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OUT-OF-EQUILIBRIUM RANDOM WALKS

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Abstract

We study the long-term behaviour of a random walker embedded in a growing sequence of graphs. We define a (generally non-Markovian) real-valued stochastic process, called the knowledge process, that represents the ratio between the number of vertices already visited by the walker and the current size of the graph. We mainly focus on the case where the underlying graph sequence is the growing sequence of complete graphs.

Keywords: Random walks; growing graphs; asymptotics

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1. Introduction

The following models of dynamics *in graphs* and *of graphs* have been extensively studied during the past decades.

Dynamic Agent in a Static World. Let G_0 be a finite connected undirected graph. Let V_0 be the set of its vertices, and assume that $|V_0| = N$. Also, let E_0 be the set of its edges. We denote the vertices of G_0 by the first N natural numbers. For a given vertex v, we denote by $\deg_{G_0}(v)$ its degree in the graph G_0. Also, if $\{u, v\} \in E_0$, we set $u \sim_{G_0} v$. We start a random walk at some given vertex drawn from a prescribed probability distribution on the set V_0. For the time being, consider the random walk to be a simple walk; i.e., the walker evolves according to the transition kernel

$$P_{G_0}(u, v) = \frac{1}{\deg_{G_0}(u)} \mathbf{1}_{\{u \sim_{G_0} v\}}.$$

In this *static* context, we have at our disposal several measures of the performance of the random walk, the **hitting times** and **cover time** being the most important for our purposes. For a random walk started at X_0 and for some arbitrary $v \in V_0$, the hitting time (or first passage time) τ_v is the first time that the walker visits v: that is, $\tau_v := \min\{n \geq 0 : X_n = v\}$. The cover time of the random walk is defined as $\mathrm{Cov}(G_0) = \max_{v \in G_0} \tau_v$. Thanks to the finiteness and connectedness of the graph, the cover time is finite almost surely (a.s.). A classical result states that for a connected simple finite graph, the expectation of the cover time is polynomially bounded above by $O(n^3)$, with the bound being achieved on the lollipop graph, and that the lowest possible order is achieved on the complete graph with expectation $O(n \log(n))$ (see the important paper [2]). Matthews has shown that for any finite connected graph, the expectation

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of the cover time satisfies

$$\mathbb{E}_{X_0}(\mathrm{Cov}(G_0)) \leq \max_{u,v\in V_0} \{\mathbb{E}_u(\tau_v)\} \sum_{j=1}^{N-1} \frac{1}{j} ,$$

with equality in the case of the complete graph (see [15] for the original result, and [13], [1] for an introduction to what David Aldous calls the Matthews method). Of course, tighter bounds can be proved as soon as we have more information about the structure of the underlying graph. See the survey [14] and the textbooks [13], [1] for a gentle introduction to this topic.

Changing World. Consider now a sequence of graphs $G_0, G_1, \ldots$, where each graph is finite and connected. In this work we will make some simplifying assumptions on the sequence. Specifically, we assume that the sequence is **growing** and **consistent**. We say that the graph sequence is growing if for each $k \geq 1$, $|V_k| \leq |V_{k+1}|$. We say that the sequence is consistent if for each $k \geq 1$, the subgraph of G_k induced by the first $|V_{k-1}|$ vertices equals G_{k-1}. Accordingly, without loss of generality, we assume that the vertices of each graph G_k are labelled with $\{1, 2, \ldots, |V_k|\}$. The sequence of graphs could be either deterministic or the product of a generative model of random graphs, the classical example being the Barabási–Albert preferential attachment model as presented in [5] and [6]. For the time being, we assume that the sequence evolves according to a (possibly degenerate) Markov kernel $P^{\mathcal{G}}(\cdot, \cdot)$, and that the aforementioned conditions of growing and consistency of the trajectories hold a.s.

Consider now the situation where the above dynamics are interwoven.

Dynamic Agent in a Changing World. A random walker of the type presented in the first item above wanders in an ever-changing sequence of graphs of the type presented in the second item. Although for technical reasons we will prefer a continuous-parameter model, at this point the discrete-parameter one is more amenable to verbal description. So, let G_0 be a finite connected graph and fix a vertex in V_0. Also, fix a parameter $0 < p < 1$ (the extremes are excluded to avoid trivialities). We declare the **current world** of the walker to be G_0. For $n \geq 1$, given X_{n-1} and given that the current world is F, with probability $1 - p$ we perform a step of the random walk according to $P^F(X_{n-1}, \cdot)$ (i.e. X_n is drawn according to the transition kernel of the current world), whereas with probability p the current world evolves according to $P^{\mathcal{G}}(F, \cdot)$ and the walker stays put (i.e. we set $X_n = X_{n-1}$).

This is the kind of dynamics we are concerned with in this work. Models akin to ours have been proposed previously. In [3], the authors consider a model termed *Random Walk in Changing Environment*, where a random walker wanders inside a maze with a changing landscape of conductances. There, the problem addressed is the determination of the recurrence or transience of the walk under the appropriate conditions of boundedness of the conductances and its dependence on the walker's state. In [4], the authors propose a model called *Evolving Graphs*, where the connection configuration undergoes mutation due to edge addition and/or deletion on a fixed set of vertices. The main quantity studied in [4] is the cover time of a simple random walk. It is shown that, in some circumstances, this quantity may greatly exceed the polynomial bound that holds on fixed graphs and may reach, for example, exponential order.

Observe that in the first work mentioned, the focus is on qualitative aspects of the walk, and the most important results are those that assume that the underlying conductances are bounded away from zero. This hypothesis is far from trivial, and accordingly there is no obvious extension to the case where the connections have a binary status. In the second paper, however rich the dynamics of the connections may be, the model is constrained to have a fixed number of vertices, and, again, there seems to be no clue as to how to adapt the results to a growing sequence of graphs.

The setting closest to the one we have termed *Dynamic Agent in a Changing World* is studied in [7], where the authors consider a random walker moving on a growing sequence of graphs. It is worth taking a closer look at the models and main results of [7]. The authors focus on two models of growth. In both, the initial graph G_1 consists of just one vertex and $m \geq 2$ loops. In Model 1, given the graph G_{t-1}, the graph G_t is built by adding a new vertex and m edges whose endpoints are chosen uniformly amongst the vertices in G_{t-1}. Model 2 is a version of the preferential attachment graph: given G_{t-1}, the new vertex throws m edges whose endpoints are chosen independently amongst the already present vertices in G_{t-1}, with probabilities proportional to their degrees. As the authors explain, this may cause self-loops and multiple edges, but for large times these perturbances are negligible. Now, while vertex t is being added, the walker is sitting at some vertex X_{t-1} of G_{t-1}. After the addition of vertex t, and before the beginning of step $t+1$, the walker performs a random walk of length l, where $l \geq 1$ is a fixed, deterministic positive integer independent of t.

Of course, in this context it is meaningless to ask about the cover time of the random walk. However, it is still reasonable to ask about the **evolution of the ratio of the number of vertices visited to the size of the underlying world**. The authors of [7] find that in the large-m regime, for $l = 1$, the expected proportion of vertices *not* visited asymptotically equals 0.57 in Model 1 and 0.59 in Model 2, and that in both cases, for large l, this proportion asymptotically equals $\frac{2}{l}$.

In the context of fixed graphs, the study of a quantity closely related to the above ratio has had a long history. Seminal work traces back to the classical research of Erdös and Dvoretzky (see [12] for the original paper, and [16] for a survey on the quantity called **range** in the context of lattice walks). More recently, and mainly in connection with problems of interest to the statistical physics community, there have been some attempts to determine this quantity when the underlying graph is a large random graph (see [9] and the references therein). In any case, in all these papers, the model assumes a fixed underlying world.

After the above discussion, it seems natural to generalise the model proposed in [7] and pose the question about the asymptotics of the aforementioned ratio in the setting we have termed *Dynamic Agent in a Changing World*. More precisely, for a given time $n \geq 1$, set $K_n = \{i \geq 1 : X_j = i \text{ for some } 0 \leq j \leq n\}$. We define the **knowledge process** as

$$\left(\frac{|K_n|}{|V_n|} : n \geq 0\right), \tag{1}$$

and we ask for its quantitative long-term behaviour when the Markov kernel that controls the growth of the world satisfies certain conditions.

In the quite general setting of the above definitions, some natural questions arise:

1. When does there exist an a.s. constant limit of (1)?
2. A quite reasonable ansatz is that, as long as $\frac{p}{1-p}$ is small, with probability 1 a positive proportion of the graph is asymptotically visited. Now, for a given transition kernel $P^{\mathcal{G}}(\cdot, \cdot)$, is there a critical value of the proportion between growth rate and walking rate above which the asymptotics vanish a.s.? This is a question about the existence of *phase transitions*.

3. A dual question to the previous one can be stated as follows: is it possible to characterise the kind of graph evolution that guarantees a positive asymptotic proportion for any growth rate and walking rate?

4. Finally: does there exist a quantitative relationship between the cover time of the sequence of graphs and the asymptotic proportion of vertices visited by the walker as the world grows? More precisely, suppose we know in advance that the Markov kernel that governs the growth is such that the graphs it generates have a cover time with expected value asymptotically equal to a certain function f, i.e. that

$$\mathbb{E}(\mathrm{Cov}(G_n)) = O(f(n)).$$

What, then, can we say about $\lim_{n\to\infty} \mathbb{E}\left(\frac{|K_n|}{|V_n|}\right)$ in terms of f?

This is the kind of question we address in the present work. As a preliminary approach to the subject, our aim is, first, to provide an appropriate theoretical setting that allows us to pose these questions, and second, to provide answers to some of these questions when the underlying graph dynamics are relatively simple. Accordingly, the rest of this work is organised as follows.

In Section 2 we define the probability spaces where the knowledge process lives. In order to allow for future developments, our exposition proposes a framework more general than is needed for our present purposes. At the end of the same section, we present the simple model on which we concentrate thereafter, with the appropriate specialisations and simplifications of the general model.

In Section 3 we study the mean and almost sure properties of the quantities of interest in our simple model.

In Section 4 we present a very simple model of growing random graphs and study the knowledge process in this changing structure.

Finally, in Section 5, we summarise the most relevant results and indicate some paths for further work in the area.

2. The model

In this section we provide the definitions, notation, and basic results regarding the general framework where our processes (random walks and graph sequences) will live. At this point, it seems fair to explain the motivations for some of our choices. First, in order to avoid trivialities due to lack of irreducibility of random walks in discrete time, we will assume that the steps of the walker are performed at the jump times of a homogeneous Poisson process. In the same vein, the evolution of the underlying graph will take place at the jump times of a pure birth Markov process (independent of the previous one). Since the main quantity we are interested in is a function of the number of vertices, and since the graph dynamics add vertices one by one, the height of the trajectories of this pure birth process is just the number of vertices modulo a constant. This continuous-time model has a second important advantage over the discrete-time version sketched in the introduction, in that it avoids the rather awkward detention of the walker while the world evolves.

