

ARTICLES

IS MORE MEMORY IN EVOLUTIONARY SELECTION (DE)STABILIZING?

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We investigate the effects of memory on the stability of evolutionary selection dynamics based on a multinomial logit model in a simple asset pricing model with heterogeneous beliefs. Whether memory is stabilizing or destabilizing depends in general on three key factors: (1) whether or not the weights on past observations are normalized; (2) the ecology or composition of forecasting rules, in particular the average trend extrapolation factor and the spread or diversity in biased forecasts; and (3) whether or not costs for information gathering of economic fundamentals have to be incurred.

Keywords: Fitness Measure, Asset Pricing, Bifurcations, Evolutionary Selection, Heterogeneous Beliefs, Memory Strength

1. INTRODUCTION

Heterogeneous expectations models are becoming increasingly popular in various fields of economic analysis, such as exchange rate models [De Grauwe et al. (1993); Da Silva (2001); De Grauwe and Grimaldi (2005, 2006)], macro-monetary policy models [Evans and Honkapohja (2003); Evans and McGough (2005); Bullard et al. (2008); Anufriev et al. (2009)], overlapping-generations models [Duffy (1994); Tuinstra (2003); Tuinstra and Wagener (2007)], and models of socioeconomic behavior [Lux (1995); Brock and Durlauf (2001); Alfarano et al. (2005)]. Yet the application with the most systematic and perhaps most promising heterogeneous expectations models seems to be asset price modeling. Contributions of, e.g., Brock and Hommes (1998), Lux and Marchesi (1999), LeBaron (2000), Chiarella and He (2002), Brock et al. (2005), and Gaunersdorfer et al. (2008) demonstrate how a simple standard asset pricing model with heterogeneous beliefs is able to lead to complex dynamics that makes it extremely hard

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to predict the coevolution of prices and forecasting strategies in asset markets. A widely used framework is adaptive belief systems (ABS), a financial market application of the evolutionary selection of expectation rules, introduced by Brock and Hommes (1997). See Hommes (2006) and LeBaron (2006) for extensive reviews of agent-based models in finance; recent overviews stressing the empirical and experimental validation of agent-based models are Hommes and Wagener (2009) and Lux (2009).

An important result in asset pricing models with heterogeneous beliefs is that nonrational traders, such as technical analysts extrapolating past price trends, may survive evolutionary competition. These results contradict the hypothesis that irrational traders will be driven out of the market by rational arbitrageurs, who trade against them and earn higher profits and accumulate higher wealth [Friedman (1953)]. In most asset pricing models with heterogeneous beliefs, irrational chartists can survive because evolutionary selection is driven by *short-run* profitability. The role of memory, time horizons, or long-run profitability in the evolutionary fitness measure underlying strategy selection has hardly been studied in the literature, however.

LeBaron (2001) and (2002) are among the few papers that have addressed the role of investors time horizons in learning and strategy selection in an agent-based financial market. It has been argued that investors' time horizons are related to whether they believe that the world is stationary or nonstationary. In a stationary world agents should use all available information in learning and strategy selection, whereas if one views the world as constantly in a state of change, then it will be better to use a shorter history of past observations. One of LeBaron's main findings is that in a world where more agents have a long memory horizon, the volatility of asset price fluctuations is smaller. Stated differently, long-horizon investors make the market more stable, whereas short-horizon investors contribute to excess volatility and prevent asset prices from converging to the rational, fundamental benchmark.

Another contribution along these lines is Brock and Hommes (1999), who use a simple, tractable asset pricing model with heterogeneous beliefs to investigate the effect of memory on the fitness measure for strategy selection. In contrast to LeBaron (2001, 2002), they find that more memory in strategy selection may destabilize asset price dynamics.¹

Honkapohja and Mitra (2003) provide analytical results for the dynamics of adaptive learning when the learning rule has finite memory. These authors focus on the case of learning a stochastic steady state. Although their work is not done in a heterogeneous agent setting, the results are interesting for our analysis. Their fundamental outcome is that the expectational stability principle, which plays a central role in the stability of adaptive learning, as discussed, e.g., in Evans and Honkapohja (2001), retains its importance in the analysis of incomplete learning, though it takes a new form. Their main result is that expectational stability guarantees stationary dynamics under learning with finite memory, with unbiased forecasts but higher price volatility than under complete learning with infinite memory.

Chiarella et al. (2006) study the effect of the time horizon in technical trading rules upon stability in a dynamic financial market model with fundamentalists

and chartists. The chartist demand is governed by the difference between the current price and a (long-run) moving average. One of their main results is that an increase of the window length of the moving average rule can destabilize an otherwise stable system, leading to more complicated, even chaotic behavior. Analysis of the corresponding stochastic model was able to explain various market price phenomena, including temporary bubbles, sudden market crashes, price resistance, and price switching between different levels.

The aim of our paper is to study the role of memory or time horizon in evolutionary strategy selection in a simple, analytically tractable asset pricing model with heterogeneous beliefs. We analyze the effects of additional memory in the fitness measure on evolutionary adaptive systems and the consequences for survival of technical trading strategies. By complementing the stability analysis with local bifurcation theory [see Kuznetsov (2004) for an extensive mathematical treatment], we will be able to analyze the effects of adding different amounts of memory to the fitness measure on stability in a standard asset pricing model with heterogeneous beliefs.

The outline of the paper is as follows. In Section 2 an adaptive belief system is presented in its general form with H different trader types. In Section 3 an ABS with two types and costs for information gathering is examined. In Section 4 we investigate the stability of the fundamental steady state in a more generalized framework without information costs. In Section 5 our theoretical findings with respect to memory are examined numerically in an example with three strategies. The final section concludes, and proofs are collected in an Appendix.

2. ADAPTIVE BELIEF SYSTEMS

An adaptive belief system is a standard discounted-value asset pricing model derived from mean–variance maximization with heterogeneous beliefs about future asset prices. We shall briefly recall the model of Brock and Hommes (1998); for a recent more detailed discussion see, e.g., Hommes and Wagener (2009).

2.1. The Asset Pricing Model

Agents can invest either in a risk-free asset or in a risky asset. The risk-free asset is in infinite elastic supply and pays a fixed rate of return r ; the risky asset is in fixed supply z^s and pays an uncertain dividend. Let p_t be the price per share of the risky asset at time t , y_t the stochastic dividend process of the risky asset, and z_t the number of shares of risky assets purchased at date t . Then wealth dynamics is given by

$$W_{t+1} = (1 + r)W_t + [p_{t+1} + y_{t+1} - (1 + r)p_t]z_t. \quad (1)$$

There are H different types of trading strategies. Let \mathbf{E}_{ht} and \mathbf{V}_{ht} denote forecasts of trader type h , with $h = 1, \dots, H$, about conditional expectation and conditional variance, which are based on a publicly available information set of past prices

and past dividends. Demand $z_{h,t}$ of a trader of type h for the risky asset is derived from myopic mean–variance maximization; i.e.,

$$\max_{z_t} \left\{ \mathbf{E}_{ht} [W_{t+1}] - \frac{a}{2} \mathbf{V}_{ht} [W_{t+1}] \right\}, \tag{2}$$

where a is the risk aversion parameter. Then the demand $z_{h,t}$ is given by

$$z_{h,t} = \frac{\mathbf{E}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\mathbf{V}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}. \tag{3}$$

Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let $n_{h,t}$ denote the fraction of type h at date t . Then equality of the demand and the supply in the market equilibrium implies

$$\sum_{h=1}^H n_{ht} \frac{\mathbf{E}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\mathbf{V}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]} = z^s. \tag{4}$$

We shall assume the conditional variance $V_{h,t} = \sigma^2$ to be constant and equal for all types²: thus the equilibrium pricing equation is given by

$$(1+r)p_t = \sum_{h=1}^H n_{h,t} \mathbf{E}_{h,t}[p_{t+1} + y_{t+1}] - a\sigma^2 z^s. \tag{5}$$

As in Brock and Hommes (1998), we focus on the case of zero outside supply, i.e., $z^s = 0$. It is well known that, if all agents are rational, the asset price is given by the discounted sum of expected future dividends,

$$p_t^* = \sum_{k=1}^{\infty} \frac{\mathbf{E}_t[\mathbf{y}_{t+k}]}{(1+r)^k}. \tag{6}$$

The price p_t^* is called the *fundamental price*. The properties of p_t^* depend on the stochastic dividend process y_t . We focus on the case of IID dividend process y_t with constant mean \bar{y} , for which the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{7}$$

It will be convenient to work with the deviation from the fundamental price,

$$x_t = p_t - p^*. \tag{8}$$

Beliefs of type h satisfy the following assumptions:

- [B1] $\mathbf{V}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t] = \sigma^2$,
- [B2] $\mathbf{E}_{h,t}[y_{t+1}] = \mathbf{E}_t[y_{t+1}] = \bar{y}$,
- [B3] $\mathbf{E}_{h,t}[p_{t+1}] = \mathbf{E}_t[p_{t+1}] + f_h(x_{t-1}, \dots, x_{t-L}) = p^* + f_h(x_{t-1}, \dots, x_{t-L})$.

Assumption [B1] says that beliefs about conditional variance are equal and constant for all types. According to assumption [B2], expectations about future dividends y_{t+1} are the same and correct for all trader types. According to assumption [B3], traders of type h believe that in a heterogeneous world the price may deviate from its fundamental value p_t^* by some function $f_h = f_h(x_{t-1}, \dots, x_{t-L})$ of past deviations. The function f_h represents agent type h 's view of the world.

Brock and Hommes (1998) investigated evolutionary competition between simple linear forecasting rules with only one lag,

$$f_{h,t} = g_h x_{t-1} + b_h, \tag{9}$$

where g_h is the *trend* and b_h is the *bias* of trader type h . If $b_h = 0$ we call an agent h a *pure trend chaser* if $g_h > 0$ and a *contrarian* if $g_h < 0$. In the special case $g_h = 0$ and $b_h = 0$, and trader of type h is a *fundamentalist*, believing that price returns to its fundamental value.

An important and convenient consequence of assumptions [B1]–[B3] is that the heterogeneous agent market equilibrium (5) can be reformulated in deviations from the fundamental price. The fact that the fundamental price satisfies $(1 + r)p^* = E_t[p_{t+1} + y_{t+1}]$ yields the equilibrium equation in deviations from the fundamental value,

$$(1 + r)x_t = \sum_{h=1}^H n_{h,t} f_{h,t}. \tag{10}$$

2.2. Evolutionary Fitness with Memory

The evolutionary part of the model describes how beliefs are updated, i.e., how the fractions $n_{h,t}$ of trader types in the market evolve over time. Fractions are updated according to an *evolutionary fitness measure* $U_{h,t}$. The fractions of agents choosing strategy h are given by the *multinomial logit* probabilities

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})}. \tag{11}$$

The *intensity of choice* parameter $\beta \geq 0$ measures how sensitive the traders are to selecting the optimal prediction strategy. The extreme case $\beta = 0$ corresponds to the case where agents do not switch and all fractions are fixed and equal $1/H$. The other extreme case $\beta = \infty$ corresponds to the case where all traders immediately switch to the optimal strategy. An increase in the intensity of choice β represents an increase in the degree of rationality with respect to evolutionary selection of trading strategies. One of the main results of Brock and Hommes (1998) is that a rational route to randomness occurs; that is, as the intensity of choice increases, the fundamental steady state becomes unstable and a bifurcation route to complicated, chaotic asset price fluctuations arises. The key question to be addressed in this

paper is whether more memory is stabilizing or destabilizing. In particular, we are interested in the question of how memory in the fitness measure affects the primary bifurcation toward instability and how it affects the rational route to randomness.

A natural candidate for evolutionary fitness is a weighted average of current realized profits π_{ht} and last period fitness $U_{h,t-1}$,

$$\begin{aligned}
 U_{h,t} &= \gamma \pi_{h,t} + w U_{h,t-1} \\
 &= \gamma \left[(p_t + y_t - R p_{t-1}) \frac{\mathbf{E}_{h,t-1}[\mathbf{p}_t + \mathbf{y}_t - \mathbf{R} \mathbf{p}_{t-1}]}{a \sigma^2} - C_h \right] + w U_{h,t-1},
 \end{aligned}
 \tag{12}$$

where $R = 1 + r$, $C_h \geq 0$ is an average per period cost of obtaining forecasting strategy h , and $w \in [0, 1)$ is a memory parameter measuring how quickly past realized fitness is discounted for strategy selection. The parameter γ in (12) has been introduced to distinguish between two important cases in the literature. Brock and Hommes (1998) proposed the case $\gamma = 1$, implying that the weights given to past profits decline exponentially; more precisely realized profit k periods ago gets weight w^k . Brock and Hommes (1998), however, as well as almost all subsequent literature, focus the analysis on the case without memory, i.e., $w = 0$, with fitness equal to current realized profit.³ An advantage of the case $\gamma = 1$ is that $w = 1$ corresponds to the benchmark where fitness equals the accumulated excess profit of the risky asset over the risk-free asset.⁴ A disadvantage, however, is that for $\gamma = 1$ the weights are not normalized, but rather sum up to $1/(1 - w)$. The second case studied in the literature assumes $\gamma = 1 - w$, corresponding to the case where the weights are normalized and add up to 1. Note that for $w = 1/T$ and $\gamma = 1 - 1/T$, this case reduces to a T -period average with fixed T [see, e.g., LeBaron (2001) and Diks and van der Weide (2005)]. We will refer to the case $\gamma = 1$ as *cumulative weights* and to the case $\gamma = 1 - w$ as *normalized weights*.⁵

