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RESEARCH ARTICLE

The nonclassical diffusion equations with time-dependent memory kernels and a new class of nonlinearities

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Abstract
In this study, we consider the nonclassical diffusion equations with time-dependent memory kernels

$$u_t - \Delta u_t - \Delta u - \int_0^\infty k_t'(s)\Delta u(t-s)ds + f(u) = g$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. Firstly, we study the existence and uniqueness of weak solutions and then, we investigate the existence of the time-dependent global attractors $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ in $H_0^1(\Omega) \times L^2_{\mu_t}(\mathbb{R}^+, H_0^1(\Omega))$. Finally, we prove that the asymptotic dynamics of our problem, when k_t approaches a multiple $m\delta_0$ of the Dirac mass at zero as $t \to \infty$, is close to the one of its formal limit

$$u_t - \Delta u_t - (1+m)\Delta u + f(u) = g.$$

The main novelty of our results is that no restriction on the upper growth of the nonlinearity is imposed and the memory kernel $k_t(\cdot)$ depends on time, which allows for instance to describe the dynamics of aging materials.

1. Introduction

In this study, we consider the following semilinear nonclassical diffusion equation with time-dependent memory

$$\begin{cases} \partial_t u - \partial_t \Delta u - \Delta u - \int_0^\infty k_t'(s)\Delta u(t-s)ds + f(u) = g, & x \in \Omega, t > \tau, \\ u(x,t) = 0, & x \in \partial\Omega, t > \tau, \\ u(x,\tau) = u_\tau(x), & x \in \Omega, \\ u(x,\tau - s) = \phi_\tau(x,s), & x \in \Omega, s > 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The first equation in (1.1) arises in the classical diffusion theory when assuming that the diffusing species behaves as a linear viscous fluid, which leads to include its velocity gradient in the constitutive laws [1, 22, 27]). In the past years, the existence and long-time behavior of solutions to nonclassical diffusion equations have been studied extensively, in both autonomous case [20, 24, 25, 28, 31, 32] and non-autonomous case [2–4, 25, 26, 33]. The time-dependent global attractor for the nonclassical diffusion equations was studied in [19, 21].

The convolution term takes into account the influence of the past history of u on its future evolution, providing a more accurate description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers (see e.g. [16]). In the past years, the existence and long-time behavior of solutions to nonclassical diffusion equations with memory have been explored in the case of the memory kernel independent on the time [5, 6, 9]. In particular, the existence of global attractors of weak solutions to a class of nonclassical diffusion equations with hereditary memory and nonlinear terms of exponential type has been studied in [5].

To the best of our knowledge, although there are several results on attractors for evolution equations with constant-in-time memory kernels, only M. Conti has studied time-dependent memory kernels [7]. In this study, we therefore build on the M. Conti's results by removing the technical conditions imposed on the memory kernels. We will prove the existence of weak solutions and existence of a time-dependent global attractor under a weak assumption on the time-dependent memory kernel $k_t(s)$ and a very large class of nonlinearities that particularly covers both above classes and even exponential nonlinearities.

When k_t approaches a multiple $m\delta_0$ of the Dirac mass at zero as $t \to \infty$, we prove that the asymptotic dynamics of our problem is close to the one of its formal limit

$$u_t - \Delta u_t - (1+m)\Delta u + f(u) = g.$$

This is the main novelty of our paper.

To study problem (1.1), we assume that the initial datum $u_\tau \in H_0^1(\Omega)$ is given, and that the nonlinearity f and the external force g satisfy the following conditions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \geq -\ell, \tag{1.2}$$

$$f(u)u \geq -\beta u^2 - C_0, \text{ for all } u \in \mathbb{R}, \tag{1.3}$$

where ℓ, β, C_0 are positive constants, $0 < \beta < \lambda_1$ with $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ in Ω with the homogeneous Dirichlet condition,

It follows from (1.2) that $0 \leq \int_0^u (f'(s)s + \ell s)ds$, and therefore by integrating by parts, we obtain

$$F(u) \leq f(u)u + \frac{\ell u^2}{2}, \text{ for all } u \in \mathbb{R}, \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$ is a primitive of f.

(H2) The external force $g \in H^{-1}(\Omega)$.

(H3) The convolution (or memory) kernel k_t is a nonnegative summable function having the explicit form

$$k_t(s) = \int_s^\infty \mu_t(r)dr, \tag{1.5}$$

where $(t, s) \mapsto \mu_t(s) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is allowed to exhibit (infinitely many) jumps. Moreover, we require that

(M1) For every fixed $t \in \mathbb{R}$, the map $s \mapsto \mu_t(s)$ is nonincreasing, absolutely continuous and summable.

We denote the total mass of μ_t by

$$\kappa(t) = \int_0^\infty \mu_t(s)ds.$$

(M2) There exists $\delta > 0$ such that

$$\partial_t \mu_t(s) + \partial_s \mu_t(s) + \delta \kappa(t) \mu_t(s) \leq 0$$

for every $t \in \mathbb{R}$ and almost every $s > 0$.

(M3) The function $t \mapsto \kappa(t)$ fulfills

$$\inf_{t \in \mathbb{R}} \kappa(t) > 0.$$

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Remark 1.1. *We recall the example of time-dependent memory kernels arising in the physical applications, already introduced in [7, 8].*

Let $\mu \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R}^+)$ be a (nonnull and nonnegative) nonincreasing function with $\mu(0) < \infty$. Given a bounded positive function $\varepsilon \in \mathcal{C}^1(\mathbb{R})$ satisfying

$$\varepsilon'(t) \le 0, \quad \forall t \in \mathbb{R},$$

we define the time-dependent rescaled kernel

$$\mu_t(s) = \frac{1}{[\varepsilon(t)]^2}\mu\left(\frac{s}{\varepsilon(t)}\right).$$

According to (1.5), the corresponding integrated memory kernel reads

$$k_t(s) = \frac{1}{\varepsilon(t)}k\left(\frac{s}{\varepsilon(t)}\right) \text{ where } k(s) = \int_s^\infty \mu(y)dy.$$

Especially, assuming k summable with total mass $m > 0$, the most interesting situation is when $\varepsilon(t) \to 0$ as $t \to \infty$. In this case, we recover the distributional convergence $k_t \to m\delta_0$ to (a multiple of) the Dirac mass at zero. As shown in [7, 8], this μ_t complies with (M1)–(M3). Here, we make two further assumptions: there exists σ such that

$$\mu'(s) + \sigma\mu(s) \le 0, \quad \forall s;$$

$$\inf_{t\in\mathbb{R}} \varepsilon'(t) > -\frac{\delta}{2}.$$

Example *For instance, a possible choice is the exponential kernel $\mu(s) = e^{-s}$ and*

$$\varepsilon(t) = c\left[\frac{\pi}{2} - \arctan(t)\right], \quad 0 < c < \frac{1}{2}.$$

The study is organized as follows. In Section 3, we prove the existence and uniqueness of weak solutions to problem (1.1) by utilizing the compactness method and weak convergence techniques in Orlicz spaces [15]. In Section 4, the existence of a time-dependent global attractor for the process associated to the problem is studied. In the final section, we show the asymptotic closeness to the nonclassical diffusion equation (without memory term) when the kernel approaches the Dirac mass.

2. Notations and preliminaries

In this section, we review some notations about function spaces and preliminary results.