2.1. The construction of the process

We start with some standard definitions and notation. A (simple, undirected) graph is a pair $G=(V, E)$, where V is a finite or countably-infinite set and E is a symmetric, antireflexive

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relation on $V \times V$ (and so E can be identified with a subset of the unordered pairs of elements of V). The elements of V are referred to as vertices, while those of E are called edges. Given a graph G, the notation $V(G)$ (resp. $E(G)$) will denote the set of its vertices (resp. the set of its edges). More often than not, we will use uppercase italics, $F, G, H, \ldots$, to identify particular graphs. We say that two vertices $u, v \in V$ are neighbours in the graph $G = (V, E)$ if $\{u, v\} \in E$, and in this case we write $u \sim_G v$. Also, for a vertex v, we write $\mathcal{N}^G(v)$ to denote the set of neighbours of v in the graph G. We write $\mathcal{G}_n$, $\mathcal{G}_{fin}$, and $\mathcal{G}_{\mathbb{N}}$ to denote the families of graphs with n vertices, the finite graphs, and the countably infinite graphs, respectively. Since the nature of the elements used as labels is entirely immaterial, we will assume that the vertices of graphs in $\mathcal{G}_n$ are labelled by $[n] = \{1, 2, \ldots, n\}$. With this convention, for a graph $G \in \mathcal{G}_n$ and for $m \leq n$, we define $G|_{[m]}$ to be the subgraph of $\mathcal{G}_n$ induced by the first m vertices.

Remark 1. We can get a more vivid image of a random walk in a growing graph sequence from the following alternative construction. Given a graph G and a vertex $v \in G$, write $\mathbb{P}^{G,v}$ to denote the law of the continuous-time simple random walk on the graph G started at v. Given an increasing sequence of graphs $G = \{G_0, G_1, \ldots\}$ and an increasing sequence of times $T = \{T_0 = 0, T_1, T_2, \ldots\}$, we build the transition probabilities of the random walk started at vertex x of the graph G_0 conditioned on (G, T) as follows: if $T_{n-1} \leq s \leq t < T_n$, then

$$\mathbb{P}_{G_0,x}(X_t = v | X_s = u) = \mathbb{P}^{G_{n-1},u}(X_{t-s} = v),$$

whereas if $T_{m-1} \leq s \leq T_m < T_{m+1} < \ldots < T_n \leq t < T_{n+1}$, then

$$\begin{aligned}\mathbb{P}_{G_0,x}(X_t = v | X_s = u) = \sum \mathbb{P}^{G_{m-1},u}(X_{T_m - s} = v_{m-1}) \mathbb{P}^{G_m, v_{m-1}}(X_{T_{m+1}-T_m} = v_m) \ldots \\ \ldots \mathbb{P}^{G_{n-1}, v_{n-2}}(X_{T_n - T_{n-1}} = v_{n-1}) \mathbb{P}^{G_n, v_{n-1}}(X_{t-T_n} = v),\end{aligned}$$

where the sum ranges over $v_{m-1} \in G_{m-1}, \ldots, v_{n-1} \in G_{n-1}$.

In order to get a Markov process on the pair (G_t, X_t), of course, we need the graph sequence to be a Markov process itself, and we need the inter-arrival times of both the jumps of the walker and the growth epochs of the graph to be exponentially distributed.

Remark 2. In order to keep the notation as simple as possible, in the remaining sections we will abuse it slightly. For example, we will write K_t for both the set of vertices visited by time t and the cardinality of this set. The same notational convention will be used for the set of vertices V_t and its cardinality. In context, no confusion will be possible.

2.2. An example: increasing finite path sequence

Let P_n be the path on n vertices. We label its vertices as $0, 1, \ldots, n-1$. For the construction below, it is worth keeping in mind the usual image of a segment of $\mathbb{R}_+$ drawn from left to right. In order to illustrate the usage of the above construction, in this subsection we consider a graph process that starts at P_N for some $N \geq 2$ and, at each of the random times $(T_k)_{k \geq 1}$, gains one vertex at its rightmost end. Thus, $G_t = P_{N+k}$ for $t \in [T_k, T_{k+1})$. Let X_t be a rate $\mu = 1$ simple random walk on this increasing sequence of graphs started at vertex 0. With the notation of the previous subsection, the law of the random walk will be denoted by $\mathbb{P}_{P_N,0}$.

Theorem 1.

$$\frac{K_t}{V_t} \to 0 \qquad \mathbb{P}_{P_N,0}\text{- } a.s.$$

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Intuitive as it is, the proof of the above result involves some technicalities. We will need to consider the times when the walker is constrained to perform a deterministic step to the left. Notice that this happens at time $t > 0$ if and only if a transition from V_t (to the left) is performed. Formally, define $l_0 = 0$, and for $n \geq 1$, let

$$l_n = \inf\{t \geq l_{n-1} : X_{t^-} = V_t,\, X_t = V_t - 1\}.$$

It may well be the case that $l_n = +\infty$ for some $n \geq 1$, but this is not an issue for the moment.

Now, let Y_t be the simple random walk on the one-sided lattice $\mathbb{Z}_+$ with reflecting barrier at 0, defined on some arbitrary probability space. Denote by R_t the range of Y_t, i.e.,

$$R_t = |\{Y_s : 0 \leq s \leq t\}|.$$

In what follows, we will regard P_n as an induced subgraph of $\mathbb{Z}_+$ (alternatively, $\mathbb{Z}_+$ can be seen as the projective limit of P_n as $n \to \infty$).

Proof. The demonstration relies on a coupling argument and well-known facts regarding the range of recurrent random walks on the (unrestricted) line.

First, given a sample path $(X_t, V_t)(\omega)$ we build a new walk $(X'_t)_{\{t\geq 0\}}$ on $\mathbb{Z}_+$ such that

1. $X_t(\omega) \leq X'_t(\omega)$ for all $t \geq 0$;
2. $X'_t \sim Y_t$ for all $t \geq 0$.

To this end, let $(\xi_n)_{n\geq 1}$ be a sequence of $\{-1, 1\}$-valued independent and identically distributed (i.i.d.) random variables, independent of the process (X_t, V_t) as well, such that for every $n \geq 1$, $(\xi_n + 1)/2 \sim \text{Bernoulli}(1/2)$. Let $X'_0 = 0$, and prescribe the following:

1. The jump times of X' are exactly the jump times of X.
2. If, for some $s \geq 0$, $X'_s = X_s$ holds, then we fix $X'_t = X_t$ for $s \leq t < t_s := \min\{l_n : l_n \geq s\}$. If $t_s = l_m$, fix $X'_{l_m} = X_{l_m^-} + \xi_m$.
3. If, on the contrary, for some $s \geq 0$, $X'_s \neq X_s$, let X' evolve as a simple random walk on the unrestricted line up to the first time after s when both processes meet.

It is easily checked that the construction above works as required. Now, since $X_t \leq X'_t$, we have that

$$K_t = |\{X_s : s \leq t\}| \leq R'_t = |\{X'_s : s \leq t\}| \sim R_t.$$

Since $\frac{V_t}{\gamma t} \to 1$ $\mathbb{P}_{P_N,0}$-a.s., we have

$$\begin{aligned}\mathbb{P}_{P_n,0}\left\{\limsup_t \frac{K_t}{V_t} > 0\right\} &= \mathbb{P}_{P_n,0}\left\{\limsup_t \frac{K_t}{\gamma t}\frac{\gamma t}{V_t} > 0\right\}\\ &\leq \mathbb{P}_{P_n,0}\left\{\limsup_t \frac{R'_t}{\gamma t}\frac{\gamma t}{V_t} > 0\right\} = 0,\end{aligned}$$

because $R' \sim R$ and $\frac{R_t}{t} \to 0$ a.s. (see, e.g., [16]) implies $\frac{R'_t}{t} \to 0$ a.s. as well.

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2.3. Specialising the model: the sequence of complete graphs

Henceforth we concentrate our efforts on the simple model we will deal with, namely the sequence of growing complete graphs.

For $n \geq 2$, let C_n be the complete graph on n vertices. Assume that $G_0 = C_N$ for some natural number $N \geq 2$ and that $P^{\mathcal{G}}(G, \cdot) = \delta_{C_{|G|+1}}$ is the (trivial) transition kernel of the graph-valued process. By symmetry, it is easily seen that in this case the pair $(|G_t|, |K_t|)$ is a Markov process. Let $b\Delta$ be the space of bounded Borel functions on Δ, and $b\Delta_0$ the linear subspace of $b\Delta$ of functions of the form

$$h(g, x, k) = h(|g|, |k|).$$

Introduce the difference operators

$$\Delta_1 f(|g|, |k|) = f(|g| + 1, |k|) - f(|g|, |k|),$$
$$\Delta_2 f(|g|, |k|) = f(|g|, |k| + 1) - f(|g|, |k|).$$

Theorem 2. *Assume that $G_0 = C_N$ for some natural number $N \geq 2$ and that $P^{\mathcal{G}}(G, \cdot) = \delta_{C_{|G|+1}}$. Then, for $h \in b\Delta_0$, the prescription*

$$\mathcal{L}h(|g|, |k|) = \gamma \Delta_1 h(|g|, |k|) + \mu \left(1 - \frac{|k| - 1}{|g| - 1}\right) \Delta_2 h(|g|, |k|) \tag{2}$$

corresponds to the action of the infinitesimal generator of the process $(|G_t|, |K_t|)$.

Proof. The reader is referred to Theorem 6 in the Appendix. Just observe that, in this case, for any $x \leq |g|$, the second term in (20) equals

$$\begin{aligned}\mu \frac{1}{|g| - 1} \left(|\mathcal{N}^g(x)| - |\mathcal{N}^g(x) \cap k|\right) \Delta_2 h(|g|, |k|) &= \mu \frac{(|g| - 1) - (|k| - 1)}{|g| - 1} \Delta_2 h(|g|, |k|) \\ &= \mu \frac{|g| - |k|}{|g| - 1} \Delta_2 h(|g|, |k|).\end{aligned}$$

In accordance with Remark 2, in order to simplify the notation, from now on we write v for the size of a graph g, and we put k instead of $|k|$. With this, the above expression can be written as

$$\mathcal{L}h(v, k) = \gamma \Delta_1 h(v, k) + \mu \left(1 - \frac{k - 1}{v - 1}\right) \Delta_2 h(v, k).$$

Remark 3. We turn back for a while to the discrete parameter model informally sketched in the introduction. What does K_n look like as the walker wanders in a fixed structure, say the complete N-graph? Since at time 0 the walker has already visited one vertex, and since our walker is not lazy, $K_0 = 1$ and $K_1 = 2$. For $n \geq 0$, let $\mathcal{F}_n$ be the σ-field generated by the random variables $X_0, X_1, \ldots, X_n$. We observe that, for $n \geq 2$, given the past $\mathcal{F}_n$, we have $K_{n+1} = K_n$ if the $(n+1)$th step takes the walker to an already visited vertex, and this happens with probability

$$\frac{K_n - 1}{N - 1}$$

(the -1 in the numerator takes care of the fact that the random walk is a simple random walk, so that the walker never stays at a given vertex). The complementary event is, of course,

$K_{n+1} = K_n + 1$, and this happens with probability

$$\frac{N - K_n}{N - 1}.$$

Rearranging terms, we get

$$\mathbb{E}(K_{n+1}|\mathcal{F}_n) = K_n\left(\frac{N-2}{N-1}\right) + \frac{N}{N-1}. \tag{3}$$

Iteration yields

$$\mathbb{E}(K_{n+1}) = N - (N-1)\left(1 - \frac{1}{N-1}\right)^{n+1}. \tag{4}$$

Now, from (3), we see that $(K_n : n \geq 0)$ is a bounded $\mathcal{F}_n$-submartingale, and hence has an a.s. limit that is, of course, $\lim_{n\to\infty} \mathbb{E}(K_n) = N$.

