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EXISTENCE OF A SOLUTION FOR A NON-LOCAL PROBLEM IN $\mathbb{R}^N$ VIA BIFURCATION THEORY

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Abstract In this paper, we study the existence of a solution for the following class of non-local problems:

$$\begin{cases} -\Delta u = \left(\lambda f(x) - \displaystyle\int_{\mathbb{R}^N} K(x,y)|u(y)|^\gamma \mathrm{d}y\right) u & \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to+\infty} u(x) = 0, \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geq 3,\ \lambda > 0, \gamma \in [1,2),\ f : \mathbb{R} \to \mathbb{R}$ is a positive continuous function and $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a non-negative function. The functions f and K satisfy some conditions that permit us to use bifurcation theory to prove the existence of a solution for (P).

Keywords: non-local logistic equations; *a priori* bounds; positive solutions

2010 *Mathematics subject classification:* Primary 35J15; 35J60; 92B05

1. Introduction and main result

The main goal of this paper is to study the existence of a positive solution for the following class of non-local problems:

$$\begin{cases} -\Delta u = \left(\lambda f(x) - \displaystyle\int_{\mathbb{R}^N} K(x,y)|u(y)|^\gamma \mathrm{d}y\right) u & \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to+\infty} u(x) = 0, \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geq 3,\ \lambda > 0, \gamma \in [1,2),\ f : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function and $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a non-negative function. The functions f and K satisfy some technical conditions that will be mentioned later.

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The motivation to study problem (P) comes from the problem of modelling the behaviour of a species inhabiting a smooth bounded domain $\Omega \subset \mathbb{R}^N$, whose classical logistic equation is given by

$$\begin{cases} -\Delta u = u(\lambda - b(x)u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u(x)$ is the population density at location $x \in \Omega$, $\lambda \in \mathbb{R}$ is the growth rate of the species and b is a positive function denoting the carrying capacity; that is, $b(x)$ describes the limiting effect of crowding of the population.

Since (1.1) is a local problem, the crowding effect of the population u at x only depends on the value of the population in the same point x. In [7], for more realistic situations, Chipot has considered that the crowding effect depends also on the value of the population around of x; that is, the crowding effect depends on the value of integral involving the function u in the ball $B_r(x)$ centred at x of radius $r > 0$. To be more precise, in [7] the following non-local problem has been studied:

$$\begin{cases} -\Delta u = \left(\lambda - \displaystyle\int_{\Omega \cap B_r(x)} b(y)u^p(y)\,\mathrm{d}y\right) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where b is a non-negative and non-trivial continuous function. After [7], special attention has been given to the problem

$$\begin{cases} -\Delta u = \left(\lambda - \displaystyle\int_{\Omega} K(x,y)u^p(y)\,\mathrm{d}y\right) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

by supposing different conditions on K; see, for example, Allegretto and Nistri [1], Alves *et al.* [2], Chen and Shi [6], Corrêa *et al.* [8], Coville [9], Leman *et al.* [13] and Sun *et al.* [15] and their references.

In [2], Alves *et al.* have considered the existence and non-existence of a solution for problem (1.3). In that paper, the authors introduced a class $\mathcal{K}$, which is formed by functions $K : \Omega \times \Omega \to \mathbb{R}$ such that:

(i) $K \in L^\infty(\Omega \times \Omega)$ and $K(x,y) \geq 0$, for all $x, y \in \Omega$;

(ii) if w is measurable and $\int_{\Omega\times\Omega} K(x,y)|w(y)|^p|w(x)|^2\,\mathrm{d}x\,\mathrm{d}y = 0$, then $w = 0$ almost everywhere in Ω.

Using bifurcation theory and by supposing that K belongs to class $\mathcal{K}$, the following result has been proved.

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Theorem 1.1. *The problem (1.3) has a positive solution if, and only if, $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Motivated by [2], at least from a mathematical point of view, it is interesting to ask whether (P) has a solution. Here, as in [2], we intend to use bifurcation theory. However, we must be careful because, in the above paper, Ω is a smooth bounded domain, and thus it is possible to use compact embeddings for Sobolev spaces and Schauder spaces. Since we are working in whole $\mathbb{R}^N$, we need to show new estimates, and to this end, our inspiration comes from some papers by Edelson and Rumbos [10, 11], where bifurcation theory has been used to study the existence of a solution for a problem of the type

$$\begin{cases} -\Delta u + u = \lambda f(x)(u + h(u)) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \quad \text{and} \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{Q}$$

where f and h are continuous functions that satisfy some technical conditions. Here we point out that many estimates in our paper are totally different from those used in [10, 11], because in the present work the problem is non-local, while in the previous papers the problem is local.

In the present paper, $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function that verifies the following conditions.

(K_0) There is $P \in C^+_{\text{rad}}(\mathbb{R}^N, \mathbb{R}) \cap L^1(\mathbb{R}^N)$ such that

$$0 \le K(x, y) \le f(x)P(y)^{\gamma/2}Q(x, y), \quad \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $Q : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable function that verifies:

(Q_1) $M = \sup\limits_{x\in\mathbb{R}^N} |Q(x, \cdot)|_{2/2-\gamma} < +\infty$;

(Q_2) given $\varepsilon > 0$, there exist $R, L > 0$ such that

$$\int_{|y|\le L} Q(x, y)^{2/2-\gamma} \, \mathrm{d}y < \varepsilon, \quad \forall x \in B^c_R(0).$$

Here, $C^+_{\text{rad}}(\mathbb{R}^N, \mathbb{R}) = \{g \in C(\mathbb{R}^N, \mathbb{R}) : g \text{ is positive and radially symmetric}\}$.

(K_1) If w is measurable and $\int_{\mathbb{R}^N\times\mathbb{R}^N} K(x, y)|w(y)|^\gamma |w(x)|^2 \, \mathrm{d}x \, \mathrm{d}y = 0$, then $w = 0$ almost everywhere in $\mathbb{R}^N$.

From (K_0)–(K_1), it follows that $K \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$.

Related to function f, we assume that:

(f_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $0 < f(x) \le P(x), \forall x \in \mathbb{R}^N$;

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(f_2) there exists $q > N/2$ such that

$$\sup_{x\in\mathbb{R}^N} |f|_{L^q(B_2(x))} < +\infty.$$

By taking $P \in C^+_{\mathrm{rad}}(\mathbb{R}^N,\mathbb{R}) \cap L^1(\mathbb{R}^N)$ and $Q(x,y) = \chi_{B_\delta(0)}(x-y)$, the function

$$K(x,y) = f(x)P(y)^{\gamma/2}Q(x,y)$$

verifies the conditions (K_0) and (K_1).

Our main result is the following.

Theorem 1.2. *Assume that (K_0)–(K_1) and (f_1)–(f_2) hold. Then, the problem (P) has a positive solution if, and only if, $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the linear problem*

$$\begin{cases} -\Delta u = \lambda f(x)u \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{AP}$$

The paper is organized as follows. In § 2 we have shown some properties of the non-local term. In § 3 we have defined two compact operators that are crucial in our approach, while in § 4 we prove Theorem 1.2.