Notice that the two different cases lead to the same distribution of relative weights over past profits, given by $(1, w, w^2, w^3, \dots)$. Stated differently, the relative contribution of past profits to overall fitness is the same for both weighting schemes. For both weighting schemes, an increase of w thus means an increase of memory in the sense that *more weight is given to more distant observations*. However, an increase of w has another, second effect that is different for the two weighting schemes. As stated above, for $\gamma = 1$ all weights add up to $1/(1 - w)$, whereas for $\gamma = 1 - w$ the weights are normalized to 1. This implies a scaling effect for $\gamma = 1$, with the sum of the weights, $1/(1 - w)$, blowing up to infinity as w approaches 1. In particular, for $\gamma = 1$ the *fitness at steady state* is multiplied by a factor $1/(1 - w)$. Hence, for the stability of a steady state, this scaling effect for $\gamma = 1$ is equivalent to an increase of the intensity of choice β by a factor $1/(1 - w)$. Because an increase of the intensity of choice may be destabilizing [Brock and Hommes (1997, 1998)], the scaling effect for $\gamma = 1$ may be a destabilizing force as w increases, not present in the case of normalized weights. Another, related

way of looking at this is to consider the direct effect of current realized profits on fitness. In the case of normalized weights, $\gamma = 1 - w$, the direct effect of current realized profits π_{ht} (getting weight $1 - w$) on fitness vanishes, i.e., tends to 0, as w tends to 1. On the other hand, in the case of cumulative weights, $\gamma = 1$, the direct effect of current realized profits π_{ht} (always getting weight 1) on fitness stays the same, and thus remains nonnegligible, independent of w . As we will see, these differences will lead to different stability results for evolutionary selection.⁶

Fitness (12) can be rewritten in deviations from the fundamental as

$$U_{h,t} = \gamma \left[(x_t - Rx_{t-1} + \delta_t) \left(\frac{g_h x_{t-2} + b_h - Rx_{t-1}}{a\sigma^2} \right) - C_h \right] + wU_{h,t-1}, \tag{13}$$

with $\delta_t = p_t^* + y_t - \mathbf{E}_{t-1}[p_t^* + \mathbf{y}_t]$ a martingale difference sequence, representing intrinsic uncertainty about economic fundamentals. The ABS with linear forecasting rules, in deviations from the fundamental, is given by

$$(1 + r)x_t = \sum_{h=1}^H n_{h,t} (g_h x_{t-1} + b_h) + \varepsilon_t, \tag{14}$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})}, \tag{15}$$

$$U_{h,t} = \gamma \left[(x_t - Rx_{t-1} + \delta_t) \left(\frac{g_h x_{t-2} + b_h - Rx_{t-1}}{a\sigma^2} \right) - C_h \right] + wU_{h,t-1}, \tag{16}$$

where an additional noise term ε_t , e.g., representing a small fraction of noise traders, has been added to the pricing equation and will be used in some stochastic simulations below. A special case, the *deterministic skeleton*, arises when all noise terms are set to zero. To understand the properties of the general stochastic model, it is important to understand the properties of the deterministic skeleton.

3. TWO TYPES OF AGENTS AND INFORMATION COSTS

Consider an ABS with two types of traders and the following forecasting rules:

$$\begin{cases} f_{1,t} = g_1 x_{t-1}, & 0 \leq g_1 < 1, \\ f_{2,t} = g_2 x_{t-1}, & 1 < g_2. \end{cases} \tag{17}$$

Type 1 believes in *mean reversion*, that the price will converge to its fundamental value. In the special case $g_1 = 0$, type 1 becomes a pure fundamentalist, as in Brock and Hommes (1998). In contrast, type 2 believes that price deviations from the fundamental are persistent and will increase.⁷ The dynamics in deviations from

the fundamental is described by the following system:

$$Rx_t = n_{1,t}g_1x_{t-1} + n_{2,t}g_2x_{t-1}, \tag{18}$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^2 \exp(\beta U_{h,t-1})}, \tag{19}$$

$$U_{h,t-1} = \gamma \left[(x_{t-1} - Rx_{t-2}) \left(\frac{g_h x_{t-3} - Rx_{t-2}}{d} \right) - C_h \right] + wU_{h,t-2}, \tag{20}$$

where $C_2 = 0$, but $C_1 = C > 0$ is the information-gathering costs for fundamentalists that agents of type 1 must pay per period. These costs reflect the effort investors incur to collect information about economic fundamentals.

We can rewrite the system above as a five-dimensional map:

$$\begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ U_{1,t-2} \\ U_{2,t-2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{R}(n_{1,t}g_1 + n_{2,t}g_2)x_{t-1} \\ x_{t-1} \\ x_{t-2} \\ \gamma\pi_{1,t-1} + wU_{1,t-2} \\ \gamma\pi_{2,t-1} + wU_{2,t-2} \end{pmatrix}. \tag{21}$$

The following theorem describes the results concerning existence and stability of the steady states (see Appendix A for the proof).

THEOREM 1 (Existence and Stability of the Steady States). *Let us denote the fundamental steady state as $x_f = 0$, and non fundamental steady states as $x_+ = x^* > 0$ and $x_- = -x^* < 0$, where*

$$x^* = \sqrt{\frac{C - \frac{1-w}{\gamma\beta} \log\left(\frac{R-g_1}{g_2-R}\right)}{(R-1)\frac{g_2-g_1}{a\sigma^2}}}, \quad C > 0. \tag{22}$$

Let

$$\beta^* = \frac{1-w}{C\gamma} \log \frac{R-g_1}{g_2-R}. \tag{23}$$

Then three cases are possible:

- (i) $1 < g_2 < R$: the fundamental steady state x_f is the unique steady state and it is globally stable;
- (ii) $R \leq g_2 < 2R - g_1$: the system displays a pitchfork bifurcation at $\beta = \beta^*$ such that
 - (a) — for $0 < \beta < \beta^*$, x_f is unique and stable;
 - (b) — for $\beta > \beta^*$, there are three steady states, x_f , x_+ , and x_- , and the fundamental steady state x_f is unstable;
- (iii) $g_2 \geq 2R - g_1$: there are always three steady states, x_f , x_+ and x_- , and the fundamental steady state x_f is unstable.

When the trend chasers extrapolate only weakly, i.e., $1 < g_2 < R$, the fundamental steady state $x_f = 0$ is globally stable. If $C = 0$, then the two types of agents are equally represented in the market; i.e., $n_1 = n_2 = 1/2$ for any value of β , because the difference in fitnesses $U_2 - U_1 = 0$ at $x = 0$. If agents on average extrapolate very strongly, i.e., $(g_1 + g_2)/2 > R$, the fundamental steady state is unstable and there are always two additional nonfundamental steady states $x = x_+ > 0$ and $x = x_- < 0$, even when there are no information costs. The case with strongly extrapolating trend chasers, i.e., $R < g_2 < 2R - g_1$, is the most interesting. If there are no information costs, $C = 0$, the fundamental steady state is stable for all values of β and agents are equally distributed over the two types due to equality of profits. But when $C > 0$, the fundamental steady state is stable only if the agents are not too sensitive to switch the prediction strategy, i.e., for $\beta < \beta^*$. As the intensity of choice increases ($\beta > \beta^*$), most of the agents switch to using the cheap prediction rule, because if the price is in a small neighborhood of its fundamental value, then due to information costs, the first type of agents have lower profits, and for large β a majority of agents switch to the trend-extrapolating strategy.

It can be seen immediately from expressions (22) and (23) how memory affects the primary bifurcation of the system. In the case with normalized weights ($\gamma = 1 - w$), memory does *not* affect the stability. However, in the case of cumulative weights ($\gamma = 1$) and positive information-gathering costs for fundamentalists, memory does affect the stability and in fact it *destabilizes* the system; i.e., with more memory the primary bifurcation occurs earlier. This is due to a scaling effect when the parameter w increases, leading to a higher effective intensity of choice and thus to an earlier bifurcation of the fundamental steady state.