As in [12], a new variable which reflects the history of (1.1) is introduced, that is

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)dr, \ s \ge 0,$$

then we can check that

$$\partial_t\eta^t(x,s) = u(x,t) - \partial_s\eta^t(x,s), \ s \ge 0.$$

Since $\mu_t(s) = -k_t'(s)$, problem (1.1) can be transformed into the following system

$$\begin{cases} \partial_t u - \partial_t\Delta u - \Delta u - \int_0^\infty \mu_t(s)\Delta\eta^t(s)ds + f(u) = g(x), & x \in \Omega, t > \tau, \\ \partial_t\eta^t(x,s) = -\partial_s\eta^t(x,s) + u(x,t), & x \in \Omega, t > \tau, s \ge 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > \tau, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+, t > \tau, \\ u(x,\tau) = u_\tau(x), & x \in \Omega, \\ \eta^\tau(x,s) = \eta_\tau(x,s) := \int_\tau^s g_0(x,r)dr, & (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{2.1}$$

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Now, let

$$z(t) = (u(t), \eta^t), \text{ and } z_\tau = (u_\tau, \eta_\tau).$$

Unless otherwise specified, it is understood that we consider spaces of functions which are defined on the domain Ω. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the $L^2(\Omega)$-inner product and $L^2(\Omega)$-norm, respectively.

Let $L^2_{\mu_t}(\mathbb{R}^+, L^2(\Omega))$ be the Hilbert space of functions $\varphi\colon \mathbb{R}^+ \to L^2(\Omega)$ endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mu_t} = \int_0^\infty \mu_t(s) \, \langle \varphi_1(s), \varphi_2(s) \rangle \, ds,$$

and let $\|\varphi\|_{\mu_t}$ be the corresponding norm. In a similar manner, we introduce the inner products $\langle \cdot, \cdot \rangle_{1,\mu_t}$, $\langle \cdot, \cdot \rangle_{2,\mu_t}$ on $L^2_{\mu_t}(\mathbb{R}^+, H^1_0(\Omega))$ and $L^2_{\mu_t}(\mathbb{R}^+, H^2(\Omega) \cap H^1_0(\Omega))$ by

$$\langle \cdot, \cdot \rangle_{1,\mu_t} = \langle \nabla \cdot, \nabla \cdot \rangle_{\mu_t} \ , \ \langle \cdot, \cdot \rangle_{2,\mu_t} = \langle \Delta \cdot, \Delta \cdot \rangle_{\mu_t} ,$$

and the corresponding norms are denoted by $\| \cdot \|_{1,\mu_t}$, $\| \cdot \|_{2,\mu_t}$.

We now introduce the following Hilbert spaces

$$\mathcal{V}_t = H^1_0(\Omega) \times L^2_{\mu_t}(\mathbb{R}^+, H^1_0(\Omega)),$$

$$\mathcal{W}_t = \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times L^2_{\mu_t}(\mathbb{R}^+, H^2(\Omega) \cap H^1_0(\Omega)),$$

which are, respectively, endowed with the inner products

$$\langle w_1, w_2 \rangle_{\mathcal{V}_t} = \langle \nabla \psi_1, \nabla \psi_2 \rangle + \langle \varphi_1, \varphi_2 \rangle_{1,\mu_t} ,$$

$$\langle w_1, w_2 \rangle_{\mathcal{W}_t} = \langle \Delta \psi_1, \Delta \psi_2 \rangle + \langle \varphi_1, \varphi_2 \rangle_{2,\mu_t} ,$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{V}_t, \mathcal{W}_t, \ \ i = 1, 2.$

The norms induced on $\mathcal{V}_t$, $\mathcal{W}_t$, are

$$\|(\psi, \varphi)\|^2_{\mathcal{V}_t} = \|\nabla \psi\|^2 + \int_0^\infty \mu_t(s) \|\nabla \varphi(s)\|^2 ds,$$

$$\|(\psi, \varphi)\|^2_{\mathcal{W}_t} = \|\Delta \psi\|^2 + \int_0^\infty \mu_t(s) \|\Delta \varphi(s)\|^2 ds.$$

The following results will be used to prove the existence of time-dependent global attractors.

For $t \in R$, let X_t be a family of normed spaces, the two-parameter family of operators

$$U(t, \tau) : X_\tau \to X_t, \quad t \geq \tau,$$

is called a *process on time-dependent spaces* (see [10, 13]), characterized by the two properties:

(i) $U(\tau, \tau)$ is the identity map on X_τ for every τ;
(ii) $U(t, \tau)U(\tau, s) = U(t, s)$ for every $t \geq \tau \geq s$.

As introduced in [10], we consider the following definitions and theorem.

Definition 2.1. *A time-dependent absorbing set for the process $U(t, \tau)$ is a uniformly bounded family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R \geq 0$ there exists $\theta_e = \theta_e(R) \geq 0$ such that*

$$\tau \leq t - \theta_e \to U(t, \tau)\mathbb{B}_\tau(R) \subset B_t.$$

Definition 2.2. *Let* $\mathbb{K} = \{\mathcal{K} = \{K_t\}_{t\in\mathbb{R}} : K_t \subset X_t \text{ compact}, \mathcal{K} \text{ pullback attracting}\}$. *The family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}} \in \mathbb{K}$ is said to be a time-dependent global attractors if $\mathcal{A}$ is the smallest element of $\mathbb{K}$ such that*

$$A_t \subset K_t, \quad \forall t \in \mathbb{R},$$

for any element $\mathcal{K} = \{K_t\}_{t\in\mathbb{R}} \in \mathbb{K}$.

We know that the minimal element of $\mathbb{K}$ exists (and it is unique) if and only if $\mathbb{K}$ is not empty.

Theorem 2.1. *If $U(t,\tau)$ is asymptotic compact, that is,*

$$\mathbb{K}=\{\mathcal{K}=\{K_t\}_{t\in\mathbb{R}} : K_t\subset X_t \textit{ compact}, \mathcal{K} \textit{ pullback attracting}\}, \mathbb{K}\neq\varnothing$$

then $\{U(t,\tau)\}$ has a unique time-dependent global attractors $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$ and

$$A_t=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)B_\tau}.$$

Moreover, if $U(t,\tau)$ is a continuous (or norm-to-weak continuous) map for all $t\ge\tau$, then $\mathcal{A}$ is invariant.

3. Existence and uniqueness of weak solutions

Definition 3.1. *A function $z=(u,\eta^t)$ is called a weak solution of problem (2.1) on the interval (τ,T) with the initial datum $z(\tau)=z_\tau\in\mathcal{V}_\tau$ if*

$$u\in C([\tau,T];H_0^1(\Omega)), f(u)\in L^1(Q_T),$$
$$\partial_t u\in L^2(\tau,T;H_0^1(\Omega)),\ \eta^t\in C([\tau,T];L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega))),$$

and

$$\langle\partial_t u,\varphi\rangle+\langle\partial_t\nabla u,\nabla\varphi\rangle+\langle\nabla u,\nabla\varphi\rangle+\langle\eta^t,\varphi\rangle_{1,\mu_t}+\langle f(u),\varphi\rangle_{L^1,L^\infty}=\langle g,\varphi\rangle_{H^{-1},H_0^1},$$
$$\langle\partial_t\eta^t+\partial_s\eta^t,\xi\rangle_{1,\mu_t}=\langle u,\xi\rangle_{1,\mu_t},$$

for all test functions $\varphi\in W=H_0^1(\Omega)\cap L^\infty(\Omega)$, $\xi\in L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega))$, and for a.e. $t\in[\tau,T]$.

We are now ready to state the existence and uniqueness result for problem (2.1).

Theorem 3.1. *Assume that hypotheses (H1)–(H3) hold. Then for any $z_\tau=(u_\tau,\eta_\tau)\in\mathcal{V}_\tau$ and $T>\tau$ given, problem (2.1) has a unique weak solution $z=(u,\eta^t)$ on the interval (τ,T) satisfying*

$$u\in C([\tau,T];H_0^1(\Omega)),\quad \eta^t\in L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega)).$$

Moreover, the weak solutions depend continuously on the initial data.