Notation

- ω_N is the volume of the unit ball in $\mathbb{R}^N$.
- Γ is the fundamental solution of the Laplace equation in $\mathbb{R}^N$.
- χ_B is the characteristic function of B.
- $B_r(x)$ denotes the ball centred at x with radius $r > 0$ in $\mathbb{R}^N$.
- $L^s(\mathbb{R}^N)$, for $1 \le s \le \infty$, denotes the Lebesgue space with the usual norm denoted by $|u|_s$.
- $L^2_P(\mathbb{R}^N)$ denotes the class of real-valued Lebesgue measurable functions u such that

$$\int_{\mathbb{R}^N} P(x)|u(x)|^2\,\mathrm{d}x < \infty.$$

$L^2_P(\mathbb{R}^N)$ is a Hilbert space endowed with the inner product

$$(u,v)_{2,P} = \int_{\mathbb{R}^N} P(x)u(x)v(x)\,\mathrm{d}x, \quad \forall u,v \in L^2_P(\mathbb{R}^N).$$

The norm associated with this inner product will be denoted by $|\ \ |_{2,P}$.

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- $D^{1,2}(\mathbb{R}^N)$ denotes the Sobolev space endowed with inner product

$$(u,v)_{1,2} = \int_{\mathbb{R}^N} \nabla u \nabla v \, \mathrm{d}x, \quad \forall u, v \in D^{1,2}(\mathbb{R}^N).$$

  The norm associated with this inner product will be denoted by $\| \ \|_{1,2}$. In [10], it was proved that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_P(\mathbb{R}^N)$ is compact.

- We denote by E the Banach space given by

$$E = \left\{ u \in C(\mathbb{R}^N); \lim_{|x|\to\infty} u(x) = 0 \right\},$$

  endowed with the norm $| \ |_\infty$. A simple computation leads to the result that the embedding $E \hookrightarrow L^2_P(\mathbb{R}^N)$ is continuous.

- If u is a measurable function, we denote by u^+ and u^- the positive and negative parts of u, respectively, which are given by

$$u^+ = \max\{u, 0\} \quad \text{and} \quad u^- = \max\{-u, 0\}.$$

2. The non-local term

In what follows, we will show some properties of the operator $\phi : L^2_P(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ given by $\phi(u) := \phi_u$, where

$$\phi_u(x) := \int_{\mathbb{R}^N} K(x,y)|u(y)|^\gamma \, \mathrm{d}y.$$

The operator ϕ is well defined, because we are assuming (K_0)–(K_1). Using the definition of ϕ, we see that $u \in D^{1,2}(\mathbb{R}^N)$ is a solution for (P) if, and only if, it is a positive solution of

$$-\Delta u + \phi_u u = \lambda f(x) u \quad \text{in } \mathbb{R}^N. \tag{EP}$$

Next, we show some important properties of the operator ϕ for future reference.

Lemma 2.1. *The operator ϕ satisfies the following properties.*

(ϕ_1) $\phi_{tu} = t^\gamma \phi_u$, $\forall (u,t) \in E \times [0, +\infty)$.

(ϕ_2) $|\phi_u(x)| \le MP(x)|u|^\gamma_{2,P}$, $\forall (u,x) \in L^2_P(\mathbb{R}^N) \times \mathbb{R}^N$ *and* $|\phi_u(x)| \le MP(x)|u|^\gamma_\infty$, $\forall (u,x) \in E \times \mathbb{R}^N$.

(ϕ_3) *For each* $u \in L^2_P(\mathbb{R}^N)$,

$$\lim_{|x|\to+\infty} \frac{\phi_u(x)}{f(x)} = 0.$$

(ϕ_4) $|\phi_u|_1 \le M|P|_1|u|^\gamma_{2,P}$, $\forall u \in L^2_P(\mathbb{R}^N)$ *and* $|\phi_u|_1 \le M|P|_1|u|^\gamma_\infty$, $\forall u \in E$.

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(ϕ_5) *$\phi : E \longrightarrow L^1(\mathbb{R}^N)$ is continuous, that is,*

$$u_n \to u \quad \text{in } E \Longrightarrow \phi_{u_n} \to \phi_u \quad \text{in } L^1(\mathbb{R}^N).$$

(ϕ_6) *Let $(u_n) \subset L^2_P(\mathbb{R}^N)$ be a sequence and $u \in E$ such that $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. Then*

$$\liminf_{n\to+\infty} \phi_{u_n}(x) \geq \phi_u(x), \quad \forall x \in \mathbb{R}^N.$$

(ϕ_7) *Let $(u_n) \subset E$ be a sequence and $u \in E$ such that $u_n \to u$ in E. Then, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$|\phi_{u_n}(x) - \phi_u(x)| \leq \epsilon P(x), \quad \forall n \geq n_0 \text{ and } \forall x \in \mathbb{R}^N.$$

(ϕ_8) *If $u \in C^1(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$ is a non-trivial solution of (EP) and $u \geq 0$ (respectively $u \leq 0$), then $u > 0$ (respectively $u < 0$).*

Proof. (ϕ_1): This property is an immediate consequence of the definition of ϕ_u.
(ϕ_2): From (K_0) and (f_1),

$$|\phi_u(x)| \leq P(x) \int_{\mathbb{R}^N} P(y)^{\gamma/2} Q(x,y)|u(y)|^\gamma \, \mathrm{d}y, \quad \forall u \in L^2_P(\mathbb{R}^N).$$

Then, using the Hölder inequality with exponents $p = 2/\gamma$ and $p' = 2/(2-\gamma)$, we get

$$|\phi_u(x)| \leq P(x)|Q(x,y)|_{2/2-\gamma}|u|^\gamma_{2,P} \leq MP(x)|u|^\gamma_{2,P}, \quad \forall u \in L^2_P(\mathbb{R}^N),$$

where $M = \sup_{x\in\mathbb{R}^N} |Q(x,.)|_{2/2-\gamma}$ was fixed in (Q_1). The last inequality combined with the continuous embedding $E \hookrightarrow L^2_P(\mathbb{R}^N)$ gives

$$|\phi_u(x)| \leq MP(x)|u|^\gamma_\infty, \quad \forall u \in E \text{ and } \forall x \in \mathbb{R}^N.$$

(ϕ_3): Repeating the same arguments explored in (ϕ_2), we have the inequality below:

$$\begin{aligned} |\phi_u(x)| \leq f(x)\Bigg[&\left(\int_{|y|\leq L} Q(x,y)^{2/(2-\gamma)} \, \mathrm{d}y\right)^{2-\gamma/2} |u|^\gamma_{2,P} \\ &+ M\left(\int_{|y|>L} P(y)|u(y)|^\gamma \, \mathrm{d}y\right)^{\gamma/2}\Bigg], \quad \forall x \in \mathbb{R}^N \text{ and } L > 0. \end{aligned}$$

Combining the fact that $u \in L^2_P(\mathbb{R}^N)$ with (Q_2), given $\epsilon > 0$, there are $R, L > 0$ such that

$$|\phi_u(x)| \leq \epsilon f(x) \quad \text{for } |x| \geq R,$$

and (ϕ_3) is proved.
(ϕ_4): This property follows from (ϕ_2), because $P \in L^1(\mathbb{R}^N)$.
(ϕ_5): This is an immediate consequence from Lebesgue's theorem together with (ϕ_2).
(ϕ_6): As $K(x,y)$ is non-negative, the property is obtained applying Fatou's lemma.

