Canad. Math. Bull. Vol. **65** (2), 2022, pp. 466–472
http://dx.doi.org/10.4153/S0008439521000424

A Radó theorem for complex spaces

Viorel Vâjâitu

Abstract. We generalize Radó's extension theorem from the complex plane to reduced complex spaces.

1 Introduction

A theorem due to Radó asserts that a continuous complex-valued function on an open subset of the complex plane is holomorphic provided that it is holomorphic off its zero set.

Essentially, this theorem was proved in [10]. Since then, many other proofs have been proposed, e.g., [2, 3, 6, 7]. The articles [1, 11, 14] give some generalizations.

Radó's statement remains true for complex manifolds (or, more generally, for normal complex spaces) as well as in the complex plane.

In this short note, we investigate a natural extension of Radó's theorem when the ambient space has (nonnormal) singularities.

Complex spaces, unless explicitly stated, are assumed to be reduced and countable at infinity. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers.

Here, we state our main results.

Proposition 1 *There is an irreducible Stein curve X and a continuous function $f : X \longrightarrow \mathbb{C}$ that is holomorphic off its zero set, but no power f^ν, $\nu \in \mathbb{N}$, is globally holomorphic.*

Theorem 1 *Let X be a complex space and $\Omega \subset X$ be a relatively compact open set. Then, there is $\nu_\Omega \in \mathbb{N}$ such that, for every continuous function $f : X \longrightarrow \mathbb{C}$ that is holomorphic off its zero set, and for every integer $\nu \geq \nu_\Omega$, the power function f^ν is holomorphic on Ω.*

Recall the following definition [15]. Let X be a complex space. A continuous, complex-valued function f defined on an open set $U \subset X$ is *c-holomorphic* if its restriction of to $\text{Reg}(X) \cap U$ is holomorphic, where $\text{Reg}(X)$ is the open set of those points of X where it is locally a manifold. The sheaf of germs of c-holomorphic functions in X is denoted by $\mathcal{O}_X^c$; it is a coherent $\mathcal{O}_X$-module.

Received by the editors March 30, 2021; revised June 2, 2021; accepted June 2, 2021.
Published online on Cambridge Core June 17, 2021.
AMS subject classification: **32D15**, 32D20, 32C15.
Keywords: Radó's theorem, complex space, c-holomorphic function.

Henceforth, the following remark will be used tacitly. For a complex space X, any continuous function $f : X \longrightarrow \mathbb{C}$ that is holomorphic off its zero set $f^{-1}(0)$ is c-holomorphic. (This results by the classical Radó theorem on complex manifolds.)

2 Proof of Proposition 1

The example of a Stein curve X is obtained by implanting generalized cusp singularities at the points $2, 3, \ldots,$ of $\mathbb{C}$, and then the existence of the function f is obtained via Cartan's vanishing theorem on Stein spaces.

In order to proceed, let p and q be coprime integers ≥ 2. Consider the cusp-like irreducible and locally irreducible complex curve:

$$\Gamma = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ z_1^p = z_2^q\} \subset \mathbb{C}^2.$$

Its normalization is $\mathbb{C}$ and $\pi : \mathbb{C} \longrightarrow \Gamma$, $t \mapsto (t^q, t^p)$, is the normalization map. Note that π is a homeomorphism.

A continuous function $h : \Gamma \longrightarrow \mathbb{C}$ that is holomorphic off its zero set, but fails to be globally holomorphic, is produced as follows.

Select natural numbers m and n with $mq - np = 1$, and define $h : \Gamma \longrightarrow \mathbb{C}$ by setting for $(z_1, z_2) \in \Gamma$,

$$h(z_1, z_2) := \begin{cases} z_1^m / z_2^n & \text{if } z_2 \neq 0, \\ 0 & \text{if } z_2 = 0. \end{cases}$$

It is easily seen that h is continuous (as π is a homeomorphism, the continuity of h follows from that of $h \circ \pi$, which is equal to the identity mapping on $\mathbb{C}$), h is holomorphic off its zero set (incidentally, here, the regular part $\mathrm{Reg}(\Gamma)$ is the complement of this zero set), and h is not holomorphic about $(0, 0)$ (use a Taylor series expansion about $(0, 0) \in \mathbb{C}^2$ of a presumably holomorphic extension).