Proof. We use the Faedo–Galerkin method. As argued in [5], because of the separability of $H_0^1(\Omega)$, one can choose a sequence $\{\omega_j\}_{j=1}^\infty$ which forms a smooth orthonormal basis in both $L^2(\Omega)$ and $H_0^1(\Omega)$ spaces. For instance, one can take a complete set of normalized eigenfunctions for $-\Delta$ in $H_0^1(\Omega)$, such that $-\Delta\omega_j=\nu_j\omega_j$, where ν_j is the eigenvalue corresponding to ω_j. Next, we want to choose an orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega))$ which also belong to $\mathcal{D}(\mathbb{R}^+,H_0^1(\Omega))$, where $\mathcal{D}(I,X)$ is the space of infinitely differentiable X-valued functions with compact support in $I\subset\mathbb{R}$. For this purpose, we select vectors of the form $l_k\omega_j$ $(k,j=1,\dots,\infty)$, where $\{l_j\}_{j=1}^\infty$ is an orthonormal basis in both $L^2_{\mu_t}(\mathbb{R}^+)$ and $\mathcal{D}(\mathbb{R}^+)$ spaces.

(i) Existence. Given an integer n, denote by P_n and Q_n the projections on the subspaces

$$\operatorname{span}(\omega_1,\dots,\omega_n)\in H_0^1(\Omega)\quad\text{and}\quad\operatorname{span}(\zeta_1,\dots,\zeta_n)\in L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega)),$$

respectively. We look for a function $z_n=(u_n,\eta_n^t)$ of the form

$$u_n(t)=\sum_{j=1}^n a_j(t)\omega_j\quad\text{and}\quad\eta_n^t(s)=\sum_{j=1}^n b_j(t)\zeta_j(s)$$

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satisfying

$$
\begin{aligned}
&\left\langle(\partial_t u_n - \Delta\partial_t u_n, \partial_t \eta_n^t), (\omega_k, \zeta_j)\right\rangle_{\mathcal{V}_t} \\
&\quad = \left\langle (\Delta u_n + \int_0^\infty \mu_t(s)\Delta\eta_n^t(s)ds + g - f(u_n), u_n - \partial_s\eta_n^t), (\omega_k, \zeta_j)\right\rangle_{\mathcal{V}_t} \\
&u_n(\tau) = P_n u_\tau \to u_\tau = \sum_{j=1}^{\infty} \alpha_j\omega_j \quad \text{ in } H_0^1(\Omega), \text{ as } n\to\infty, \\
&\eta_n^t(\tau) = Q_n\eta_\tau \to \eta_\tau = \sum_{j=1}^{\infty} \beta_j\zeta_j(s) \quad \text{ in } L^2_{\mu_t}(\mathbb{R}^+, H_0^1(\Omega)), \text{ as } n\to\infty,
\end{aligned}
\tag{3.1}
$$

for a.e. $\tau \le t \le T$, for every $k, j = 0, \ldots, n$, where ω_0 and ζ_0 are the zero vectors in the respective spaces. Taking (ω_k, ζ_0) and (ω_0, ζ_k) in (3.1), and applying the divergence theorem to the term

$$
\left\langle \int_0^\infty \mu_t(s)\Delta\eta_n^t(s)ds, \omega_k\right\rangle,
$$

we get a system of Ordinary Differential Equation (ODE) in the variables $a_k(t)$ and $b_k(t)$ of the form

$$
\begin{aligned}
&\frac{d}{dt}\Big((1+\nu_k)a_k\Big) = -\nu_k a_k - \sum_{j=1}^{n} b_j\left\langle\zeta_j, \omega_k\right\rangle_{1,\mu_t} + \langle g, \omega_k\rangle - \langle f(u_n), \omega_k\rangle, \\
&\frac{d}{dt}b_k = \sum_{j=1}^{n} a_j\left\langle\omega_j, \zeta_k\right\rangle_{1,\mu_t} - \sum_{j=1}^{n} b_j\left\langle\zeta_j', \zeta_k\right\rangle_{1,\mu_t},
\end{aligned}
\tag{3.2}
$$

subject to the initial conditions

$$
\begin{aligned}
a_k(\tau) &= \langle u_\tau, \omega_k\rangle_{H_0^1(\Omega)}, \\
b_k(\tau) &= \langle \eta_\tau, \zeta_k\rangle_{1,\mu_\tau}.
\end{aligned}
\tag{3.3}
$$

According to standard existence theory for ODEs, there exists a continuous solution (a_k, b_k) of (3.2)–(3.3) on some interval (τ, T_n) for each n. The *a priori* estimates below imply that in fact $T_n = +\infty$.

Multiplying the first equation of (3.2) by a_k and the second by b_k, then summing over k and adding the results, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z_n\|^2_{\mathcal{V}_t} = -\|\nabla u_n\|^2 - \left\langle\partial_s\eta_n^t, \eta_n^t\right\rangle_{1,\mu_t} + \langle g, u_n\rangle_{H^{-1},H_0^1} \\
- \langle f(u_n), u_n\rangle + \int_0^\infty \partial_t\mu_t(s)\|\nabla\eta_n^t(s)\|^2 ds.
\end{aligned}
\tag{3.4}
$$

Using (1.3) and the Cauchy inequality, we have

$$
\langle g, u_n\rangle_{H^{-1},H_0^1} - \langle f(u_n), u_n\rangle \le \varepsilon\|\nabla u_n\|^2 + \frac{1}{4\varepsilon}\|g\|^2_{H^{-1}(\Omega)} + \beta\|u_n\|^2 + C_0|\Omega|,
\tag{3.5}
$$

where $\varepsilon > 0$ will be chosen later. From (3.4) and (3.5) we have

$$
\begin{aligned}
&\frac{d}{dt}\|z_n\|^2_{\mathcal{V}_t} + 2\left\langle\partial_s\eta_n^t, \eta_n^t\right\rangle_{1,\mu_t} - 2\int_0^\infty \partial_t\mu_t(s)\|\nabla\eta_n^t(s)\|^2 ds + 2\left(1 - \frac{\beta}{\lambda_1} - \varepsilon\right)\|\nabla u_n\|^2 \\
&\le \frac{1}{2\varepsilon}\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|.
\end{aligned}
$$

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Integrating by parts and using (M2), we get

$$\begin{aligned} &2\left\langle\partial_s\eta_n^t,\eta_n^t\right\rangle_{1,\mu_t}-2\int_0^\infty\partial_t\mu_t(s)\|\nabla\eta_n^t(s)\|^2ds \\ &= -2\int_0^\infty(\partial_s\mu_t(s)+\partial_t\mu_t(s))\|\nabla\eta_n^t(s)\|^2ds\geq 0. \end{aligned} \tag{3.6}$$

Thus,

$$\frac{d}{dt}\|z_n\|^2_{\mathcal{V}_t}+2(1-\frac{\beta}{\lambda_1}-\varepsilon)\|\nabla u_n\|^2\leq C(\|g\|^2_{H^{-1}(\Omega)}+1).$$

Choosing $\varepsilon>0$ small enough so that $1-\dfrac{\beta}{\lambda_1}-\varepsilon>0$ and integrating on (τ,t), $t\in(\tau,T)$, lead to the following estimate

$$\|z_n(t)\|^2_{\mathcal{V}_t}+2\left(1-\frac{\beta}{\lambda_1}-\varepsilon\right)\int_\tau^t\|\nabla u_n(r)\|^2dr\leq\|z_n(\tau)\|^2_{\mathcal{V}_\tau}+CT(\|g\|^2_{H^{-1}(\Omega)}+1).$$

Hence, in particular, we see that

$$\begin{aligned} &\{u_n\} \quad \text{is bounded in} \quad L^\infty(\tau,T;H_0^1(\Omega)),\\ &\{\eta_n^t\} \quad \text{is bounded in} \quad L^\infty(\tau,T;L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega))). \end{aligned} \tag{3.7}$$