(ϕ_7): Using (f_1) and the definitions of ϕ_{u_n} and ϕ_u, we get

$$|\phi_{u_n}(x) - \phi_u(x)| \leq P(x) \int_{\mathbb{R}^N} P(y)^{\gamma/2} Q(x,y) ||u_n(y)|^\gamma - |u(y)|^\gamma| \, \mathrm{d}y,$$

and so,

$$|\phi_{u_n}(x) - \phi_u(x)| \leq MP(x)|P|_1^{\gamma/2} ||u_n|^\gamma - |u|^\gamma|_\infty, \quad \forall x \in \mathbb{R}^N \text{ and } \forall n \in \mathbb{N}.$$

As $u_n \to u$ in E, we have

$$||u_n|^\gamma - |u|^\gamma|_\infty \to 0 \quad \text{as } n \to +\infty.$$

Hence, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|\phi_{u_n}(x) - \phi_u(x)| \leq \epsilon P(x), \quad \forall x \in \mathbb{R}^N \text{ and } n \geq n_0.$$

(ϕ_8): This is an immediate consequence of the maximum principles. □

2.1. The weight P

The aim of this section is to study the existence and regularity of some linear problems, which will later be used in the proof of some lemmas.

If $P \in C^+_{rad}(\mathbb{R}^N, \mathbb{R})$ and φ is the weak solution of the problem

$$\begin{cases} -\Delta u = P(x) \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x| \to +\infty} u(x) = 0, \end{cases} \tag{LP}$$

then $\varphi \in D^{1,2}_{rad}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ and

$$-(r^{N-1}\varphi'(r))' = r^{N-1}P(r) \quad \text{for } r > 0 \quad \text{and} \quad \lim_{r \to \infty} \varphi(r) = 0. \tag{2.1}$$

The lemma below shows the behaviour of φ at infinity. A similar result has been proved [4] but with a different argument.

Lemma 2.2. *The function φ is decreasing, positive and*

$$\lim_{r \to +\infty} r^{N-2}\varphi(r) = \frac{|P|_1}{\omega_N(N-2)}.$$

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Proof. Indeed, by (2.1),

$$-r^{N-1}\varphi'(r) = \int_0^r s^{N-1}P(s)\,\mathrm{d}s, \quad \forall r > 0.$$

As P is positive, it follows that $\varphi'(r) < 0$ for $r > 0$, and then φ is decreasing. Moreover,

$$\lim_{r \longrightarrow \infty} -r^{N-1}\varphi'(r) = \int_0^\infty s^{N-1}P(s)\,\mathrm{d}s = \frac{|P|_1}{\omega_N}$$

and

$$0 < \varphi(r) = -\int_r^\infty \varphi'(s)\,\mathrm{d}s \le \frac{|P|_1}{\omega_N}\int_r^\infty s^{1-N}\,\mathrm{d}s = \frac{|P|_1}{\omega_N(N-2)}r^{2-N},$$

leading to

$$\limsup_{r \to +\infty} r^{N-2}\varphi(r) \le \frac{|P|_1}{\omega_N(N-2)}. \tag{2.2}$$

On the other hand, given $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$-r^{N-1}\varphi'(r) \ge \frac{|P|_1 - \varepsilon}{\omega_N} \quad \text{for } r > r_0.$$

Therefore,

$$\varphi(r) = -\int_r^\infty \varphi'(s)\,\mathrm{d}s \ge \frac{|P|_1 - \varepsilon}{\omega_N}r^{2-N} \quad \text{for } r > r_0.$$

Since ε is arbitrary, we ensure that

$$\liminf_{r \to +\infty} r^{N-2}\varphi(r) \ge \frac{|P|_1}{\omega_N(N-2)}. \tag{2.3}$$

Now, the lemma follows combining (2.2) and (2.3). □

The last lemma combined with some arguments found in [**10**, pp. 225–226] permits us to conclude that

$$\varphi(x) = -\int_{\mathbb{R}^N} \Gamma(x-y)P(y)\,\mathrm{d}y, \quad \forall x \in \mathbb{R}^N. \tag{2.4}$$

2.2. Some regularity results

Let $F : \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfy the following condition:

$$|F(x)| \le c_0 P(x), \quad \forall x \in \mathbb{R}^N, \tag{H}$$

for some positive constant c_0.

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Since $w(x) \equiv 1 \in L^2_P(\mathbb{R}^N)$, we can guarantee that the functional $\Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Psi(v) := \int_{\mathbb{R}^N} F(x) v \, \mathrm{d}x$$

is continuous, because

$$\Psi(v) = \int_{\mathbb{R}^N} F(x) v \, \mathrm{d}x \leq c_0 \int_{\mathbb{R}^N} P(x) v w \, \mathrm{d}x \leq c_0 |v|_{2,P} |P|_1^{1/2} \leq C |P|_1^{1/2} \|v\|_{1,2}.$$

Using Riesz's theorem, there exists unique $u \in D^{1,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u \nabla v \, \mathrm{d}x = \int_{\mathbb{R}^N} F(x) v \, \mathrm{d}x, \quad \forall v \in D^{1,2}(\mathbb{R}^N) \quad \text{and} \quad \|u\|_{1,2} \leq C |P|_1^{1/2}.$$

Furthermore, by regularity theory, $u \in D^{1,2}(\mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ for all $p \geq 1$ and it is a strong solution of

$$-\Delta u = F(x) \quad \text{in } \mathbb{R}^N.$$

Using the above notation, we are able to prove the following result.