Furthermore, h^k is globally holomorphic provided that $k \geq (p-1)(q-1)$. (Because every integer at least $(p-1)(q-1)$ can be written in the form $\alpha p + \beta q$ with $\alpha, \beta \in \{0, 1, 2, \ldots\}$, and because h^p and h^q are holomorphic being the restrictions of z_2 and z_1 to Γ, respectively.)

In addition, $z_1^a z_2^b h$ is holomorphic on Γ provided that $q\lfloor (m+a)/p \rfloor + b \geq n$, where $\lfloor \cdot \rfloor$ is the floor function.

It is interesting to note that the stalk of germs of c-holomorphic functions $\mathcal{O}_0^c$ at 0 is generated as an $\mathcal{O}_0$-module by the germs at 0 of $1, h, \ldots, h^r$, where $r = \min\{p, q\} - 1$.

Now, for each integer $k \geq 2$, let $\Gamma_k := \{(z_1, z_2) \in \mathbb{C}^2 \ ; \ z_1^k = z_2^{k+1}\}$. As previously noted, Γ_k is an irreducible curve whose normalization map is $\pi_k : \mathbb{C} \longrightarrow \Gamma_k$, $t \mapsto (t^{k+1}, t^k)$, and the function $h_k : \Gamma_k \longrightarrow \mathbb{C}$ defined for $(z_1, z_2) \in \Gamma_k$ by:

$$h_k(z_1, z_2) := \begin{cases} z_1 / z_2 & \text{if } z_2 \neq 0, \\ 0 & \text{if } z_2 = 0, \end{cases}$$

has the following properties:

a$_k$) The function h_k is c-holomorphic.
b$_k$) The power h_k^{k-1} is not holomorphic.
c$_k$) The function $z_1^{k-1} h_k$ is holomorphic, because it is the restriction of z_2^k to Γ_k.

https://doi.org/10.4153/S0008439521000424 Published online by Cambridge University Press

Here, with these examples of singularities at hand, we change the standard complex structure of $\mathbb{C}$ at the discrete analytic set $\{2,3,\dots\}$ by complex surgery, in order to obtain an irreducible Stein complex curve X and a discrete subset $\Lambda = \{x_k : k = 2,3,\dots\}$ such that, at the level of germs, (X, x_k) is biholomorphic to $(\Gamma_k, 0)$.

The surgery, that we recall for the commodity of the reader (because, in some monographs like [8], the subsequent condition $(\star)$ is missing), goes as follows.

Let Y and U' be complex spaces together with analytic subsets A and A' of Y and U', respectively, such that there is an open neighborhood U of A in Y and $\varphi : U \setminus A \longrightarrow U' \setminus A'$ that is biholomorphic.

Then, define:

$$X := (Y \setminus A) \sqcup_\varphi U' := (Y \setminus A) \sqcup U'/_\sim$$

by means of the equivalence relation $U \setminus A \ni y \sim \varphi(y) \in U' \setminus A'$.

Then, there exists exactly one complex structure on X such that U' and $Y \setminus A$ can be viewed as open subsets of X in a canonical way provided that the following condition is satisfied:

($\star$) For every $y \in \partial U$ and $a' \in A'$, there are open neighborhoods D of y in Y, $D \cap A = \varnothing$, and B of a' in U' such that $\varphi(D \cap U) \cap B \subseteq A'$.

Thus, X is formed from Y by "replacing" A with A'.

In practice, the condition $(\star)$ is fulfilled if $\varphi^{-1} : U' \setminus A' \longrightarrow U \setminus A$ extends to a continuous function $\psi : U' \longrightarrow U$ such that $\psi(A') = A$. In this case, if D and V are disjoint open neighborhoods of ∂U and A in Y, respectively, then $B = A' \cup \varphi(V \setminus A)$ is open in U', because it equals $\psi^{-1}(V)$ and $(\star)$ follows immediately. (This process is employed, for instance, in the construction of the blowup of a point in a complex manifold!)

Coming back to the construction of the example proving Proposition 1, consider $Y = \mathbb{C}$, $A = \{2,3,\dots\}$, and for each $k = 2,3,\dots$, let $\Delta(k, 1/3)$ be the disk in $\mathbb{C}$ centered at k of radius $1/3$ that is mapped holomorphically onto an open neighborhood U_k of $(0,0) \in \Gamma_k$ through the holomorphic map $t \mapsto \pi_k(z-k)$. Applying surgery, we get an irreducible Stein curve X and the discrete subset Λ with the aforementioned properties.

