Orbital shadowing, internal chain transitivity and ω-limit sets

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Abstract. Let $f : X \to X$ be a continuous map on a compact metric space, let ω_f be the collection of ω-limit sets of f and let $ICT(f)$ be the collection of closed internally chain transitive subsets. Provided that f has shadowing, it is known that the closure of ω_f in the Hausdorff metric coincides with $ICT(f)$. In this paper, we prove that $\omega_f = ICT(f)$ if and only if f satisfies Pilyugin's notion of orbital limit shadowing. We also characterize those maps for which $\overline{\omega_f} = ICT(f)$ in terms of a variation of orbital shadowing.

1. Introduction

Let $f : X \to X$ be a continuous map on a compact metric space X. Each $x \in X$ has an associated ω-limit set $\omega(x)$ which is defined to be the set of limit points of the orbit of x. The set ω_f is the collection of all ω-limit sets of f, i.e. $\omega_f = \{A \subseteq X : \exists x \in X \text{ with } A = \omega(x)\}$. While it is relatively easy to compute the ω-limit set of a point, it is often quite difficult to determine whether a given subset A of X belongs to ω_f.

As such, finding an alternative characterization of ω-limit sets is desirable. And indeed, in many contexts, other characterizations exist. Of particular prominence is the notion of *internal chain transitivity*. Briefly, a closed set A is internally chain transitive provided that for all $\epsilon > 0$ and any pair $x, y \in A$ there exists a sequence $x = x_0, x_1, \ldots x_n = y$ in A satisfying $d(f(x_i), x_{i+1}) < \epsilon$. We denote the collection of internally chain transitive sets by $ICT(f)$. It has been shown [11] that every ω-limit set is internally chain transitive, and the converse has also been shown in a variety of contexts, including Axiom A diffeomorphisms [7], shifts of finite type [2], topologically hyperbolic maps [3], and in certain Julia sets [5, 4].

More recently, it has been demonstrated that for systems with the *shadowing property*, ω-limit sets are completely characterized by internal chain transitivity if and only if ω_f is closed with respect to the Hausdorff topology [15]. A map $f : X \to X$ has the *shadowing property* provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle$ (i.e. a sequence satisfying $d(f(x_i), x_{i+1}) < \delta$) there exists a point $z \in X$ which shadows it (i.e. $d(f^i(z), x_i) < \epsilon$).

Meddaugh and Raines [15] establish that, for maps f with shadowing, $\overline{\omega_f} = ICT(f)$. Since ω_f is known to be closed in a variety of maps (maps of the interval [6], circle [19], and other finite graphs [14]), in these systems, ω-limit sets are completely characterized by internal chain transitivity. However, it is made clear that there are maps for which shadowing does not hold, and yet the collections of ω-limit sets and internally chain transitive sets coincide in this manner.

The purpose of this paper is to find a topological characterization of those systems in which the coincidence of ω_f and $ICT(f)$ occurs. In particular, §3 explores examples of systems in which $\omega_f = ICT(f)$ occurs. These examples each exhibit a weaker type of shadowing property that is also sufficient for the equality of $\overline{\omega_f}$ and $ICT(f)$. However, it is also demonstrated that neither type is necessary for the equality to hold.

Section 4 proposes a novel type of shadowing, that of *eventual strong orbital shadowing*. It is then demonstrated that this type of shadowing is necessary and sufficient for $\overline{\omega_f}$ to be equal to $ICT(f)$. The final section considers *asymptotic* variants of these shadowing properties and provides an analogous characterization of maps for which $\omega_f = ICT(f)$.

2. *Preliminaries*

For the purposes of this paper, a *dynamical system* consists of a compact metric space X with metric d and a continuous map $f : X \to X$. For each $x \in X$, the ω-limit set of x is the set

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^i(x) : i \geq n\}},$$

i.e. the set of limit points of the sequence $\langle f^i(x) \rangle_{i \in \mathbb{N}}$. The properties of ω-limit sets are well-studied.

