Combinatorics, Probability and Computing (2020), **29**, pp. 128–136
doi:10.1017/S0963548319000269

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ARTICLE

Extensions of the Erdős–Gallai theorem and Luo's theorem

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(Received 8 February 2018; revised 6 June 2019; first published online 8 October 2019)

Abstract

The famous Erdős–Gallai theorem on the Turán number of paths states that every graph with n vertices and m edges contains a path with at least $(2m)/n$ edges. In this note, we first establish a simple but novel extension of the Erdős–Gallai theorem by proving that every graph G contains a path with at least

$$\frac{(s+1)N_{s+1}(G)}{N_s(G)}+s-1$$

edges, where $N_j(G)$ denotes the number of j-cliques in G for $1 \leqslant j \leqslant \omega(G)$. We also construct a family of graphs which shows our extension improves the estimate given by the Erdős–Gallai theorem. Among applications, we show, for example, that the main results of [20], which are on the maximum possible number of s-cliques in an n-vertex graph without a path with ℓ vertices (and without cycles of length at least c), can be easily deduced from this extension. Indeed, to prove these results, Luo [20] generalized a classical theorem of Kopylov and established a tight upper bound on the number of s-cliques in an n-vertex 2-connected graph with circumference less than c. We prove a similar result for an n-vertex 2-connected graph with circumference less than c and large minimum degree. We conclude this paper with an application of our results to a problem from spectral extremal graph theory on consecutive lengths of cycles in graphs.

2010 MSC Codes: 05C35, 05C38

1. Erdős–Gallai theorem and an extension

Let $\mathcal{H}$ be a family of graphs. The *Turán number* $\mathrm{ex}(n, \mathcal{H})$ is the largest possible number of edges in an n-vertex graph G which contains no member of $\mathcal{H}$ as a subgraph. If $\mathcal{H} = \{H\}$, then we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \mathcal{H})$. We use P_ℓ to denote a path with ℓ vertices. In this case, we say P_ℓ is of length $\ell - 1$.

Erdős and Gallai [9] proved the following celebrated theorems on Turán numbers of cycles and paths.

[†]Supported by the NSFC grants (11601379, 11771141, 11971346) and the Seed Foundation of Tianjin University (2018XRG-0025).
[‡]Supported by the NSFC grant (11601380) and the Seed Foundation of Tianjin University (2017XRX-0011)

Theorem 1.1 (Erdős and Gallai [9]).

$$\mathrm{ex}(n, \mathcal{C}_{\geqslant \ell}) \leqslant \frac{(\ell-1)(n-1)}{2},$$

where $\ell \geqslant 3$ *and* $\mathcal{C}_{\geqslant \ell}$ *is the set of all cycles of length at least* ℓ.

Theorem 1.2 (Erdős and Gallai [9]).

$$\mathrm{ex}(n, P_\ell) \leqslant \frac{(\ell-2)n}{2},$$

where $\ell \geqslant 2$.

For the tightness of Theorem 1.1, one can check the graph consisting of $(n-1)/(\ell-2)$ cliques of size $\ell-1$ with a common vertex, where $n-1$ is divisible by $\ell-2$. The tightness of Theorem 1.2 is shown by the graph with $n/(\ell-1)$ disjoint $K_{\ell-1}$, where n is divisible by $\ell-1$. For more improvements and extensions of the Erdős–Gallai theorems, see [4, 5, 6, 11, 13, 19, 22, 23]. We refer the reader to an excellent survey on related topics by Füredi and Simonovits [14].

For a graph G, let $\omega(G)$ be the *clique number* of G, that is, the size of a largest clique in G. For $1 \leqslant j \leqslant \omega(G)$, we use $N_j(G)$ to denote the number of copies of K_j in G. Theorem 1.2 can be rephrased as each graph contains a path of length at least $2N_2/N_1$. The main purpose of this note is to prove the following extension of Theorem 1.2 and present several applications of this result. We will prove Theorem 1.3 immediately after its statement as the proof is very short.