As a typical example, consider an ABS with the following two prediction rules:

$$f_{1,t} = 0.5x_{t-1}, \tag{24}$$

$$f_{2,t} = 1.2x_{t-1}. \tag{25}$$

Traders of the first type believe that the next-period deviation of the price from the fundamental will be two times less than in the current period, whereas traders of the second type predict an increase in the deviation of the price from fundamental.

It follows from Theorem 1 that the fundamental steady state $x_f = 0$ is unique and stable for $\beta \in (0, \beta^*)$, with $\beta^*(w) = 1.79(1 - w)/\gamma$. When the parameter β passes the critical value β^* , the fundamental steady state loses stability due to a pitchfork bifurcation and two new stable equilibria of the price dynamics appear.

Next consider the two different cases: cumulative versus normalized weights.

3.1. Cumulative Weights ($\gamma = 1$)

In the case with accumulated profits, i.e., when $\gamma = 1$, the pitchfork bifurcation curve is given by $\beta^*(w) = 1.79(1 - w)$, which is declining with respect to the

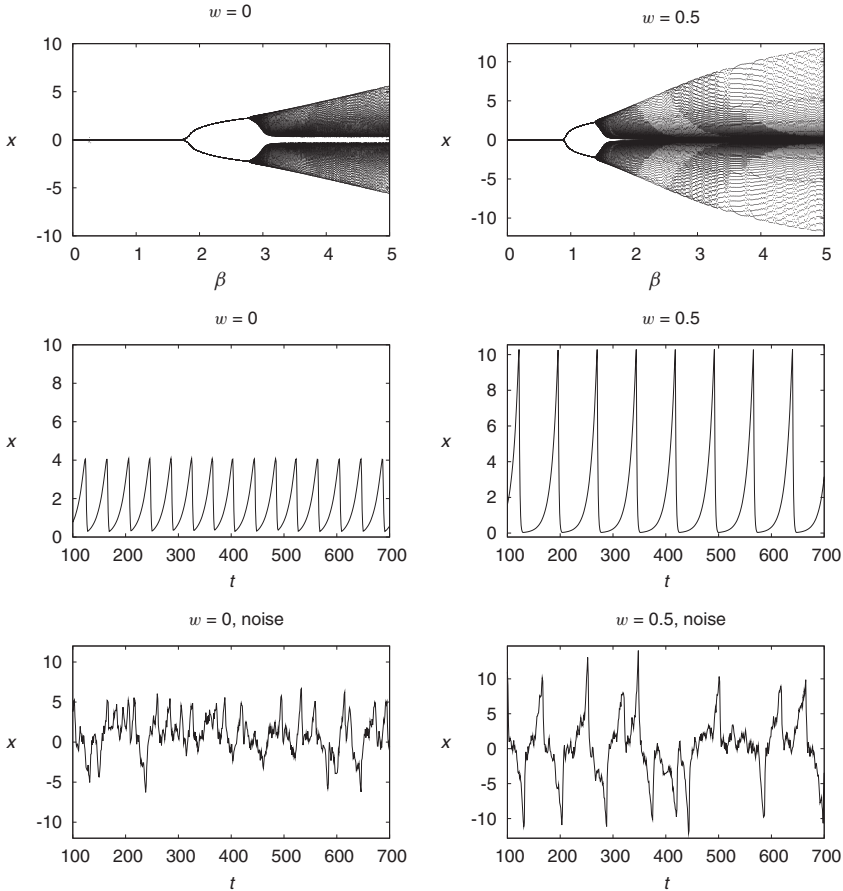


FIGURE 1. The case of two types of prediction rules and accumulated profits ($\gamma = 1$). The left column corresponds to $w = 0$; the right column corresponds to $w = 0.5$. Upper figures display bifurcation diagrams with respect to β . Time series of the price deviation are represented by the middle figures (without noise) and the lower figures (with noise). Belief parameters are $g_1 = 0.5$ and $g_2 = 1.2$; the other parameters are $\beta = 4$, $R = 1.1$, $C = 1$, and $d = 1$.

memory parameter. This means that memory destabilizes the price dynamics: the larger w the earlier the primary bifurcation occurs.

Figure 1 illustrates the dynamics without memory ($w = 0$, left panel) and with memory ($w = 0.5$, right panel). In both cases a rational route to randomness, that is, a bifurcation route to complicated dynamics as the intensity of choice increases, occurs. Notice that, with memory in the fitness measure, the temporary bubbles and crashes in the price series occur less frequently, but when they occur they last longer, with much larger deviations from the fundamental benchmark.

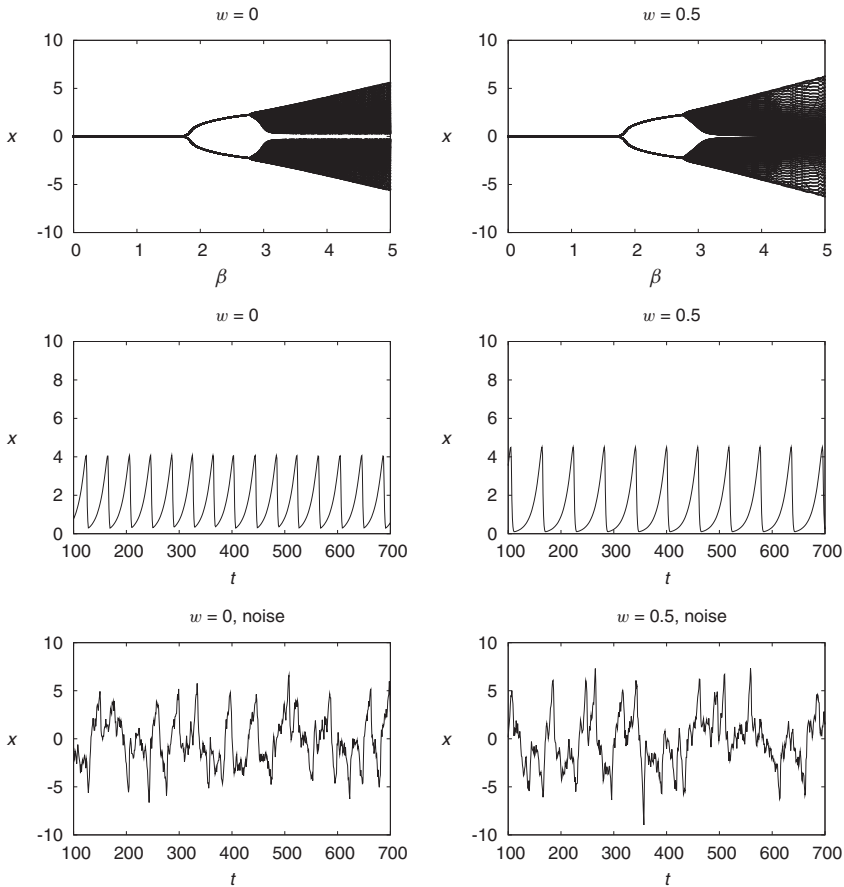


FIGURE 2. The normalized fitness measure case ($\gamma = 1 - w$): time series of the price deviation from its fundamental value for different levels of the memory. Belief parameters are $g_1 = 0.5$ and $g_2 = 1.2$; the other parameters are $\beta = 4$, $R = 1.1$, $C = 1$, and $d = 1$.