Proposition 2.3. *Assume that F satisfies the condition (H). Then, there exists unique $u \in C^1(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$ with*

$$\int_{\mathbb{R}^N} \nabla u \nabla v \, \mathrm{d}x = \int_{\mathbb{R}^N} F(x) v \, \mathrm{d}x, \quad \forall v \in D^{1,2}(\mathbb{R}^N)$$

and

$$|u(x)| \leq \frac{c_0 |P|_1}{\omega_N (N-2)} |x|^{2-N}, \quad \forall x \in \mathbb{R}^N.$$

Moreover,

$$\|\nabla u\|_{C(B_R)} \leq \left[\frac{N}{R} \|u\|_{C(B_{2R})} + \frac{12R}{N-2} \|F\|_{C(B_{2R})} \right], \quad \forall R > 0.$$

Proof. In what follows, we denote by $\rho : \mathbb{R}^N \to \mathbb{R}$ the function given by

$$\rho(x) = \begin{cases} \mathrm{e}^{1/(|x|^2-1)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

It is well known that $\rho \in C_0^\infty(\mathbb{R}^N)$ with $\mathrm{supp}\rho \subset \overline{B}_1(0)$. Using the function ρ, for each for each $n \in \mathbb{N}$ we set

$$u_n(x) := \int_{\mathbb{R}^N} \rho_n(x-y) u(y) \, \mathrm{d}y,$$

where $\rho_n(x) = C n^N \rho(nx)$ with $C = (\int_{\mathbb{R}^N} \rho(y) \, \mathrm{d}y)^{-1}$. Applying some results found by [3], we know that (u_n) and $(\partial u_n / \partial x_i)$ converge uniformly in compact sets of $\mathbb{R}^N$ to u and

$\partial u/\partial x_i$ respectively, for all $i \in \{1, \ldots, N\}$. Moreover, fixing

$$F_n(x) = \int_{\mathbb{R}^N} \rho_n(x-y)F(y)\,\mathrm{d}y,$$

we derive that u_n verifies the equality below in the classical sense:

$$-\Delta u_n = F_n(x) \quad \text{in } \mathbb{R}^N.$$

By *a priori* estimates found in [**12**], for each $R > 0$ we have

$$\|\nabla u_n\|_{C(B_R)} \leq \left[\frac{N}{R}\|u_n\|_{C(B_{2R})} + \frac{12R}{N-2}\|F_n\|_{C(B_{2R})}\right], \quad \forall n \in \mathbb{N}.$$

As (F_n) converges uniformly for F in compact sets of the space, by passing to the limit $n \to +\infty$ in the last inequality we deduce

$$\|\nabla u\|_{C(B_R)} \leq \left[\frac{N}{R}\|u\|_{C(B_{2R})} + \frac{12R}{N-2}\|F\|_{C(B_{2R})}\right].$$

Using the Green's function G_R of the ball B_R with $R > |x|$, we know that

$$u_n(x) = -\int_{B_R} G_R(x,y)F_n(y)\,\mathrm{d}y + \frac{(R^2-|x|^2)}{N\omega_N R}\int_{\partial B_R} \frac{u_n(\xi)}{|x-\xi|^N}\,\mathrm{d}\sigma_\xi.$$

Passing to the limit $n \to +\infty$, it follows that

$$u(x) = -\int_{B_R} G_R(x,y)F(y)\,\mathrm{d}y + \frac{(R^2-|x|^2)}{N\omega_N R}\int_{\partial B_R} \frac{u(\xi)}{|x-\xi|^N}\,\mathrm{d}\sigma_\xi \quad \text{for } R > |x|.$$

Proceeding as in [**10**, pp. 225–226], we get

$$u(x) = -\int_{\mathbb{R}^N} \Gamma(x-y)F(y)\,\mathrm{d}y.$$

Gathering the last equality with (2.4), we find

$$\begin{aligned} |u(x)| &\leq \int_{\mathbb{R}^N} |\Gamma(x-y)||F(y)|\,\mathrm{d}y \\ &\leq c_0\int_{\mathbb{R}^N} |\Gamma(x-y)|P(y)\,\mathrm{d}y = c_0|\varphi(x)| \leq \frac{c_0|P|_1}{\omega_N(N-2)}|x|^{2-N}. \end{aligned}$$

The proof of the lemma is complete. □

3. A linear solution operator

In this section, we study the existence and properties of an important operator, which will be used to prove the existence of a solution for the problem (P).

In what follows, we fix $f \in C(\mathbb{R}^N)$ with $0 < f(x) \le P(x)$. Then, for each $v \in L^2_P(\mathbb{R}^N)$, the compact embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_P(\mathbb{R}^N)$ together with Riesz's theorem yields the fact that there is a unique solution $u \in D^{1,2}(\mathbb{R}^N)$ of the problem

$$\begin{cases} -\Delta u = f(x)v \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{$(WLP)_{\text{v}}$}$$

From this, we can define a *solution operator* $S : L^2_P(\mathbb{R}^N) \to L^2_P(\mathbb{R}^N)$ such that $S(v) = u$, where u is the unique solution of the above weight linear problem $(WLP)_v$. By using well-known arguments, S is a compact self-adjoint operator; thus by spectral theory there exists a complete orthonormal basis $\{u_n\}$ of $L^2_P(\mathbb{R}^N)$ and a corresponding sequence of positive real numbers $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ such that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

and

$$-\Delta u_n = \lambda_n f(x) u_n \quad \text{in } \mathbb{R}^N.$$

Moreover, using the Lagrange multiplier it is possible to prove the following characterization for λ_1:

$$\lambda_1 = \inf_{v \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} f(x)|v(x)|^2 \, \mathrm{d}x}.$$

The above identity is crucial to show that λ_1 is a simple eigenvalue and that a corresponding eigenfunction φ_1 can be chosen positive in $\mathbb{R}^N$. Moreover, we also have the following property:

$$\liminf_{|x|\to\infty} |x|^{N-2}\varphi_1(x) > 0. \tag{3.1}$$

The above limit is a consequence of the lemma below.

Lemma 3.1. *Let $u \in E \cap D^{1,2}(\mathbb{R}^N)$ be a positive function and $R > 0$ such that*

$$\int_{\mathbb{R}^N} \nabla u \nabla \psi \, \mathrm{d}x \ge 0, \quad \forall \psi \in D^{1,2}(\mathbb{R}^N), \quad \text{supp}\, \psi \subset B^c_R(0) \quad \textit{and} \quad \psi \ge 0.$$

Then,

$$\liminf_{|x|\to+\infty} |x|^{N-2} u(x) > 0.$$

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Proof. First of all, as u is a positive continuous function, there exists $\varepsilon > 0$ such that

$$u(x) \geq \varepsilon > 0, \quad \forall x \in \overline{B}_R(0).$$

Setting

$$w(x) = u(x) - \frac{\varepsilon R^{N-2}}{2}|x|^{2-N}$$

and

$$\tilde{w}(x) = \begin{cases} 0 & \text{if } |x| \leq R, \\ w^-(x) & \text{if } |x| > R, \end{cases}$$

we have $\tilde{w} \in D^{1,2}(\mathbb{R}^N), \operatorname{supp} \tilde{w} \subset B_R^c(0)$ and $\tilde{w} \geq 0$. Then, we must have

$$\int_{\mathbb{R}^N} \nabla w \nabla \tilde{w} \, \mathrm{d}x = \int_{\mathbb{R}^N} \nabla u \nabla \tilde{w} \, \mathrm{d}x \geq 0,$$