It remains to produce the function f as stated. For this, we let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent ideal sheaf with support Λ and such that $\mathcal{I}_{x_k} = \mathfrak{m}_{x_k}^{k-1}$ for $k = 2,3\dots$, where $\mathfrak{m}_{x_k}$ is the maximal ideal of the analytic algebra of the stalk of $\mathcal{O}_X$ at x_k.

From the exact sequence:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}^c \longrightarrow \mathcal{O}^c/\mathcal{I} \longrightarrow 0,$$

we obtain a c-holomorphic function f on X such that, for each $k = 2,3,\dots$, at germs level, f equals $h_k \pmod{\mathcal{I}_{x_k}}$.

By properties $\mathbf{a}_k$), $\mathbf{b}_k$), and $\mathbf{c}_k$) from above, it follows that there does exist $\nu \in \mathbb{N}$ such that f^ν becomes holomorphic on X. (For instance, if $f = h_k + g_k^{k-1}$, for certain $g_k \in \mathfrak{m}_{x_k}$, then f^{k-1} is not holomorphic about x_k.)

https://doi.org/10.4153/S0008439521000424 Published online by Cambridge University Press

3 Proof of Theorem 1

This is divided into four steps. In Step 1, we recall, following [4], the multiplicity of an analytic set at a point. Then, in Step 2, we estimate the vanishing order of a c-holomorphic function germ at a point of its zero set in terms of the multiplicity of the analytic germ where it is defined. In Step 3, we collect some useful facts about O^N-approximability due to Spallek [13] and Siu [12]. Eventually, the proof of theorem is achieved in the fourth step.

Step 1. Let A be a pure k-dimensional locally analytic subset of $\mathbb{C}^n$. Let $a \in A$ and select an $(n-k)$-dimensional complex subspace $L \subset \mathbb{C}^n$ such that a is an isolated point of the set $A \cap (\{a\} + L)$. Then, as we know, there is a domain $U \ni a$ in $\mathbb{C}^n$ such that $A \cap U \cap (\{a\} + L) = \{a\}$ and such that the projection $\pi_L : A \cap U \longrightarrow U'_L \subset L^\perp$ along L is a d-sheeted analytic cover, for some $d \in \mathbb{N}$, where $L^\perp$ is the orthogonal of L with respect to the canonical scalar product in $\mathbb{C}^n$.

The critical analytic set Σ of this cover does not partition the domain U'_L and is nowhere dense in it; therefore, the number of sheets of this cover does not change when shrinking U. Furthermore, if z' is the projection of z in $L^\perp$ and $z' \in U'_L \setminus \Sigma$, then,

$$\sharp\ A \cap U \cap (\{z\} + L) = d,$$

and all d points of the fiber above z' tend to a as $z' \to a'$. This number is called the *multiplicity of the projection* $\pi_L|_A$ at a, and is denoted by $\mu_a(\pi_L|_A)$.

For any point $x \in A$ in the above indicated small neighborhood $U \ni a$, the number of sheets of the cover $A \cap U \longrightarrow U'_L$ does not exceed d in a neighborhood of x (it may be less); hence, the function $\mu_x(\pi_L|_A)$ is upper semicontinuous on $A \cap U$. See [4, p. 102].

Thus, for every $(n-p)$-dimensional complex plane $L \subset \mathbb{C}^n$ such that a is an isolated point in $A \cap (\{a\} + L)$, the multiplicity of the projection $\mu_a(\pi_L|_A)$ is finite. The minimum of these numbers over all $L \in \mathrm{Gr}(n-p, n)$ as above is denoted $\mu_a(A)$ and is called the *multiplicity of A at a*.

Furthermore, it can be shown that the multiplicity $\mu_a(A)$ does not depend on how A is locally embedded at a into a complex euclidean space.

Altogether, we get a function $A \ni x \mapsto \mu_x(A) \in \mathbb{N}$ that is upper semicontinuous. See [4, p. 120].

Step 2. For the sake of simplicity, let $a = 0$, and for the complex subspace $L = \{0\} \times \mathbb{C}^{n-k}$, the projection $\pi_L|_A$ realizes $\mu_0(A)$, namely $\mu_0(\pi_L|_A) = \mu_0(A)$.

