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FP-INJECTIVE DIMENSIONS AND GORENSTEIN HOMOLOGY

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Abstract Let R be a left coherent ring. It is proven that if an R-module M has a finite FP-injective dimension, then the Gorenstein projective (resp. Gorenstein flat) dimension and the projective (resp. flat) dimension coincide. Also, we obtain that the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ forms a projective cotorsion pair under some mild conditions.

Keywords: FP-injective modules; Gorenstein projective modules; Gorenstein flat modules

2020 *Mathematics subject classification:* 18G25; 18G20

Introduction

Inspired by the work of Auslander and Bridger [1], Enochs, Jenda, and Torrecillas introduced the concept of Gorenstein projective, Gorenstein injective and Gorenstein flat modules [9, 12] for any ring, and then established Gorenstein homological algebra. Such a relative homological algebra has been developed rapidly during the past several years and become a rich theory. In this paper, we will consider the following two topics:

(1) The triviality of Gorenstein projective and Gorenstein flat dimensions.

(2) The complete hereditary cotorsion pair induced by the class of Gorenstein projective modules.

For the first topic, it is well known that the Gorenstein projective dimension $\mathrm{Gpd}_R(-)$ (resp. Gorenstein flat dimension $\mathrm{Gfd}_R(-)$) is a refinement of the usual projective dimension $\mathrm{pd}_R(-)$ (resp. flat dimension $\mathrm{fd}_R(-)$), that is, for any R-module M there is an inequality $\mathrm{Gpd}_R M \leq \mathrm{pd}_R M$ (resp. $\mathrm{Gfd}_R M \leq \mathrm{fd}_R M$), and if $\mathrm{pd}_R M$ (resp. $\mathrm{fd}_R M$) is finite, then there is an equality $\mathrm{Gpd}_R M = \mathrm{pd}_R M$ (resp. $\mathrm{Gfd}_R M = \mathrm{fd}_R M$). On the other hand, the equalities are closely related with injective dimensions. Holm proved in [18, Theorem 2.2] that the equality $\mathrm{Gpd}_R M = \mathrm{pd}_R M$ holds when M has a finite injective dimension;

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Holm showed in [18, Theorem 2.6] that if R is a left and right coherent ring with finite RightFPD(R), then the equality $\mathrm{Gfd}_R M = \mathrm{fd}_R M$ holds when M has a finite injective dimension. Here RightFPD(R) denotes the right finitistic projective dimension of R and is defined as the supremum of the projective dimensions of all right R-modules with finite projective dimension.

The first motivation of the paper is to link the above equalities with FP-injective dimensions. The main results, being shown in § 2, are as follows:

Theorem A. *Assume that R is a left coherent ring. If $\text{FP-id}_R M < \infty$, then there is an equality $\mathrm{Gpd}_R M = \mathrm{pd}_R M$.*

Theorem B. *Assume that R is a left coherent ring. If $\text{FP-id}_R M < \infty$, then there is an equality $\mathrm{Gfd}_R M = \mathrm{fd}_R M$.*

Let us turn our attention to the second topic. A classical result over Gorenstein rings tells us that, over such a ring R, the following claims hold, where $\mathcal{GP}$ (resp. $\mathcal{GI}$ and $\mathcal{GF}$) stands for the class of all Gorenstein projective (resp. Gorenstein injective and Gorenstein flat) left R-modules:

(2.1) The pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ forms a complete hereditary cotorsion pair.

(2.2) The pair $(\mathcal{GF}, \mathcal{GF}^{\perp})$ forms a complete hereditary cotorsion pair.

(2.3) The pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ forms a complete hereditary cotorsion pair.

In order that Gorenstein homological algebra should work, many authors in [5, 13, 15, 21, 23, 25, 27, 32] seek more general rings such that (2.1)–(2.3) hold. It is worth mentioning that, by virtue of the work of Šaroch and Št'ovíček [27], the claims (2.1) and (2.2) hold for any ring. However, the claim (2.3) for an arbitrary ring remains unknown.

Let R be a ring such that any level left R-module has a finite projective dimension. The class of such rings includes strictly the one of right coherent rings R such that any flat left R-module has a finite projective dimension (see Remark 3.13). By establishing a link between Gorenstein AC-projective and Gorenstein projective modules, Bravo, Gillespie and Hovey [5, 15] proved that the claim (2.3) holds true. To compare these facts, the second motivation of the paper is to obtain a condition of left coherent rings such that the claim (2.3) holds.

