tag{A.14}$$

Due to the definition of the proposed dividend strategy we can write

$$\begin{aligned}\mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\,dD_{\pi_e}(s)\right] &= \mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\mathbb{1}_{\{X_{\pi_e}(t)\geq b_e^*\}}\,dD_{\pi_e}(s)\right] \\ &= \mathbb{E}^x\left[\eta\int_{0-}^t e^{-\delta s}\,dD_{\pi_e}(s)\right].\end{aligned} \tag{A.15}$$

Similarly, using point 6 of the verification lemma,

$$\begin{aligned}\mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\,dE_{\pi_e}(s)\right] &= \mathbb{E}^x\left[\int_{0-}^t e^{-\delta s}H'(X_{\pi_e}(s))\mathbb{1}_{\{X_{\pi_e}(t)=0\}}\,dE_{\pi_e}(s)\right] \\ &= \mathbb{E}^x\left[\kappa\int_{0-}^t e^{-\delta s}\,dE_{\pi_e}(s)\right].\end{aligned} \tag{A.16}$$

Substituting these into (A.14), rearranging and letting $t \to \infty$, it follows that

$$H(x) = \mathbb{E}^x\left[\limsup_{t\to\infty}\left(\eta\int_{0-}^t e^{-\delta s}\,dD_{\pi_e}(s) - \kappa\int_{0-}^t e^{-\delta s}\,dE_{\pi_e}(s)\right)\right]. \tag{A.17}$$

□

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B. Proof of Concavity of Value Function

The following proof is an adaptation to the dual model with diffusion of the proof of concavity provided in Kulenko and Schmidli (2008, in the Cramér-Lundberg model). This proof holds for any jump distribution, albeit only when ruin is guaranteed not to occur.

Consider two surplus processes $Y(t)$ and $Z(t)$ of type (1.1) with identical parameters but for their initial surpluses $y \geq 0$ and $z \geq 0$, respectively. Furthermore, consider the admissible strategies $\pi_{e,y} = (D_{\pi_{e,y}}, E_{\pi_{e,y}})$, $\pi_{e,z} = (D_{\pi_{e,z}}, E_{\pi_{e,z}}) \in \Pi_e$ and let $\alpha_y, \alpha_z \in (0, 1)$ with $\alpha_y + \alpha_z = 1$. Define

$$D_{\pi_{e,w}}(t) := \alpha_y D_{\pi_{e,y}}(t) + \alpha_z D_{\pi_{e,z}}(t), \text{ and} \tag{B.1}$$

$$E_{\pi_{e,w}}(t) := \alpha_y E_{\pi_{e,y}}(t) + \alpha_z E_{\pi_{e,z}}(t). \tag{B.2}$$

Note that $D_{\pi_{e,w}}(t) = D_{\pi_{e,\alpha_y y+\alpha_z z}}(t)$ but that in general $E_{\pi_{e,w}}(t) \neq E_{\pi_{e,\alpha_y y+\alpha_z z}}(t)$. We have

$$\begin{aligned} &\alpha_y Y(t) + \alpha_z Z(t) - D_{\pi_{e,w}}(t) + E_{\pi_{e,w}}(t) = \\ &\alpha_y \underbrace{\left(Y(t) - D_{\pi_{e,y}}(t) + E_{\pi_{e,y}}(t)\right)}_{\geq 0} + \alpha_z \underbrace{\left(Z(t) - D_{\pi_{e,z}}(t) + E_{\pi_{e,z}}(t)\right)}_{\geq 0} \geq 0. \end{aligned} \tag{B.3}$$

Thus the strategy $\pi_w = (D_{\pi_{e,w}}, E_{\pi_{e,w}})$ is admissible. In addition, we must have $E_{\pi_{e,\alpha_y y+\alpha_z z}}(t) \leq \alpha_y E_{\pi_{e,y}}(t) + \alpha_z E_{\pi_{e,z}}(t)$, otherwise we reach the contradiction that the value of the strategy $(D_{\pi_{e,w}}, E_{\pi_{e,\alpha_y y+\alpha_z z}})$ is inferior to the value of the strategy $(D_{\pi_{e,w}}, E_{\pi_{e,w}})$. Then

$$\begin{aligned} &V(\alpha_y y + \alpha_z z; \pi_e^*) \\ &\geq \mathbb{E}\left[\limsup_{t\to\infty}\left(\eta \int_{0-}^{t} e^{-\delta s} dD_{\pi_{e,w}}(s) - \kappa \int_{0-}^{t} e^{-\delta s} dE_{\pi_{e,\alpha_y y+\alpha_z z}}(s)\right)\right] \\ &\geq \mathbb{E}\left[\limsup_{t\to\infty}\left(\eta \int_{0-}^{t} e^{-\delta s}\left(\alpha_y dD_{\pi_{e,y}}(s) + \alpha_z dD_{\pi_{e,z}}(s)\right) - \kappa \int_{0-}^{t} e^{-\delta s}\left(\alpha_y dE_{\pi_{e,y}}(s) + \alpha_z dE_{\pi_{e,z}}(s)\right)\right)\right] \\ &= \mathbb{E}\left[\limsup_{t\to\infty}\left(\alpha_y \int_{0-}^{t} e^{-\delta s}\left(\eta dD_{\pi_{e,y}}(s) - \kappa dE_{\pi_{e,y}}(s)\right) + \alpha_z \int_{0-}^{t} e^{-\delta s}\left(\eta dD_{\pi_{e,z}}(s) - \kappa dE_{\pi_{e,z}}(s)\right)\right)\right] \\ &= \alpha_y J(y; \pi_{e,y}) + \alpha_z J(z; \pi_{e,z}). \end{aligned} \tag{B.4}$$

Let $\Pi_{e,x}$ denote the set of admissible strategies for the initial capital x such that ruin is guaranteed not to occur. Taking the supremum over all admissible strategies we find

$$V(\alpha_y y + \alpha_z z; \pi_e^*) \geq \alpha_y \sup_{\pi_{e,y} \in \Pi_{e,y}} J(y; \pi_{e,y}) + \alpha_z \sup_{\pi_{e,z} \in \Pi_{e,z}} J(z; \pi_{e,z}) \tag{B.5}$$

$$= \alpha_y V(y; \pi_{e,y}^*) + \alpha_z V(z; \pi_{e,z}^*). \tag{B.6}$$

Remark B.1. *When ruin is allowed to occur (such as in Sections 2 and 4), the upper bounds in* (B.4) *become functions of* t, $\tau_{\pi_{e,w}}$, $\tau_{\pi_{e,y}}$ *and* $\tau_{\pi_{e,z}}$ *and the last inequality cannot be guaranteed any more.*

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