With the necessary changes, by Step 1, we arrive at the following setup.

The set A is analytic in $D \times \mathbb{C}^{n-k}$ with D a domain of $\mathbb{C}^k$, the map $\pi : A \longrightarrow D$ is induced by the first projection from $\mathbb{C}^k_z \times \mathbb{C}^{n-k}_w$ onto $\mathbb{C}^k_z$, $A \ni x = (z, w) \mapsto \pi(x) = z$, such that π is a (finite) branched covering with image D, covering number $d := \mu_0(A)$, critical set Σ, which is a nowhere dense analytic subset of D, and $\pi^{-1}(0) = \{0\}$.

Now, let $h : A \longrightarrow \mathbb{C}$ be any c-holomorphic function. For every point $x = (z, w) \in (D \setminus \Sigma) \times \mathbb{C}^{n-k}$, we define the polynomial:

$$\omega(x, t) = \prod_{\pi(x')=z} (t - h(x')) = t^d + a_1(x)t^{d-1} + \cdots + a_d(x).$$

https://doi.org/10.4153/S0008439521000424 Published online by Cambridge University Press

Because h is holomorphic on the regular part $\mathrm{Reg}(A)$ of A and h is continuous on A, *a fortiori* h is bounded on any compact subset of A (in particular, on $\pi^{-1}(K)$, for every compact set K of D), the coefficients a_j are naturally holomorphic on $(D \setminus \Sigma) \times \mathbb{C}^{n-k}$ and locally bounded on $D \times \mathbb{C}^k$. Thus, granting Riemann's extension theorem, they extend holomorphically to $D \times \mathbb{C}^{n-k}$ (we keep the same notations for the extensions). If, furthermore, $h(0) = 0$, then all coefficients $a_j(0) = 0$, because π is proper and $\pi^{-1}(0) = \{0\}$.

Therefore, we obtain a distinguished Weierstrass polynomial of degree d, $W(x,t) = t^d + a_1(x)t^{d-1} + \cdots + a_d(x)$, which is the unique extension of ω to $D \times \mathbb{C}^{n-k}$ and such that $W(x, h(x)) = 0$ for all $x \in A$.

Note that, if $W(x,t) = 0$, then the identity $|t|^d = \mathrm{O}(\|x\|)$ holds true as $(x,t) \to 0$ because

$|a_j(x)| = \mathrm{O}(\|x\|)$, or equivalently:

$$|t| = \mathrm{O}(\|x\|^{1/d}) \text{ as } (x,t) \to 0,$$

meaning that there are positive constants M and ε such that, if $W(x,t) = 0$ and $\max\{|t|, \|x\|\} < \varepsilon$, then $|t| \leq M\|x\|^{1/d}$.

To sum up, coming back to the general setting, and using that for two real numbers α and β, one has $s^\alpha = \mathrm{O}(s^\beta)$ as $(0,\infty) \ni s \to 0$ if and only if $\alpha \geq \beta$, by routine arguments, from Step 1 and the above discussion, we get the following fact.

(†) Let A be a locally analytic subset of $\mathbb{C}^n$ of pure dimension. Then, the multiplicity function $\mu_x(A)$ on $x \in A$ is upper semicontinuous. Furthermore, any point $a \in A$ admits an open neighborhood U in A such that, for every point $x_0 \in U$ and every nonconstant, c-holomorphic germ $h : (A, x_0) \longrightarrow (\mathbb{C}, 0)$, one has,

$$|h(x)| = \mathrm{O}(\|x - x_0\|^\alpha) \text{ as } A \ni x \to x_0,$$

where $\alpha = 1/\mu_a(A)$.

In general, if $(A,x) = \cup_j (A_j, x)$ is the decomposition of the germ (A,x) into its finitely many irreducible components, whose number might depend on $x \in A$, then we set $\mu_x(A) = \max_j \mu_x(A_j)$. The multiplicity function thus defined is upper semicontinuous on A, and the above "identity" in (†) holds for the exponent α given by $1/\alpha = \max_j \mu_a(A_j)$.

For the commodity of the reader, we mention that, for any complex space X, we get a natural multiplicity function $X \ni x \mapsto \mu_x(X) \in \mathbb{N}$ that is upper semicontinuous, although this information is not used hereafter.