Theorem 1.3. *Let G be a graph. For each positive integer s with $1 \leqslant s \leqslant \omega(G)$, there is a path of length at least*

$$\frac{(s+1)N_{s+1}(G)}{N_s(G)} + s - 1 \text{ in } G.$$

Proof. We prove the theorem by induction on s. The case of $s=1$ is Theorem 1.2. Suppose it is true for $s = k-1$, where $s \leqslant \omega(G)-1$. For each vertex $x \in V(G)$, let G_x be the subgraph induced by $N_G(x)$, and ℓ_x be the length of a longest path in G_x. By induction hypothesis, for each vertex $x \in V(G)$ with $N_{k-1}(G_x) \neq 0$,

$$\ell_x \geqslant \frac{kN_k(G_x)}{N_{k-1}(G_x)} + k - 2.$$

Equivalently, $(\ell_x - k + 2)N_{k-1}(G_x) \geqslant kN_k(G_x)$. Let $\ell_{\max} = \max\{\ell_x : x \in V(G)\}$. Then

$$(\ell_{\max} - k + 2)N_{k-1}(G_x) \geqslant kN_k(G_x) \tag{1.1}$$

holds for each x. Summing inequality (1.1) over all $x \in V(G)$, we get

$$(\ell_{\max} - k + 2) \sum_{x\in V(G)} N_{k-1}(G_x) \geqslant k \sum_{x \in V(G)} N_k(G_x).$$

Note that $\sum_{x\in V(G)} N_{k-1}(G_x) = kN_k(G)$ and $\sum_{x\in V(G)} N_k(G_x) = (k+1)N_{k+1}(G)$. We get

$$kN_k(G)(\ell_{\max} - k + 2) \geqslant k \sum_{x\in V(G)} N_k(G_x) = k(k+1)N_{k+1}(G).$$

So

$$\ell_{\max} \geqslant \frac{(k+1)N_{k+1}(G)}{N_k(G)} + k - 2.$$

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This implies that there exists a vertex v such that G_v contains a path P_v of length at least

$$\frac{(k+1)N_{k+1}(G)}{N_k(G)}+k-2.$$

Therefore, there is a path of length at least

$$\frac{(k+1)N_{k+1}(G)}{N_k(G)}+k-1 \text{ in } G. \qquad \square$$

The following family of graphs shows our extension improving the estimate given by Theorem 1.2. Let G be an n-vertex graph which consists of a K_{n-2} and two pendant edges sharing an endpoint from the K_{n-2}. Theorem 1.2 implies that G contains a path of length at least

$$\frac{2N_2(G_1)}{N_1(G_1)}=n-5+\frac{10}{n},$$

while Theorem 1.3 tells us that G contains a path of length at least

$$\frac{(n-2)N_{n-2}(G)}{N_{n-3}(G)}+n-4=n-3,$$

where we choose $s=n-3$.

For two graphs G and H, we write $G\vee H$ for their *join* defined by $V(G\vee H)=V(G)\cup V(H)$ and $E(G\vee H)=E(G)\cup E(H)\cup\{xy: x\in V(G), y\in V(H)\}$. The proof of Theorem 1.3 gives the following result.

Theorem 1.4. *Let G be a graph and k be an integer. If $\omega(G)\geqslant k\geqslant 2$, then G contains a subgraph $P_\ell\vee K_1$, where*

$$\ell\geqslant\frac{(k+1)N_{k+1}(G)}{N_k(G)}+k-1.$$

In particular, G contains cycles of lengths from 3 to

$$\left\lceil\frac{(k+1)N_{k+1}(G)}{N_k(G)}\right\rceil+k.$$

2. Short proofs of two theorems of Luo

Before we present applications of Theorems 1.3 and 1.4 to the generalized Turán number, we recall a few definitions. Let T be a graph and $\mathcal{H}$ be a family of graphs. The *generalized Turán number* $\mathrm{ex}(n,T,\mathcal{H})$ is the maximum possible number of copies of T in an n-vertex graph which is H-free for each $H\in\mathcal{H}$. When $\mathcal{H}=\{H\}$, we write $\mathrm{ex}(n,T,H)$ instead of $\mathrm{ex}(n,T,\{H\})$. If $T=K_2$, then $\mathrm{ex}(n,K_2,H)=\mathrm{ex}(n,H)$ is the classical Turán number of H.

The generalized Turán number has received a lot of attention recently. There are several notable and nice papers concerning the generalized Turán number $\mathrm{ex}(n,T,H)$ (see [1, 3, 8, 10, 14, 15, 20]). Erdős [8] first determined $\mathrm{ex}(n,K_t,K_r)$ for all $t<r$. Bollobás and Győri [3] determined the order of magnitude of $\mathrm{ex}(n,C_3,C_5)$. Their estimate was improved by Alon and Shikhelman [1] and recently by Ergemlidze, Győri, Methuku and Salia [10]. Alon and Shikhelman obtained a number of results on $\mathrm{ex}(n,T,H)$ for different T and H and posed several open problems in [1].