Let R be a left coherent ring and n be a non-negative integer. Note from Remark 3.1 that $\mathrm{Ext}^1_R(G, M) = 0$ for all $G \in \mathcal{GP}$ and all $M \in \mathcal{FI}_n$, where $\mathcal{FI}_n$ denotes the class of all left R-modules M with $\text{FP-id}_R(M) \leq n$. Our main results in § 3 is as follows, which checks when such two classes form a complete hereditary cotorsion pair, where FFPID(R) denotes the *left fintistic FP-injective dimension* of R, it is defined as the supremum of the FP-injective dimensions of all left R-modules with finite FP-injective dimension.

Theorem C. *Let R be a left coherent ring and n a non-negative integer. Then the following conditions are equivalent.*

(1) *The pair $(\mathcal{GP}, \mathcal{FI}_n)$ forms a projective cotorsion pair.*

(2) $\max\{\text{FP-id}_R(R), \mathrm{FFPID}(R)\} \leq n$.

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As an immediate consequence of Theorem C, we have the claim (2.3) hold whenever R is a left coherent ring with both $\text{FP-id}_R(R) < \infty$ and $\text{FFPID}(R) < \infty$. Note that there exists a ring R with both $\text{FP-id}_R(R) < \infty$ and $\text{FFPID}(R) < \infty$ which neither satisfies that level left R-module has a finite projective dimension nor is right coherent (See Remarks 3.10 and 3.11).

As we all know, the famous “Auslander's last theorem” announced that over an arbitrary Gorenstein ring any finitely generated module has a Gorenstein projective cover. We give a sufficient and necessary condition that all modules have a Gorenstein projective cover over a left coherent ring R with both $\text{FP-id}_R(R) < \infty$ and $\text{FFPID}(R) < \infty$. Thus, Gorenstein rings can not guarantee that over them all modules have a Gorenstein projective cover (see Remark 3.6). Furthermore, we obtain that, over such a ring, Gorenstein projective and Ding projective modules coincide with each other, which are exactly modules as some kernel of an exact complex of projective modules (see Proposition 3.9). This improves [16, Theorem 1.1(1)]. We then compare the two conditions “R is a left coherent ring with both $\text{FP-id}_R(R) < \infty$ and $\text{FFPID}(R) < \infty''$ and $''R$ is a ring such that any level left R-module has finite projective dimension” as well as other well-known conditions such that the claim (2.3) holds (see Remarks 3.12–3.14).

As another application of Theorem C, in § 4 we characterize Gorenstein global dimension of a left coherent ring.

1. Notation and terminology

Throughout the paper, R denotes an associative ring with identity. An R-module will mean a left R-module, unless stated otherwise. We also refer to right R-modules as modules over the opposite ring R^o. The category of R-modules will be denoted by R-Mod.

Recall from [11] that an R-module M is called *Gorenstein projective* if there exists an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

of projective R-modules with $M = \text{Ker}(P_0 \to P_{-1})$, such that it remains exact after applying $\text{Hom}(-, Q)$ for any projective R-module Q. An R-module N is *Gorenstein flat* if there exists an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

of flat R-modules with $N = \text{Ker}(F_0 \to F_{-1})$ such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective R^o-module. As usual, we use the symbol $\text{pd}_R(-)$, $\text{fd}_R(-)$, $\text{Gpd}_R(-)$, and $\text{Gfd}_R(-)$ to denote the projective, flat, Gorenstein projective and Gorenstein flat dimension, respectively. We use $\mathcal{GP}$ and $\mathcal{GF}$ to denote the class of Gorenstein projective, and Gorenstein flat modules, respectively. We say that $\mathcal{GP}$ is *special precovering* (resp. *covering*) if every module has a *special Gorenstein projective precover* (resp. *Gorenstein projective cover*), that is, if every module has a special $\mathcal{GP}$-precover (resp. $\mathcal{GP}$-cover).

Recall from [29] that an R-module E is called *FP-injective* if $\text{Ext}^1_R(A, E) = 0$ for all finitely presented R-modules A. The *FP-injective dimension* of a module B, denoted by FP-id$_R(B)$, is defined to be the least integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(A, B) = 0$ for all finitely presented R-modules A. If no such n exists, set FP-id$(B) = \infty$. FP-injective modules and dimension have a nice description over coherent rings, see Enochs–Jenda's book [11].