Step 3. From Spallek [13], we recall the following notion. Let $A \subset \mathbb{C}^n$ be a set and a a point of A. We say that a germ function $\varphi : (A, a) \longrightarrow (\mathbb{C}, \varphi(a))$ is O^N-*approximable at* a if there exists a polynomial $P(z, \bar{z})$ of degree at most $N-1$ in the variables $z_j - a_j, \overline{z_j - a_j}$, $j = 1, \ldots, n$, such that,

$$|\varphi(z) - P(z,\bar{z})| = \mathrm{O}(\|z - a\|^N) \text{ as } A \ni z \to a.$$

Example 1 If φ is the restriction of a $\mathcal{C}^\infty$-smooth, complex-valued function defined on a neighborhood of a in $\mathbb{C}^n$, then using Taylor's formula, one has that φ is O^N-*approximable at* a for all positive integers N.

Example 2 Let A be locally analytic at the point a, and $\nu, N \in \mathbb{N}$ that satisfy $\nu > \mu_a(A)N$. Then, by (†), it follows that for any germ of a c-holomorphic map $h : (A, a) \longrightarrow (\mathbb{C}, 0)$, Re h^ν and Im h^ν are O^N*-approximable at* a.

The following result due to Siu [12] improves onto Spallek's similar one from [13].

Proposition 2 *For every compact set K of a complex space X, there exists a positive integer $N = N(K)$ depending on K such that, if f is a c-holomorphic function germ at $x \in K$ and Re f is O^N-approximable at any point in some neighborhood of x, then f is a holomorphic germ at x.*

Step 4. To conclude the theorem, because the assertion to be proved is local, without any loss in generality, we may assume that X is an analytic subset of some open set of $\mathbb{C}^n$.

Now, let K be a compact set of X. We claim that there is $\nu_K \in \mathbb{N}$ such that, for any c-holomorphic function f on X that is holomorphic off its zero set $f^{-1}(0)$, the power f^ν is holomorphic about K for all integers $\nu \geq \nu_K$.

For this, consider a compact neighborhood K^* of K in X. Because the function $X \ni x \mapsto \mu_x(X) \in \mathbb{N}$ is upper semicontinuous, there exists a natural number d such that $\mu_x(X) < d$ for all $x \in K^*$.

We show that $\nu_K = dN$ is as desired, where N is selected according to Proposition 2 corresponding to the compact K of X.

Indeed, in order to show that f^ν is holomorphic about K for $\nu \in \mathbb{N}$ that satisfies $\nu \geq \nu_K$, we apply Proposition 2, and for this, we need to check that the function Ref^ν is O^N-approximable at any point $x \in K^*$.

This follows by case analysis.

If $f(x) \neq 0$, because f is holomorphic on the open set $X \setminus f^{-1}(0)$ of X, so that Ref and Imf are C^∞-smooth there, by Example 1, it follows that Ref^ν is O^N-approximable at x.

If $f(x) = 0$, then by Example 1, the function Ref^ν is O^N-approximable at x, because $\nu \geq dN = \nu_K$.

This completes the proof of the theorem.

4 A final remark

Below we answer a question raised by Th. Peternell at the XXIV Conference on Complex Analysis and Geometry, held in Levico Terme, June 10–14, 2019. He asked whether or not a similar statement like Theorem 1 does hold for nonreduced complex spaces.

More specifically, let $(X, \mathcal{O}_X)$ be a not necessarily reduced complex space and $f : X \longrightarrow \mathbb{C}$ be continuous such that, if A denotes the zero set of f, then $X \setminus A$ is dense in X, and there is a section $\sigma \in \Gamma(X \setminus A, \mathcal{O}_X)$ whose reduction Red(σ) equals $f|_{X \setminus A}$.

Is it true that, for every relatively compact open subset D of X, there is a positive integer n such that σ^n extends to a section in $\Gamma(D, \mathcal{O}_X)$?

We show that the answer is "No."

https://doi.org/10.4153/S0008439521000424 Published online by Cambridge University Press

In order to do this, recall that, if R is a commutative ring with unit and M is an R-module, we can endow the direct sum $R \oplus M$ with a ring structure with the obvious addition, and multiplication defined by:

$$(r, m) \cdot (r', m') = (rr', rm' + r'm).$$

This is the Nagata ring structure from algebra [9].