Therefore, by the Banach–Alaoglu theorem, there exists a function $z=(u,\eta^t)$ such that

$$\begin{aligned} &u_n\rightharpoonup u \quad \text{weakly star in} \quad L^\infty(\tau,T;H_0^1(\Omega)),\\ &\eta_n^t\rightharpoonup\eta^t \quad \text{weakly star in} \quad L^\infty(\tau,T;L^2_{\mu_t}(\mathbb{R}^+,H_0^1(\Omega))), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\Delta u_n\rightharpoonup\Delta u \quad \text{weakly in} \quad L^2(\tau,T;H^{-1}(\Omega)),\\ &\Delta\eta_n^t\rightharpoonup\Delta\eta^t \quad \text{weakly in} \quad L^2(\tau,T;L^2_{\mu_t}(\mathbb{R}^+,H^{-1}(\Omega))), \end{aligned} \tag{3.9}$$

up to a subsequence. Now, we estimate $\partial_t z_n$. From (3.4) and (3.6), we get

$$\frac{d}{dt}\|z_n\|^2_{\mathcal{V}_t}+\|\nabla u_n\|^2+2\int_\Omega f(u_n)u_ndx\leq\|g\|^2_{H^{-1}(\Omega)}. \tag{3.10}$$

Integrating (3.10) from τ to T, we obtain

$$\|z_n(T)\|^2_{\mathcal{V}_T}+\int_\tau^T\|\nabla u_n(t)\|^2dt+2\int_{Q_T}f(u_n)u_ndxdt\leq\|z_\tau\|^2_{\mathcal{V}_n(\tau)}+T\|g\|^2_{H^{-1}(\Omega)}.$$

In particular,

$$\int_{Q_T}f(u_n)u_ndxdt\leq C. \tag{3.11}$$

Multiplying the first equation of (3.2) by $a_k+\varepsilon a_k'$ and the second by b_k, then summing over k and adding the results, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2+(1+\varepsilon)\|\nabla u_n\|^2+\int_0^\infty\mu_t(s)\|\eta_n^t\|^2ds+2\varepsilon\langle F(u_n),1\rangle\right)+\|\nabla u_n\|^2\\ &+\varepsilon\left(\|\partial_t u_n\|^2+\|\nabla\partial_t u_n\|^2\right)+\langle f(u_n),u_n\rangle-\int_0^\infty(\partial_s\mu_t(s)+\partial_t\mu_t(s))\|\nabla\eta_n^t(s)\|^2ds\\ &=-\int_0^\infty\mu_t(s)\langle\nabla\eta_n^t(s),\nabla\partial_t u_n\rangle ds+\langle g,u_n+\varepsilon\partial_t u_n\rangle_{H^{-1},H_0^1}. \end{aligned} \tag{3.12}$$

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Using Young inequality and $\kappa(t) = \int_0^\infty \mu_t(s)ds$, we have

$$-\int_0^\infty \mu_t(s)\langle \nabla\eta_n^t(s), \nabla\partial_t u_n\rangle ds \le \int_0^\infty \mu_t(s)\|\nabla\eta_n^t(s)\|\|\nabla\partial_t u_n\|ds$$
$$\le \sqrt{\varepsilon}\delta\kappa(t)\int_0^\infty \mu_t(s)\|\nabla\eta_n^t(s)\|^2 ds + \frac{\varepsilon\sqrt{\varepsilon}}{\delta}\|\nabla\partial_t u_n\|^2,$$

and

$$\langle g, u_n + \varepsilon\partial_t u_n\rangle_{H^{-1},H_0^1} \le \frac{1}{2}\|\nabla u_n\|^2 + \frac{\varepsilon}{2}\|\partial_t\nabla u_n\|^2 + C(\varepsilon)\|g\|^2_{H^{-1}(\Omega)}. \tag{3.13}$$

Combining (3.12)–(3.13), and owing to (M2) and (1.4), we get

$$\frac{d}{dt}\left(\|u_n\|^2 + (1+\varepsilon)\|\nabla u_n\|^2 + \int_0^\infty \mu_t(s)\|\eta_n^t\|^2 ds + 2\varepsilon\langle F(u_n), 1\rangle\right) + \|\nabla u_n\|^2$$
$$+ 2\left(\varepsilon - \frac{\varepsilon\sqrt{\varepsilon}}{\delta}\right)\left(\|\partial_t u_n\|^2 + \|\nabla\partial_t u_n\|^2\right) + 2\langle f(u_n), u_n\rangle \tag{3.14}$$
$$+ 2(1-\sqrt{\varepsilon})\delta\kappa(t)\int_0^\infty \mu_t(s)\|\nabla\eta_n^t(s)\|^2 ds \le C(\varepsilon)\|g\|^2_{H^{-1}(\Omega)}.$$

Integrating (3.14) from τ to t and using (1.4), (3.11) and (3.7), we can deduce that

$$\{\partial_t u_n\} \quad \text{is bounded in} \quad L^2(\tau, T; H_0^1(\Omega)),$$

so, up to a subsequence,

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly in} \quad L^2(\tau, T; H_0^1(\Omega)),$$
$$\partial_t\Delta u_n \rightharpoonup \partial_t\Delta u \quad \text{weakly in} \quad L^2(\tau, T; H^{-1}(\Omega)). \tag{3.15}$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(Q_T)$ where $Q_T = \Omega\times(\tau, T)$. Putting $h(s) = f(s) - f(0) + \gamma s$, where $\gamma > \ell$. Note that $h(s)s = (f(s) - f(0))s + \gamma s^2 = f'(c)s^2 + \gamma s^2 \ge (\gamma - \ell)s^2 \ge 0$ for all $s\in\mathbb{R}$, we have

$$\int_{Q_T} |h(u_n)|\, dxdt \le \int_{Q_T\cap\{|u_n|>1\}} |h(u_n)u_n|\, dxdt + \int_{Q_T\cap\{|u_n|\le 1\}} |h(u_n)|\, dxdt$$
$$\le \int_{Q_T} h(u_n)u_n dxdt + \sup_{|s|\le 1}|h(s)|\,|Q_T|$$
$$\le \int_{Q_T} f(u_n)u_n dxdt + |f(0)|\|u_n\|_{L^1(Q_T)} + \gamma\|u_n\|^2_{L^2(Q_T)}$$
$$+ \sup_{|s|\le 1}|h(s)|\,|Q_T|$$
$$\le C,$$

where we have used (3.7), (3.11) and the boundedness of $\{u_n\}$ in $L^\infty(\tau, T; H_0^1(\Omega))$. Hence it implies that $\{h(u_n)\}$, and therefore $\{f(u_n)\}$ is bounded in $L^1(Q_T)$.

Using the Aubin–Lions lemma in [18], we can suppose that $u_n \to u$ strongly in $L^2(\tau, T; L^2(\Omega))$. Hence $u_n \to u$ a.e. in Q_T, up to a subsequence. Besides, using the definition of $h(s)$ and (3.10), (3.7), we have

$$\int_{Q_T} h(u_n)u_n dxdt \le C.$$

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Therefore, by Lemma 6.1 in [14], we obtain that $h(u) \in L^1(Q_T)$ and for all test functions $\varphi \in C_0^\infty([\tau, T]; H_0^1(\Omega) \cap L^\infty(\Omega))$,

$$\int_{Q_T} h(u_n)\varphi dxdt \to \int_{Q_T} h(u)\varphi dxdt.$$

Hence, $f(u) \in L^1(Q_T)$ and

$$\int_{Q_T} f(u_n)\varphi dxdt \to \int_{Q_T} f(u)\varphi dxdt, \text{ for all } \varphi \in C_0^\infty([\tau, T]; H_0^1(\Omega) \cap L^\infty(\Omega)). \tag{3.16}$$