Let $\mathcal{C}$ be an abelian category, and $\mathcal{A}$ and $\mathcal{B}$ be classes of objects in $\mathcal{C}$. We think of a *cotorsion pair* $(\mathcal{A}, \mathcal{B})$ as being "orthogonal with respect to the functor $\mathrm{Ext}_{\mathcal{C}}^1$". This is often expressed with the notation $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *complete* if for any object X there are exact sequences $0 \to X \to B \to A \to 0$ and $0 \to \widetilde{B} \to \widetilde{A} \to X \to 0$ with $B, \widetilde{B} \in \mathcal{B}$ and $A, \widetilde{A} \in \mathcal{A}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *cogenerated* by a set $\mathcal{S} \subseteq \mathcal{A}$ whenever $B \in \mathcal{B}$ if and only if $\mathrm{Ext}^1(S, B) = 0$ for all $S \in \mathcal{S}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *hereditary* if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathcal{A}$ then A' is also in $\mathcal{A}$, or equivalently, if $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathcal{B}$ then B'' is also in $\mathcal{B}$. The notion of a cotorsion pair was first introduced by Salce in [26] and rediscovered by Enochs and coauthors in 1990's (see [11]). Its importance in homological algebra has been shown by its use in the proof of the existence of flat covers of modules over any ring [3].

2. The triviality of Gorenstein projective and Gorenstein flat dimensions

Let R be a left coherent ring, and P an acyclic complex of projective R-modules. It is proved by Gillespie [16, Theorem 3.6] that $\mathrm{Ext}_R^1(\mathbf{Z}_n P, A) = 0$ for any FP-injective R-module A, where $\mathbf{Z}_n P$ is the nth cycle of the complex P. We shall use this result to show the following lemma.

Lemma 2.1. *Let R be a left coherent ring and G a Gorenstein projective R-module. Then $\mathrm{Ext}_R^{i\geq 1}(G, B) = 0$ for any R-module B of finite FP-injective dimension.*

Proof. If B is FP-injective, then the result follows directly by [16, Theorem 3.6]. Now we assume that FP-id$_R B = n > 0$. By [29, Lemma 3.1] there exists an exact sequence $0 \to B \to E^0 \to \cdots \to E^n \to 0$ of R-modules with each E^i FP-injective. As G is Gorenstein projective, there is an exact sequence

$$\mathbf{P} =: \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

of projective R-modules, such that $G \cong \mathrm{Ker}(P_0 \to P_{-1})$ and each kernel of $\mathbf{P}$ is Gorenstein projective, and it remains exact after applying the functor $\mathrm{Hom}_R(-, Q)$ for every projective R-module Q. We get an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R(\mathbf{P}, B) \longrightarrow \mathrm{Hom}_R(\mathbf{P}, E^0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(\mathbf{P}, E^n) \longrightarrow 0\ .$$

It is obvious that each complex $\mathrm{Hom}_R(\mathbf{P}, E^j)$ is acyclic since $\mathrm{Ext}_R^{i\geq 1}(N, E^j) = 0$ for any kernel N of $\mathbf{P}$ and any E^j. So we conclude that the complex $\mathrm{Hom}_R(\mathbf{P}, B)$ is also acyclic, this implies that $\mathrm{Ext}_R^{i\geq 1}(G, B) = 0$, as desired. □

It is proved by Holm [18, Theorem 2.2] that if M is an R-module with $\mathrm{id}_R M < \infty$, then $\mathrm{Gpd}_R M = \mathrm{pd}_R M$. We extend the result as the following.

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Theorem 2.2. *Assume that R is a left coherent ring and let M be an R-module. If* $\text{FP-id}_R M < \infty$, *then* $\text{Gpd}_R M = \text{pd}_R M$.