Of particular import is the well-known fact that for each $x \in X$, $\omega(x)$ is a compact subset of X and, as such, belongs to the hyperspace of compact subsets of X. This hyperspace is a metric space in its own right, using the *Hausdorff metric* induced by the metric d; given compact subsets A and B of X, the *Hausdorff distance* between A and B is given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

In this paper, we are primarily concerned with two specific subsets of this hyperspace. The first is *the ω-limit space of f*, denoted by ω_f, which is the collection of all ω-limit sets of points in X. The structure of this set is intrinsically related to the dynamics of the map f. Of particular interest is when this set is closed with respect to the Hausdorff metric.

In [6], Blokh *et al* demonstrated that for an interval map $f : I \to I$, the set ω_f is closed with respect to this metric. It has also been shown that dynamical systems on circles [19] and on graphs [14] have the property that ω_f is closed. It is not, however, the case that ω_f is always closed. Examples of systems for which ω_f is not closed include certain maps on dendrites [12] and the unit square [13].

The second subset of the hyperspace of compact subsets of X that we are interested in is the collection of internally chain transitive sets. A closed subset A of X is *internally chain transitive with respect to f*, or ICT, provided that, for all $\epsilon > 0$ and each pair $a, b \in A$,

there exists an *ε-chain from a to b in A*, i.e. a sequence $x_0 = a, x_1, \ldots, x_n = b$ in A satisfying $d(f(x_i), x_{i+1}) < \epsilon$ for each $i < n$. Note that it is an immediate consequence of the definition that compact, internally chain transitive sets are invariant, i.e. $f(A) = A$, and that the closure of an internally chain transitive set is internally chain transitive. We will denote the collection of (closed) internally chain transitive sets by $ICT(f)$.

Internally chain transitive sets have also been fairly well-studied. Hirsch [11] demonstrated that in any dynamical system, the ω-limit sets are internally chain transitive, i.e. $\omega_f \subseteq ICT(f)$. It is also the case that in specific dynamical systems, ω_f is equal to $ICT(f)$. In particular, this is true for shifts of finite type [2], Julia sets for certain quadratic maps [4, 5] and certain classes of interval maps [1]. Another class of maps for which this equivalence holds is that of Axiom A diffeomorphisms [7], although in that paper the term *abstract ω-limit set* is used rather than internal chain transitivity.

In many of the systems for which $ICT(f) = \omega_f$, it is observed that the system in question has the *shadowing property* (sometimes referred to as the *pseudo-orbit tracing property*). To define shadowing we first define the notion of a *pseudo-orbit*. For $\delta > 0$, a δ-pseudo-orbit is a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ for which $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{N}$. It is occasionally useful to talk about the ω-limit set of a pseudo-orbit, given by

$$\omega(\langle x_i \rangle_{i \in \mathbb{N}}) = \bigcap_{n \in \mathbb{N}} \overline{\{x_i : i > n\}}.$$

We say that a map f has the *shadowing property* provided that for all $\epsilon > 0$ there exists $\delta > 0$ such that for each δ-pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ for which $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{N}$. That is, for every δ-pseudo-orbit, there is a point whose orbit shadows it.

Maps with shadowing are also well-studied in a variety of contexts, including in the context of Axiom A diffeomorphisms [7], shifts of finite type [2], and in interval maps [9]. There have also been many recent results concerning variations of the shadowing property, including such notions as ergodic shadowing [10], limit shadowing [3, 17], orbital shadowing [18, 17], and various others [8, 16].

Of particular relevance is the fact that in many systems with shadowing, ω-limit sets and ICT sets coincide [1, 2, 4, 5]. It has recently been demonstrated, however, that this coincidence of sets is not a general phenomenon [20].

However, in [15], Meddaugh and Raines prove that, under the assumption that f has the shadowing property, ω_f is closed precisely when it is equal to the set of internally chain transitive sets of f. The paper demonstrates this by effectively proving the following two results (the second of which is only implicitly proven in the original paper).

LEMMA 1. (Meddaugh and Raines [15]) *Let* $f : X \to X$ *be a dynamical system. Then* $ICT(f)$ *is closed with respect to the Hausdorff metric.*

THEOREM 2. (Meddaugh and Raines [15]) *Let* $f : X \to X$ *be a dynamical system with the shadowing property. Then the closure of* ω_f *is equal to* $ICT(f)$.

While the first result is perfectly general, the second requires that the system exhibits the shadowing property. In addition, Meddaugh and Raines [15] note that there are systems for which $\omega_f = ICT(f)$ but which do not exhibit the shadowing property.