3.2. Normalized Weights ($\gamma = 1 - w$)

In the case with normalized weights, i.e., when $\gamma = 1 - w$, the pitchfork bifurcation curve is given by $\beta^*(w) = 1.79$. Hence, memory does not affect the stability of the fundamental steady state. Figure 2 illustrates the dynamics without memory ($w = 0$, left panel) and with memory ($w = 0.8$, right panel). Although less pronounced, memory has a similar effect on price fluctuations: with memory in the fitness measure, the temporary bubbles and crashes in the price series occur less frequently, but once started, bubbles last longer, with larger swings away from the fundamental benchmark.

The bottom panels of Figures 1 and 2 contain time series simulations in the presence of noise, represented by a small fraction of noise traders. Whereas in

the deterministic simulations the chaotic bubbles and crashes are still somewhat predictable, in the presence of noise they become very irregular and highly unpredictable.

4. STABILITY IN THE MODEL WITH H TYPES

Brock and Hommes (1998) stressed the importance of *simple* forecasting rules, because it is unlikely that enough traders will coordinate on a complicated rule for it to have an impact in real markets. The learning-to-forecast laboratory experiments of Hommes et al. (2005) show that simple, linear forecasting rules with only a few lags describe individual forecasting behavior surprisingly well. In this section, we investigate the role of memory in an ABS with an arbitrary number H of linear forecasting rules with one lag, i.e.,

$$f_{i,t} = g_i x_{t-1} + b_i, \quad g_i, b_i \in \mathbf{R}, \quad i = 1, \dots, H, \tag{26}$$

and without information-gathering costs, i.e., $C_i = 0$ for all $i = 1, \dots, H$. The coevolution prices and beliefs are described by the following difference equation:

$$Rx_t = \sum_{h=1}^H n_{h,t} (g_h x_{t-1} + b_h), \tag{27}$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})}, \tag{28}$$

$$\begin{aligned} U_{h,t-1} &= \gamma \left[(x_{t-1} - Rx_{t-2}) \left(\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{d} \right) \right] + wU_{h,t-2} \\ &= \gamma \pi_{h,t} + wU_{h,t-2}, \end{aligned} \tag{29}$$

with $d = a\sigma^2$. Equation (27) can be rewritten as an $(H+3)$ -dimensional map

$$\begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ U_{1,t-2} \\ \dots \\ U_{H,t-2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{R} \sum_{h=1}^H n_{h,t} (g_h x_{t-1} + b_h) \\ x_{t-1} \\ x_{t-2} \\ \gamma \pi_{1,t-1} + wU_{1,t-2} \\ \dots \\ \gamma \pi_{H,t-1} + wU_{H,t-2} \end{pmatrix}. \tag{30}$$

The following theorem describes the results concerning existence and stability of the fundamental steady state (see Appendix B for the proof).

THEOREM 2 (Existence and Stability of the Fundamental Steady State). *Assume that*

1. *The average bias equals zero, i.e., $\sum_{i=1}^H b_i = 0$.*
2. *There is at least one non-zero bias, i.e., $V = \frac{1}{H} \sum_{i=1}^H b_i^2 > 0$.*
3. *The mean trend is not too strong, i.e., $|\bar{g}| = |\frac{1}{H} \sum_{i=1}^H g_i| < R$.*

Then the fundamental price $x_f = 0$ is a steady state of (30). The fundamental steady state is stable for $0 \leq \beta < \beta_{NS}$, where

$$\beta_{NS} = \frac{a\sigma^2}{V\gamma} \left(1 - \frac{\bar{g}}{R} w \right) > 0. \tag{31}$$

*At the value $\beta = \beta_{NS}$ the steady state loses stability due to a Neimark–Sacker bifurcation. For $\beta > \beta_{NS}$ the fundamental steady state is unstable.*⁸

The assumption that the average bias is zero seems reasonable, as there is no a priori reason that the average bias would be negative or positive.⁹ The other two assumptions, that there is at least one nonzero bias and that the average trend over all rules is not too strong, also seem plausible. The theorem says that, under these assumptions, the dynamic behavior of the price of the risky asset is independent of the number of agent’s strategies, but depends on the mean value \bar{g} of the trend-extrapolating coefficients g_h and the diversity or spread V of the biases b_h . The larger the absolute average trend $|\bar{g}|$, the lower β_{NS} and the earlier the primary bifurcation occurs; if the trend chasers on average extrapolate more heavily away from the fundamentals, the system destabilizes faster. Similarly, the greater the variance V in biases, the lower β_{NS} and again the bifurcation occurs earlier; if there is more variability among biased traders, the price dynamics becomes unstable earlier. Note that for the special case $\bar{g} = 0$ and $\gamma = 1$, memory does not affect the stability of the fundamental steady state, because $\beta_{NS} = a\sigma^2/V$ [cf. Brock and Hommes (1998)].

In the case $\gamma = 1$, i.e., in the case of cumulative weights, the Neimark–Sacker bifurcation curve (31) becomes a straight line,

$$\beta_{NS} = \frac{a\sigma^2}{V} \left(1 - \frac{\bar{g}}{R} w \right), \tag{32}$$

as illustrated in Figure 3 (left panel). The slope of the line depends on the sign of the average trend extrapolation \bar{g} . If agents on the average extrapolate positively, then the line is decreasing and the bifurcation w.r.t. β comes earlier with more memory. The intuition is that positive trend extrapolation reinforces market movements away from the fundamentals and the system destabilizes faster. On the other hand, if agents on average are contrarians extrapolating negatively, then (32) is an increasing line and the bifurcation w.r.t. β comes later with more memory. Here the intuition is that contrarian behavior counterbalances market movements away from the fundamentals and the system destabilizes more slowly.

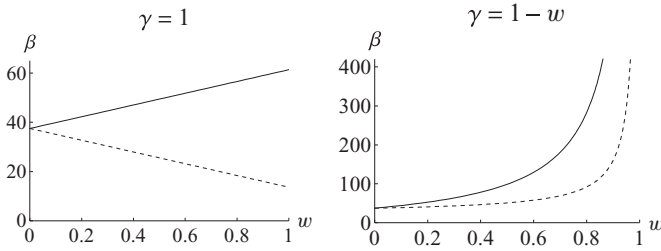


FIGURE 3. Neimark–Sacker bifurcation curves β_{NS} in (31) for different values of the parameters γ and \bar{g} : dotted lines correspond to the case $\bar{g} > 0$, whereas solid lines correspond to the case $\bar{g} < 0$. For the case with $\gamma = 1$ (left panel) the bifurcation curves are straight lines, whereas for $\gamma = 1 - w$ (right panel) they are hyperbolas. In the case $\gamma = 1$ (left panel) and $\bar{g} > 0$, memory has a destabilizing effect on the dynamics; i.e., the bifurcation w.r.t. β comes earlier. In contrast, in the case $\gamma = 1 - w$ (right panel), more memory always has a stabilizing effect.