Proof. Clearly $\text{Gpd}_R M \leq \text{pd}_R M$ always holds true (for any ring R and any R-module M). To see the inverse inequality, we may assume that $\text{Gpd}_R M = n < \infty$. Then [6, Lemma 2.17] yields a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow P \longrightarrow G \longrightarrow 0$$

with $\text{pd}_R P = n < \infty$ and G Gorenstein projective. By Lemma 2.1 one has $\text{Ext}^1_R(G, M) = 0$ since $\text{FP-id}_R M < \infty$. It follows that the above short exact sequence is split, and hence $\text{pd}_R M \leq \text{pd}_R P = \text{Gpd}_R M$, as desired. □

By introducing the notion of projectively coresolved Gorenstein flat modules, Šaroch and Šťovíček [27, Theorem 4.11] have shown that any ring R is *GF-closed*, that is, the class, $\mathcal{GF}$, of all Gorenstein flat R-modules is closed under extensions. Here a left R-module M is called *projectively coresolved Gorenstein flat* [27] if there exists an exact sequence of projective R-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

such that $M \cong \text{Ker}(P_0 \to P_{-1})$, and such that it remains exact after applying the functor $I \otimes_R -$ for every injective R^o-module I. It is trivial that any projectively coresolved Gorenstein flat module is Gorenstein flat. Note that any projectively coresolved Gorenstein flat module is also Gorenstein projective [27, Theorem 4.4].

Lemma 2.3. *Let R be a ring and n a non-negative integer. If M is an R-module with* $\text{Gfd}_R(M) \leq n$, *then there is an exact sequence of R-modules*

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$$

with $\text{fd}_R(F) \leq n$ *and G Gorenstein projective.*

Proof. Let M be an R-module with $\text{Gfd}_R(M) \leq n$. Since the class $\mathcal{GF}$ is closed under extensions as mentioned above, it follows from [4, Definition 1] and [4, Theorem 4] that there is a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$$

with $\text{fd}_R(E) \leq n$ and $N \in \mathcal{GF}$. According to [27, Theorem 4.11], there exists another short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow G \longrightarrow N \longrightarrow 0$$

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with K flat and G projectively coresolved Gorenstein flat. Now we consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since, in the middle column, K is flat and $\mathrm{fd}_R(E) \le n$, it follows that $\mathrm{fd}_R(F) \le n$. Besides, $G \in \mathcal{GP}$ as any projectively coresolved Gorenstein flat R-module is always Gorenstein projective (see [27, Theorem 4.4]). Therefore, the middle row is the desired exact sequence. □

Assume that R is a left and right coherent ring with finite RightFPD(R). Here RightFPD(R) is the right finitistic projective dimension of R, it is defined as the supremum of the projective dimensions of all right R-modules with finite projective dimension. It is proved by Holm [18, Theorem 2.6] that if $\mathrm{id}_R M < \infty$, then $\mathrm{Gfd}_R M = \mathrm{fd}_R M$. We improve this result by removing the assumption of the finiteness of RightFPD(R) and the right coherence of R as the following, which is the " flat version" of the previous Theorem 2.2.

Theorem 2.4. *Assume that R is a left coherent ring and let M be an R-module. If FP-$\mathrm{id}_R M < \infty$, then $\mathrm{Gfd}_R M = \mathrm{fd}_R M$.*

Proof. Suppose FP-$\mathrm{id}_R M < \infty$. It is sufficient to show $\mathrm{fd}_R(M) \le \mathrm{Gfd}_R(M)$ since the inverse inequality is clear. For this, let $\mathrm{Gfd}_R(M) = n < \infty$. Then Lemma 2.3 yields a short exact sequence $0 \to M \to F \to G \to 0$ with $\mathrm{fd}_R(F) \le n$ and G Gorenstein projective. Thus, by Lemma 2.1, one gets that the above sequence is split. It follows easily that $\mathrm{fd}_R(M) \le \mathrm{fd}_R(F) \le n$, as desired. □

3. Cotorsion pairs induced by gorenstein projective modules

As mentioned in the introduction, it is still open whether the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ forms a complete hereditary cotorsion pair over an arbitrary ring. Let R be a right coherent ring such that all flat R-modules have finite projective dimension. It is known from [15, Fact 10.2] that, over such a ring the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ forms a complete hereditary cotorsion pair. In this section, we will obtain the same result for certain left coherent rings (see Corollary 3.4). We also compare some conditions of rings (see Remarks 3.12, 3.13, and 3.14).

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In what follows, for any non-negative integer n, we will denote by $\mathcal{FI}_n$ the class of all R-modules M with $\text{FP-id}_R(M) \leq n$. Let us start with a fact proved by Mao and Ding.

Remark 3.1. Let R be a left coherent ring. It follows by [24, Proposition 3.6(2)] that $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$ is a complete hereditary cotorsion pair.