In this paper, we will develop an appropriate notion of shadowing which will characterize those systems for which $\overline{\omega_f} = ICT(f)$.

3. *Variations of shadowing*

The concept of shadowing is well-studied and, as mentioned in the previous section, there are a number of variations of shadowing that are interesting in their own right. The first variation of shadowing we will consider is the notion of *eventual shadowing*.

Definition 3. A system $f : X \to X$ has the *eventual shadowing property* provided that for all $\epsilon > 0$ there exists $\delta > 0$ such that for each δ-pseudo-orbit $\langle x_i \rangle$, there exists $z \in X$ and $N \in \mathbb{N}$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \geq N$.

This property is equivalent to the $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing property of Oprocha [16]. In [16], it is demonstrated that shadowing implies eventual shadowing. Despite the fact that eventual shadowing is weaker than shadowing, it is still sufficient to prove the following result.

THEOREM 4. *Let $f : X \to X$ have the eventual shadowing property. Then $\overline{\omega_f} = ICT(f)$.*

Proof. Let $A \in ICT(f)$. Since A is internally chain transitive, as in [15], define a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ of points in A such that for all $N \in \mathbb{N}$, $\langle x_i \rangle_{i \geq N}$ is dense in A and $d(f(x_i), x_{i+1})$ converges to zero.

Let $\epsilon > 0$ and choose $\delta > 0$ to witness the eventual shadowing property for $\epsilon/2$. Since $d(f(x_i), x_{i+1})$ converges to zero, there exists $M \in \mathbb{N}$ with $\langle x_{M+i} \rangle_{i \in \mathbb{N}}$ a δ-pseudo-orbit. Let $z \in X$ and $N \in \mathbb{N}$ as given by eventual shadowing so that $d(f^i(z), x_{M+i}) < \epsilon/2$ for all $i \geq N$. It then follows that

$$d_H(\omega(z), \omega(\langle x_{M+i} \rangle_{i \in \mathbb{N}})) < \epsilon,$$

and since $\omega(\langle x_{M+i} \rangle_{i \in \mathbb{N}}) = A$, we have

$$d_H(\omega(z), A) < \epsilon.$$

Thus, $A \in \overline{\omega_f}$.

Since $\omega_f \subseteq ICT(f)$ by [11], $\overline{\omega_f} = ICT(f)$.   □

Note that the example of a map without shadowing but with $\overline{\omega_f} = ICT(f)$ described in [15] does indeed exhibit the eventual shadowing property.

Example 5. The function $f : [-1, 1] \to [-1, 1]$ be given as follows:

$$f(x) = \begin{cases} x^3, & -1 \leq x \leq 0, \\ 2x, & 0 \leq x \leq 1/2, \\ 2(1-x), & 1/2 \leq x \leq 1, \end{cases}$$

with graph shown in Figure 4 has the eventual shadowing property.

This can be observed by noting that for any sufficiently small δ, any δ-pseudo-orbit eventually lies in a δ neighborhood of the fixed point -1 or a δ neighborhood of the interval $[0, 1]$. Since f restricted to either $[-1, 0]$ or $[0, 1]$ has shadowing, the property follows.

However, it is not the case that eventual shadowing characterizes those maps with $\overline{\omega_f} = ICT(f)$. To see this, consider the following example.
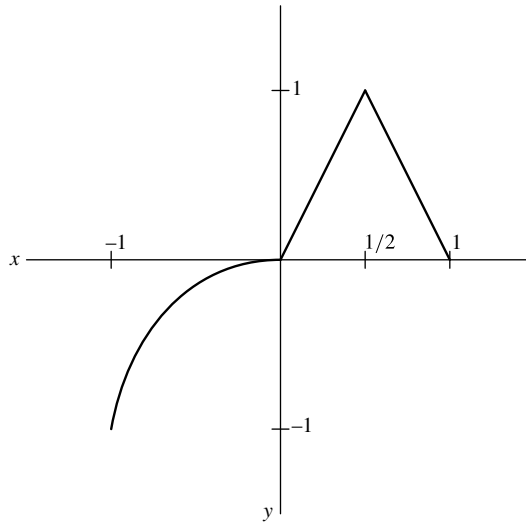
FIGURE 1. The graph of a function $f : [-1, 1] \to [-1, 1]$ which satisfies $ICT(f) = \omega_f$ but does not exhibit shadowing.