Trivial as it is, the point of this observation is that, to some extent, we can rely on analogous reasoning to understand the knowledge process as the complete graph grows. To grasp this connection, let $[N] = \{1, 2, \ldots, N\}$, and consider an $[N]$-valued pure birth process, say $(Z_s : s \geq 0)$, started at $Z_0 = 1$, with absorbing barrier at N, and with transition rates given by

$$\nu_{i,i+1} = \mu\frac{N-i}{N-1}, \qquad 1 \leq i < N. \tag{5}$$

We see that in this case, Z_s and (the continuous counterpart of) the walker of the first paragraph have the same law. In other words, in the complete graph, the knowledge process is a pure birth Markov process with rates of the type given in (5). Now, by looking at (2), we see clearly what we mentioned before. By identifying a pair $(|g|, |k|)$ with the corresponding point of the lattice $\mathbb{N}^2$, we can consider a derived random walk on Δ that starts at $(N, 1)$ and whose transitions are as follows. At the jump times of a Poisson process of intensity γ, the derived walker performs a unit step from $(|g|, |k|)$ to $(|g| + 1, |k|)$. This is a process of constant drift towards the east. Independently of the above drift, when the walker is at $(|g|, \cdot)$, a pure birth process of rate

$$\mu\frac{|g| - |k|}{|g| - 1}$$

controls the transition from $(|g|, |k|)$ to $(|g|, |k| + 1)$. This is a process of decreasing drift towards the north. The overall behaviour is that of a walker travelling northeast. In this setting, if there exists a random variable

$$\lim_{s\to\infty} \frac{|K_s|}{|G_s|},$$

this corresponds to the asymptotic slope of the walker in the wedge. This is the topic of the next section.

Later, we will use the expression at (2) to derive limit theorems. Meanwhile, as a direct consequence of the above theorem, we provide a semimartingale representation of the knowledge process.

Corollary 1. *Let $f \in b\Delta$ be defined by*

$$f_0(v, k) = \frac{k-1}{v-1}.$$

Then

$$M_t := f_0(V_t, K_t) + \gamma \int_0^t \frac{f_0(V_s, K_s)}{V_s} ds - \mu \int_0^t \frac{1 - f_0(V_s, K_s)}{V_s - 1} ds$$

is an L^2-bounded $\mathcal{F}_t$-martingale.

Proof. A straightforward application of (2) shows that M_t is a centred local martingale. An easy computation shows that the *carré du champ* associated to $f \in b\Delta$ is given by

$$\begin{aligned}\Gamma(f, f) &:= \mathcal{L}(f^2) - 2f\mathcal{L}(f) \\ &= \gamma(\Delta_1 f)^2 + \mu(1 - f_0)(\Delta_2 f)^2.\end{aligned}$$

Hence, the unique predictable process $(\langle M, M\rangle_t : t \geq 0)$ such that $(M_t^2 - \langle M, M\rangle_t : t \geq 0)$ is a centred local martingale is given by

$$\begin{aligned}\langle M, M\rangle_t &= \int_0^t \Gamma(f_0, f_0)(V_s, K_s) ds \\ &= \int_0^t \gamma f_0(V_s, K_s)^2 \left(\frac{1}{V_s}\right)^2 + \mu(1 - f_0(V_s, K_s)) \left(\frac{1}{V_s - 1}\right)^2 ds.\end{aligned}$$

For a fixed time $t \geq 0$, let T_t be the first jump time of V after t, and let N_t be the number of jumps up to and including time T_t. Since $f_0 \in [0, 1]$ a.s., for some constant C independent of t, we have the estimate

$$\begin{aligned}\langle M, M\rangle_t &\leq C \int_0^t \frac{1}{(V_s - 1)^2} ds \\ &= C \sum_{k=0}^{N_t} \frac{1}{(N + k)^2}(T_{k+1} - T_k) \\ &\leq C \sum_{k \geq 0} \frac{1}{(N + k)^2}(T_{k+1} - T_k).\end{aligned} \tag{6}$$

Since $(T_k - T_{k-1} : k \geq 1)$ is an i.i.d. sequence of random variables with distribution $\text{Exp}(\gamma)$, we have

$$\mathbb{E}(\langle M, M\rangle_t) \leq C \sum_{k \geq 0} \frac{1}{\gamma} \frac{1}{(N + k)^2} \leq \tilde{C},$$

where $\tilde{C}$ is some constant independent of t. Now, let $(U_n : n \geq 0)$ be a localising sequence for the local martingale M, i.e., an increasing sequence of stopping times satisfying $U_n \to \infty$ a.s. and such that $(M_t^{U_n} := M_{t \wedge U_n} : t \geq 0)$ is a martingale for each n. We have

$$\mathbb{E}\big((M_t^{U_n})^2\big) = \mathbb{E}\big(\langle M^{U_n}, M^{U_n}\rangle_t\big) = \mathbb{E}(\langle M, M\rangle_{t \wedge U_n}) \leq \tilde{C},$$

and taking the limit as $n \to \infty$, we obtain

$$\mathbb{E}(M_t^2) \leq \tilde{C}.$$

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3. Covering properties on the complete graph sequence

In this section we apply the previous Markov model to the problem of finding asymptotics of $|K_t|$ when the underlying graph sequence is simple enough to yield analytically solvable expressions. As at the end of the previous section, we set $V_s = |G_s|$, and we drop the bars in expressions such as $|K_s|$ and simply put K_s. Recall the notation: $\mathbb{P}^{G,v}$ is the probability measure induced by the continuous-time simple random walk on the fixed graph G started from v, and for a given graph G and vertex $v \in G$, we write $\mathbb{P}_{G_0,v}$ for the law of the Markov process (G_t, X_t) started at the vertex v of the graph G. Let $T_0 = 0 < T_1 < \ldots$ be the jump times of the structure, and let $S_0 = 0 < S_1 < \ldots$ be the jump times of the walker.

3.1. Hitting times on the complete graph sequence

We first study the behaviour of the hitting times on the sequence of growing complete graphs. Just for this subsection, we set $\mu = 1$, and G_N refers to the complete graph on $N > 2$ vertices. Fix arbitrarily a vertex of G_N, say v, and let τ be the hitting time of v, i.e.,

$$\tau = \min\{t \geq 0 : X_t = v\}.$$

For fixed $u \in G_N$ different from v, we are interested in the mean hitting time

$$\mathbb{E}_{G_N,u}(\tau).$$

It is elementary that, in the case of a fixed graph, $\mathbb{E}_{G_N,u}(\tau) = N - 1$. In the case of a growing sequence, the situation is much more interesting.

We need the following simple result, whose proof is omitted.

Lemma 1. *Let $X \sim$ Exp (μ_1), $Y \sim \mathrm{Exp}(\mu_2)$ be two independent random variables defined on the same probability space $(\Omega, \mathbb{P})$. Then*

$$\mathbb{E}(X|X < Y) = \mathbb{E}(Y|Y < X).$$

We now observe that for $u \neq v$ under $\mathbb{P}^{G_{N+k},u}$ the random variable τ is exponentially distributed with rate $\frac{1}{N+k-1}$. Thus, by the previous result,

$$\mathbb{E}^{G_{N+k},u}(\tau|\tau < (T_{k+1} - T_k)) = \mathbb{E}^{G_{N+k},u}(T_{k+1} - T_k|\tau > (T_{k+1} - T_k)) = \frac{N+k-1}{(N+k-1)\gamma + 1}.$$

Lemma 2. *For $u \neq v$,*

$$\mathbb{P}^{G_{N+k},u}(\tau < T_{k+1} - T_k) = \frac{1}{1 + \gamma(N+k-1)}.$$

Proof. Put $n = N + k$, $T = T_{k+1} - T_k$, and let $\tilde{S}_0, \tilde{S}_1, \ldots$ be the jump times of the walker on G_n. Under $\mathbb{P}^{G_n,u}$, $u \neq v$, we have $\tilde{S}_m \sim \Gamma(m, 1)$. Plainly,

$$\begin{aligned}
\mathbb{P}^{G_n,u}(\tau < T) &= \sum_{m\geq 1} \mathbb{P}^{G_n,u}(\tau < T|v \text{ is reached for the first time in exactly } m \text{ steps}) \\
&\quad \times \left(\frac{n-2}{n-1}\right)^{m-1} \frac{1}{n-1} \\
&= \frac{1}{n-1} \sum_{m\geq 1} \mathbb{P}_u^{G_n}(\tilde{S}_m < T) \left(\frac{n-2}{n-1}\right)^{m-1}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{n-1} \sum_{m\geq 1} \int_0^\infty \gamma e^{-\gamma t} \int_0^t \frac{s^{m-1}e^{-s}}{(m-1)!} \left(\frac{n-2}{n-1}\right)^{m-1} dsdt \\
&= \frac{1}{n-1} \int_0^\infty \gamma e^{-\gamma t} \int_0^t \sum_{m\geq 1} \frac{s^{m-1}e^{-s}}{(m-1)!} \left(\frac{n-2}{n-1}\right)^{m-1} dsdt \\
&= \frac{1}{n-1} \int_0^\infty \gamma e^{-\gamma t} \int_0^t e^{-\frac{s}{n-1}} dsdt \\
&= \int_0^\infty \gamma e^{-\gamma t}(1 - e^{-\frac{t}{n-1}})dt \\
&= \frac{1}{1+\gamma(n-1)}.
\end{aligned}
$$

Lemma 3. *For $k \geq 0$,*

$$
\mathbb{P}_{G_N,u}(\tau \in [T_k, T_{k+1})) = \frac{1}{\gamma} \frac{\Gamma(N+k-1)}{\Gamma(N-1)} \frac{\Gamma(N+\gamma^{-1}-1)}{\Gamma(N+\gamma^{-1}+k)}. \tag{7}
$$

Proof. Clearly, on the set $\{\tau > T_j\}$, we have $X_{T_j} \neq v$, and by symmetry,

$$
\mathbb{P}^{G_j, X_{T_j}}(\tau > (T_{j+1} - T_j)) = \mathbb{P}^{G_j,u}(\tau > (T_{j+1} - T_j))
$$

whenever $u \neq v$. Thus,

$$
\begin{aligned}
\mathbb{P}_{G_N,u}(\tau \in [T_k, T_{k+1})) &= \mathbb{P}_{G_N,u}(\tau < T_{k+1}, \tau > T_k, \tau > T_{k-1} \ldots, \tau > T_1) \\
&= \mathbb{P}_{G_N,u}(\tau < T_{k+1} | \tau > T_k)\mathbb{P}_{G_N,u}(\tau > T_k | \tau > T_{k-1}) \ldots \\
&\quad \ldots \mathbb{P}_{G_N,u}(\tau > T_2 | \tau > T_1)\mathbb{P}_{G_N,u}(\tau > T_1) \\
&= \mathbb{P}^{G_{N+k},u}(\tau < T_{k+1} - T_k)\mathbb{P}^{G_{N+k-1},u}(\tau > T_k - T_{k-1}) \ldots \\
&\quad \ldots \mathbb{P}^{G_{N+1},u}(\tau > T_2 - T_1)\mathbb{P}^{G_N,u}(\tau > T_1) \\
&= \frac{1}{1+\gamma(N+k-1)} \prod_{j=0}^{k-1} \frac{N-1+j}{N-1+\gamma^{-1}+j},
\end{aligned} \tag{8}
$$

where in the last line we have used the previous lemma. The result follows after rearranging.