which leads to

$$-\int_{\mathbb{R}^N \setminus B_R(0)} |\nabla w^-|^2 \, \mathrm{d}x \geq 0.$$

Then, $w^- = 0$ in $\mathbb{R}^N \setminus B_R(0)$, from which it follows that

$$u(x) \geq \frac{\varepsilon R^{N-2}}{2}|x|^{2-N} \quad \text{for } |x| \geq R,$$

and the proof is complete. □

Since $E \subset L^2_P(\mathbb{R}^N)$, we intend to prove that $S : E \to E$ is well defined and it is a linear compact operator. To see why, we will consider the subspace E_0 of E given by

$$E_0 := \left\{ u \in C^1(\mathbb{R}^N);\ \sup_{x \in \mathbb{R}^N} [|x|^{N-2} u(x)] < \infty \right\},$$

endowed with the following norm:

$$\|u\| := \sup \left\{ |x|^{N-2}|u(x)| : x \in \mathbb{R}^N \right\}.$$

Using the space E_0, we claim that $S(E) \subset E_0$. Indeed, for each $v \in E$ we set $F(x) = f(x)v(x)$. Then, the Proposition 2.3 ensures the existence of a unique $u \in C^1(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u \nabla w \, \mathrm{d}x = \int_{\mathbb{R}^N} f(x) v w \, \mathrm{d}x, \quad \forall w \in D^{1,2}(\mathbb{R}^N) \tag{3.2}$$

and

$$|u(x)| \leq \frac{|P|_1}{\omega_N (N-2)} |v|_\infty |x|^{2-N}, \quad \forall x \in \mathbb{R}^N. \tag{3.3}$$

Therefore, $S(v) = u \in E_0$. As $v \in E$ is arbitrary, we can guarantee that $S(E) \subset E_0$.

Next, we show an important result of convergence of sequences, which will be used to prove that $S : E \to E$ is compact.

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Lemma 3.2. *Let (u_n) be a bounded sequence in E_0. If for each compact $A \subset \mathbb{R}^N$ the sequence $(\|u_n\|_{C^1(A)})$ is also bounded, then (u_n) admits a convergent subsequence in E.*

Proof. From boundedness of (u_n) in E_0, there exists $R_1 > 0$ such that

$$|u_n(x) - u_m(x)| \leq M|x|^{2-N} < 1 \quad \text{for } |x| \geq R_1 \text{ and } \forall n, m \in \mathbb{N}.$$

On the other hand, using the hypothesis that (u_n) is bounded in $C^1(\overline{B}_{R_1}(0))$, it follows that

$$|u_n(x) - u_n(y)| \leq M|x - y|, \quad \forall x, y \in \overline{B}_{R_1} \text{ and } \forall n \in \mathbb{N}.$$

Applying Arzelà–Ascoli's theorem, there exists $\mathbb{N}_1 \subset \mathbb{N}$, such that $(u_n)_{n\in\mathbb{N}_1}$ is a Cauchy sequence in $C(\overline{B}_{R_1})$ with

$$|u_n(x) - u_m(x)| < 1 \quad \text{for } |x| > R_1 \text{ and } n, m \in \mathbb{N}_1.$$

Repeating the above arguments, there exists $R_2 > R_1$, such that

$$|u_n(x) - u_m(x)| \leq M|x|^{2-N} < 1/2 \quad \text{for } |x| > R_2,\ \forall n, m \in \mathbb{N}_1.$$

Once (u_n) is bounded in $C^1(\overline{B}_{R_2}(0))$, we derive

$$|u_n(x) - u_n(y)| \leq M|x - y|, \quad \forall x, y \in \overline{B}_{R_2} \text{ and } n \in \mathbb{N}_1.$$

Applying again Arzelà–Ascoli's theorem, there exists $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $(u_n)_{n\in\mathbb{N}_2}$ is a Cauchy sequence in $C(\overline{B}_{R_2}(0))$ and

$$|u_n(x) - u_m(x)| < 1/2 \quad \text{for } |x| > R_2 \text{ and } n, m \in \mathbb{N}_2.$$

Repeating the above argument, we will find an increasing sequence $(R_k) \subset \mathbb{R}$ with $R_k \to +\infty$ and sets $\mathbb{N} \supseteq \mathbb{N}_1 \supseteq \mathbb{N}_2 \supseteq \cdots \supseteq \mathbb{N}_k \supseteq \cdots$ such that

$$|u_n(x) - u_m(x)| < 1/k, \text{ always that } |x| > R_k, \text{ with } n, m \in \mathbb{N}_k.$$

Thereby, there is a subsequence of (u_n), still denoted by (u_n), such that given $\varepsilon > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ verifying

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for } |x| > R \text{ and } n, m \geq n_0.$$

On the other hand, the boundedness of (u_n) in $C^1(\overline{B}_R(0))$ permits one to assume, changing the subsequence if necessary, the inequality below:

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for } n, m \geq n_0 \text{ and } x \in \overline{B}_R(0).$$

Therefore,

$$|u_n - u_m|_\infty < \varepsilon \quad \forall n, m \geq n_0,$$

from where it follows that (u_n) is a Cauchy subsequence in E. When E is a Banach space, (u_n) is convergent in E, which is the desired conclusion. □

Now, we are ready to prove the compactness of S.

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Lemma 3.3. *The operator $S: E \to E$ is compact.*

Proof. Let (v_n) be a bounded sequence in E and $u_n = S(v_n)$. By (3.3), $u_n \in E_0$ and

$$\|u_n\| \leq \frac{|v_n|_\infty |P|_1}{\omega_N(N-2)}, \quad \forall n \in \mathbb{N}.$$

Moreover, considering $F_n(x) = f(x)v_n(x)$ and fixing $R > 0$, the Proposition 2.3 guarantees that

$$\|\nabla u_n\|_{C(B_R)} \leq \left[\frac{N}{R}\|u_n\|_{C(B_{2R})} + \frac{12R}{N-2}\|F_n\|_{C(B_{2R})}\right], \quad \forall n \in \mathbb{N}.$$

Using (f_2) and bootstrap argument, we know that (u_n) is also bounded in $C(\overline{B}_{2R}(0))$. As (F_n) is also bounded in $C(\overline{B}_{2R}(0))$, because (v_n) is bounded in E, the right side of the last inequality is bounded. Thereby, we can apply Lemma 3.2 to infer that (u_n) possesses a convergent subsequence in E, and the lemma follows. □

Next, we show a result which will be used in the proof of Theorem 1.2.

Lemma 3.4. *Let $u \in E$ be a positive solution of $u = \lambda_1 S(u)$ and $\sigma, R > 0$ satisfy*

$$|x|^{N-2}u(x) \geq \sigma \quad \text{for } |x| \geq R.$$

Let $v \in E$ be a weak solution of

$$-\Delta v + b(x)v = \lambda f(x)v \quad \text{in } \mathbb{R}^N,$$

where b is a continuous function. Then, there exists $\varepsilon > 0$ such that, if $|\lambda - \lambda_1| + |u - v|_\infty < \varepsilon$ and $|b(x)| \leq \varepsilon P(x)$ for all $x \in \mathbb{R}^N$, then the function v is positive and $2|x|^{N-2}v(x) \geq \sigma$ for $|x| \geq R$.