We are now ready to show that the limit $z = (u, \eta^t)$ is a weak solution of (2.1). Choose an arbitrary test function

$$\phi = (\varphi, \xi) \in \mathcal{D}([\tau, T], H_0^1(\Omega) \cap L^\infty(\Omega)) \times \mathcal{D}([\tau, T], \mathcal{D}(\mathbb{R}^+, H_0^1(\Omega)))$$

of the form

$$\varphi(t) = \sum_{j=1}^m a_j(t)\omega_j \quad \text{and} \quad \xi(t) = \sum_{j=1}^m b_j(t)\zeta_j,$$

where m is a fixed integer, $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^m$ are given functions in $\mathcal{D}((\tau, T))$. Then (3.1) holds with $(v(t), \xi(t))$ in place of (ω_k, ζ_j). Integrating the resulting equation over (τ, T) and passing to the limits, in view of (3.8), (3.9), (3.15) and (3.16), we get

$$\begin{aligned}\int_\tau^T \Big[\langle \partial_t u, \varphi\rangle + \langle \partial_t \nabla u, \nabla\varphi\rangle + \langle \partial_t \eta^t, \xi\rangle_{1,\mu_t}\Big]dt \\ = -\int_\tau^T \Big[\langle \nabla u, \nabla\varphi\rangle + \langle \eta^t, \varphi\rangle_{1,\mu_t}\Big]dt \\ -\int_\tau^T \Big[\int_\Omega f(u)\varphi dx - \langle g, \varphi\rangle_{H^{-1},H_0^1}\Big]dt \\ +\int_\tau^T \Big[-\langle \partial_s \eta^t, \xi\rangle_{1,\mu_t} + \langle u, \xi\rangle_{1,\mu_t}\Big]dt.\end{aligned}$$

Using a density argument, we conclude that $z = (u, \eta^t)$ satisfies the equation in the weak sense. By standard arguments, we can check that z satisfies the initial condition $z(\tau) = z_\tau$. This implies that $z(\cdot)$ is a weak solution of problem (2.1).

(ii) *Uniqueness and continuous dependence on the initial data.* We assume that $z_1 = (u_1, \eta_1^t)$ and $z_2 = (u_2, \eta_2^t)$ are two solutions of (2.1) with initial data $z_{1\tau}$ and $z_{2\tau}$, respectively. Denote $w = z_1 - z_2 = (u_3, \eta_3^t)$, then

$$\partial_t u_3 - \partial_t \Delta u_3 - \Delta u_3 - \int_0^\infty \mu_t(s)\Delta\eta_3^t(x, s)ds + (\hat{f}(u_1(t) - \hat{f}(u_2)) - \ell u_3 = 0, \text{ for all } t > 0, \tag{3.17}$$

where $\hat{f}(s) = f(s) + \ell s$. Here, because $u_3(t)$ does not belong to $W = H_0^1(\Omega) \cap L^\infty(\Omega)$, we cannot choose $u_3(t)$ as a test function. Consequently, the proof will be more involved than that in [9, 29, 30].

We use some ideas in [15]. Let

$$B_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \le k, \\ -k & \text{if } s < -k. \end{cases}$$

Consider the corresponding Nemytskii mapping $\hat{B}_k : W \to W$ defined as follows

$$\hat{B}_k(u_3)(x) = B_k(u_3(x)), \text{ for all } x \in \Omega.$$

By Theorem 4.7 in [17] (see also Lemma 2.3 in [15]), we have that $\|\hat{B}_k(u_3) - u_3\|_W \to 0$ as $k \to \infty$. Now multiplying (3.17) by $\hat{B}_k(u_3)$, then integrating over Ω we get

$$\frac{d}{dt}\left(\int_\Omega u_3\hat{B}_k(u_3)dx + \int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)dx - \frac{1}{2}\left(\|\hat{B}_k(u_3)\|^2 + \|\nabla\hat{B}_k(u_3)\|^2\right)\right)$$
$$+ \int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)dx + \int_0^\infty \mu_t(s)\int_\Omega \nabla\eta_3^t\nabla\hat{B}_k(u_3)dxds$$
$$+ \int_\Omega (\hat{f}(u_1) - \hat{f}(u_2))\hat{B}_k(u_3)dx - \ell\int_\Omega u_3\hat{B}_k(u_3)dx = 0.$$

Thus,

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u_3\hat{B}_k(u_3)dx + \int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)dx - \frac{1}{2}\left(\|\hat{B}_k(u_3)\|^2 + \|\nabla\hat{B}_k(u_3)\|^2\right)\right)\\ &+ \int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} |\nabla u_3|^2dx + \int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \nabla\eta_3^t\nabla u_3dxds\\ &+ \int_\Omega \hat{f}'(\xi)u_3\hat{B}_k(u_3)dx = \ell\int_\Omega u_3\hat{B}_k(u_3)dx. \end{aligned} \tag{3.18}$$

Note that $\hat{f}'(s) \ge 0$ and $sB_k(s) \ge 0$ for all $s \in \mathbb{R}$, we have

$$\int_\Omega \hat{f}'(\xi)u_3\hat{B}_k(u_3)dx \ge 0.$$

Moreover,

$$\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} |\nabla u_3|^2dx \ge 0,$$

and

$$\begin{aligned} &\int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \nabla\eta_3^t\nabla u_3dxds\\ &\quad= \int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \nabla\eta_3^t\partial_t\nabla\eta_3^tdxds\\ &\qquad+ \int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \nabla\eta_3^t\partial_s\nabla\eta_3^tdxds\\ &\quad= \int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \nabla\eta_3^t\partial_t\nabla\eta_3^tdxds\\ &\qquad- \frac{1}{2}\int_0^\infty \partial_s\mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} |\nabla\eta_3^t|^2dxds. \end{aligned}$$

From the above inequalities we deduce from (3.18) that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u_3\hat{B}_k(u_3)dx + \int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)dx - \frac{1}{2}\left(\|\hat{B}_k(u_3)\|^2 + \|\nabla\hat{B}_k(u_3)\|^2\right)\right)\\ &+ \frac{1}{2}\int_0^\infty \mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} \frac{d}{dt}|\nabla\eta_3^t|^2dxds\\ &\le \ell\int_\Omega u_3\hat{B}_k(u_3)dx + \frac{1}{2}\int_0^\infty \partial_s\mu_t(s)\int_{\{x\in\Omega\,:\,|u_3(x,t)|\le k\}} |\nabla\eta_3^t|^2dxds. \end{aligned}$$

Integrating from τ to t, where $t\in(\tau,T)$, then letting $k\to\infty$, we obtain

$$\begin{aligned}
&\|u_3(t)\|^2+\|\nabla u_3(t)\|^2+\|\eta_3^t\|^2_{1,\mu_t}\\
\le\ & \|u_3(\tau)\|^2+\|\nabla u_3(\tau)\|^2+\|\eta_3^\tau\|^2_{1,\mu_\tau}+2\ell\int_\tau^t\|u_3(s)\|^2ds\\
&+\int_\tau^t\left(\int_0^\infty(\partial_s\mu_r(s)+\partial_r\mu_r(s))\|\nabla\eta_3^t\|^2ds\right)dr\\
\le\ & \|u_3(\tau)\|^2+\|\nabla u_3(\tau)\|^2+\|\eta_3^\tau\|^2_{1,\mu_\tau}+2\ell\int_\tau^t(\|u_3(s)\|^2+\|\nabla u_3(s)\|^2+\|\eta_3^s\|^2_{1,\mu_s})ds.
\end{aligned}$$

By the Gronwall inequality of integral form, we get

$$\|w(t)\|^2_{\mathcal{V}_t}\le\|w(\tau)\|^2_{\mathcal{V}_\tau}e^{2\ell(t-\tau)}\le\|w(\tau)\|^2_{\mathcal{V}_\tau}e^{2\ell(T-\tau)},\ \text{ for all } t\in[\tau,T].$$

Hence we get the continuous dependence on the initial data of the solutions, and in particular, the uniqueness when $w(\tau)=0$. □

4. Existence of a time-dependent global attractor

Theorem 3.1 allows us to define a process on time-dependent spaces $U(t,\tau):\mathcal{V}_\tau\to\mathcal{V}_t$ associated to problem (2.1) by the formula

$$U(t,\tau)z_\tau:=z(t),$$

where $z(\cdot)$ is the unique global weak solution of (2.1) with the initial datum $z_\tau\in\mathcal{V}_\tau$.