Luo [20] recently proved upper bounds for $\mathrm{ex}(n,K_s,\mathcal{C}_{\geqslant\ell})$ and $\mathrm{ex}(n,K_s,P_\ell)$ which are generalizations of Theorems 1.1 and 1.2.

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Theorem 2.1 **(Luo [20]).**

$$\mathrm{ex}(n, K_s, \mathcal{C}_{\geqslant \ell}) \leqslant \frac{n-1}{\ell-2}\binom{\ell-1}{s},$$

where $\ell \geqslant 3$ *and* $s \geqslant 2$.

Theorem 2.2 **(Luo [20]).**

$$\mathrm{ex}(n, K_s, P_\ell) \leqslant \frac{n}{\ell-1}\binom{\ell-1}{s},$$

where $\ell \geqslant 2$ *and* $s \geqslant 2$.

Luo's result turned out to be useful for investigating Turán-type problems in hypergraphs. For example, Győri, Methuku, Salia, Tompkins and Vizer [16] applied Theorem 2.1 to study the maximum number of hyperedges in a connected r-uniform n-vertex hypergraph without a Berge path of length k.

We next give very short proofs of Theorems 2.1 and 2.2 by applying Theorems 1.4 and 1.3 respectively.

Short proof of Theorem 2.1. Let c be the length of a longest cycle in G. By Theorem 1.4 and the condition in Theorem 2.1, we have

$$\frac{kN_k(G)}{N_{k-1}(G)} + k - 1 \leqslant c \leqslant \ell - 1,$$

where $3 \leqslant k \leqslant s$. This implies

$$N_k(G) \leqslant \frac{\ell-k}{k} N_{k-1}(G)$$

holds for $3 \leqslant k \leqslant s$. We apply the inequality recursively and get

$$N_s(G) \leqslant \frac{(\ell-s)(\ell-s+1)\cdots(\ell-3)}{s(s-1)\cdots 3} N_2(G).$$

By Theorem 1.1, we have

$$N_2(G) \leqslant \frac{(n-1)(\ell-1)}{2},$$

and thus

$$N_s(G) \leqslant \frac{n-1}{\ell-2}\binom{\ell-1}{s}.$$

□

Short proof of Theorem 2.2. Since G is P_ℓ-free, the length of a longest path P in G is at most $\ell - 2$. By Theorem 1.3, we have

$$\ell - 2 \geqslant \frac{kN_k(G)}{N_{k-1}(G)} + k - 2$$

whenever $2 \leqslant k \leqslant s$. It follows that

$$N_k(G) \leqslant \frac{\ell-k}{k} N_{k-1}(G)$$

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for $2 \leqslant k \leqslant s$. Recursively applying this inequality, we get

$$N_s(G) \leqslant \frac{(\ell-s)(\ell-s+1)\cdots(\ell-2)}{s(s-1)\cdots 2} N_1(G) = \frac{n}{\ell-1}\binom{\ell-1}{s}. \qquad \square$$

3. Extension of Luo's theorem

In order to prove Theorems 2.1 and 2.2, Luo [20] extended some classical theorems due to Kopylov [18]. Let $H(n,k,c)$ be the graph obtained from K_{c-k} by connecting each vertex of a set of $n-(c-k)$ isolated vertices to the same k vertices chosen from K_{c-k}. Let $f_s(n,k,c)$ be the number of K_s in $H(n,k,c)$. Namely,

$$f_s(n,k,c) = \binom{c-k}{s} + \binom{k}{s-1}(n-(c-k)).$$

When $s=2$, it equals the number of edges in $H(n,k,c)$. The *circumference* of a graph G is the length of a longest cycle in G. Improving Theorem 1.1, Kopylov [18] proved the following.

Theorem 3.1 (Kopylov [18]). *Let $n \geqslant c \geqslant 5$ and G be a 2-connected graph on n vertices with circumference less than c. Then*

$$N_2(G) \leqslant \max\left\{f_2(n,2,c), f_2\left(n, \left\lfloor \frac{c-1}{2} \right\rfloor, c\right)\right\}.$$

Kopylov's theorem was re-proved by Fan, Lv and Wang in [12] who indeed proved a slightly stronger result with the aid of another result of Woodall [23]. In the same paper [23], Woodall posed a conjecture which is a generalization of a previous result on non-Hamiltonian graphs due to Erdős [7].