Theorem 3. *For every $u \neq v$,*

$$
\mathbb{E}_{G_N,u}(\tau) = \frac{1}{\gamma} \sum_{k\geq 0} \left(\sum_{j=0}^{k} \frac{N+j-1}{\gamma(N+j-1)+1} \right) \left(\frac{\Gamma(N+k-1)}{\Gamma(N-1)} \frac{\Gamma(N+\gamma^{-1}-1)}{\Gamma(N+\gamma^{-1}+k)} \right). \tag{9}
$$

In particular,

$$
\mathbb{E}_{G_n,u}(\tau) < +\infty \textit{ if and only if } \gamma < 1.
$$

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Proof. Let $p_{N,k}$ be the expression in the right-hand side of (7). We have

$$\begin{aligned}
\mathbb{E}_{G_n,u}(\tau) &= \sum_{k\geq 0} \mathbb{E}_{G_N,u}(\tau \mathbf{1}_{\{\tau\in[T_k,T_{k+1})\}}) \\
&= \sum_{k\geq 0} \mathbb{E}_{G_N,u}(\tau | \tau \in [T_k, T_{k+1})) p_k \\
&= \sum_{k\geq 0} \mathbb{E}_{G_N,u}((\tau - T_k) + (T_k - T_{k-1} + \ldots (T_1 - T_0)) | \tau \in [T_k, T_{k+1})) p_k \\
&= \sum_{k\geq 0} \left(\sum_{j=1}^{k} \mathbb{E}_{G_N,u}(T_j - T_{j-1} | \tau \in [T_k, T_{k+1})) + \mathbb{E}_{G_N,u}(\tau - T_k | \tau \in [T_k, T_{k+1})) \right) p_k.
\end{aligned}$$

But for $1 \leq j \leq k$,

$$\begin{aligned}
\mathbb{E}_{G_N,u}(T_j - T_{j-1} | \tau \in [T_k, T_{k+1})) &= \mathbb{E}^{G_{N+j-1},u}(T_j - T_{j-1} | \tau > (T_j - T_{j-1})) \\
&= \frac{N+j-2}{\gamma(N+j-2)+1},
\end{aligned}$$

and analogously, using the result in Lemma 2,

$$\begin{aligned}
\mathbb{E}_{G_N,u}(\tau - T_k | \tau \in [T_k, T_{k+1})) &= \mathbb{E}^{G_{N+k},u}(\tau | \tau < (T_{k+1} - T_k)) \\
&= \frac{N+k-1}{\gamma(N+k-1)+1}.
\end{aligned}$$

The explicit expression for the expectation in (9) follows at once.

Finally, observe that $p_k = O(k^{-1-1/\gamma})$, and that the inner sum in (9) is $O(k)$. Thus, the general term of the outer sum is $O(k^{-1/\gamma})$. This proves the last claim of the theorem.

3.2. L^1 asymptotics for the knowledge process

In this subsection, we set ourselves in the special framework described by (2) and the remark after it, i.e., the random walk on an increasing sequence of complete graphs. Let $\rho = \mu/\gamma$. Figure 3.2 shows a simulation of three sample paths for K_t/V_t up to horizon $T = 50$, with parameters $\rho = 1$ (left) and $\rho = 0.5$ (right).

Proposition 1.

$$\lim_{k\to\infty} \mathbb{E}\left(\frac{K_{T_k}}{V_{T_k}}\right) = \frac{\mu}{\mu+\gamma} := \nu.$$

Proof. We proceed as in the remark after Theorem 2, but now we compute conditional expectations with respect to the stopped filtration $(\mathcal{F}_{T_k} : k \geq 0)$. Let $M_k = |\{i : T_{k-1} < S_i < T_k\}|$ be the number of steps performed by the walker in the time interval (T_{k-1}, T_k), and let $R_1, R_2, \ldots, R_{M_k}$ be the times when those steps take place. Some observations are in order. First, with probability 1, no step of the walker coincides with a bump of the graph. Second, if there is no step of the walk between T_{k-1} and T_k, we obviously have $K_{T_k} = K_{T_{k-1}}$. Otherwise, for $M_k \geq 1$, we have that K_{T_k} is exactly $K_{R_{M_k}}$. Third, the random variable $z_k := T_k - T_{k-1}$ is

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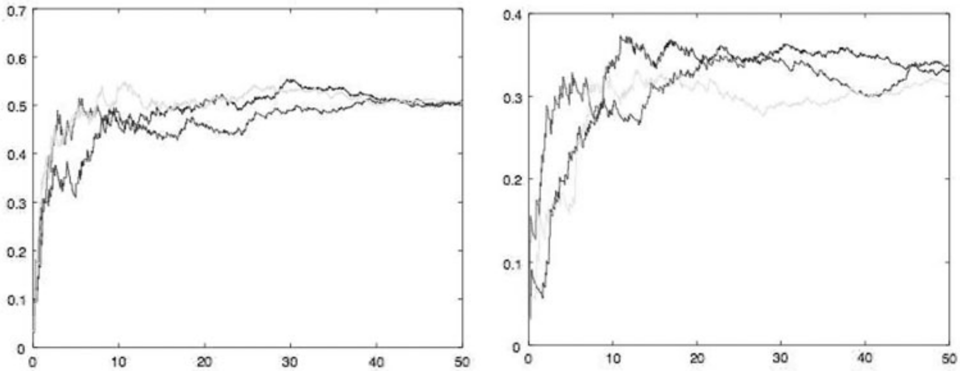

FIGURE 1: Knowledge process on complete graphs: left, $\rho = 1$; right, $\rho = 0.5$.

independent of $\mathcal{F}_{T_{k-1}}$ and has an $\mathrm{Exp}(\gamma)$ distribution. Putting together these observations, we get

$$\begin{aligned}
&\mathbb{E}(K_{T_k}|\mathcal{F}_{T_{k-1}})\\
&\quad= \sum_{n\ge 0} \mathbb{E}(K_{T_k}\mathbf{1}_{\{M_k=n\}}|\mathcal{F}_{T_{k-1}})\\
&\quad= K_{T_{k-1}}\mathbb{E}(e^{-\mu(T_k-T_{k-1})}|\mathcal{F}_{T_{k-1}}) + \sum_{n\ge 1}\mathbb{E}(K_{R_{M_k}}\mathbf{1}_{\{M_k=n\}}|\mathcal{F}_{T_{k-1}})\\
&\quad= K_{T_{k-1}}\mathbb{E}(e^{-\mu z_k}) + \sum_{n\ge 1}\mathbb{E}\left\{\left[K_{R_{M_k}-1}\left(\frac{N+k-3}{N+k-2}\right)+\frac{N+k-1}{N+k-2}\right]\mathbf{1}_{\{M_k=n\}}|\mathcal{F}_{T_{k-1}}\right\}\\
&\quad= K_{T_{k-1}}\mathbb{E}(e^{-\mu z_k}) +\\
&\qquad \sum_{n\ge 1}\mathbb{E}\left\{\left[K_{T_{k-1}}\left(\frac{N+k-3}{N+k-2}\right)^n+\frac{N+k-1}{N+k-2}\sum_{j=0}^{n-1}\left(\frac{N+k-3}{N+k-2}\right)^j\right]\mathbf{1}_{\{M_k=n\}}|\mathcal{F}_{T_{k-1}}\right\}\\
&\quad= K_{T_{k-1}}\mathbb{E}(e^{-\mu z_k}) +\\
&\qquad \sum_{n\ge 1}\left[K_{T_{k-1}}\left(\frac{N+k-3}{N+k-2}\right)^n+\frac{N+k-1}{N+k-2}\sum_{j=0}^{n-1}\left(\frac{N+k-3}{N+k-2}\right)^j\right]\mathbb{E}\left(e^{-\mu z_k}\frac{(\mu z_k)^n}{n!}\right),
\end{aligned}$$

where in the last line we have used the independence of z_k and $\mathcal{F}_{T_{k-1}}$ and the fact that M_k has law $\mathrm{Poisson}(\mu z_k)$. Now, for $n \ge 0$,

$$\mathbb{E}(e^{-\mu z_k} z_k^n) = (-1)^n \frac{d^n}{d\mu^n}\mathbb{E}(e^{-\mu z_k}) = n!\frac{\gamma}{(\mu+\gamma)^{n+1}},$$

but then we can write

$$\begin{aligned}
\mathbb{E}(K_{T_k}|\mathcal{F}_{T_{k-1}}) = K_{T_{k-1}}\frac{\gamma}{\mu+\gamma}\sum_{n\ge 0}\left(\frac{\mu(N+k-3)}{(\mu+\gamma)(N+k-2)}\right)^n +&\\
(N+k-1)\frac{\gamma}{\mu+\gamma}\sum_{n\ge 1}\left[\left(\frac{\mu}{\mu+\gamma}\right)^n - \left(\frac{\mu(N+k-3)}{(\mu+\gamma)(N+k-2)}\right)^n\right].&
\end{aligned}$$

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After some algebra we get

$$\begin{aligned}\mathbb{E}(K_{T_k}|\mathcal{F}_{T_{k-1}}) &= K_{T_{k-1}}\frac{\gamma(N+k-2)}{\mu+\gamma(N+k-2)} + \frac{\mu(N+k-1)}{\mu+\gamma(N+k.-2)}\\ &= K_{T_{k-1}}\frac{N+k-2}{N+k-2+\rho} + \rho\frac{N+k-1}{N+k-2+\rho},\end{aligned} \tag{10}$$

and hence

$$M_k := K_{T_k}\frac{\Gamma(N+k-1+\rho)}{\Gamma(N+k-1)} - \rho\sum_{j=N}^{N+K-1} j\frac{\Gamma(j-1+\rho)}{\Gamma(j)} \tag{11}$$

is an $\mathcal{F}_{T_k}$-martingale. So

$$\mathbb{E}(K_{T_k}) = \frac{\Gamma(N+k-1)}{\Gamma(N+k-1+\rho)}\left(\frac{\Gamma(N-1+\rho)}{\Gamma(N-1)} + \rho\sum_{j=N}^{N+K-1} j\frac{\Gamma(j-1+\rho)}{\Gamma(j)}\right). \tag{12}$$

Now, the usual approximation procedures (application of Stirling's formula and comparison of the sum with the corresponding integral) allow us to obtain the asymptotic ratio

$$\lim_{k\to\infty}\frac{\mathbb{E}(K_{T_k})}{N+k} = \frac{\rho}{1+\rho} = \frac{\mu}{\mu+\gamma}.$$

3.3. L^1 asymptotics on the complete graph sequence: variable rate of growth

With minor modifications, the growth rate of the underlying world can be made size-dependent. Of course, the important case is when the vertex-addition rate decreases. In this case, the fundamental recurrence equation becomes

$$\mathbb{E}(K_{T_k}) = \mathbb{E}(K_{T_{k-1}})\frac{N+k-2}{N+k-2+\rho_k} + \rho_k\frac{N+k-1}{N+k-2+\rho_k}.$$