Proof. Indeed, for $w = u - v$ we have

$$\int_{\mathbb{R}^N} \nabla w \nabla \psi \,\mathrm{d}x = \int_{\mathbb{R}^N} F(x)\psi \,\mathrm{d}x, \quad \forall \psi \in D^{1,2}(\mathbb{R}^N),$$

where

$$F(x) = (\lambda_1 - \lambda)f(x)v + \lambda_1 f(x)w - b(x)w + b(x)u, \quad \forall x \in \mathbb{R}^N.$$

Using the hypotheses, we find that

$$|F(x)| \leq C\varepsilon P(x), \quad \forall x \in \mathbb{R}^N.$$

Thus, by choosing small enough $\varepsilon > 0$, the Proposition 2.3 gives

$$|x|^{N-2}|w(x)| \leq \frac{C\varepsilon|P|_1}{\omega_N(N-2)} < \frac{\sigma}{2}, \quad \forall x \in \mathbb{R}^N,$$

and so

$$2|x|^{N-2}|w(x)| \leq \sigma, \quad \forall x \in \mathbb{R}^N.$$

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Consequently

$$|x|^{N-2}v(x) \geq |x|^{N-2}u(x) - |x|^{N-2}w(x) \geq \frac{\sigma}{2} \quad \text{for } |x| \geq R,$$

from where it follows that v is positive for $|x| \geq R$. Now, for $|x| \leq R$, decreasing ε if necessary, the positiveness of u means that v is also positive for $|x| \leq R$. This completes the proof. □

3.1. A nonlinear compact operator

In this subsection, we will study the properties of another compact operator.

For each $v \in E$, using the notation of § 2, there exists $C > 0$ such that

$$|-\phi_v(x)v(x)| \leq C|v|_\infty^{\gamma+1}P(x), \quad \forall x \in \mathbb{R}^N.$$

Thus, applying Proposition 2.3 with $F(x) = -\phi_v(x)v(x)$, there exists unique $u \in E_0 \cap D^{1,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u \nabla w \,\mathrm{d}x = \int_{\mathbb{R}^N} -\phi_v(x)vw \,\mathrm{d}x, \quad \forall w \in D^{1,2}(\mathbb{R}^N). \tag{3.4}$$

Moreover, we also have

$$|u(x)| \leq \int_{\mathbb{R}^N} |\Gamma(x-y)||F(y)|\mathrm{d}y \leq C|v|_\infty^{\gamma+1} \int_{\mathbb{R}^N} \Gamma(x-y)P(y)\mathrm{d}y \leq C|v|_\infty^{\gamma+1}|\varphi(x)|,$$

where φ was given in (2.4). Once φ is bounded, we get

$$|u(x)| \leq C|v|_\infty^{\gamma+1}, \quad \forall x \in \mathbb{R}^N. \tag{3.5}$$

From the previous analysis, we can define the nonlinear operator $G : E \to E_0 \subset E$ given by $G(v) = u$, where $u \in E_0 \cap D^{1,2}(\mathbb{R}^N)$ is the unique solution of (3.4).

The lemma below establishes that G is a compact operator. Since the proof of this fact follows with the same type of arguments explored in the proof of Lemma 3.3, we omit its proof.

Lemma 3.5. *The operator $G : E \to E$ is compact.*

Using the definition of G, (3.5) yields

$$|G(v)|_\infty \leq C|v|_\infty^{\gamma+1}, \quad \forall v \in E,$$

from which it follows that

$$\lim_{|v|_\infty \to +\infty} \frac{G(v)}{|v|_\infty} = 0, \tag{3.6}$$

that is,

$$G(v) = o(|v|_\infty). \tag{3.7}$$

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4. Proof of Theorem 1.2

Using the definitions of S and G, it is easy to check that $(\lambda, u) \in \mathbb{R} \times D^{1,2}(\mathbb{R}^N)$ solves (P) if, and only if,

$$u = F(\lambda, u) := \lambda S(u) + G(u). \tag{4.1}$$

In what follows, we will apply the following result obtained by Rabinowitz [14].

Theorem 4.1 (global bifurcation). *Let E be a Banach space. Suppose that L is a compact linear operator and $\lambda^{-1} \in \sigma(L)$ has odd algebraic multiplicity. If Ψ is a compact operator and*

$$\lim_{\|u\| \to 0} \frac{\Psi(u)}{\|u\|} = 0,$$

then the set

$$\Sigma = \overline{\{(\lambda, u) \in \mathbb{R} \times E : u = \lambda L(u) + \Psi(u), u \neq 0\}}$$

has a closed connected component $\mathcal{C} = \mathcal{C}_\lambda$ such that $(\lambda, 0) \in \mathcal{C}$ and

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*

(ii) *there exists $\hat{\lambda} \neq \lambda$, such that $(\hat{\lambda}, 0) \in \mathcal{C}$ and $\hat{\lambda}^{-1} \in \sigma(L)$.*

In what follows, we will apply the above theorem with $L = S$ and $\Psi = G$. By the previous results, we know that there is a first positive eigenfunction φ_1 associated with λ_1. Moreover, λ_1^{-1} is an eigenvalue of S with multiplicity equal to 1. From the global bifurcation theorem, there exists a closed connected component $\mathcal{C} = \mathcal{C}_{\lambda_1}$ of solutions for (P) that satisfies (i) or (ii). We claim that (ii) does not occur. To show this claim, we need Lemma 4.2 below.

Lemma 4.2. *There exists $\delta > 0$ such that if $(\lambda, u) \in \mathcal{C}$ with $|\lambda - \lambda_1| + |u|_\infty < \delta$ and $u \neq 0$, then u has defined signal, that is,*

$$u(x) > 0, \quad \forall x \in \mathbb{R}^N \quad \text{or} \quad u(x) < 0, \quad \forall x \in \mathbb{R}^N.$$

Proof. It is sufficient to prove that for any two sequences $(u_n) \subset E$ and $\lambda_n \to \lambda_1$ with

$$u_n \neq 0, \quad |u_n|_\infty \to 0 \quad \text{and} \quad u_n = F(\lambda_n, u_n) = \lambda_n S(u_n) + G(u_n),$$

u_n has defined signal for large enough n.