In the case with normalized weights, $\gamma = 1 - w$, the Neimark–Sacker bifurcation curve (31) becomes a “hyperbola” for both positive and negative values of \bar{g} (see Figure 3, right panel):

$$\beta_{NS} = \frac{a\sigma^2}{V(1-w)} \left(1 - \frac{\bar{g}}{R} w \right). \tag{33}$$

In the case of normalized weights, memory is always stabilizing (independent of the average extrapolation factor \bar{g}). Notice that the Neimark–Sacker bifurcation values (32) and (33) only differ by a factor $(1 - w)$ in the denominator of (33) representing the scaling effect when weights are not normalized. Comparing the left and right panels of Figure 3, this scaling effect dominates when average trend extrapolation $\bar{g} > 0$ and destabilizes the system when the memory parameter w increases in the case of cumulative weights (i.e., $\gamma = 1$).

5. NUMERICAL SIMULATION OF A THREE-TYPE EXAMPLE

In this section we discuss a simple, but typical ABS with three types of traders in order to illustrate the differences in impact of the memory strength on the stability of the fundamental price in the two cases of cumulative weights ($\gamma = 1$) and normalized weights ($\gamma = 1 - w$).

Consider the ABS with the following three types of prediction rules:

$$f_{1,t} = 0, \tag{34}$$

$$f_{2,t} = 1.2x_{t-1} - 0.2, \tag{35}$$

$$f_{3,t} = 0.9x_{t-1} + 0.2. \tag{36}$$

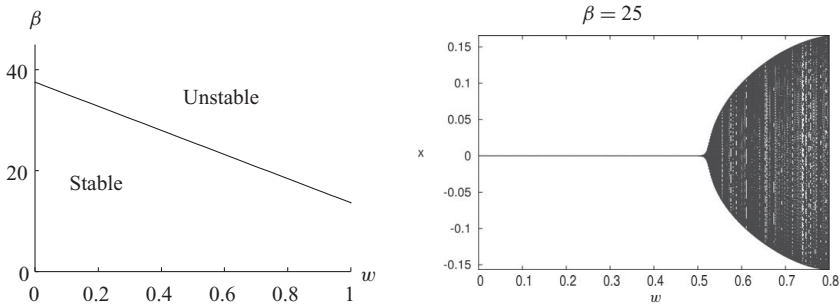


FIGURE 4. Neimark–Sacker bifurcation curve (left panel) and bifurcation diagram with respect to the memory parameter w (right panel) for the model with three types of agents and fitness given by accumulated profits, i.e., $\gamma = 1$. Belief parameters are $g_1 = 0, b_1 = 0; g_2 = 1.1, b_2 = -0.2;$ and $g_3 = 0.9, b_3 = 0.2$. Other parameters are $R = 1.1, a\sigma^2 = 1,$ and $\beta = 25$ (for the right panel). The Neimark–Sacker bifurcation curve divides the (w, β) –plane into two regions; for the parameter values in the upper region the fundamental steady state is unstable, whereas for the parameter values in the lower region it is stable.

The second and third types are symmetrically opposite–biased positive trend extrapolators; the first type are fundamentalists. The remaining parameters are fixed at $R = 1.1, a\sigma^2 = 1$. Because $\bar{g} = 0.7 < R, V = 0.08/3 \neq 0,$ and biases sum up to zero, according to Theorem 2, the fundamental steady state loses stability in a Neimark–Sacker bifurcation at $\beta = \beta_{NS}$:

$$\beta_{NS} = \frac{37.5 - 23.9w}{\gamma}. \tag{37}$$

5.1. The Case $\gamma = 1$

In the case with cumulative weights, i.e., when $\gamma = 1,$ the Neimark–Sacker bifurcation curve is a declining straight line:

$$\beta_{NS} = 37.5 - 23.9w. \tag{38}$$

As can be seen from Figure 4, in this case memory destabilizes the price dynamics; with higher memory strength the bifurcation occurs earlier, i.e., for smaller values of β . Because both nonfundamentalist agents extrapolate positively, and thus the average trend extrapolation is also positive, in accordance with our findings from Section 4, the extrapolation of trend reinforces market movements away from the fundamentals and the bifurcation line is thus decreasing. In addition, it can be observed in the bifurcation diagram of Figure 4 (right panel) how, for a fixed β -value, the fundamental steady state becomes unstable and complicated, chaotic price movements arise as the memory parameter w increases. Figure 4 (right panel) also illustrates that the amplitude of price fluctuation increases as memory

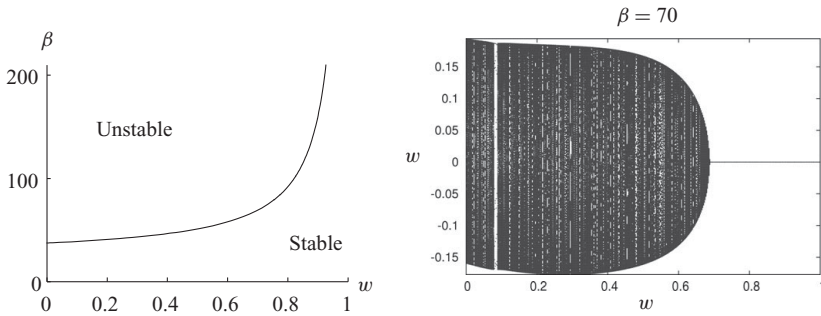


FIGURE 5. Neimark–Sacker bifurcation curve (left) and bifurcation diagram with respect to the memory (right) for the model with three types of agents’ strategies and normalized fitness measure, i.e., $\gamma = 1 - w$. Belief parameters are $g_1 = 0, b_1 = 0; g_2 = 1.1, b_2 = -0.2;$ and $g_3 = 0.9, b_3 = 0.2$. Other parameters are $R = 1.1, d = 1,$ and $\beta = 70$ (for the right figure). The Neimark–Sacker bifurcation curve divides the (w, β) –plane into two regions; for the parameter values in the upper region the fundamental steady state is unstable, whereas for the parameter values in the lower region it is stable.

increases, in accordance with our earlier finding that bubbles last longer with more memory.

5.2. The Case $\gamma = 1 - w$

In the case with normalized weights, i.e., when $\gamma = 1 - w$, the Neimark–Sacker bifurcation curve (37) becomes a “hyperbola”:

$$\beta_{NS} = \frac{37.5 - 23.9w}{1 - w}. \tag{39}$$

As can be seen from Figure 5 (left panel), more memory now stabilizes the price dynamics; an increase in the memory strength makes the bifurcation occur later, i.e., for larger values of β . Even when the traders are on average positive trend extrapolators (with some bias), if the weight on cumulative past fitness (the memory strength w) is high enough compared to the weight on current realized profits ($\gamma = 1 - w$), the dynamics is stable. Indeed, the bifurcation diagram in Figure 5 (right panel) shows that, for a given β , the dynamics stabilizes from chaotic movements (interspersed with stable cycles) for low values of the memory parameter w to a stable fundamental steady state when memory w is sufficiently large.