Conjecture 3.2 (Woodall [23][1]). Let $n \geqslant c \geqslant 5$. If G is a 2-connected graph on n vertices with circumference less than c and minimum degree $\delta(G) \geqslant k$, then

$$N_2(G) \leqslant \max\left\{f_2(n,k,c), f_2\left(n, \left\lfloor \frac{c-1}{2} \right\rfloor, c\right)\right\}.$$

One can easily find that Kopylov's theorem confirmed Woodall's conjecture for $k=2$. Generalizing Kopylov's result, Luo [20] proved the following theorem.

Theorem 3.3 (Luo [20]). *Let $n \geqslant c \geqslant 5$ and $s \geqslant 2$. If G is a 2-connected graph on n vertices with circumference less than c, then*

$$N_s(G) \leqslant \max\left\{f_s(n,2,c), f_s\left(n, \left\lfloor \frac{c-1}{2} \right\rfloor, c\right)\right\}.$$

We present an extension of Theorem 3.3, which is in the spirit of Kopylov's remark (see the footnote).

[1] It should be mentioned that, in the last part of his paper [18], Kopylov wrote: 'we remark that a proof of Woodall's conjecture can be obtained by a minor modification of the solution to Problem D.'

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Theorem 3.4. *Let $n \geqslant c \geqslant 5$ and $s \geqslant 2$. If G is a 2-connected graph on n vertices with circumference less than c and minimum degree $\delta(G) \geqslant k \geqslant 2$, then*

$$N_s(G) \leqslant \max\left\{f_s(n,k,c), f_s\left(n, \left\lfloor \frac{c-1}{2} \right\rfloor, c\right)\right\}.$$

To prove Theorem 3.4, we need the following lemma whose proof is omitted in [18]. We would like to mention that this generalizes Bondy's lemma on longest cycles whose proof is implicitly included in the proof of Lemma 1 in [4].

Lemma 3.5 (Kopylov [18]). *Let G be a 2-connected n-vertex graph containing a path P of m edges with endpoints x and y. For $v \in V(G)$, let $d_P(v) = |N(v) \cap V(P)|$. Then G contains a cycle of length at least* $\min\{m+1, d_P(x) + d_P(y)\}$.

We also need a definition from Kopylov [18].

Definition (α-disintegration of a graph, Kopylov [18]). Let G be a graph and α be a natural number. Delete all vertices of degree at most α from G; for the resulting graph G', we again delete all vertices of degree at most α from G'. We keep running this process until we finally get a graph, denoted by $H(G;\alpha)$, such that all vertices are of degree greater than α.

Note that $H(G;\alpha)$ is now commonly called the $(\alpha+1)$-core of G. Our proof is very similar to Kopylov's proof [18] of Theorem 3.1 and the proof of Theorem 3.3 in [20]. We give only a sketch and omit the details. We split the proof into five steps.

Sketch of the proof of Theorem 3.4. Let G be a counter-example such that G is edge maximal, that is, adding each non-edge creates a cycle of length at least c. Thus each pair of non-adjacent vertices is connected by a path of length at least $c-1$. Let $t = \lfloor (c-1)/2 \rfloor$ and $H = H(G;t)$.

Claim 1 ([20]). *H is not empty.*

Proof. Suppose not. For the first $n-t$ vertices in the process of getting $H(G;t)$, each of them has degree at most t and then it is contained in at most $\binom{t}{s-1}$ copies of K_s. The number of copies of K_s in the subgraph induced by the last t vertices is bounded from above by $\binom{t}{s}$. Thus we have the following upper bound on $N_s(G)$:

$$N_s(G) \leqslant (n-t)\binom{t}{s-1} + \binom{t}{s} \leqslant f_s(n,t,c),$$

which is a contradiction.

Claim 2 ([18]). *H is a clique.*

The main differences come from Claims 3 and 4, whose proofs need the minimum degree condition and a new function.

Claim 3. *Let $r = |V(H)|$. Then $k \leqslant c - r \leqslant t$.*

Proof. As $H = H(G;t)$ is a clique, $r \geqslant t+2$. We first claim $r \leqslant c-k$, where $\delta(G) \geqslant k$. Suppose $r \geqslant c-k+1$. If $x \in V(G) \setminus V(H)$, then x is not adjacent to at least one vertex in H. Otherwise, $x \in H$. We pick $x \in V(G) \setminus V(H)$ and $y \in V(H)$ satisfying the following two conditions: (a) x and