Proposition 2. *With the above notation, if* $\alpha \in (0, 1)$ *and* $\rho_k = \rho k^\alpha$, *then*

$$\begin{aligned}\mathbb{E}\left(\frac{K_{T_k}}{V_{T_k}}\right) &\sim \frac{e^{-\frac{\rho}{\alpha}k^\alpha}}{k}\int_1^k \rho x^\alpha e^{\frac{\rho}{\alpha}x^\alpha}\,dx\\ &\sim 1-\frac{1}{\rho k^\alpha}\left[1+\frac{\alpha-1}{\rho k^\alpha}+\frac{(\alpha-1)(2\alpha-1)}{\rho^2k^{2\alpha}}+\frac{(\alpha-1)(2\alpha-1)(3\alpha-1)}{\rho^3k^{3\alpha}}+\ldots\right],\end{aligned}$$

whereas if $\rho_k = \rho\ln(k)$, *then*

$$\begin{aligned}\mathbb{E}\left(\frac{K_{T_k}}{V_{T_k}}\right) &\sim \frac{e^{-\frac{\rho}{2}\ln^2(k)}}{k}\int_1^k \rho\ln(x)e^{\frac{\rho}{2}\ln^2(x)}dx\\ &\sim 1-\frac{1}{(1+\rho\ln(k))}\left[1+\frac{1\cdot\rho}{(1+\rho\ln(k))^2}+\frac{3\cdot 1\cdot\rho^2}{(1+\rho\ln(k))^4}+\frac{5\cdot 3\cdot 1\cdot\rho^3}{(1+\rho\ln(k))^6}+\ldots\right],\end{aligned}$$

where an expression of the form $a_k \sim b_k$ *stands for* $\lim_{k\to\infty}\dfrac{a_k}{b_k} = 1$.

Proof. We prove the asymptotics for the power decay rate, since the logarithmic case is demonstrated along the same lines. Write $e_k = \mathbb{E}\left(K_{T_k}\right)$. Observe that

$$e_k = e_{k-1}(1-\rho k^{\alpha-1}) + \rho k^{\alpha} + \delta_k, \tag{13}$$

where $\delta_k = O(k^{3\alpha-2})$. Since $\alpha < 1$, we have $3\alpha - 2 < \alpha$, and thus $\delta_k = o(k^{\alpha})$. Choose $k_0 > 1$ such that $\rho k_0^{\alpha-1} < 1$, and for $k \geq k_0$ define

$$\phi(k) := (1-\rho k^{\alpha-1})(1-\rho(k-1)^{\alpha-1}) \ldots (1-\rho k_0^{\alpha-1}) = \prod_{j=k_0}^{k} (1-\rho j^{\alpha-1}).$$

Iterating the relation (13), we can write

$$e_k = \frac{\phi(k)}{\phi(k_0)} e_{k_0} + \sum_{j=k_0+1}^{k} \rho j^{\alpha} \frac{\phi(k)}{\phi(j)} + \sum_{j=k_0+1}^{k} \delta_j \frac{\phi(k)}{\phi(j)}. \tag{14}$$

On the other hand,

$$\ln(\phi(j)) = \sum_{i=k_0}^{j} \ln(1-\rho i^{\alpha-1}) = -\sum_{i=k_0}^{j} \rho i^{\alpha-1} + O(j^{2\alpha-1}) = -\sum_{i=k_0}^{j} \rho i^{\alpha-1} + o(j^{\alpha}),$$

where the last equality follows since $\alpha < 1$. Thus,

$$\phi(j) = e^{-\sum_{i=k_0}^{j} \rho i^{\alpha-1} + o(j^{\alpha})}.$$

Observe that

$$\int_{k_0}^{j} x^{\alpha-1} dx \leq \sum_{i=k_0}^{j} i^{\alpha-1} \leq \int_{k_0-1}^{j} x^{\alpha-1} dx,$$

and hence, for some constant $C(k_0)$ depending only on k_0,

$$\begin{aligned}\phi(j) &= C(k_0)(1+o(1)) \exp\left(-\int_{k_0}^{j} \rho x^{\alpha-1} dx + o(j^{\alpha})\right)\\ &= C(k_0)(1+o(1)) \exp\left(-\frac{\rho}{\alpha} j^{\alpha} + o(j^{\alpha})\right).\end{aligned}$$

Then, from (14) we obtain

$$\begin{aligned}\frac{e_k}{k} &= \frac{e^{-\frac{\rho}{\alpha}k^{\alpha}+o(k^{\alpha})}}{k} + \frac{e^{-\frac{\rho}{\alpha}k^{\alpha}+o(k^{\alpha})}}{k} \sum_{j=k_0+1}^{k} \rho j^{\alpha} e^{\frac{\rho}{\alpha}j^{\alpha}+o(j^{\alpha})}\\ &\quad + \frac{e^{-\frac{\rho}{\alpha}k^{\alpha}+o(k^{\alpha})}}{k} \sum_{j=k_0+1}^{k} \delta_j e^{\frac{\rho}{\alpha}j^{\alpha}+o(j^{\alpha})}.\end{aligned} \tag{15}$$

Since $\delta_j = o(j^{\alpha})$, we see that the leading term in the above expression is the second of the three summands. Thus,

$$\frac{e_k}{k} \sim \frac{e^{-\frac{\rho}{\alpha}k^{\alpha}}}{k} \sum_{j=k_0+1}^{k} \rho j^{\alpha} e^{\frac{\rho}{\alpha}j^{\alpha}}.$$

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A new comparison of the sum with the corresponding integral yields

$$\frac{e_k}{k} \sim \frac{e^{-\frac{\rho}{\alpha}k^\alpha}}{k}\int_{k_0}^{k} \rho x^\alpha e^{\frac{\rho}{\alpha}x^\alpha}\,dx \sim \frac{e^{-\frac{\rho}{\alpha}k^\alpha}}{k}\int_{1}^{k} \rho x^\alpha e^{\frac{\rho}{\alpha}x^\alpha}\,dx,$$

since any finite interval of integration is asymptotically immaterial. This proves the first line of our claim. The second line follows easily by integration by parts.

The next result is a direct consequence of the above proposition, L'Hôpital's rule, and Markov's inequality.

Corollary 2. *For the power and logarithmic decay,*

$$\frac{K_{T_k}}{V_{T_k}} \to 1 \qquad \textit{in probability.}$$

Remark 4. As a second extension to the basic model, we can assume that at each time $t \geq 0$ there are L independent walkers investigating the growing structure. Here, $L \geq 1$ is the number of walkers that collaborate to achieve the goal of knowing as much of the network as possible; i.e., the knowledge at time $t \geq 0$ increases if and only if at least one (and, a.s., exactly one) amongst the L walkers jumps at time t into a non-visited vertex. Of course, we can regard these L walkers as a single walker $(\mathbf{X_t} = (X_t^{(i)})_{1\leq i\leq L} : t \geq 0)$ such that for each $t \geq 0$, $\mathbf{X_t} \in V_t^L$. For $1 \leq i \leq L$, let μ_i be the transition rate of the ith walker. Thanks to independence, the transitions of $\mathbf{X_t}$ occur at rate $\mu = \sum_{i=1}^{L} \mu_i$. This fact leads to a pleasant additive property of the knowledge process: regardless of which walker is performing a transition at time t, a jump of K occurs at this time if and only if this walker jumps into one of the $V_{t^-} - K_{t^-}$ non-visited vertices. This said, the next result, whose proof is omitted, should be almost obvious.

Proposition 3. *With μ redefined as above, in the multiple-walker case,*

$$\lim_{k\to\infty} \mathbb{E}\left(\frac{K_{T_k}}{V_{T_k}}\right) = \frac{\mu}{\mu+\gamma}$$

holds as well.

3.4. Almost sure properties for the knowledge process on the complete graph sequence

We turn back to the model of the growing sequence of complete graphs *with constant rate of growth*. Recall the generator (2). In what follows, $\mathcal{L}^*$ denotes the adjoint operator of $\mathcal{L}$.

Lemma 4. *Let $h \in b\Delta^*$ be defined as*

$$h(v, k) = \binom{v-1}{k-1} v^{k-1}(1-v)^{v-k}, \qquad 1 \leq k \leq v.$$

Then $\mathcal{L}^* h = 0$.

Proof. Based on (2), a routine computation shows that $\mathcal{L}^*$ is given by

$$\mathcal{L}^* g(v, k) = \gamma\left[g(v-1, k) - g(v, k)\right] + \mu\left[g(v, k-1)\frac{v-k+1}{v-1} - g(v, k)\frac{v-k}{v-1}\right]. \qquad (16)$$

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When this is applied to h, we obtain

$$\begin{aligned}\mathcal{L}^*h(v,k) &= \gamma\left[\binom{v-2}{k-1}\nu^{k-1}(1-\nu)^{v-k-1} - \binom{v-1}{k-1}\nu^{k-1}(1-\nu)^{v-k}\right] + \\ &\quad + \mu\left[\binom{v-1}{k-2}\nu^{k-2}(1-\nu)^{v-k+1}\frac{v-k+1}{v-1} - \binom{v-1}{k-1}\nu^{k-1}(1-\nu)^{v-k}\frac{v-k}{v-1}\right] \\ &= \gamma\frac{\nu}{1-\nu}\binom{v-2}{k-1}\nu^{k-2}(1-\nu)^{v-k} - \gamma\binom{v-1}{k-1}\nu^{k-1}(1-\nu)^{v-k} \\ &\quad + \mu\frac{1-\nu}{\nu}\nu^{k-1}(1-\nu)^{v-k}\binom{v-1}{k-2}\frac{v-k+1}{v-1} - \mu\nu^{k-1}(1-\nu)^{v-k}\binom{v-1}{k-1}\frac{v-k}{v-1} \\ &= \mu\nu^{k-2}(1-\nu)^{v-k}\binom{v-2}{k-1} \\ &\quad + \nu^{k-1}(1-\nu)^{v-k}\left[\gamma\binom{v-1}{k-2}\frac{v-k+1}{v-1} - \mu\binom{v-1}{k-1}\frac{v-k}{v-1} - \gamma\binom{v-1}{k-1}\right].\end{aligned}$$

But the expression inside the square brackets equals

$$\gamma\left(\binom{v-2}{k-2} - \binom{v-1}{k-1}\right) - \mu\binom{v-2}{k-1} = -(\mu+\gamma)\binom{v-2}{k-1},$$

and thus

$$\begin{aligned}\mathcal{L}^*h(v,k) &= \mu\nu^{k-2}(1-\nu)^{v-k}\binom{v-2}{k-1} - (\mu+\gamma)\nu^{k-1}(1-\nu)^{v-k}\binom{v-2}{k-1} \\ &= 0,\end{aligned}$$

as claimed.

For $k \le v$, set $p_v(k) = h(v,k)$.