Now, if $(X, \mathcal{O}_X)$ is a complex space and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then $\mathcal{H} := \mathcal{O}_X \oplus \mathcal{F}$ becomes a coherent sheaf of analytic algebras and $(X, \mathcal{H})$ a complex space [5, Satz 2.3].

The example is as follows. Let ${}_n\mathcal{O}$ denote the structural sheaf of $\mathbb{C}^n$. The above discussion produces a complex space $(\mathbb{C}, \mathcal{H})$ such that $\mathcal{H} = {}_1\mathcal{O} \oplus {}_1\mathcal{O}$, which can be written in a suggestive way $\mathcal{H} = {}_1\mathcal{O} + \varepsilon \cdot {}_1\mathcal{O}$, where ε is a symbol with $\varepsilon^2 = 0$. As a matter of fact, if we consider $\mathbb{C}^2$ with complex coordinates (z, w) and the coherent ideal $\mathcal{I}$ generated by w^2, then $\mathcal{H}$ is the analytic restriction of the quotient ${}_2\mathcal{O}/\mathcal{I}$ to $\mathbb{C}$.

The reduction of $(\mathbb{C}, \mathcal{H})$ is $(\mathbb{C}, {}_1\mathcal{O})$. A holomorphic section of $\mathcal{H}$ over an open set $U \subset \mathbb{C}$ consists of couple of ordinary holomorphic functions on U.

Now, take f the identity function id on $\mathbb{C}$, and the holomorphic section $\sigma \in \Gamma(\mathbb{C}^\star, \mathcal{H})$ given by $\sigma = \mathrm{id} + \varepsilon g$, where g is holomorphic on $\mathbb{C}^\star$ having a singularity at 0, for instance, $g(z) = 1/z$.

Obviously, the reduction of σ is the restriction of id on $\mathbb{C}^\star$, and no power σ^k of σ extends across 0 to a section in $\Gamma(\mathbb{C}, \mathcal{H})$, because $\sigma^k = \mathrm{id} + \varepsilon k g$ and g does not extend holomorphically across $0 \in \mathbb{C}$.

Acknowledgment I would like to thank the anonymous reviewer for critical reading and suggestions that helped to improve on earlier drafts of the manuscript.

References

[1] B. Aupetit, *Une généralisation du théorème d'extension de Radó*. Manuscripta Math. **23**(1977/78), 319–323.

[2] H. Behnke and K. Stein, *Modifikation komplexer Mannigfaltigkeiten und Riemannscher Gebiete*. Math. Ann. **124**(1951), 1–16.

[3] H. Cartan, *Sur une extension d'un théorème de Radó*. Math. Ann. **125**(1952), 49–50.

[4] E. M. Chirka, *Complex analytic sets*, Kluwer Academic Publishers Group, Dordrecht, 1989.

[5] O. Forster, *Zur Theorie der Steinschen Algebren und Moduln*. Math. Z. **97**(1967), 376–405.

[6] E. Heinz, *Ein elementarer Beweis des Satzes von Radó–Behnke–Stein–Cartan über analytische Funktionen*. Math. Ann. **131**(1956), 258–259.

[7] R. Kaufman, *A theorem of Radó*. Math. Ann. **169**(1967), 282.

[8] L. Kaup and B. Kaup, *Holomorphic functions of several variables*, de Gruyter Studies in Mathematics, 3, De Gruyter, Berlin, 1983.

[9] M. Nagata, *Local rings*, Interscience, New York, 1962.

[10] T. Radó, *Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit*. Math. Z. **20**(1924), 1–6.

[11] J. Riihentaus, *A note concerning Radó's theorem*. Math. Z. **182**(1983), 159–165.

[12] Y. T. Siu, *O^N-approximable and holomorphic functions on complex spaces*. Duke Math. J. **36**(1969), 451–454.

[13] K. Spallek, *Differenzierbare und holomorphe Funktionen auf analytischen Mengen*. Math. Ann. **161**(1965), 143–162.

[14] E. L. Stout, *A generalization of a theorem of Radó*. Math. Ann. **177**(1968), 339–340.

[15] H. Whitney, *Complex analytic varieties*, Addison, Wesley, 1972.

Université de Lille, U.F.R. de Mathématiques, F-59655 Villeneuve d'Ascq Cedex, France
e-mail: viorel.vajaitu@univ-lille.fr

https://doi.org/10.4153/S0008439521000424 Published online by Cambridge University Press