Let $\mathcal{A}$ be an abelian category. We say that a class $\mathcal{X}$ of objects in $\mathcal{A}$ is *thick* if $\mathcal{X}$ is closed under direct summands and such that if any two out of three of the terms in a short exact sequence are in $\mathcal{X}$, then so is the third.

If $\mathcal{A}$ has enough projectives, then we say that a pair $(\mathcal{X}, \mathcal{Y})$ of objects in $\mathcal{A}$ is a *projective cotorsion pair* if $(\mathcal{X}, \mathcal{Y})$ forms a complete cotorsion pair such that $\mathcal{Y}$ is thick and $\mathcal{X} \cap \mathcal{Y}$ coincides with the subcategory of projective objects. Equivalently, if $(\mathcal{X}, \mathcal{Y})$ forms a hereditary complete cotorsion pair such that $\mathcal{X} \cap \mathcal{Y}$ coincides with the subcategory of projective objects (see [15, Definition 3.4] and [15, Proposition 3.7]).

The left finitistic FP-injective dimension $\text{FFPID}(R)$ of the ring R, is defined as the supremum of the FP-injective dimensions of those R-modules that have finite FP-injective dimension.

Lemma 3.2. *Let R be a left coherent ring and n a non-negative integer. Then the following conditions are equivalent.*

(1) *The cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$ is projective.*

(2) $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$.

Proof. (1) $\Rightarrow$ (2). Suppose that the cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$ is projective. Then by [15, Theorem 5.4], one has ${}^{\perp}\mathcal{FI}_n \subseteq \mathcal{GP}$. Using the completeness of the cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$, one gets a short exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow E \longrightarrow N \longrightarrow 0$$

with $E \in \mathcal{FI}_n$ and $N \in {}^{\perp}\mathcal{FI}_n \subseteq \mathcal{GP}$. Note that the short exact sequence is split since $\text{Ext}^1_R(N, R) = 0$, and hence $\text{FP-id}_R(R) \leq n$.

Let M be an R-module with $\text{FP-id}_R(M) < \infty$. It follows from Lemma 2.1 that $\text{Ext}^1_R(G, M) = 0$ for any $G \in \mathcal{GP}$, and then $\text{Ext}^1_R(X, M) = 0$ for any $X \in {}^{\perp}\mathcal{FI}_n$ as ${}^{\perp}\mathcal{FI}_n \subseteq \mathcal{GP}$ by [15, Theorem 5.4] and the assumption. This yields that $M \in \mathcal{FI}_n$ via the cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$. That is, $\text{FP-id}_R(M) \leq n$.

(2) $\Rightarrow$ (1). Assume $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$. By Remark 3.1, the pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$ is a complete hereditary cotorsion pair. Note that $\mathcal{P} \subseteq {}^{\perp}\mathcal{FI}_n \cap \mathcal{FI}_n$ since $\mathcal{P} \subseteq {}^{\perp}\mathcal{FI}_n$ is clear and $\mathcal{P} \subseteq \mathcal{FI}_n$ follows by the equality

$$\text{FP-id}_R(R) = \sup\{\text{FP-id}_R(P) \mid P \text{ is a projective } R\text{-module}\}$$

and the assumption $\text{FP-id}_R(R) \leq n$. Now, let $M \in {}^{\perp}\mathcal{FI}_n \cap \mathcal{FI}_n$. Consider the short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Noticing that P and M have finite FP-injective dimension, one gets that K has a finite FP-injective dimension as well. The assumption $\text{FFPID}(R) \leq n$

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yields $K \in \mathcal{FI}_n$ and so $\mathrm{Ext}^1_R(M, K) = 0$. Thus, the short exact sequence above splits and so M is projective. Therefore, the cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$ is projective by [15, Proposition 3.7]. □

Now we are able to prove the following theorem.

Theorem 3.3. *Let R be a left coherent ring and n a non-negative integer. Then the following conditions are equivalent.*

(1) *The pair $(\mathcal{GP}, \mathcal{FI}_n)$ forms a projective cotorsion pair.*

(2) $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$.

Proof. (1) ⇒ (2). Suppose that the pair $(\mathcal{GP}, \mathcal{FI}_n)$ forms a projective cotorsion pair. Then one has ${}^{\perp}\mathcal{FI}_n = \mathcal{GP}$ by Remark 3.1. So according to Lemma 3.2, one has $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$‘.