Theorem 4. *We have*

$$\frac{K_t}{V_t} \xrightarrow[t\to\infty]{} \nu \qquad \mathbb{P}_{C_N, K_0\sim p_N}\text{-a.s.}$$

Proof. Let $(P_t : t \ge 0)$ be the Markov semigroup associated to the process $(V_t, K_t)_{t\ge 0}$, and P_t^* its adjoint. Lemma 4 says that p_v is harmonic for P_t^*; i.e.,

$$P_t^*(p_v)(\cdot) = p_v(\cdot). \tag{17}$$

For $n \ge 1$ consider any measurable function $f : [1, n] \to \mathbb{R}$, and define the function $g \in b\Delta$ by

$$g(v,k) = f(k)\mathbf{1}_{\{v=n\}}.$$

If we use integrals instead of the corresponding finite sums, a.s. we have

$$\begin{aligned}\mathbb{E}_{K_0\sim p_{V_0}}(g(V_t,K_t)) &= \int P_t(f\mathbf{1}_{\{\cdot=n\}})(V_0,k)p_{V_0}(dk) \\ &= \int f(k)\mathbf{1}_{\{V_0=n\}}(P_t^* p_{V_0})(dk),\end{aligned}$$

and by the harmonic property (16), a.s. the last line equals

$$\int f(k)\mathbf{1}_{\{V_0=n\}}p_{V_0}(dk) = \mathbf{1}_{\{V_0=n\}}\mathbb{E}_{K_0\sim p_{V_0}}(f(K_0)).$$

In other words, we have proved that

$$\mathbb{E}_{K_0\sim p_{V_0}}(f(K_t)|V_t) = \mathbb{E}_{K_t\sim p_{V_t}}(f(K_t)) \quad \text{a.s.,}$$

or

$$(K_t - 1)|V_t \sim \text{Binomial}(V_t - 1, \nu) \quad \text{a.s.}$$

The rest of the proof relies on the classical estimate for trials of i.i.d. random variables with finite fourth moment. Indeed, there exists a constant c, depending on ν but not on t, such that

$$\mathbb{E}(\,|(K_t-1)-\nu(V_t-1)|^4\,|V_t) \le c(V_t-1)^2,$$

and thus

$$\mathbb{E}\biggl(\,\biggl|\frac{K_t-1}{V_t-1}-\nu\biggr|^4\,|V_t\biggr) \le c(V_t-1)^{-2},$$

so that

$$\mathbb{E}\biggl(\,\biggl|\frac{K_t-1}{V_t-1}-\nu\biggr|^4\,\biggr) \le \mathbb{E}(c(V_t-1)^{-2}).$$

An application of the Fubini–Tonelli theorem to the left-hand side of this inequality yields

$$\mathbb{E}\biggl(\int_0^\infty \biggl|\frac{K_t-1}{V_t-1}-\nu\biggr|^4 dt\biggr) \le \int_0^\infty \mathbb{E}(c(V_t-1)^{-2})dt.$$

The expression on the right-hand side is easily seen to be convergent. Thus, the integral on the left-hand side converges a.s. In particular, the integrand vanishes a.s. as t goes to infinity, and this implies our claim.

4. Complete bipartite graph sequence

As a final application, in this section we consider a simple rule for the growth of a bipartite graph. Let $p^{(j)}$, for $j=0, 1$, be positive parameters such that $p^{(0)}+p^{(1)}=1$. We start the graph sequence with G_0 equal to the complete graph on two vertices. Call the initial vertices 0 and 1. For $t\ge 0$, the set V_t is split into two components, $V_t = V_t^{(0)} \cup V_t^{(1)}$, and initially each component contains one vertex. Just as before, at each time of renewal of a Poisson process of parameter γ a new vertex arrives. Suppose the new vertex, say v, arrives at time $t>0$. Then with probability $p^{(j)}$ it is assigned to the jth component (i.e. we set $V_t^{(j)} = V_{t-}^{(j)} \cup \{v\}$), and, at the same time, we declare the edges of v to be exactly those of the old vertices in the component it was assigned to.

Just as in the previous section, let $(X_t : t \ge 0)$ be the simple random walk on the growing graph $(G_t : t\ge 0)$. Let $Y_t = \mathbf{1}_{\{X_t\in V_t^{(1)}\}}$; i.e., Y_t is the component X_t belongs to. Let π be the invariant distribution of a simple random walk on G_0. Obviously, π is Y_t-invariant too.

We introduce some new notation related to the splitting of the growing graph into two components. Fix $j \in \{0, 1\}$. For $t \geq 0$, let $K_t^{(j)} = K_t \cap V_t^{(j)}$; i.e., $K_t^{(j)}$ is the set of vertices in $V_t^{(j)}$ visited by time t. Let $Z_0^{(j)} = 0$, and for $k \geq 1$, let

$$Z_k^{(j)} := \min \left\{ t \geq Z_{k-1}^{(j)} : V_{t^-}^{(j)} \neq V_t^{(j)} \right\};$$

write $I_k^{(j)} = [Z_{k-1}^{(j)}, Z_k^j)$ and $l_k^{(j)} = |I_k^{(j)}|$. Finally, let $J_k^{(j)}$ be the number of transitions to some vertex in the component j performed by the walker in the interval $I_k^{(j)}$.

Some observations are in order. First, the processes $N_t^{(j)} := |\{k : Z_k^j \leq t\}|$ are (up to a constant shift) Poisson processes of intensity $\gamma p^{(j)}$. In particular, $l_k^{(j)}$ has distribution $\mathrm{Exp}(\gamma p^{(j)})$ for every $k \geq 1$. Second, under the assumption that $X_0 \sim \pi$, we have that for fixed j, the random variables $(J_k^{(j)} : k \geq 1)$ are identically distributed. So, for $j = 0, 1$, let $J^{(j)}$ be a random variable distributed according to this common law, defined on an otherwise arbitrary probability space $(\Omega_j, \mathbb{P}_j)$.

Proposition 4. *For $k \geq 1$, let $r_k = \left(1 - \dfrac{1}{k}\right)$, and for $j = 0, 1$, let $C_k^{(j)} := \mathbb{E}(r_k^{J^{(j)}})$. Then, for every $k \geq 1$,*

$$\mathbb{E}_\pi \left(K_{Z_k^{(j)}}^{(j)} \right) = \mathbb{E}_\pi \left(K_{Z_{k-1}^{(j)}}^{(j)} \right) C_k^{(j)} + k(1 - C_k^{(j)}). \tag{18}$$

Proof. Let $\mathbb{P}^{Z^{(j)}}(J_k^{(j)} = \cdot)$ be the law of $J_k^{(j)}$ given the sequence $Z^{(j)} := (Z_k^{(j)} : k \geq 1)$. For $n \geq 0$, define

$$P_n^{(j)} = \mathbb{E}_\pi \left(\mathbb{P}^{Z^{(j)}}(J_k^{(j)} = n) | \mathcal{F}_{Z_{k-1}^{(j)}} \right).$$

The definition is consistent. Indeed, $J_k^{(j)}$ depends only on the length l_k, and conditional on $\mathcal{F}_{Z_{k-1}}$, this length is distributed as $\mathrm{Exp}(\gamma p^{(j)})$ independently of the σ-field $\mathcal{F}_{Z_{k-1}^{(j)}}$. So $P_n^{(j)}$ is a constant a.s., and furthermore,

$$P_n^{(j)} = \mathbb{P}_j(J^{(j)} = n),$$

where $\mathbb{P}_j$ refers to the probabilty measure in the space Ω_j where $J^{(j)}$ has been defined.

Now, and in the same vein as in the proof of our result for the complete graph sequence, we compute conditional expectations, but this time we condition on $\mathcal{F}_{Z_{k-1}^{(j)}}$ for $k \geq 1$. In order to simplify the notation, just for the next computation, we drop the (j) superindices. On the set $J_k = n$, let $R_1, R_2, \ldots, R_n$ be the times in the interval I_k where a transition to the state in the component j takes place. Then

$$\begin{aligned}
\mathbb{E}(K_{Z_k} | \mathcal{F}_{Z_{k-1}}) &= \mathbb{E}(K_{Z_k} \mathbf{1}_{\{J_k = 0\}} | \mathcal{F}_{Z_{k-1}}) + \sum_{n \geq 1} \mathbb{E}(K_{Z_k} \mathbf{1}_{\{J_k = n\}} | \mathcal{F}_{Z_{k-1}}) \\
&= K_{Z_{k-1}} P_0 + \sum_{n \geq 1} \mathbb{E}(K_{R_n} \mathbf{1}_{\{J_k = n\}} | \mathcal{F}_{Z_{k-1}}) \\
&= K_{Z_{k-1}} P_0 + \sum_{n \geq 1} \mathbb{E} \left\{ \left(K_{R_{n-1}} \frac{K_{R_{n-1}}}{k} + (K_{R_{n-1}} + 1) \frac{k - K_{R_{n-1}}}{k} \right) \mathbf{1}_{\{J_k = n\}} | \mathcal{F}_{Z_{k-1}} \right\}
\end{aligned}$$

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$$
\begin{aligned}
&= K_{Z_{k-1}} P_0 + \sum_{n\geq 1} \mathbb{E}\left\{ \left(K_{R_{n-1}} \frac{k-1}{k} + 1 \right) \mathbf{1}_{\{J_k=n\}} | \mathcal{F}_{Z_{k-1}} \right\} \\
&= K_{Z_{k-1}} p_0 + \sum_{n\geq 1} \mathbb{E}\left\{ \left(K_{Z_{k-1}} \left(\frac{k-1}{k}\right)^n + \sum_{i=0}^{n-1} \left(\frac{k-1}{k}\right)^i \right) \mathbf{1}_{\{J_k=n\}} | \mathcal{F}_{Z_{k-1}} \right\},
\end{aligned}
$$

and this line equals

$$
K_{Z_{k-1}} \left(\sum_{n\geq 0} \left(1 - \frac{1}{k}\right)^n P_n \right) + k \left(1 - \sum_{n\geq 0} \left(1 - \frac{1}{k}\right)^n P_n \right).
$$

After restoring the (j) superindices and taking expectations, we obtain (18).

The constants $C_k^{(j)}$ can be computed exactly. As an example, when $p^{(0)} = p^{(1)} = 1/2$ (and hence, the growth of the structure can be defined as a sequence of uniformly growing complete bipartite graphs), we obtain that

$$
C_k^{(0)} = C_k^{(1)} = \frac{3\rho+1}{(2\rho+1)^2} + \rho \left(\frac{4\rho+1}{2\rho+1} \right)^2 \frac{k-1}{k(4\rho+1)+4\rho^2} = 1 - \frac{\rho}{k} + O\left(\frac{1}{k^2}\right).
$$

It turns out that we do not need the exact value of these constants.

Theorem 5. *For the model of the growing bipartite graph with parameters $p^{(0)}$, $p^{(1)}$,*

$$
\mathbb{E}\left(\frac{K^{(j)}_{Z_k^{(j)}}}{V^{(j)}_{Z_k^{(j)}}} \right) \to \frac{\rho^{(j)}}{1+\rho^{(j)}},
$$

where $\rho^{(j)} = \dfrac{\rho}{2p^{(j)}}$.