6. CONCLUSION

We investigated how memory affects the stability of evolutionary selection dynamics in a simple, analytically tractable asset-pricing model with heterogeneous beliefs. By complementing the stability analysis with local bifurcation theory,

we were able to analyze the effects of adding different amounts of memory to the fitness measure on the stability of the fundamental steady state. Whether memory is stabilizing or destabilizing depends on three key factors: (1) whether we have a fitness measure with cumulative weights or normalized weights; (2) the ecology (i.e., the composition of the set) of forecasting rules, and in particular the average strength of trend extrapolation and the spread in biased forecasts; and (3) whether or not costs for information gathering of economic fundamentals have to be incurred.

When there are costs for gathering fundamental information, more memory in the fitness measure does *not* stabilize the dynamics. In the case with normalized weights, due to the information-gathering costs, memory has no effect on stability; in the case of cumulative weights, when there are information-gathering costs for fundamentalists, more memory is destabilizing due to a scaling effect, leading to a larger effective intensity of choice.

We have also studied the model with an arbitrary number of linear forecasting rules with one lag and no costs for information gathering. The stability depends critically on the ecology of forecasting rules. In particular, the system may become unstable more easily when the average trend parameter and/or the variability of biased forecasts become larger. How memory affects the stability of the fundamental steady state depends again on whether we have cumulative weights or normalized weights. In the case of normalized weights, more memory is always stabilizing: with more memory the first bifurcation toward instability comes later. In the case of cumulative weights the effect of memory on the stability depends on the direction of average trend extrapolation. If agents on the average are contrarians, extrapolating negatively, more memory stabilizes the system; if on the other hand agents on the average extrapolate positively, memory destabilizes the system. This is due to a dominant scaling effect on fitness at steady state, when weights are cumulative, which destabilizes the system if average trend extrapolation is positive.

Our analysis yields a precise mathematical classification of the stability of evolutionary selection for cumulative versus normalized weights in the fitness measure within a very simple modeling framework. Which of these two fitness measures is more relevant in reality is an empirical and behavioral question. Is individual choice, for example individual portfolio selection in financial markets, driven by cumulative fitness (e.g., accumulated wealth) or by normalized fitness (e.g., average realized returns)? In particular, how much weight do individuals put on the most recently observed fitness? Our theoretical results show that the more weight they put on the most recent observation, the more easily the system may destabilize. Future research with laboratory experiments with human subjects may shed light on which behavioral assumptions fit individual decision making in strategy selection more closely and, in particular, how much weight individuals put on the most recent observations.

The difference between cumulative versus normalized weights, as expressed through the weighting coefficients $\gamma = 1$ versus $\gamma = 1 - w$, is related to the more

general issue of whether one should use a cumulative or normalized fitness measure in strategy-switching models. An advantage of normalization is that one can compare the magnitude of the intensity-of-choice parameter across different normalized fitness measures and market settings. The intensity-of-choice parameter is notoriously hard to estimate and only few significant results have been obtained. Boswijk et al. (2007) estimate the intensity of choice in an asset-pricing model with heterogeneous beliefs using yearly S&P500 data, whereas Goldbaum and Mizrach (2008) estimate the intensity of choice in mutual fund allocation decisions. Our results stress the importance of normalization of the fitness measure in empirical applications. But in general it is not clear how exactly a fitness measure should be normalized, especially when the fitness (such as realized profits) may attain (arbitrarily large) positive as well as negative values. The normalization itself may affect, e.g., the primary bifurcation toward instability. Laboratory experiments on individual selection among different strategies with a normalized fitness measure may give useful estimates of the intensity of choice of individual strategy selection across different market settings.

NOTES

1. Another related paper is Levy et al. (1994), who simulate an agent-based microscopic stock market model with a fixed memory length of 10.

2. Gaunersdorfer (2000) investigates the case with time-varying beliefs about variances and shows that the asset price dynamics are quite similar. Chiarella and He (2002) and Chiarella et al. (2003) investigate the model with heterogeneous risk aversion coefficients.

3. It is interesting to note that Anufriev and Hommes (in press) fit an evolutionary selection model to data from laboratory experiments and use a memory parameter $w = 0.7$.

4. There is a large related literature on wealth-driven selection models with heterogeneous investors, with fractions of each type determined by relative wealth. See, e.g., Anufriev and Bottazzi (2006) and Anufriev (2008) for some recent contributions and Chiarella et al. (2009) and Evstigneev et al. (2009) for extensive up-to-date reviews.

5. This terminology is similar to that used in the experience-weighted attraction (EWA) learning in games literature [e.g., Camerer and Ho (1999) and Camerer (2003)], where a parameter moves from 0 to 1 between the extremes of cumulative and average reinforcement.

6. The difference between cumulative weights versus normalized weights as expressed through the weighting coefficients $\gamma = 1$ versus $\gamma = 1 - w$ is related to the more general issue of cumulative versus normalized fitness measure $U_{h,t}$; see the final section for more discussion.

7. Boswijk et al. (2007) estimated this ABS with two types of investors using yearly S&P 500 data and found coefficients of $g_1 \approx 0.8$ and $g_2 \approx 1.15$, thus suggesting behavioral heterogeneity.

8. Note that in the special case $V = 0$ all biases equal zero, and if $|\bar{g}| < R$ the fundamental steady state is stable for all values of β and w .

9. If the average bias is nonzero but close to 0, the fundamental price is not a steady state but the system has a steady state close to the fundamental. In that case, a stability analysis becomes much more cumbersome, however.

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APPENDIX A: PROOF OF THEOREM 3.1

The steady states of the map (21) satisfy the equation

$$Rx = x \left[\frac{g_1}{1 + \exp(\beta\Delta)} + \frac{g_2}{1 + \exp(-\beta\Delta)} \right], \quad (\text{A.1})$$

where $\Delta = \frac{\gamma}{1-w} [(1-R)(\frac{g_2-g_1}{d})x^2 + C]$.

It is easy to see that the fundamental steady state $x_f = 0$ always exists. The other (nonfundamental) steady state is a solution of the equation

$$\exp \left\{ \beta \frac{\gamma}{1-w} \left[(1-R) \frac{g_2 - g_1}{d} x^2 + C \right] \right\} = \frac{R - g_1}{g_2 - R}. \tag{A.2}$$

Note that if $(R - g_1)/(g_2 - R) \leq 0$ there are no solutions for this equation. If we take into account that $g_1 < 1$ then we can conclude that for $1 < g_2 < R$ the map (18)–(20) is contracting and has a unique globally stable steady state $x_f = 0$.

Assume now that $g_2 > R$; then we can obtain nonfundamental steady states from the equation

$$x^2 = \frac{C - \frac{1-w}{\beta\gamma} \ln \frac{R - g_1}{g_2 - R}}{(R - 1) \frac{g_2 - g_1}{d}}, \tag{A.3}$$

which has solutions $x = \pm x^*$, when its right-hand side is positive. It is satisfied for $\beta > \beta^*$ in (23) if $R \leq g_2 < 2R - g_1$, and for any positive β if $g_2 \geq 2R - g_1$. This proves the statements about existence of equilibria in (i), (ii), and (iii).