Setting $w_n = u_n / |u_n|_\infty$, we have

$$w_n = \lambda_n S(w_n) + \frac{G(u_n)}{|u_n|_\infty} = \lambda_n S(w_n) + o_n(1). \tag{4.2}$$

From compactness of the operator S, we can assume that $(S(w_n))$ is convergent. Then, $w_n \to w$ in E for some $w \in E$ with $|w|_\infty = 1$. Consequently,

$$w = \lambda_1 S(w)$$

or, equivalently,

$$-\Delta w = \lambda_1 f(x) w \quad \text{in } \mathbb{R}^N.$$

Thereby, w is an eigenfunction associated with λ_1. Then,

$$w(x) > 0, \quad \forall x \in \mathbb{R}^N \quad \text{or} \quad w(x) < 0, \quad \forall x \in \mathbb{R}^N.$$

Without loss of generality, we assume that $w(x) > 0$ for all $x \in \mathbb{R}^N$. Therefore, by Lemma 3.1 there exist $\sigma, R > 0$ such that

$$|x|^{N-2} w(x) \geq \sigma \quad \text{for} \quad |x| \geq R.$$

We know that given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda_n - \lambda_1| + |w - w_n|_\infty < \varepsilon \quad \text{for } n \geq n_0$$

and

$$|\phi_{u_n}(x)| \leq \varepsilon P(x), \quad \forall n \geq n_0 \text{ and } x \in \mathbb{R}^N.$$

Since, by (4.2),

$$-\Delta w_n + \phi_{u_n}(x).w_n = \lambda_n f(x) w_n \text{ in } \mathbb{R}^N,$$

Lemma 3.4 gives

$$w_n(x) > 0, \quad \forall x \in \mathbb{R}^N,$$

for large enough n. When u_n and w_n have the same signal, u_n is also positive, which is the desired conclusion. □

It is easy to check that if $(\lambda, u) \in \Sigma$, then the pair $(\lambda, -u)$ is also in Σ. In the lemma below, we show that $\mathcal{C}$ can be decomposed into two important sets.

Lemma 4.3. *Consider the sets*

$$\mathcal{C}^+ = \{(\lambda, u) \in \mathcal{C} : u(x) > 0, \ \forall x \in \mathbb{R}^N\} \cup \{(\lambda_1, 0)\}$$

and

$$\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : u(x) < 0, \ \forall x \in \mathbb{R}^N\} \cup \{(\lambda_1, 0)\}.$$

Then,

$$\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-. \tag{4.3}$$

Moreover, note that $\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : (\lambda, -u) \in \mathcal{C}^+\}$, $\mathcal{C}^+ \cap \mathcal{C}^- = \{(\lambda_1, 0)\}$ and $\mathcal{C}^+$ is unbounded if, and only if, $\mathcal{C}^-$ is also unbounded.

Proof. In what follows, we fix

$$\mathcal{C}^\pm = \{(\lambda, u) \in \mathcal{C} : u^\pm \neq 0\}.$$

By observing that

$$\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \cup \mathcal{C}^\pm,$$

it becomes clear that to prove (4.3) it is sufficient to show that $\overline{\mathcal{C}^\pm} = \emptyset$. Supposing by contradiction that $\overline{\mathcal{C}^\pm} \neq \emptyset$; as $\mathcal{C}$ is a connected set in $\mathbb{R} \times E$ and $\mathcal{C}^+ \cup \mathcal{C}^-$ is a closed

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non-empty set with $(\mathcal{C}^+ \cup \mathcal{C}^-) \cap \mathcal{C}^\pm = \emptyset$, we must have

$$(\mathcal{C}^+ \cup \mathcal{C}^-) \cap \overline{\mathcal{C}^\pm} \neq \emptyset.$$

Therefore, there is a solution (λ, u) of (P) and sequences $(\lambda_n, u_n) \subset \mathcal{C}^+ \cup \mathcal{C}^-$ and $(s_n, w_n) \subset \mathcal{C}^\pm$ such that

$$\lambda_n, s_n \to \lambda \quad \text{in } \mathbb{R}, \quad u_n \to u \quad \text{in } E \quad \text{and} \quad w_n \to u \quad \text{in } E.$$

Consequently, $u \geq 0$ in $\mathbb{R}^N$ or $u \leq 0$ in $\mathbb{R}^N$, and then by Lemma 4.2, $u \neq 0$. Supposing that $u \geq 0$ and $u \neq 0$, then the property (ϕ_8) ensures that $u(x) > 0$ in $\mathbb{R}^N$. Now, by (ϕ_3), there exists $R > 0$ such that

$$-\Delta u = (\lambda f(x) - \phi_u(x))u \geq 0 \quad \text{for } |x| \geq R,$$

in the weak sense, that is,

$$\int_{\mathbb{R}^N} \nabla u \nabla \psi \, \mathrm{d}x \geq 0, \quad \forall \psi \in D^{1,2}(\mathbb{R}^N), \quad \text{supp } \psi \subset B_R^c(0) \quad \text{and} \quad \psi \geq 0.$$

Applying Lemma 3.1, we get

$$\liminf_{|x| \to +\infty} |x|^{2-N} u(x) > 0. \tag{4.4}$$

Now, setting

$$F_n(x) = (\lambda_n - \lambda) f(x) w_n + \lambda f(x)(w_n - u) + (\phi_{w_n} - \phi_u) w_n + \phi_u (w_n - u),$$

and given $\epsilon > 0$, the proprieties (ϕ_2) and (ϕ_7) guarantee that there exists $n_0 \in \mathbb{N}$ such that

$$|F_n(x)| < \epsilon P(x), \quad \forall n \geq n_0 \quad \text{and} \quad \forall x \in \mathbb{R}^N.$$

The above inequality combined with (4.4) permits us to repeat the same arguments used in the proof of Lemma 3.4 to conclude that w_n is positive for large enough n, resulting in a contradiction. Thereby, $\overline{\mathcal{C}^\pm} = \emptyset$, and the proof is complete. □

Now, we are able to prove that (ii) does not hold.

Lemma 4.4. *$\mathcal{C}^+$ is unbounded.*

Proof. Suppose by contradiction that $\mathcal{C}^+$ is bounded. Then, $\mathcal{C}$ is also bounded. From the global bifurcation theorem, there exists $(\hat{\lambda}, 0) \in \mathcal{C}$, where $\hat{\lambda} \neq \lambda_1$ and $\hat{\lambda}^{-1} \in \sigma(S)$. Hence, there exists (u_n) in E and $\lambda_n \to \hat{\lambda}$, such that

$$u_n \neq 0, \quad |u_n|_\infty \to 0 \quad \text{and} \quad u_n = F(\lambda_n, u_n).$$

Considering $w_n = u_n/|u_n|_\infty$, we know that (4.2) is also satisfied. Moreover, as in the proof of Lemma 4.2, passing to a subsequence if necessary, (w_n) converges to w in E, which is a non-trivial solution of the problem

$$-\Delta w = \hat{\lambda} f(x) w \quad \text{in } \mathbb{R}^N,$$

showing that w is an eigenfunction related to $\hat{\lambda}$. Since $\hat{\lambda} \neq \lambda_1$, w must change signal. Then, for large enough n, w_n must change signal, implying that u_n also changes signal, which is absurd, because $(\lambda_n, u_n) \in \mathcal{C}^+$ or $(\lambda_n, u_n) \in \mathcal{C}^-$. □

From the previous lemma, the connected component $\mathcal{C}^+$ is unbounded. Now, our goal is to show that this component intersects any hyperplane $\{\lambda\} \times E$, for $\lambda > \lambda_1$. To see this, we need the following *a priori* estimate.