Example 6. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an irrational rotation of the circle. Then f satisfies $\overline{\omega_f} = ICT(f)$ but does not have the eventual shadowing property.

To see this, first, observe that the only ω-limit set for such a map is the entire space X. Furthermore, observe that, as internally chain transitive sets are invariant, if A is internally chain transitive, and $z \in A$, then $\omega(z) \subseteq A$, and hence $A = X$. Thus, $\omega_f = ICT(f) = \{X\}$. Note that this same argument demonstrates this equality for any minimal system.

However, the system does not have the eventual shadowing property. To see this, consider $\mathbb{S}^1$ as $X = \mathbb{R}/\mathbb{Z}$ and let $r \in (0, 1)$ be the rotation constant. Consider $\epsilon < 1/8$ and let $\delta > 0$. Find a number r' such that $r > r' > r - \delta$ and so that there exists $N \in \mathbb{N}$ with $3/4 > N(r - r') > 1/4$.

Now, consider the δ-pseudo-orbit $\langle ir' \rangle$. Suppose that $z \in \mathbb{S}^1$ satisfies $d(f^i(z), ir')$ $< 1/8$. Then $|ir + z - ir'| < 1/8$. But then $d(f^{i+N}(z), (i + N)r') = |(i + N)r + z - (i + N)r'| = |ir + z - ir' + N(r - r')| \geq |N(r - r')| - |ir + z - ir'| \geq 1/8$.

Example 6, however, does have another form of shadowing, specifically, it has the *orbital shadowing property* as discussed in [**18**, **17**]. The orbital shadowing property is concerned with the structure of pseudo-orbits and orbits as *sets* rather than as *sequences*.

Definition 7. A system $f : X \to X$ has the *orbital shadowing property* provided for all $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ with

$$d_H(\overline{\{x_i\}_{i \in \mathbb{N}}}, \overline{\{f^i(z)\}_{i \in \mathbb{N}}}) < \epsilon.$$

While the irrational rotation of the circle as in Example 6 does exhibit orbital shadowing, there are systems with orbital shadowing that do not have the property that $\overline{\omega_f} = ICT(f)$.

Example 8. Consider the two-sided shift space $\Sigma = \{0, 1\}^{\mathbb{Z}}$ with the usual metric and shift map f. Let X be the subshift of Σ with language consisting of those words in which the symbol '1' appears no more than once. Then $f|_X$ has orbital shadowing but does not have $\overline{\omega_f} = ICT(f)$.

For this map, it is easy to verify that $ICT(f) = \{X, \{\overline{0}\}\}$ whereas $\omega_f = \{\{\overline{0}\}\}$ where $\overline{0}$ denotes the word consisting of only zeros. Thus, $\overline{\omega_f} \neq ICT(f)$ for this system. That f has the orbital shadowing property is observed by noting that for any $\delta > 0$, a δ-pseudo-orbit $\langle x_i \rangle$ is either eventually contained in the δ ball around $\overline{0}$ or is δ-dense in X (i.e. every δ ball contains a point of the pseudo-orbit). In the former case, there is a point z in X which shadows the pseudo-orbit in the traditional sense, and in the latter case, any point $z \in X$ whose orbit is δ dense in X orbitally shadows $\langle x_i \rangle$.

However, a stronger form of orbital shadowing is sufficient for $\overline{\omega_f} = ICT(f)$.

Definition 9. A system $f : X \to X$ has the *strong orbital shadowing property* provided for all $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ with

$$d_H(\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon$$

for all $N \in \mathbb{N}$.

THEOREM 10. *Let $f : X \to X$ have the strong orbital shadowing property. Then $\overline{\omega_f} = ICT(f)$.*

Proof. Let $A \in ICT(f)$. Since A is internally chain transitive, as in [**15**], define a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ of points in A such that for all $N \in \mathbb{N}$, $\langle x_i \rangle_{i \geq N}$ is dense in A and $d(f(x_i), x_{i+1})$ converges to zero.