4.1. Existence of a time-dependent absorbing set

Lemma 4.1. *Under assumptions (H1)–(H3), there exists a time-dependent absorbing set in $\mathcal{V}_t$ for the process $U(t,\tau)$.*

Proof. Multiplying the first equation of (2.1) by $u(t)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|\nabla u\|^2\right)+\|\nabla u\|^2+\int_\Omega f(u)udx+\int_0^\infty\mu_t(s)\langle\nabla\eta^t(s),\nabla u\rangle ds = \langle g,u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{4.1}$$

Using the hypothesis (1.3) and the Cauchy inequality, we have

$$\begin{aligned}
\int_\Omega f(u)udx&\ge-\beta\|u\|^2-C_0|\Omega|,\\
\langle g,u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}&\le\varepsilon_0\|\nabla u\|^2+\frac{1}{4\varepsilon_0}\|g\|^2_{H^{-1}(\Omega)}.
\end{aligned}\tag{4.2}$$

Recalling that $u(t)=\partial_t\eta^t(s)+\partial_s\eta^t(s)$, we have

$$\begin{aligned}
&\int_0^\infty\mu_t(s)\langle\nabla\eta^t(s),\nabla u\rangle ds\\
&=\int_0^\infty\mu_t(s)\langle\nabla\eta^t(s),\nabla\partial_s\eta^t(s)\rangle ds+\int_0^\infty\mu_t(s)\langle\nabla\eta^t(s),\nabla\partial_t\eta^t(s)\rangle ds\\
&=-\frac{1}{2}\int_0^\infty\partial_s\mu_t(s)\|\eta^t(s)\|^2ds+\frac{1}{2}\frac{d}{dt}\|\eta^t(s)\|^2-\frac{1}{2}\int_0^\infty\partial_t\mu_t(s)\|\eta^t(s)\|^2ds\\
&=-\frac{1}{2}\int_0^\infty(\partial_s\mu_t(s)+\partial_t\mu_t(s))\,\|\eta^t(s)\|^2ds+\frac{1}{2}\frac{d}{dt}\|\eta^t\|^2_{1,\mu_t}.
\end{aligned}\tag{4.3}$$

From (4.3) and (M2), we get

$$\int_0^\infty \mu_t(s)\langle\nabla\eta^t(s),\nabla u\rangle ds \geq \frac{1}{2}\delta\kappa(t)\|\eta^t\|^2_{1,\mu_t} + \frac{1}{2}\frac{d}{dt}\|\eta^t\|^2_{1,\mu_t}. \tag{4.4}$$

Combining (4.2), (4.4) and (4.1), we get

$$\frac{d}{dt}\left(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu_t}\right) + 2\left(1 - \frac{\beta}{\lambda_1} - \varepsilon_0\right)\|\nabla u\|^2 + \delta\kappa(t)\|\eta^t(s)\|^2_{1,\mu_t} \leq \frac{1}{2\varepsilon}\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|.$$

Using hypothesis (M3), we deduce $\exists\delta_0 > 0$, so that $\kappa(t) \geq \delta_0 > 0$ $\forall t \in \mathbb{R}$. Therefore

$$\frac{d}{dt}\left(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu_t}\right) + \gamma\left(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu_t}\right) \leq C\left(\|g\|^2_{H^{-1}(\Omega)} + 1\right).$$

By the Gronwall inequality, we get

$$y(t) \leq e^{-\gamma(t-\tau)}y(\tau) + C\left(\|g\|^2_{H^{-1}(\Omega)} + 1\right),$$

where

$$y(t) = \|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu_t}.$$

Hence, there exists $\rho_0 > 0$ such that

$$\|z(t)\|^2_{\mathcal{V}_t} \leq \rho_0, \tag{4.5}$$

for all $z_\tau \in B$ and $t \geq t_\tau = t_\tau(B)$, where B is an arbitrary bounded subset of $\mathcal{V}_t$. This completes the proof. □

4.2 Asymptotic compactness

Recall that in this paper we only assume the external force $g \in H^{-1}(\Omega)$. However, we know that for any $g \in H^{-1}(\Omega)$ and $\varepsilon > 0$ given, there is a $g^\varepsilon \in L^2(\Omega)$, which depends on g and ε, such that

$$\|g - g^\varepsilon\|_{H^{-1}(\Omega)} < \varepsilon. \tag{4.6}$$

Now, in order to show that the process is asymptotically compact, we shall exhibit a pullback attracting family of compact sets. To this aim, the strategy classically consists in finding a suitable decomposition of the process in the sum of a decaying part and of a compact one.

4.2.1 Decomposition of the equation

Since $\mathcal{B} = \{\mathbb{B}_t(R)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for $U(t,\tau)z_\tau$, then for each initial data $z_\tau \in \mathbb{B}_t(R)$, we decompose $U(t,\tau)z_\tau$ as follows

$$U(t,\tau)z_\tau = U_1(t,\tau)z_\tau + U_2(t,\tau)z_\tau,$$

where $U_1(t,\tau)z_\tau = z_1(t)$ and $U_2(t,\tau)z_\tau = z_2(t)$, that is, $z = (u,\eta^t) = z_1 + z_2$, the decomposition is of the following form

$$u = v^\varepsilon + w^\varepsilon, \quad \eta^t = \zeta^{t\varepsilon} + \xi^{t\varepsilon},$$

$$z_1 = (v^\varepsilon, \zeta^{t\varepsilon}), \quad z_2 = (w^\varepsilon, \xi^{t\varepsilon}),$$

where $z_1(t)$ is the unique solution of the following problem

$$\begin{cases} \partial_t v^\varepsilon - \partial_t \Delta v^\varepsilon - \Delta v^\varepsilon + f(u) - f(w^\varepsilon) - \int_0^\infty \mu_t(s)\Delta\zeta^{t\varepsilon}(s)ds + \lambda v^\varepsilon = g - g^\varepsilon,\ \lambda > \ell, \\ \partial_t \zeta^{t\varepsilon} = -\partial_s \zeta^{t\varepsilon} + v^\varepsilon, \\ v^\varepsilon(x,t)|_{\partial\Omega} = 0,\ v^\varepsilon(x,t)|_{t=\tau} = u_\tau(x), \\ \zeta^{t\varepsilon}(x,s)|_{\partial\Omega} = 0,\ \zeta^\tau(x,s) = \zeta_\tau(x,s) := \int\limits_\tau^s g_0(x,r)dr, \end{cases} \tag{4.7}$$

and $z_2(t)$ is the unique solution of the following problem

$$\begin{cases} \partial_t w^\varepsilon - \partial_t \Delta w^\varepsilon - \Delta w^\varepsilon + f(w^\varepsilon) - \int_0^\infty \mu_t(s)\Delta\xi^{t\varepsilon}(s)ds - \lambda(u - w^\varepsilon) = g^\varepsilon,\ \lambda > \ell, \\ \partial_t \xi^{t\varepsilon} = -\partial_s \xi^{t\varepsilon} + w^\varepsilon, \\ w^\varepsilon(x,t)|_{\partial\Omega} = 0,\ w^\varepsilon(x,t)|_{t=\tau} = 0, \\ \xi^{t\varepsilon}(x,s)|_{\partial\Omega} = 0,\ \xi^\tau(x,s) = \xi_\tau(x,s) = 0. \end{cases} \tag{4.8}$$

By using similar arguments as in the proof of Theorem 3.1, one can prove the existence and uniqueness of solutions to problems (4.7) and (4.8). Moreover, for problem (4.8), because the external force $g^\varepsilon \in L^2(\Omega)$ and the initial data are zero (so it belong to $\mathcal{W}_t := \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times L^2_{\mu_t}(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega))$), we can show that the solution $(w^\varepsilon, \xi^{t\varepsilon})$ is in fact a strong solution. In particular, we will have $w^\varepsilon \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ for any $T > 0$. This will be used in the proof of Lemma 4.3 below.