Lemma 4.5 (*a priori* estimate). *For any $\Lambda > 0$, there exists $r > 0$ such that, if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \in [0, \Lambda]$, then $|u|_\infty \leq r$.*

Proof. We start the proof with the following claim.

Claim 4.6. *For any $\Lambda > 0$, there exists $r > 0$ such that, if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \in [0, \Lambda]$, we must have $\|u\|_{1,2} \leq r$. Consequently, by Sobolev embedding there exists $r_1 > 0$ such that $|u|_{2^*} \leq r_1$.*

Indeed, if the claim does not hold, there are (u_n) in $D^{1,2}(\mathbb{R}^N)$ and $(\lambda_n) \subset [0, \Lambda]$ such that $\|u_n\|_{1,2} \to \infty$ and $u_n = F(\lambda_n, u_n)$. Considering $w_n = u_n/\|u_n\|_{1,2}$, it follows that

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \psi \, \mathrm{d}x + \int_{\mathbb{R}^N} \phi_{u_n}(x) w_n \psi \, \mathrm{d}x = \lambda_n \int_{\mathbb{R}^N} f(x) w_n \psi \, \mathrm{d}x, \quad \forall \psi \in D^{1,2}(\mathbb{R}^N). \tag{4.5}$$

Once (w_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, without loss of generality, we can suppose that there exists $w \in D^{1,2}(\mathbb{R}^N)$, such that $w_n \rightharpoonup w$ in $D^{1,2}(\mathbb{R}^N)$. Consequently, for some subsequence, the Sobolev embedding and (ϕ_5) combine to give

$$w_n(x) \to w(x) \quad \text{almost everywhere in } \mathbb{R}^N \quad \text{and} \quad \liminf_{n \to \infty} \phi_{w_n}(x) \geq \phi_w(x), \quad \forall x \in \mathbb{R}^N. \tag{4.6}$$

Setting $\psi = u_n/\|u_n\|_{1,2}^{\gamma+1}$ as a function test in (4.5) and using (ϕ_1), we get

$$\frac{1}{\|u_n\|_{1,2}^{\gamma}} + \int_{\mathbb{R}^N} \phi_{w_n}(x) w_n^2 \, \mathrm{d}x = \frac{\lambda_n}{\|u_n\|_{1,2}^{\gamma}} \int_{\mathbb{R}^N} f(x) w_n^2 \, \mathrm{d}x, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \phi_{w_n}(x) w_n^2 \, \mathrm{d}x = 0.$$

Then, Fatou's lemma together with (4.6) leads to

$$0 \leq \int_{\mathbb{R}^N} \phi_w(x) w^2(x) \, \mathrm{d}x \leq \lim_n \int_{\mathbb{R}^N} \phi_{w_n}(x) w_n^2 \, \mathrm{d}x = 0,$$

that is,

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} K(x, y) |w(y)|^\gamma |w(x)|^2 \, \mathrm{d}y \, \mathrm{d}x = 0.$$

Therefore, from (K_1), it follows that $w \equiv 0$, and so $w_n \to 0$ in $L_P^2(\mathbb{R}^N)$. Now, fixing $\psi = w_n$ as a test function in (4.5), we obtain

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} \phi_{u_n}(x) w_n^2 \, \mathrm{d}x = \lambda_n \int_{\mathbb{R}^N} f(x) w_n^2 \, \mathrm{d}x,$$

from where it follows that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 \,\mathrm{d}x \le \Lambda \int_{\mathbb{R}^N} P(x) w_n^2 \,\mathrm{d}x \to 0.$$

Thus,

$$\|w_n\|_{1,2}^2 \to 0$$

which is absurd, because $\|w_n\|_{1,2} = 1$ for all $n \in \mathbb{N}$, which proves the claim.

To obtain an *a priori* estimate, we need a good estimate from above for the norm $|\ \ |_\infty$. To this end, we will use the lemma below, whose proof will be omitted because it is a small modification of the Moser iteration, similar to that found in [**5**].

Lemma 4.7. *Let $h : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a non-negative measurable function verifying*

$$\sup_{x \in \mathbb{R}^N} |h|_{L^q(B_2(x))} < \infty$$

with $q > N/2$ and $v \in D^{1,2}(\mathbb{R}^N)$ be a weak solution of the problem

$$-\Delta v + b(x) v = H(x, v) \quad \text{in } \mathbb{R}^N, \tag{4.7}$$

where $b : \mathbb{R}^N \to \mathbb{R}$ is a non-negative function and $H : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function verifying

$$|H(x, s)| \le h(x)|s|, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}.$$

Then, $v \in E$ and there exists a constant $C := C(q, h) > 0$ such that

$$|v|_\infty \le C|v|_{2^*}.$$

To conclude the proof of Lemma 4.5, it is sufficient to apply Lemma 4.7 with

$$b(x) = \phi_{u_n}(x), \quad H(x, s) = \lambda_n f(x) s \quad \text{and} \quad h(x) = \Lambda f(x). \qquad \square$$

5. Conclusion of the proof of Theorem 1.2

From Lemma 4.5, for all $\lambda > \lambda_1$, we have $(\{\lambda\} \times E) \cap \mathcal{C}^+ \ne \emptyset$, that is, $\mathcal{C}^+$ crosses the hyperplane $\{\lambda\} \times E$. Indeed, otherwise there exists $\Lambda > \lambda_1$ such that $\mathcal{C}^+$ does not cross the hyperplane $\{\lambda\} \times E$, and thus by Lemma 4.5 there exists $R > 0$ such that $(\lambda, u) \in \mathcal{C}^+$, $\lambda \in [0, \Lambda]$, and $|u|_\infty \le R$. Therefore, $\mathcal{C}^+$ would be ounded, which contradicts Lemma 4.4.

To finalize the proof of Theorem 1.2, we must show that there is no solution for (P) when $\lambda \le \lambda_1$. Indeed, arguing by contradiction, if (λ, u) is a solution of (P), taking $\psi = \varphi_1$ as a test function in (P), we derive

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi_1 \,\mathrm{d}x + \int_{\mathbb{R}^N} \phi_u u \varphi_1 \,\mathrm{d}x = \lambda \int_{\mathbb{R}^N} f(x) u \varphi_1 \,\mathrm{d}x$$

from which it follows that

$$\lambda_1 \int_{\mathbb{R}^N} f(x) u \varphi_1 < \lambda \int_{\mathbb{R}^N} f(x) u \varphi_1,$$

or equivalently $\lambda_1 < \lambda$. This proves the theorem.

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