(2) ⇒ (1). Assume $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$. Then Lemma 3.2 yields a projective cotorsion pair $({}^{\perp}\mathcal{FI}_n, \mathcal{FI}_n)$. We have to show that ${}^{\perp}\mathcal{FI}_n = \mathcal{GP}$. Indeed, on the one hand, applying [15, Theorem 5.4] to the above projective cotorsion pair, one has ${}^{\perp}\mathcal{FI}_n \subseteq \mathcal{GP}$. On the other hand, Lemma 2.1 yields $\mathcal{GP} \subseteq {}^{\perp}\mathcal{FI}_n$. Thus, ${}^{\perp}\mathcal{FI}_n = \mathcal{GP}$, as desired. □

Using Theorem 3.3, one can deduce easily the following corollary.

Corollary 3.4. *Let R be a left coherent ring with both* $\text{FP-id}_R(R)$ *and* $\text{FFPID}(R)$ *finite. Then $(\mathcal{GP}, \mathcal{W})$ forms a projective cotorsion pair, where $\mathcal{W}$ denotes the class of R-modules of finite FP-injective dimension. Moreover, every R-module has a special Gorenstein projective precover.*

Using Theorem 3.3, we can also characterize Gorenstein projective modules as follows.

Corollary 3.5. *Let R be a left coherent ring with both* $\text{FP-id}_R(R)$ *and* $\text{FFPID}(R)$ *finite. Then the following conditions are equivalent for any R-module M.*

(1) *M is Gorenstein projective.*

(2) *There is an exact sequence of R-modules $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i projective.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1). Assume that $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} = n < \infty$. Then Theorem 3.3 yields a projective cotorsion pair $(\mathcal{GP}, \mathcal{FI}_n)$. Now suppose that we are given an exact sequence of R-modules $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i projective. To see M is Gorenstein projective, it is sufficient to show $\mathrm{Ext}^1_R(M, F) = 0$ for all $F \in \mathcal{FI}_n$ by the above cotorsion pair.

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Indeed, combining the exact sequence in (2) with a projective resolution of M, one can get an exact sequence of projective R-modules

$$\mathbf{P} =: \ \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

such that $M = \mathrm{Im}(P_0 \to P^0)$. For any $F \in \mathcal{FI}_n$, using [16, Theorem 3.6.] and applying the arguments used in the proof of Lemma 2.1, one has $\mathrm{Ext}^1_R(M, F) = 0$, as desired. □

Let R be a left coherent ring with both FP-id$_R(R)$ and FFPID(R) finite. Then Corollary 3.4 shows that the class $\mathcal{GP}$ is special precovering. The next result considers when the class $\mathcal{GP}$ is covering.

Corollary 3.6. *Let R be a left coherent ring with both* FP-id$_R(R)$ *and* FFPID(R) *finite and $\mathcal{W}$ be as above. Then the following are equivalent*:

(1) *The class $\mathcal{GP}$ is covering.*

(2) *The class $\mathcal{GP}$ is closed under direct limits.*

(3) *The pair $(\mathcal{GP}, \mathcal{W})$ is a perfect cotorsion pair.*

(4) *Every Gorenstein flat R-module M has a Gorenstein projective cover.*

(5) *Every flat R-module M has a Gorenstein projective cover.*

(6) *R is left perfect.*

Proof. Note that (2) ⇒ (3) follows from Corollary 3.4 and [11, Theorem 7.2.6], (3) ⇒ (1) ⇒ (4) ⇒ (5) are trivial.

(5) ⟹ (6). Let F be a flat R-module and $\alpha : P \longrightarrow F$ a Gorenstein projective cover of F. So α must epic since the class $\mathcal{GP}$ contains all projective R-modules. Consider the short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\alpha} F \longrightarrow 0$$

with $K = \mathrm{Ker}\alpha$. Then K belongs to $\mathcal{W}$ by Wakamutsu's Lemma since the class $\mathcal{GP}$ is closed under extensions (see [19, Theorem 2.5]). As R is a left coherent ring with FP-id$_R(R) < \infty$, it is easy to see that

$$\text{FP-id}_R(R) = \sup\{\text{FP-id}_R(M) \mid M \text{ is a flat } R\text{-module}\}.$$

This means every flat R-module $F \in \mathcal{W}$. Thus, one has $P \in \mathcal{W}$, and so $P \in \mathcal{W} \cap \mathcal{GP} = \mathcal{GP}^{\perp} \cap \mathcal{GP} = \mathcal{P}$. It follows that $\alpha : P \longrightarrow F$ is also a projective cover of F. Therefore, R is left perfect by the proof of (3) ⇒ (1) in [11, Theorem 5.3.2].