Let $\epsilon > 0$ and choose $\delta > 0$ to witness the strong orbital shadowing property for $\epsilon/2$. Since $d(f(x_i), x_{i+1})$ converges to zero, there exists $M \in \mathbb{N}$ with $\langle x_{M+i} \rangle_{i \in \mathbb{N}}$ a δ-pseudo-orbit. Let $z \in X$ as given by strong orbital shadowing so that

$$d_H(\overline{\{x_{M+N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon/2$$

for all $N \in \mathbb{N}$. It then follows that

$$d_H(\omega(\langle x_{M+i} \rangle_{i \in \mathbb{N}}), \omega(z)) < \epsilon,$$

and since $\omega(\langle x_{M+i} \rangle_{i \in \mathbb{N}}) = A$, we have

$$d_H(A, \omega(z)) < \epsilon.$$

Thus, $A \in \overline{\omega_f}$.

Since $\omega_f \subseteq ICT(f)$ by [**11**], $\overline{\omega_f} = ICT(f)$.   □

However, as with eventual shadowing, there are systems with $\overline{\omega_f} = ICT(f)$ but without strong orbital shadowing. In particular, the system in Example 5 is such a system. Thus, neither eventual shadowing nor strong orbital shadowing is necessary for a system to have $\overline{\omega_f} = ICT(f)$.

4. Characterizing $\overline{\omega_f} = ICT(f)$

While each of eventual shadowing and strong orbital shadowing imply that $\overline{\omega_f} = ICT(f)$, neither is necessary for this property to appear. Thus, if we are to find a shadowing property which is necessary and sufficient for $\overline{\omega_f} = ICT(f)$, we must look for a weaker notion of shadowing. Strong orbital shadowing requires that z can be chosen so that the sets $\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}$ and $\overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}$ be close for all $N \in \mathbb{N}$, whereas eventual shadowing only requires that z can be chosen so that $f^i(z)$ and x_i are close for all i larger than some K. By combining these, we have the following notion of shadowing.

Definition 11. A system $f : X \to X$ has the *eventual strong orbital shadowing property* provided that for all $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ and $K \in \mathbb{N}$ with

$$d_H(\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon$$

for all $N \geq K$.

In other words, f has the eventual strong orbital shadowing property if z can be chosen so that the sets $\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}$ and $\overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}$ are close to for all but finitely many N. Another form of shadowing can be defined by requiring only that z can be chosen so that the sets are close for infinitely many $N \in \mathbb{N}$.

Definition 12. A system $f : X \to X$ has the *cofinal orbital shadowing property* provided that for all $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ such that for all $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}, \overline{\{x_{N+i}\}_{i \in \mathbb{N}}}) < \epsilon.$$

While it is immediate that the eventual strong orbital shadowing property implies the cofinal orbital shadowing property, the converse is actually true as well. In fact, these types of shadowing are precisely the ones which characterize the property of $\overline{\omega_f} = ICT(f)$.

THEOREM 13. *Let $f : X \to X$ be a dynamical system. Then the following are equivalent:*
(1) *f has the eventual strong orbital shadowing property;*
(2) *f has the cofinal orbital shadowing property;*
(3) *$\overline{\omega_f} = ICT(f)$.*

Proof. Since eventual strong orbital shadowing implies cofinal orbital shadowing, we first establish that a system with the cofinal orbital shadowing property exhibits $\overline{\omega_f} = ICT(f)$.

Indeed, let $A \in ICT(f)$. As before, since A is internally chain transitive, as in [15], define a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ of points in A such that for all $N \in \mathbb{N}$, $\langle x_i \rangle_{i \geq N}$ is dense in A and $d(f(x_i), x_{i+1})$ converges to zero.

Let $\epsilon > 0$ and choose $\delta > 0$ to witness cofinal orbital shadowing for $\epsilon/2$. Since $d(f(x_i), x_{i+1})$ converges to zero, there exists $M \in \mathbb{N}$ with $\langle x_{M+i} \rangle_{i \in \mathbb{N}}$ a δ-pseudo-orbit. Let $z \in X$ be given by cofinal orbital shadowing so that

$$d_H(\overline{\{x_{M+N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon/2$$

for infinitely many $N \geq K$. It then follows that

$$d_H(\omega(\langle x_{M+i} \rangle_{i \in \mathbb{N}}), \omega(z)) < \epsilon,$$

and since $\omega(\langle x_{M+i}\rangle_{i\in\mathbb{N}}) = A$, we have

$$d_H(A, \omega(z)) < \epsilon.$$

Thus, $A \in \overline{\omega_f}$. Since $\omega_f \subseteq ICT(f)$ by [**11**], $\overline{\omega_f} = ICT(f)$.