To explore the stability of the fundamental steady state, we need to compute eigenvalues of the Jacobian matrix

$$J(x_f) = \begin{pmatrix} g_1 + g_2 \exp \left(\frac{C\beta\gamma}{1-w} \right) & 0 & 0 & 0 & 0 \\ \frac{1}{\left[1 + \exp \left(\frac{C\beta\gamma}{1-w} \right) \right]} R & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & w \end{pmatrix}. \tag{A.4}$$

The characteristic equation is given by

$$(w - \lambda)^2 \lambda^2 \left\{ g_1 \exp \left(\frac{-C\gamma\beta}{1-w} \right) + g_2 - R\lambda \left[1 + \exp \left(\frac{-C\gamma\beta}{1-w} \right) \right] \right\} \tag{A.5}$$

and thus

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = w, \quad \lambda_5 = \frac{g_1 \exp \left(\frac{-C\gamma\beta}{1-w} \right) + g_2}{R \left[1 + \exp \left(\frac{-C\gamma\beta}{1-w} \right) \right]} > 0. \tag{A.6}$$

Note that all eigenvalues are real and nonnegative, so the only bifurcation that can occur is a pitchfork bifurcation, which happens if

$$\lambda_5 = 1 \Leftrightarrow \beta = \beta^*. \tag{A.7}$$

This means that if $g_2 \in [R, 2R - g_1]$ for $\beta \in (0, \beta^*)$, there exists a unique stable fundamental steady state, and at the critical parameter value $\beta = \beta^*$, two nonfundamental steady states occur due to a pitchfork bifurcation.

APPENDIX B: PROOF OF THEOREM 4.1

Note that at the fundamental steady state all fitnesses are equal to zero; i.e., $U_h^* = 0$ for $h = 1, \dots, H$, which implies that all fractions are equal, $n_h^* = 1/H$. Therefore the steady state price satisfies the equation

$$Rx^* = \frac{1}{H} \sum_{h=1}^H (g_h x^* + b_h) \tag{B.1}$$

and thus

$$x^* (R - \bar{g}) = \frac{1}{H} \sum_{h=1}^H b_h. \tag{B.2}$$

It is clear that the fundamental steady state exists if and only if $\sum_{h=1}^H b_h = 0$.

The Jacobian of (30) computed at the fundamental steady state is given by

$$\begin{pmatrix} \frac{d\bar{g} + V\gamma\beta}{d} & -\frac{V\gamma\beta}{d} & 0 & J_{1,1} & \dots & & J_{1,H} \\ 1 & 0 & 0 & 0 & \dots & & 0 \\ 0 & 1 & 0 & 0 & \dots & & 0 \\ \frac{b_1\gamma}{d} & -\frac{b_1R\gamma}{d} & 0 & w & 0 & \dots & 0 \\ \frac{b_2\gamma}{d} & -\frac{b_2R\gamma}{d} & 0 & 0 & w & 0 & \dots & 0 \\ \vdots & \ddots & & & & & & \\ \frac{b_H\gamma}{d} & -\frac{b_HR\gamma}{d} & 0 & 0 & \dots & 0 & w \end{pmatrix},$$

where $d = a\sigma^2$ and

$$J_{1,s} = -\frac{b_s w \beta}{HR}, \quad s = 1, \dots, H.$$

The characteristic equation for the fundamental steady state is given by

$$\lambda^2(w - \lambda)^{H-1} \underbrace{\{dw\bar{g} + R\beta V\gamma + [-d(\bar{g} + R w) - \beta V\gamma]\lambda + dR\lambda^2\}}_{p(\lambda)} = 0. \tag{B.3}$$

The characteristic equation (B.3) has $H+3$ roots, where $H+1$ of them are inside the unit circle, $\lambda_3 = \lambda_4 = 0$ and $\lambda_5 = \dots = \lambda_{H+3} = w < 1$, whereas the other two are roots of the polynomial $p(\lambda)$ and thus they determine stability of the steady state. If $p(\lambda)$ has at least one root outside of the unit circle, the steady state is unstable. We denote roots of $p(\lambda)$ as λ_1 and λ_2 .

Let us now explore three cases where one or two roots of $p(\lambda)$ are crossing a unit circle:

1. $\lambda_1 = 1$, *pitchfork bifurcation*,

$$p(1) = 9d(R - \bar{g})(1 - w) + 9V(R - 1)\gamma\beta.$$

If $V = 0$ then $p(1) > 0$ for $w \in [0, 1)$ and $|\bar{g}| < R$. If $V > 0$ then

$$p(1) = 0 \Leftrightarrow \beta = \frac{d(1-w)(\bar{g}-R)}{V(R-1)\gamma} < 0 \text{ for } \bar{g} < R, \tag{B.4}$$

which means that this type of bifurcation cannot occur in the system.

2. $\lambda_1 = -1$, *period-doubling bifurcation*,

$$p(-1) = 9d(R + \bar{g})(1 + w) + 9V(R + 1)\gamma\beta.$$

If $V = 0$ then $p(-1) > 0$ for $w \in [0, 1)$ and $|\bar{g}| < R$. If $V > 0$ then

$$p(-1) = 0 \Leftrightarrow \beta = \beta_{PD} = -\frac{4(\bar{g} + R)(1 + w)}{V(1 + R)(1 - w)} < 0,$$

which means that this type of bifurcation cannot occur in the system either.

3. $\lambda_{1,2} = \mu_1 \pm \mu_2 i$, where $\mu_2 > 0$ and $\mu_1^2 + \mu_2^2 = 1$, *Neimark–Sacker bifurcation*.

Using Vieta’s formula, we get

$$\mu_1^2 + \mu_2^2 = \lambda_1 \lambda_2 = \frac{d\bar{g}w + RV\beta\gamma}{dR} = 1. \tag{B.5}$$

If $V = 0$, the equation (B.5) does not have solutions for $w \in [0, 1)$ and $|\bar{g}| < R$. Therefore all eigenvalues corresponding to the fundamental steady state are inside the unit circle and thus the steady state is stable for $w \in [0, 1)$ and $\beta \geq 0$.

If $V > 0$, we obtain from (B.5) the equation of the Neimark–Sacker bifurcation curve

$$\beta_{NS} = \frac{d}{V\gamma} \left(1 - \frac{\bar{g}}{R} w \right). \tag{B.6}$$

We have to make sure that $\mu_2 \neq 0$ or equally $\mu_2^2 > 0$. Because $\mu_1^2 + \mu_2^2 = 1$, the latter inequality holds if $\mu_1^2 < 1$. Using again Vieta’s formula, we have

$$\mu_1 = \frac{\lambda_1 + \lambda_2}{2} = \frac{d(\bar{g} + R w) + \beta V \gamma}{2dR} > 0.$$

To make sure that $\mu_1^2 < 1$, we need to check the inequality

$$\frac{d(\bar{g} + R w) + V\beta\gamma}{2dR} < 1.$$

Together with (B.6), it implies

$$w(R^2 - \bar{g}) < R(2R - 1 - \bar{g}), \tag{B.7}$$

which is satisfied for $|\bar{g}| < R$ and any value of $w \in [0, 1)$.

Our analysis shows that the Neimark–Sacker bifurcation is the only bifurcation that occurs in the system. It happens for $\beta = \beta_{NS}$ as in (B.6) and leads to a loss of stability of the fundamental steady state. ■