We begin with the decay estimate for solutions of (4.7).

Lemma 4.2. *For any $\varepsilon > 0$, the solutions of equation (4.7) satisfy the following estimates: there is a constant d_0 which depends on λ_1, ℓ, such that for every $t \geq 0$,*

$$\|U_1(t,\tau)z_\tau\|^2_{\mathcal{V}_t} \leq Q(\|z_\tau\|_{\mathcal{V}_t})e^{-d_0 t} + \varepsilon.$$

Proof. Multiplying the first equation of (4.7) by v^ε we get

$$\frac{1}{2}\frac{d}{dt}\left(\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2\right) + \lambda\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2 + \int_0^\infty \mu_t(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla v^\varepsilon dxds$$
$$+\langle f(u) - f(w^\varepsilon), v^\varepsilon\rangle = \langle g - g^\varepsilon, v^\varepsilon\rangle_{H^{-1},H_0^1}.$$

Applying Cauchy inequality, we get

$$\langle g - g^\varepsilon, v^\varepsilon\rangle_{H^{-1},H_0^1} \leq \frac{1}{2}\|\nabla v^\varepsilon\|^2 + \frac{1}{2}\|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

Noting that $\partial_t\zeta^{t\varepsilon} = -\partial_s\zeta^{t\varepsilon} + v^\varepsilon$ and reasoning exactly as in (4.3), (4.4), we obtain

$$\int_0^\infty \mu_t(s)\langle\nabla\zeta^{t\varepsilon}(s), \nabla v^\varepsilon\rangle ds \geq \frac{1}{2}\delta\kappa(t)\|\zeta^{t\varepsilon}(s)\|^2_{1,\mu_t} + \frac{1}{2}\frac{d}{dt}\|\zeta^{t\varepsilon}(s)\|^2_{1,\mu_t}. \tag{4.9}$$

Therefore, because $f'(\xi) \geq -\ell$, we have

$$\frac{d}{dt}\left(\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2 + \|\zeta^{t\varepsilon}\|^2_{1,\mu_t}\right) + \|\nabla v^\varepsilon\|^2 + 2(\lambda - \ell)\|v^\varepsilon\|^2 + \delta\kappa(t)\|\zeta^{t\varepsilon}(s)\|^2_{1,\mu_t}$$
$$\leq \|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

Using hypothesis (M3), we deduce $\exists\delta_0 > 0$, so that $\kappa(t) \geq \delta_0 > 0\ \forall t \in \mathbb{R}$.

Thus, similarly to the proof of Lemma 4.1, we obtain for some $d_0 > 0$,

$$\|U_1(t,\tau)z_\tau\|^2_{\mathcal{V}_t} \leq Q(\|z_\tau\|_{\mathcal{V}_t})e^{-d_0 t} + \frac{1}{C}\|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

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Taking $\varepsilon^2 \le C\varepsilon$ in (4.6) we have

$$\|U_1(t,\tau)z_\tau\|^2_{\mathcal{V}_t} \le Q(\|z_\tau\|_{\mathcal{V}_t})e^{-d_0 t} + \varepsilon.$$

□

About the solution $z_2(t)$ of (4.8), we have the following lemma.

Lemma 4.3. *For any $\varepsilon > 0$, there is $M > 0$ such that for any $z_\tau \in \mathcal{W}_t$, there exists $T > 0$ large enough, which depends on $\|g\|^2_{H^{-1}(\Omega)}$, ε, such that*

$$\|U_2(t,\tau)z_\tau\|^2_{\mathcal{W}_t} \le M, \text{ for all } t \ge T. \tag{4.10}$$

Proof. Multiplying the first equation of (4.8) by $-\Delta w^\varepsilon$, then using (1.2), (M2) and the Cauchy inequality, we have

$$\frac{d}{dt}\left(\|\nabla w^\varepsilon\|^2 + \|\Delta w^\varepsilon\|^2 + \|\xi^{t\varepsilon}\|^2_{2,\mu_t}\right) + \|\Delta w^\varepsilon\|^2 + 2(\lambda-\ell)\|\nabla w^\varepsilon\|^2$$
$$+\delta\kappa(t)\|\xi^{t\varepsilon}(s)\|^2_{1,\mu_t} \le \frac{1}{2}\left(\|g^\varepsilon\|^2_{H^{-1}(\Omega)} + \|u\|^2_{H^1_0(\Omega)}\right) \le C(\|g^\varepsilon\|^2_{H^{-1}(\Omega)} + \rho_0)$$

when $t \ge t_0(B)$. Note that we used the estimate (4.5) for this expression.

Hence, similarly to the proof of Lemma 4.1, we obtain a number $T > 0$ large enough such that

$$\|U_2(t,\tau)z_\tau\|^2_{\mathcal{W}_t} \le M, \text{ for all } t \ge T.$$

The proof is complete. □

Remark 4.1. *From Lemma 4.3, we immediately have the following regularity result: $\mathcal{A}_t$ is bounded in $\mathcal{W}_t$ (with a bound independent of t).*

Since $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set, collecting Lemmas 4.2 and 4.3 we infer that the family of $\mathcal{W}_t$-ball $\mathcal{K} = \{K_t(r)\}_{t\in\mathbb{R}}$ is pullback attracting provided that $r > 0$ is sufficiently large, for

$$dist_{\mathcal{V}_t}(U(t,\tau)B_\tau, K_t(r)) \le \sup_{z_\tau\in B_\tau} \|D(t,\tau)z_\tau\|_{\mathcal{V}_t} \le Ce^{-\frac{\gamma}{2}(t-\tau)}.$$

Unfortunately, there are not enough conditions to conclude the existence of the unique time-dependent global attractor. Indeed, although closed balls of $\mathcal{W}_t$ are uniformly bounded by the embedding constant of $\mathcal{W}_t \hookrightarrow \mathcal{V}_t$ is independent of t, they fail to be compact in $\mathcal{V}_t$ due to the lack of compactness of the embedding $L^2_{\mu_t}(\mathbb{R}^+, H^1_0(\Omega)) \hookrightarrow L^2_{\mu_t}(\mathbb{R}^+, H^2(\Omega)\cap H^1_0(\Omega))$. The argument as in the proof of [30, Theorem 3.13], where the same model is considered for a constant-in-time memory kernel, we get that the process $U(t,\tau)$ is asymptotically compact, which proves the existence of the unique time-dependent global attractor in below.