Proof. For fixed j and k, consider first the random variable $J_k^{(j)}$, and let $\tau = \tau_k^{(j)}$ be the number of steps of the process Y_t during the time interval $I_k^{(j)}$. Then $\mathbb{E}_\pi(\tau) = \dfrac{\mu}{\gamma p^{(j)}}$. If we write $\tilde{Y}_1, \tilde{Y}_2, \ldots$ for the steps of the discrete parameter chain associated to the process Y during the same time interval $I_k^{(j)}$, we have

$$
\begin{aligned}
\mathbb{E}_\pi(J_k^{(j)}) &= \mathbb{E}_\pi \left(\sum_{i=1}^{\tau} \mathbf{1}_{\{\tilde{Y}_i = j\}} \right) \\
&= \sum_{m\geq 0} \mathbb{E}_\pi \left(\sum_{i=1}^{\tau} \mathbf{1}_{\{\tilde{Y}_i = j\}} | \tau = m \right) \mathbb{P}_\pi(\tau = m) \\
&= \sum_{m\geq 0} \pi_j m \mathbb{P}_\pi(\tau = m) \\
&= \pi_j \mathbb{E}_\pi(\tau) \\
&= \frac{\rho}{2p^{(j)}} \\
&= \rho^{(j)}.
\end{aligned}
$$

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Hence

$$C_k^{(j)} = \mathbb{E}\left\{\left(1-\frac{1}{k}\right)^{J^{(j)}}\right\} = \mathbb{E}\left\{\exp\left(\ln\left(1-\frac{1}{k}\right)J^{(j)}\right)\right\}$$
$$= 1 + \ln\left(1-\frac{1}{k}\right)\mathbb{E}(J^{(j)}) + O\left(\frac{1}{k^2}\right)$$
$$= 1 - \frac{\mathbb{E}(J^{(j)})}{k} + O\left(\frac{1}{k^2}\right)$$
$$= 1 - \frac{\rho^{(j)}}{k} + O\left(\frac{1}{k^2}\right).$$

Thus, (18) becomes

$$\mathbb{E}_\pi\left(K^{(j)}_{Z_k^{(j)}}\right) = \mathbb{E}_\pi\left(K^{(j)}_{Z_{k-1}^{(j)}}\right)\left(1-\frac{\rho^{(j)}}{k}\right) + \rho^{(j)} + d_k, \tag{19}$$

where $d_k = O\left(\frac{1}{k}\right)$. Now, fix $k_0 \geq 1$ such that $\frac{\rho^{(j)}}{k_0} < 1$. Iterating the relation (19), after a bit of algebra we get for $k \geq k_0$ that

$$\mathbb{E}_\pi\left(K^{(j)}_{Z_k^{(j)}}\right) = \mathbb{E}_\pi\left(K^{(j)}_{Z_{k_0}^{(j)}}\right)\frac{\Gamma(k+1-\rho^{(j)})\Gamma(k_0+2)}{\Gamma(k_0+2-\rho^{(j)})\Gamma(k+1)} + \rho^{(j)}\sum_{i=k_0+1}^{k}\frac{\Gamma(k+1-\rho^{(j)})\Gamma(i+1)}{\Gamma(k+1)\Gamma(i+1-\rho^{(j)})}$$
$$+ \sum_{i=k_0+1}^{k} d_i\frac{\Gamma(k+1-\rho^{(j)})\Gamma(i+1)}{\Gamma(k+1)\Gamma(i+1-\rho^{(j)})},$$

and once again, the usual approximations and the fact that $d_k \to 0$ as $k \to \infty$ yield

$$\mathbb{E}\left(\frac{K^{(j)}_{Z_k^{(j)}}}{V^{(j)}_{Z_k^{(j)}}}\right) \to \frac{\rho^{(j)}}{1+\rho^{(j)}}.$$

5. Final remarks

We have conducted a preliminary study of the behaviour of the asymptotic proportion of vertices visited by a random walker embedded in growing structures. We can think of this limit ratio as an index of the fitness of a dynamic structure aimed at supporting a randomised algorithm capable of taking advantage of new processing units as they become available. In fact, questions regarding the efficiency of autonomous, non-centralised crawling of a growing graph, as described in [7], largely served as inspiration for the present work. It must be stressed that any quantification of this efficiency is not likely to be based on other well-known indices (e.g. functionals of hitting times) as we have shown in the case of the complete graphs (Section 3, Subsection 2.1): in a fast-growing regime, τ turns out to be non-integrable.

The model we have proposed in Section 2 is motivated by the need to answer the question posed in the introduction. Thus, it necessarily differs from the much simpler model presented in [3], where the question about recurrence/transience of random walks in changing landscapes of conductances is qualitative in nature. The cost to be paid is that, at this stage, our research can only give an account of very simple (equivalently, highly symmetrical) growing strucures.

We have found that when the underlying world is the growing finite path (Subsection 2.2) or the complete graph sequence (Section 3), the asymptotic proportion is a.s. a constant, 0 in the first case and a.s. positive in the second case. In other words, we have found two simple classes of graph evolution that give an answer in the positive sense to Question 1 of the introduction. In the case of a naïve random structure (the complete bipartite graph sequence of Section 4) we were able to exhibit L^1 asymptotics for the aforementioned proportion in each of the two components of the graph. The question of whether this limit is an a.s. limit remains open, and the same holds for the other questions we posed in the introduction.

In our opinion, there are at least two lines of further research that are worth pursuing. The first one is to prove or disprove the conjecture below (or a related one) regarding the asymptotic order of the cover times of the structures and its relation with the asymptotic expected value of the knowledge process, as described in the last question of the introduction.

Conjecture 1. *Let $P^{\mathcal{G}}$ be a Markov kernel on $\mathcal{G}_{fin}$, and $(G_n : n \geq 0)$ a graph process, as in Section 2. Set $c_n := \dfrac{\mathbb{E}(\mathrm{Cov}(G_n))}{n \ln n}$. Then*

$$\lim_{t\to\infty} \mathbb{E}\left(\frac{K_t}{V_t}\right) = \frac{\mu}{\mu + \gamma \lim_{n\to\infty} c_n}.$$

Some support for this still-to be-proved proposition can be found not only in the cases we have analysed in this work, but also in a simple comparison of the main results in [7] and [8].

The second line of further research is related to the regularity of the limits. More precisely, assume that for a given transition kernel $P^{\mathcal{G}}$ and for some choice of μ and γ, there exists a constant $\nu(\mu, \gamma)$ such that $\lim_{t\to\infty} K_t/V_t = \nu(\mu, \gamma)$ a.s. In the interpretation of this limit as an efficiency index for the growing structure intended to support a crawler, it would be desirable to have at our disposal some kind of *fine-tuning theorem* that gives conditions on $P^{\mathcal{G}}$ such that this limit is a smooth function of the natural parameters of the model. For example, it is far from trivial that the existence of an a.s. positive constant limit for a given choice of (μ, γ) guarantees the existence of an a.s. positive constant limit for some open range of the parameters.

We are currently working on both of these lines of research.

Appendix

In this short appendix, we rigorously construct the Markov process Ξ_t of Section 2 and compute its generator. We adopt the notation introduced therein.

First, we observe that the set $\mathcal{G}_{fin}$ can be turned into a metric space. For two graphs $F, G \in \mathcal{G}_{fin}$ we define $N_0(F, G) = \inf\{k : F|_{[k]} \neq G|_{[k]}\}$, with the convention $\inf \emptyset = +\infty$, and set

$$d_{\mathcal{G}_{fin}}(F, G) := 2^{-N_0(F,G)}.$$

It is easy to verify that this metric induces the cylinder topology on $\mathcal{G}_{fin}$, and that under $d_{\mathcal{G}_{fin}}$, $\mathcal{G}_{fin}$ is a Polish space.

We give a topology to the set of finite subsets of $\mathbb{N}$, denoted $\mathcal{P}_{fin}(\mathbb{N})$, in such a way that the cardinal application

$$\begin{aligned} |\cdot| : \mathcal{P}_{fin}(\mathbb{N}) &\to \mathbb{R}_+ \\ A &\mapsto |A| \end{aligned}$$

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and set-union operation

$$\cup : \mathcal{P}_{fin}(\mathbb{N}) \times \mathcal{P}_{fin}(\mathbb{N}) \to \mathcal{P}_{fin}(\mathbb{N})$$
$$(A, B) \mapsto A \cup B$$

become continuous functions in this topology. The simplest way to do this is to metrise $\mathcal{P}_{fin}(\mathbb{N})$ with the discrete metric $d_{\mathcal{P}_{fin}(\mathbb{N})} = \mathbf{1}_{\{A \neq B\}}$. Observe that under the assumption of the Axiom of Countable Choice, this space becomes trivially a separable space that is complete for the discrete metric.

Finally, we endow $\mathbb{N}$ with the discrete topology as well.

For each of the spaces $\mathcal{G}_{fin}$, $\mathbb{N}$, and $\mathcal{P}_{fin}(\mathbb{N})$ endowed with their respective metrics, we consider their Borel σ-fields, $\mathfrak{B}(\mathcal{G}_{fin})$, $\mathfrak{B}(\mathbb{N})$, and $\mathfrak{B}(\mathcal{P}_{fin}(\mathbb{N}))$, and set $\mathfrak{B} := \mathfrak{B}(\mathcal{G}_{fin}) \otimes \mathfrak{B}(\mathbb{N}) \otimes \mathfrak{B}(\mathcal{P}_{fin}(\mathbb{N}))$.

Having established the elementary topological and measure-theoretic ingredients described above, we are now ready to build a Markov process taking values on a subspace of $\mathcal{S} := \mathcal{G}_{fin} \times \mathbb{N} \times \mathcal{P}_{fin}(\mathbb{N})$, namely on the wedge

$$\Delta := \{(g, w, k) \in \mathcal{S} : w \leq |g|, k \in 2^{|g|}, w \in k\}.$$

Every point of Δ will represent a state of the Markov process, and we give the following interpretation to the coordinates: the first coordinate represents the current world, the second coordinate represents the position of the walker in the current world, and the third coordinate keeps track of the vertices already visited by the walker.

The construction of the process proceeds as follows.

1. Let a kernel $P^{\mathcal{G}} : \mathcal{G}_{fin} \times \mathfrak{B}(\mathcal{G}_{fin}) \to [0, 1]$ be given, and assume that it satisfies the following:

   - $P^{\mathcal{G}}(\cdot, A)$ is a measurable application for every $A \in \mathfrak{B}(\mathcal{G}_{fin})$.
   - $P^{\mathcal{G}}(G, \cdot)$ is a probability measure for every $G \in \mathcal{G}_{fin}$.
   - For every $G \in \mathcal{G}_n$, the measure $P^{\mathcal{G}}(G, \cdot)$ is concentrated in $\mathcal{G}_{n+1} \cap \{H : H|_{[n]} = G\}$.

The choices regarding the metric on $\mathcal{G}_{fin}$ guarantee that $\mathcal{G}_n$ is a Borel set of $\mathcal{G}_{fin}$ for each $n \in \mathbb{N}$, and so the above conditions are consistent. Given any initial graph G on $\mathcal{G}_{fin}$, Kolmogorov's consistency theorem guarantees the existence of a probability measure $P^{\mathcal{G}}_G$ on $(\Omega^{\mathcal{G}} := \mathcal{G}_{fin}^{\mathbb{N}}, \mathfrak{B}(\mathcal{G}_{fin})^{\otimes \mathbb{N}})$ and a process $(G_n : n \geq 0)$ defined on it with $G_0 = G$ whose one-step transitions are given by the kernel $P^{\mathcal{G}}$.