Now, let us establish that a system with $\overline{\omega_f} = ICT(f)$ must have the eventual strong orbital shadowing property. Suppose to the contrary that $f : X \to X$ does not exhibit the eventual strong orbital shadowing property, and let $\epsilon > 0$ witness this. Then, for each $n \in \mathbb{N}$ there exists a $1/2^n$-pseudo-orbit $\langle x_i^n \rangle$ such that for all $z \in X$ and all $K \in \mathbb{N}$ there exists $N \geq K$ with $d_H(\{f^{N+i}(z)\}_{i\in\mathbb{N}}, \overline{\{x_{N+i}^n\}_{i\in\mathbb{N}}}) \geq \epsilon$.

For each $n \in \mathbb{N}$, let $W_n = \omega(\langle x_i^n \rangle)$. Without loss, we may assume that the sequence $\langle W_n \rangle_{n\in\mathbb{N}}$ is convergent, and let $W = \lim W_n$. As W is a limit of compact sets, it is itself compact. We will show that W is ICT but is not in $\overline{\omega_f}$.

To see that $W \in ICT(f)$, let $a, b \in W$ and let $\xi > 0$. By uniform continuity, choose $\eta > 0$ such that $d(p, q) < \eta$ implies that $d(f(p), f(q)) < \xi/2$. Without loss, take $\eta < \xi/2$.

Choose N sufficiently large so that $1/2^N < \eta/3$ and $d_H(W_N, W) < \eta/3$. Also, choose a $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{K+i}^N\}_{i\in\mathbb{N}}}, W_N) < \eta/3,$$

and so we have

$$d_H(\overline{\{x_{K+i}^N\}_{i\in\mathbb{N}}}, W) < 2\eta/3.$$

Now, choose $j \in \mathbb{N}$ so that $d(x_{K+j}^N, a) < 2\eta/3$ and choose $k > j$ with $d(x_{K+j}^N, b) < 2\eta/3$. Now, let $z_0 = a$, $z_{k-j} = b$ and for each $i < k - j$, choose $z_i \in B_{2\eta/3}(x_{K+j+i}^N) \cap W$.

Then for all $i < k - j$

$$
\begin{aligned}
d(f(z_i), z_{i+1}) &\leq d(f(z_i), f(x_{K+j+i}^N)) + d(f(x_{K+j+i}^N), x_{K+j+i+1}^N) + d(x_{K+j+i+1}^N, z_{i+1}) \\
&< \xi/2 + \eta/3 + 2\eta/3 \\
&< \xi.
\end{aligned}
$$

Thus, for all $a, b \in W$ and all $\xi > 0$, there is a ξ-chain in W from a to b, and since W is compact, $W \in ICT(f)$.

To see that $W \notin \overline{\omega_f}$, suppose to the contrary, i.e. that $W \in \overline{\omega_f}$. Then we can find $z \in X$ with $d_H(\omega(z), W) < \epsilon/4$. We can also choose an $N \in \mathbb{N}$ such that $d_H(W_N, W) < \epsilon/4$. Finally, we can choose a $K \in \mathbb{N}$ so that both $d_H(\overline{\{f^{k+i}(z)\}_{i\in\mathbb{N}}}, \omega(z)) < \epsilon/4$ and $d_H(\{x_{k+i}^N\}_{i\in\mathbb{N}}, W_N) < \epsilon/4$ for all $k \geq K$. Then, for all $k \geq K$,

$$
\begin{aligned}
d_H(\overline{\{f^{k+i}(z)\}_{i\in\mathbb{N}}}, \overline{\{x_{k+i}^N\}_{i\in\mathbb{N}}}) &\leq d_H(\overline{\{f^{k+i}(z)\}_{i\in\mathbb{N}}}, \omega(z)) + d_H(\omega(z), W) \\
&\quad + d_H(W_N, W) + d_H(\overline{\{x_{k+i}^N\}_{i\in\mathbb{N}}}, W_N) \\
&< \epsilon
\end{aligned}
$$

which contradicts our choice of $\langle x_i^N \rangle$.