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y are not adjacent, and (b) a longest path in G from x to y contains the largest number of edges among such non-adjacent pairs. Let P be a longest path in G from x to y. Clearly, $|V(P)| \geqslant c$ as G is edge maximal. We next show $N_G(x) \subseteq V(P)$. Suppose not. Let $z \in N_G(x)$ and $z \notin V(P)$. If z and y are not adjacent, then there is a longer path from z to y, a contradiction to the selection of x and y. If z and y are adjacent, then there is a cycle of length at least $c+1$, a contradiction to the assumption of G. Similarly, we can show $N_H(y) \subseteq V(P)$. By Lemma 3.5, there is a cycle with length at least $\min\{c, d_P(x) + d_P(y)\} \geqslant \min\{c, k + c - k\} = c$, a contradiction. Thus $r \leqslant c - k$. Recall $t + 2 \leqslant r \leqslant c - k$. We get $k \leqslant c - r \leqslant c - t - 2 \leqslant t$. This proves Claim 3.

Claim 4. *Let $H' = H(G; c - r)$. Then $H \neq H'$.*

Proof. Suppose $H = H'$. We next show an upper bound on $N_s(G)$. Firstly, the number of K_s contained in $V(H') = V(H)$ is at most $\binom{r}{s}$. Secondly, each vertex from $V(G) \setminus V(H')$ is contained in at most $\binom{c-r}{s-1}$ cliques of size s as its degree is at most $c - r$. Therefore,

$$N_s(G) \leqslant (n-r)\binom{c-r}{s-1} + \binom{r}{s} = f_s(n, c-r, c) \leqslant \max\{f_s(n,k,c), f_s(n,t,c)\},$$

as the function $f_s(n, x, c)$ is convex for $x \in [k, t]$ and $k \leqslant c - r \leqslant t$. This is a contradiction and Claim 4 follows.

Claim 5. *G contains a cycle of length at least c.*

The proof of the claim above is the same as Kopylov's proof and we skip it. The proof of Theorem 3.4 is complete. □

Similar to Theorem 3.4, we have the following result and skip the details of the proof.

Theorem 3.6. *If G is an n-vertex connected graph containing no P_ℓ and having minimum degree $\delta(G) \geqslant k$, where $n \geqslant \ell \geqslant 4$, then*

$$N_s(G) \leqslant \max\left\{f_s(n, k, \ell-1), f_s\left(n, \left\lfloor \frac{\ell}{2} \right\rfloor - 1, \ell - 1\right)\right\}.$$

4. Consecutive lengths of cycles

For a graph G, let $\mu(G)$ be the *spectral radius* of G which is the largest eigenvalue of the adjacency matrix. Nikiforov [21] proved the following: If G is a graph of sufficiently large order n and the spectral radius $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$, then G contains a cycle of length t for every $t \leqslant n/320$. We slightly improve Nikiforov's result as follows.

Theorem 4.1. *Let G be a graph of sufficiently large order n with $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$. Then G contains a cycle of length t for every $t \leqslant n/160$.*

Notice that Theorem 1.4 implies the following fact.

Fact 4.2. *A graph G contains all cycles of length $t \in [3, \ell]$, where*

$$\ell = \left\lceil \frac{3N_3(G)}{N_2(G)} \right\rceil + 2.$$

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Sketch of the proof of Theorem 4.1. Compared with the original proof in [21], the improvement comes from the fact mentioned above. In [21], it is shown that for n sufficiently large, there exists an induced subgraph $H \subset G$ with $|H| > n/2$ satisfying one of the following conditions:

(i) $\mu(H) > (1/2 + 1/80)|H|$,
(ii) $\mu(H) > |H|/2$ and $\delta(H) > 2|H|/5$.

For case (i), it is shown in [21] that

$$N_3(H) \geqslant \frac{1}{960}|H|^3.$$

In this case, if $e(H) = N_2(H) > |H|^2/4$, then a theorem of Bollobás [2] implies there are cycles of lengths from 3 to $|H|/2$ in H. Thus there are cycles of length t for each $3 \leqslant t \leqslant n/4$. We assume $e(H) \leqslant |H|^2/4$. By Fact 4.2, H contains all cycles of length

$$\ell \in \left[3, \frac{3|H|^3/960}{|H|^2/4}\right].$$

Since

$$\frac{3|H|^3/960}{|H|^2/4} \geqslant \frac{1}{160}n,$$

we proved the result for the case (i). The proof for case (ii) follows from Nikiforov's original proof (see [21, page 1497]).

Acknowledgement

The authors are grateful to both referees for their valuable comments on an earlier version of this paper.

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Cite this article: Ning B and Peng X (2020). Extensions of the Erdős–Gallai theorem and Luo's theorem. *Combinatorics, Probability and Computing* **29**, 128–136. https://doi.org/10.1017/S0963548319000026