Theorem 4.4. *Assume that (H1)–(H3) hold. Then the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) admits an invariant time-dependent global attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$.*

5. Recovering nonclassical diffusion equation

In the last section, using the idea of Conti et al. in [7], we discuss the case when $k_t \to m\delta_0$ for some $m > 0$, that is,

$$\lim_{t\to\infty}\int_\varepsilon^\infty k_t(s)ds = \begin{cases} m & \text{if } \varepsilon = 0, \\ 0 & \text{if } \varepsilon > 0. \end{cases} \tag{5.1}$$

Then, for long times our problem becomes the nonclassical diffusion equation (without memory kernel)

$$u_t - \Delta u_t - (1+m)\Delta u + f(u) = g. \tag{5.2}$$

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Equation (5.2) generates a $\mathcal{C}_0$-semigroup of solutions

$$S(t): H_0^1(\Omega) \to H_0^1(\Omega),$$

possessing the global attractor $\hat{A}$ in the classical sense. Besides, $\hat{A}$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega)$, and coincides with the sections at (any) time $t_0 \in \mathbb{R}$ of the set of all *complete bounded trajectories* (CBT) of $S(t)$ (see, [28]). Namely, for any fixed $t_0 \in \mathbb{R}$,

$$\hat{A} = \{\hat{z}(t_0) : \hat{z} \text{ CBT of } S(t)\},$$

where a CBT the semigroup $S(t)$ is a map

$$t \mapsto \hat{z}(t) = \hat{u}(t) \in H_0^1(\Omega)$$

satisfying

$$\sup_{t\in\mathbb{R}} \|\hat{z}(t)\|_{H_0^1(\Omega)} < \infty \quad \text{and} \quad \hat{z}(t) = S(t)\hat{z}(\tau) \ \forall t \geq 0, \forall \tau \in \mathbb{R}.$$

On the other hand, according to Theorem 4.4 and [11, Theorem 3.2], the invariant time-dependent global attractor is characterized as the set of all CBT of the process, that is,

$$A = \{z(t) : z \text{ CBT of } U(t,\tau)\},$$

where a CBT of $U(t,\tau)$ is a map

$$t \mapsto z(t) = (u(t), \partial_t u(t), \eta^t) \in \mathcal{V}_t$$

satisfying

$$\sup_{t\in\mathbb{R}} \|z(t)\|_{\mathcal{V}_t} < \infty \quad \text{and} \quad z(t) = U(t,\tau)z(\tau) \ \forall t \geq \tau \in \mathbb{R}.$$

Using ideas as in [7], we have the following theorem which establishes the closeness of the long-term dynamics of (1.1) to the one of the "limit problem" (5.2) when $k_t \to m\delta_0$.

Theorem 5.1. *Assume that (5.1) hold. Then, for any sequence (u_n, η_n^t) of CBT of $U(t,\tau)$ and any $t_n \to \infty$, there exists a CBT $\hat{u}$ of $S(t)$ such that the convergence*

$$\sup_{t\in[-T,T]} \|u_n(t+t_n) - \hat{u}(t)\|_{H_0^1(\Omega)} \to 0 \tag{5.3}$$

holds up to a subsequence as $n \to \infty$ for every $T > 0$.

Defining the canonical projection $\mathbb{P}_t : \mathcal{V}_t \to H_0^1(\Omega)$ by $\mathbb{P}_t(u,\eta) = u$, and denoting the Hausdorff semidistance of two (nonempty) sets $B, C \subset X$ by

$$dist_X(B,C) = \sup_{x\in B} \inf_{x\in C} \|x-y\|_X,$$

then we have an immediate corollary.

Corollary 5.1. *Assume that (5.1) hold. Then, we have the convergence*

$$\lim_{t\to\infty} \left[\mathrm{dist}_{H^2(\Omega)\cap H_0^1(\Omega)}(\mathbb{P}_t A_t, \hat{A}) \right] = 0.$$

Indeed, from (5.3), we deduce that for every $t_n \to \infty$ the convergence

$$\mathrm{dist}_{H^2(\Omega)\cap H_0^1(\Omega)}(\mathbb{P}_t A_t, \hat{A}) \to 0$$

holds (up to a sequence) as $n \to \infty$. This is clearly enough to draw the desired conclusion.

*Proof of Theorem 5.1.*Let $w_n(\cdot) = u_n(\cdot + t_n)$, for every n. Then the function w_n fulfills the equation

$$\partial_t w_n(t) - \Delta\partial_t w_n(t) - \Delta w_n(t) - \int_0^\infty k_{t+t_n}(s)\Delta w_n(t-s)ds + f(w_n(t)) = g. \tag{5.4}$$

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From the estimates of Lemmas 4.2 and 4.3, we get that

$$w_n \text{ is bounded in } L^\infty(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)). \tag{5.5}$$

Then, there exists $\hat{u} \in L^\infty(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega))$ such that, up to a subsequence,

$$w_n \rightharpoonup \hat{u} \text{ weakly-star in } L^\infty(\mathbb{R}, H^2(\Omega) \cap H_0^1(\Omega)).$$

Since $\lim_{n\to\infty} k_{t+t_n} = m\delta_0$ in the sense of (5.1), we can see that, for every $T > 0$,

$$\sup_{t\in[-T,T]} \int_0^\infty k_{t+t_n}(s)ds \leq 2m,$$

for every n sufficiently large (depending on T).

Multiplying equation (5.4) by $w_n + \partial_t w_n$ then integrating from $-T$ to T we conclude that

$$\partial_t w_n \text{ is bounded in } L^\infty(-T, T; H_0^1(\Omega)). \tag{5.6}$$

From (5.5) and (5.6) and applying the classical Simon-Aubin Theorem [23], the strong convergence

$$w_n \to \hat{u} \text{ in } \mathcal{C}([-T, T], H_0^1(\Omega))$$

holds (up to a subsequence), implying in particular (5.3).

Now, we are left to prove that $\hat{u}$ solves the nonclassical diffusion equation, namely

$$\partial_t\hat{u} - \Delta\partial_t\hat{u} - (1+m)\Delta\hat{u} + f(\hat{u}) = g.$$

Indeed, we will prove that the equality above is recovered when passing to the limit as $n \to \infty$ in (5.4), the only nonstandard convergence being

$$-\int_0^\infty k_{t+t_n}(s)\Delta w_n(t-s)ds \to -m\Delta\hat{u}(t).$$

Let ϕ be fixed, and

$$\Lambda_n(t) = \int_0^\infty k_{t+t_n}(s)\langle w_n(t-s), -\Delta\phi\rangle ds.$$

Then, we decompose the function $\Lambda_n(t)$ into the sum

$$\Lambda_n(t) = \Lambda_n^1(t) + \Lambda_n^2(t) + \Lambda_n^3(t),$$

where

$$\Lambda_n^1(t) = \int_0^1 k_{t+t_n}(s)\langle \hat{u}(t-s), -\Delta\phi\rangle ds,$$
$$\Lambda_n^2(t) = \int_0^1 k_{t+t_n}(s)\langle w_n(t-s) - \hat{u}(t-s), -\Delta\phi\rangle ds,$$
$$\Lambda_n^3(t) = \int_1^\infty k_{t+t_n}(s)\langle w_n(t-s), -\Delta\phi\rangle ds.$$

Since $\langle w_n, -\Delta\phi\rangle \in L^\infty(\mathbb{R})$ uniformly with respect to n, and $\langle w_n, -\Delta\phi\rangle \to \langle \hat{u}, -\Delta\phi\rangle$ in $C(I)$ for every closed interval I, and using (5.1) once again

$$|\Lambda_n^2(t) + \Lambda_n^3(t)| \leq \|\langle w_n - \hat{u}, -\Delta\phi\rangle\|_{\mathcal{C}([t-1,t])} \int_0^1 k_{t+t_n}(s)ds$$
$$+ \|\langle w_n, -\Delta\phi\rangle\|_{L^\infty(\mathbb{R})} \int_1^\infty k_{t+t_n}(s)ds \to 0,$$

while

$$\Lambda_n^1(t) \to m\langle \hat{u}, -\Delta\phi\rangle.$$

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Therefore, we conclude that

$$\Lambda_n(t) = \int_0^\infty k_{t+t_n}(s)\langle w_n(t-s), -\Delta\phi\rangle ds \to m\langle \hat{u}, -\Delta\phi\rangle,$$

for any sufficiently regular ϕ. Thus

$$-\int_0^\infty k_{t+t_n}(s)\Delta w_n(t-s)ds \to -m\Delta\hat{u}(t).$$

This completes the proof. □

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