Thus, $W \in ICT(f) \setminus \overline{\omega_f}$.   □

It should be noted that in light of the above, we have the following relations among the various types of shadowing.

Remark 14. Let $f : X \rightarrow X$ be a continuous map. Then the following implications hold.
(1) If f has shadowing, then f has strong orbital shadowing.
(2) If f has strong orbital shadowing, then f has orbital shadowing.
(3) If f has shadowing, then f has eventual strong orbital shadowing (cofinal orbital shadowing).
(4) If f has strong orbital shadowing, then f has eventual strong orbital shadowing (cofinal orbital shadowing).
(5) If f has eventual shadowing, then f has eventual strong orbital shadowing (cofinal orbital shadowing).

However, as witnessed by Examples 5, 6, and 8 above, the converse of each of the above is false. In addition, there are maps with eventual strong orbital shadowing which have none of the other types of shadowing. One such map can be realized by taking the system which is the disjoint union of the systems from Examples 5 and 6.

5. *Asymptotic shadowing types and characterizations of $\omega_f = ICT(f)$*
In [15], the authors set out to provide conditions under which $\omega_f = ICT(f)$. As demonstrated in that paper, under the assumption of shadowing, this occurs if and only if ω_f is closed in the Hausdorff metric. In light of the results of the previous section, we have the following corollary to Theorem 13.

COROLLARY 15. *Let $f : X \rightarrow X$ be a dynamical system in which ω_f is closed. Then $\omega_f = ICT(f)$ if and only if f has the eventual strong orbital shadowing property.*

It has been demonstrated that for maps of the finite graphs [14] and, in particular, the interval [6], the collection of ω-limit sets is closed and, thus, in these contexts, the eventual strong orbital shadowing property completely characterizes the property of $\omega_f = ICT(f)$.

COROLLARY 16. *Let $f : X \rightarrow X$ be a continuous map on a finite graph. Then $\omega_f = ICT(f)$ if and only if f has the eventual strong orbital shadowing property.*

However, there are many more spaces in which ω_f is not necessarily closed. In particular, there are dendrite maps [12], maps of the square [13], and many others.

In these spaces a more careful characterization is required. In [3], Barwell *et al* explore the property of asymptotic shadowing, or limit shadowing as it is sometimes called. A sequence $\langle x_i \rangle$ in X is an *asymptotic pseudo-orbit for f* provided that $\lim d(f(x_i), x_{i+1}) = 0$.

Definition 17. A system $f : X \rightarrow X$ has *asymptotic shadowing (limit shadowing)* provided that for each asymptotic pseudo-orbit $\langle x_i \rangle$, there exists a point $z \in X$ with

$$\lim d(f^i(z), x_i) = 0.$$

THEOREM 18. (Barwell *et al* [3]) *Let $f : X \rightarrow X$ be a dynamical system with asymptotic shadowing. Then $\omega_f = ICT(f)$.*

However, as with the results of [**15**], asymptotic shadowing is sufficient for $\omega_f = ICT(f)$, but it is not necessary. The irrational rotation of the circle as in Example 6 has $\omega_f = ICT(f)$ but fails to have asymptotic shadowing [**17**].

In light of these observations, it seems sensible that an asymptotic version of the eventual strong orbital shadowing property might characterize the property of $\omega_f = ICT(f)$. Indeed, we can develop asymptotic versions of each shadowing property we have already discussed. We might begin with an asymptotic version of eventual shadowing, but it is immediately clear that such a shadowing property is equivalent to the usual asymptotic shadowing property. As for the orbital shadowing properties, we offer the following two definitions.

Definition 19. A system $f : X \to X$ has the *asymptotic orbital shadowing property* provided that for every asymptotic pseudo-orbit $\langle x_i \rangle$ there exists a point $z \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon.$$

Definition 20. A system $f : X \to X$ has the *asymptotic strong orbital shadowing property* provided that for every asymptotic pseudo-orbit $\langle x_i \rangle$ there exists a point $z \in X$ such that for all $\epsilon > 0$ there exists a $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon$$

for all $N \geq K$.

Note that, while we could similarly define notions of asymptotic eventual strong orbital shadowing or asymptotic cofinal orbital shadowing, they would be essentially immediately equivalent to the asymptotic strong orbital shadowing property.