2. On an arbitrary space $(\Omega^0, \mathcal{F}^0, P^0)$ let there be given two independent, time-homogeneous Poisson processes, $(N^{walk}_s : s \geq 0)$ and $(N^{grow}_s : s \geq 0)$. The first one (which dictates the pace of the walker) is a Poisson process of intensity μ, and the second one (which controls the inflation of the graph) is a Poisson process of intensity γ. Endow Ω^0 with the natural filtration $(\mathcal{F}^0_t)$ induced by the pair (N^{walk}_s, N^{grow}_s). Let $S_0 = T_0 = 0$, and for $n \geq 1$ let S_n (resp. T_n) be the nth jump time of N^{walk} (resp. N^{grow}).

3. Let $G_0 \in \mathcal{G}_{fin}$ be a fixed initial graph. Given (ω, ω') drawn from $\Omega^{\mathcal{G}} \times \Omega^0$ according to $P^{\mathcal{G}} \otimes P^0$, for $s \geq 0$ we define

$$G_s(\omega, \omega') := G_k(\omega) \quad \text{on } \{T_k(\omega') \leq s < T_{k+1}(\omega')\}.$$

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4. We build an $\mathbb{N}$-valued Markov process conditioned on $G=(G_s=G_s(\omega,\omega'):s\geq 0)$ and on $(N^{walk}, N^{grow})=((N_s^{walk}(\omega), N_s^{grow}(\omega)):s\geq 0)$. For each fnite graph $G=(V,E)\in\mathcal{G}_{fin}$, let P^G be its one-step transition kernel associated to the simple random walk taking values on the set V. Given an initial vertex $X_0\in[|G_0|]$, for $n\geq 1$ we draw $X_n\sim P^{G_{S_n}}(X_{n-1},\cdot)$. Again, Kolmogorov's consistency theorem guarantees the existence of a probability measure $P^{\mathcal{X}}=P^{\mathcal{X}}(\omega,\omega')$ on $(\Omega^{\mathcal{X}}:=\mathbb{N}^{\mathbb{N}}, \mathfrak{B}(\mathbb{N})^{\otimes\mathbb{N}})$ such that $(X_n:n\geq 0)$ is a (non-time-homogeneous) Markov chain with the prescribed probability transition. Let $\omega''=\omega''(\omega,\omega')$ denote a typical sequence on this space, i.e., $X_n(\omega'')=\omega''_n$. Set $X_t=X_0$ on $0\leq t\leq S_1$ and
$$X_s(\omega,\omega'')=X_n(\omega'') \qquad \text{on } \{S_n(\omega)\leq s<S_{n+1}(\omega)\}.$$
5. Finally, given a set $K_0\subset[|G_0|]$ such that K_0 contains X_0, define
$$K_t:=K_0\cup\{v\in\mathbb{N}:\exists s\leq t \text{ such that } X_s=v\}.$$

Let $\Xi:=(\Xi_t=(G_t,X_t,K_t):t\geq 0)$, and consider the filtrations
$$(\mathcal{F}_t^{\mathcal{G}}:=\sigma(G_s:0\leq s\leq t):t\geq 0),$$
$$(\mathcal{F}_t^{\mathcal{W}}:=\sigma(X_s:0\leq s\leq t):t\geq 0),$$
$$(\mathcal{F}_t^{\mathcal{K}}:=\sigma(K_s:0\leq s\leq t):t\geq 0),$$
$$\mathcal{F}_t=\mathcal{F}_t^{\mathcal{G}}\vee\mathcal{F}_t^{\mathcal{W}}\vee\mathcal{F}_t^{\mathcal{K}}.$$

The following properties are direct consequences of our definitions:

- $\mathcal{F}_t^0\subset\mathcal{F}_t$.
- $((G_s,X_s):s\geq 0)$ is a $(\mathcal{G}_{fin}\times\mathbb{N})$-valued $\mathcal{F}_t$-Markov process defined on the probability space $\Omega:=\Omega^0\times\Omega^{\mathcal{G}}\times\Omega^{\mathcal{X}}$; the law $\mathbb{P}_{g,x}$ of the process started at vertex x in the graph g is computed by the formula
$$\mathbb{P}_{g,x}((G,X)\in A\times B)=P^{\mathcal{X}}(X^{-1}(B)|G^{-1}(A))(P^0\otimes P^{\mathcal{G}})(G^{-1}(A))$$
for all Borel sets $A\in\mathfrak{B}(\mathcal{G}_{fin})^{\otimes\mathbb{N}}$, $B\in\mathfrak{B}(\mathbb{N})^{\otimes\mathbb{N}}$.

Let $b\Delta$ be the Banach space of bounded measurable real-valued functions on Δ, and let $\mathbb{E}$ be the expectation operator associated to the probability $\mathbb{P}_{G_0,v}$. For a function $f\in b\Delta$, we use the notation
$$P^{\mathcal{G}}f(g,x,k)=\sum_{H\in\mathcal{G}_{|g|+1}}P^{\mathcal{G}}(g,H)f(H,x,k),$$
and if $g\in\mathcal{G}_{fin}$, we write
$$P^g f(g,x,k)=\sum_{y\sim_g x}P^g(x,y)f(g,y,k).$$

Theorem 6. *The process $(\Xi_t:t\geq 0)$ is a Δ-valued Markov process whose infinitesimal generator acts on $f\in b\Delta$ as*
$$\mathcal{L}f(g,x,k)=\gamma\left[P^{\mathcal{G}}f(g,x,k)-f(g,x,k)\right]+\mu\sum_{y\sim_g x}P^g(x,y)\left[f(g,y,k\cup\{y\})-f(g,x,k)\right]. \tag{20}$$

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Proof. For $0 \le s \le t$, let $w_{s,t}$ be the number of walking steps between s and t, and let $g_{s,t}$ be the number of growing steps (i.e. added vertices) between s and t. Using Landau's little-o notation, for f as in our claim we compute as follows:

$$\begin{aligned}
\mathbb{E}(f(G_t, X_t, K_t)|\mathcal{F}_s) &= \mathbb{E}(f(G_t, X_t, K_t)|(G_s, X_s, K_s)) \\
&= \sum_{m\ge 0}\sum_{n\ge 0} \mathbb{E}(f(G_t, X_t, K_t)\mathbf{1}_{\{g_{s,t}=m\}}\mathbf{1}_{\{w_{s,t}=n\}}|(G_s, X_s, K_s)) \\
&= e^{-(\mu+\gamma)(t-s)} f(G_s, X_s, K_s) \\
&\quad + e^{-\mu(t-s)}\gamma(t-s)e^{-\gamma(t-s)} \sum_{H\in\mathcal{G}_{fin}} P^{\mathcal{G}}(G_s, H) f(H, X_s, K_s) \\
&\quad + e^{-\gamma(t-s)}\mu(t-s)e^{-\mu(t-s)} \\
&\quad \times \sum_{y\sim_{G_s}X_s} P^{G_s}(X_s, y)(f(G_s, y, K_s)\mathbf{1}_{\{y\in K_s\}} + f(G_s, y, K_s\cup\{y\})\mathbf{1}_{\{y\notin K_s\}}) \\
&\quad + o(t-s) \\
&= e^{-(\mu+\gamma)(t-s)} f(G_s, X_s, K_s) \\
&\quad + e^{-\mu(t-s)}\gamma(t-s)e^{-\gamma(t-s)} \sum_{H\in\mathcal{G}_{fin}} P^{\mathcal{G}}(G_s, H) f(H, X_s, K_s) \\
&\quad + e^{-\gamma(t-s)}\mu(t-s)e^{-\mu(t-s)} \sum_{y\sim_{G_s}X_s} P^{G_s}(X_s, y) f(G_s, y, K_s\cup\{y\}) \\
&\quad + o(t-s),
\end{aligned}$$

where the last equality follows from the fact that $K_s = K_s \cup \{y\}$ on $y \in K_s$. Then we have

$$\begin{aligned}
\lim_{t\to s} \frac{\mathbb{E}(f(G_t, X_t, K_t)|(G_s, X_s, K_s)) - f(G_s, X_s, K_s)}{t-s} &= -(\mu+\gamma) f(G_s, X_s, K_s) \\
&\quad + \gamma P^{\mathcal{G}} f(G_s, X_s, K_s) + \mu \sum_{y\sim_{G_s}X_s} P^{G_s}(X_s, y) f(G_s, y, K_s\cup\{y\}),
\end{aligned}$$

and our claim follows after rearranging terms and observing that

$$\sum_{y\sim_{G_s}X_s} P^{G_s}(X_s, y) = 1.$$

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References

[1] ALDOUS, K. AND FILL, J. (1999). *Reversible Markov Chains and Random Walks on Graphs*. Unpublished manuscript. Available at http://www.stat.berkeley.edu/ aldous/RWG/book.html.
[2] ALELIUNAS, R. et al. (1979). Random walks, universal traversal sequences, and the complexity of maze problems. In *Proceedings of the 20th Annual Symposium on Foundations of Computer Science*, IEEE Computer Science, Washington, DC, pp. 218–223.
[3] AMIR, G., BENJAMINI, I., GUREL-GUREVICH, O. AND KOZMA, G. (2017). Random walks in changing environment. Preprint. Available at https://arxiv.org/abs/1504.04870.
[4] AVIN, C., KOUCKÝ, M. AND LOTKER, Z. (2008). How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). In *Automata, Languages and Programming*, Springer, Berlin, pp. 121–132.
[5] BARABÁSI, A.-L. AND ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286**, 509–512.
[6] BOLLOBÁS, B., RIORDAN, O., SPENCER, J. AND TUSNÁDY, G. (2001). The degree sequence of a scale-free random graph process. *Random Structures Algorithms* **18**, 279–290.
[7] COOPER, C. AND FRIEZE, A. (2003). Crawling on simple models of Web graphs. *Internet Math.* **1**, 57–90.
[8] COOPER, C. AND FRIEZE, A. (2007). The cover time of the preferential attachment graph. *J. Combinatorial Theory* **97**, 269–290.
[9] DE BACCO, C., MAJUMDAR, S. AND SOLLICH, P. (2015). The average number of distinct sites visited by a random walker on random graphs. Preprint. Available at https://arxiv.org/abs/1501.01528v2.
[10] DEMBO, A., HUANG, R. AND SIDORAVICIUS, V. (2014). Walking within growing domains: recurrence versus transience. *Electron. J. Prob.* **19**, 1–20.
[11] DING, J., LEE, J. R. AND PERES, Y. (2012). Cover times, blanket times, and majorizing measures. *Ann. Math.* **175**, 1409–1471.
[12] DVORETZKY, A. AND ERDŐS, P. (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, pp. 353–367.
[13] LEVIN, D., PERES, Y. AND WILMER, E. (2017). *Markov Chains and Mixing Times*, 2nd edn. American Mathematical Society, Providence, RI.
[14] LOVÁSZ, L. (1996). Random walks on graphs: a survey. In *Combinatorics, Paul Erdős is Eighty*, János Bolyai Mathematical Society, Budapest, pp. 353–398.
[15] MATTHEWS, P. (1988). Covering problems for Brownian motion on spheres. *Ann. Prob.* **16**, 189–199.
[16] SPITZER, F. (1976). *Principles of Random Walks*. Springer, New York.