(6) ⇒ (2). Suppose that R is left perfect. To see that the class $\mathcal{GP}$ is closed under direct limits, let $\{(G_j)_{j\in J}\}$ be a family of Gorenstein projective R-modules. Then for

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each $j \in J$, there is an exact sequence of R-modules

$$0 \longrightarrow G_j \longrightarrow P_j^0 \longrightarrow P_j^1 \longrightarrow \cdots$$

with all $P_j^i \in \mathcal{P}$. Hence, one can obtain an exact sequence of R-modules

$$0 \longrightarrow \varinjlim G_j \longrightarrow \varinjlim P_j^0 \longrightarrow \varinjlim P_j^1 \longrightarrow \cdots$$

with all $\varinjlim P_j^i \in \mathcal{P}$ since R is left perfect. Thus, $\varinjlim G_j$ is Gorenstein projective by Corollary 3.5. □

Recall that a ring R is called an *n-FC ring* (resp. *n-Gorenstein ring*) if it is both left and right coherent (resp. Noether) and FP-$\mathrm{id}_R(R)$ and FP-$\mathrm{id}_{R^o}(R)$ (resp. $\mathrm{id}_R(R)$ and $\mathrm{id}_{R^o}(R)$) are both less than or equal to n. A ring R is called *Ding-Chen* (resp. *Gorenstein*) if it is an n-FC (resp. n-Gorenstein) ring for some non-negative integer n. Ding-Chen rings are natural generalizations of Gorenstein rings.

Recall from [10, 14] that an R-module M is called *Ding projective* if there exists an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

of projective R-modules with $M = \mathrm{Ker}(P_0 \to P_{-1})$, such that it remains exact after applying $\mathrm{Hom}(-, Q)$ for any flat R-module Q. It is clear that Ding projective modules are always Gorenstein projective, conversely, it is not known.

Let R be a Ding-Chen ring. Then Gillespie [14] showed that $(\mathcal{DP}, \mathcal{W})$ is a projective cotorsion pair, where $\mathcal{DP}$ denotes the class of Ding projective R-modules, and $\mathcal{W}$ denotes the class of R-modules of finite FP-injective dimension. Moreover, Gillespie [16, Theorem 1.1] has shown recently that the class $\mathcal{DP}$ coincides with $\mathcal{GP}$. It yields the following corollary. Meanwhile, this is obvious by Corollary 3.4 as any Ding-Chen ring R satisfies $\mathrm{FFPID}(R) < \infty$ by [8, Proposition 3.16].

Corollary 3.7. *Let R be a Ding-Chen ring. Then the pair $(\mathcal{GP}, \mathcal{W})$ forms a projective cotorsion pair, where $\mathcal{W}$ is as above.*

Remark 3.8. (1) Let R be a Ding-Chen ring (in particular the case for R is Gorenstein). Then Corollary 3.7 shows that the class $\mathcal{GP}$ is special precovering. Applying Corollary 3.6, one gets that the class $\mathcal{GP}$ is covering if and only if R is left perfect.

(2) Let R be a Gorenstein ring. Then the famous "Auslander's last theorem" (see [11, Theorem 11.6.9]) tells us that every finitely generated R-module has a Gorenstein projective cover. However, there exists a Gorenstein ring such that not all modules have a Gorenstein projective cover. Indeed, as we all know, the ring $\mathbb{Z}$ is a commutative 1-Gorenstein ring which is not perfect. Then by (1), over $\mathbb{Z}$ the class $\mathcal{GP}$ is not covering.

The following proposition improves the Gillespie's result [16, Theorem 1.1(1)] from a Ding-Chen ring to a left coherent ring R with both FP-$\mathrm{id}_R(R)$ and $\mathrm{FFPID}(R)$ finite (see Remark 3.10 for the existence of that a left coherent ring R with both $\mathrm{FP}\text{-}\mathrm{id}_R(R)$ and $\mathrm{FFPID}(R)$ finite may not be Ding-Chen ring).