It is also worth noting that there is a shadowing property known as the *orbital limit shadowing property*, which has been studied by Pilyugin and others [**17**].

Definition 21. A system $f : X \to X$ has the orbital limit shadowing property provided that for every asymptotic pseudo-orbit $\langle x_i \rangle$ there exists a point z such that $\omega(z) = \omega(\langle x_i \rangle)$.

It turns out that all of these properties are equivalent, and indeed also characterize the property of $\omega_f = ICT(f)$.

THEOREM 22. *Let $f : X \to X$ be a dynamical system. Then the following are equivalent:*
(1)   *f has the asymptotic orbital shadowing property;*
(2)   *f has the asymptotic strong orbital shadowing property;*
(3)   *f has the orbital limit shadowing property; and*
(4)   *$\omega_f = ICT(f)$.*

Proof. That (2) implies (1) is immediate.

Now, assume that (1) holds, and let $A \in ICT(f)$. Since A is internally chain transitive, let $\langle x_i \rangle$ be an asymptotic pseudo-orbit contained in, and dense in A. By (1), there exists a point z such that for all $\epsilon > 0$, there exists N with

$$d_H(\overline{\{x_{N+i}\}_{i \in \mathbb{N}}}, \overline{\{f^{N+i}(z)\}_{i \in \mathbb{N}}}) < \epsilon.$$

But $\overline{\{x_{N+i}\}_{i\in\mathbb{N}}} = A$, and so

$$d_H(A, \overline{\{f^{N+i}(z)\}_{i\in\mathbb{N}}}) < \epsilon.$$

Now, for each $\epsilon > 0$ there is such an $N \in \mathbb{N}$ so there are two possibilities. Either there is an increasing sequence N_j such that

$$\lim_{j\to\infty} d_H(A, \overline{\{f^{N_j+i}(z)\}_{i\in\mathbb{N}}}) = 0$$

in which case $\omega(z) = A$, or there exists some $K \in \mathbb{N}$ for which

$$d_H(A, \overline{\{f^{K+i}(z)\}_{i\in\mathbb{N}}}) < \epsilon$$

for all $\epsilon > 0$. But in this latter case, we would have

$$d_H(A, \overline{\{f^{K+i}(z)\}_{i\in\mathbb{N}}}) = 0,$$

i.e. $A = \overline{\{f^{K+i}(z)\}_{i\in\mathbb{N}}}$, and since A is invariant under f, it follows that

$$A = f(A) = f(\overline{\{f^{K+i}(z)\}_{i\in\mathbb{N}}}) = \overline{\{f^{K+1+i}(z)\}_{i\in\mathbb{N}}}$$

and we would have $A = \omega(z)$.

In either case, $A \in \omega_f$, and so $ICT(f) \subseteq \omega_f$. By [11], the opposite inclusion holds, and we have property (4).

Now, assume property (4) holds, and let $\langle x_i \rangle$ be an asymptotic pseudo-orbit. Then $\omega(\langle x_i \rangle) \in ICT(f)$, and hence $\omega(\langle x_i \rangle) \in \omega_f$. Thus there exists $z \in X$ with $\omega(z) = \omega(\langle x_i \rangle)$. So, (4) implies (3).

Finally, we need to show (3) implies (2). So, let $\langle x_i \rangle$ be an asymptotic pseudo-orbit. By (3) there is some $z \in X$ with $\omega(z) = \omega(\langle x_i \rangle)$. In particular,

$$\lim_{N\to\infty} d_H(\overline{\{f^{N+i}(z)\}_{i\in\mathbb{N}}}, \overline{\{x_{N+i}\}_{i\in\mathbb{N}}}) = d_H(\omega(z), \omega(\langle x_i \rangle)) = 0.$$

Thus, for all $\epsilon > 0$, there exists K such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i\in\mathbb{N}}}, \overline{\{x_{N+i}\}_{i\in\mathbb{N}}}) < \epsilon$$

for all $N \geq K$. This establishes (2). □

Remark 23. In the event that ω_f is closed, the following are equivalent:
(1) orbital limit shadowing;
(2) eventual strong orbital shadowing (cofinal orbital shadowing);
(3) asymptotic (strong) orbital shadowing.

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