Proposition 3.9. *Let R be a left coherent ring with both $\mathrm{FP}\text{-}\mathrm{id}_R(R)$ and $\mathrm{FFPID}(R)$ finite. Then Ding projective R-modules are exactly the Gorenstein projective R-modules, and hence are exactly some kernel of an exact complex of projective R-modules.*

Proof. Since R is a left coherent ring with $\mathrm{FP}\text{-}\mathrm{id}_R(R) < \infty$, it is easy to see that $\mathrm{FP}\text{-}\mathrm{id}_R(R) = \sup\{\mathrm{FP}\text{-}\mathrm{id}_R(M) \mid M \text{ is a flat } R\text{-module}\}$. This means every flat R-module $F \in \mathcal{GP}^{\perp} = \mathcal{W}$ by Corollary 3.4. Therefore, it follows by [20, Lemma 4] that any Gorenstein projective R-module is Ding projective. The last assertion follows then by Corollary 3.5. □

Remark 3.10. We notice that there exists a left coherent ring R with both $\mathrm{FP}\text{-}\mathrm{id}_R(R)$ and $\mathrm{FFPID}(R)$ finite which is not right coherent. For example, let R_1 be the oppositive ring of T, where $T = \begin{pmatrix} A & F \\ 0 & A/L \end{pmatrix}$is constructed by Small [28], in which A is an algebra over a field F, L is a left ideal of A with $\mathrm{pd}_A(L) = 1$. By virtue of [28, Theorem 1], T is a right hereditary ring with $l.\mathrm{gldim}(T) = 3$ which is not left coherent. Hence $R_1 = T^o$ is a left hereditary ring (hence a left coherent ring) with $\mathrm{FP}\text{-}\mathrm{id}_{R_1}(R_1) \leq \mathrm{id}_{R_1}(R_1) \leq 1$ and $\mathrm{FFPID}(R_1) \leq 1$ which is not right coherent. In particular, R_1 is not a Ding-Chen ring.

Recall from [5] that a right R-module F is of *type FP_∞* if F has a projective resolution consisting of finitely generated modules, and that a left R-module M is *level* if $\mathrm{Tor}_1^R(F, M) = 0$ for all right R-modules F of type FP_∞.

Remark 3.11. We notice that there exists a left coherent ring R admitting both finite left self-FP-injective dimension and finite left finitistic FP-injective dimension which is neither right coherent nor satisfies that all level left R-modules have finite projective dimension.

We first consider $R_2 = F_\alpha$, the free Boolean ring on $\aleph_\alpha$ generators with α an infinite cardinality. By [30, Example 3.3], R_2 is a commutative von Neumann regular ring (hence a commutative coherent ring with $\mathrm{FP}\text{-}\mathrm{id}_{R_2}(R_2) = 0$ and $\mathrm{FFPID}(R_2) = 0$) which has infinite (Gorenstein) global dimension. Note that over a commutative von Neumann regular ring every module is flat, and that the global dimension of a commutative ring is finite if and only if every module has a finite projective dimension. These facts imply that there is a flat left R_2-module M with infinite projective dimension. On the other hand, note from [5, Corollary 2.11] that, over a right coherent ring the level modules are exactly flat modules. Thus there is a level left R_2-module M with infinite projective dimension.

Now we consider $R = R_1 \times R_2$, the product ring of R_1 and R_2, where R_1 is as in Remark 3.10. Then R admits both finite left self-FP-injective dimension and finite left finitistic FP-injective dimension since so do R_1 and R_2. However, R is neither right coherent nor satisfies that all level left R-modules have finite projective dimension since R_1 is not right coherent and R_2 does not satisfy that all level left R_2-modules have finite projective dimension.

As mentioned above, whether the pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ forms a complete and hereditary cotorsion pair over any associative ring is still open. It is known that over a (commutative)

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Gorenstein ring the answer of the question is affirmative (see [11, Remark 11.5.10]). Later, many authors extended the result to a more general ring:

(1) For the commutative settings, it is extended to the commutative Noether rings with a dualizing complex or with finite Krull dimension (see [21, Proposition 1.9, Theorem 1.0 and Corollary 2.13]).

(2) For the possibly not commutative settings, it is extended to the following:

(C1) rings such that any level module has a finite projective dimension (see [15, Fact 10.2]). Note that such class of rings contains right coherent rings such that any flat module has a finite projective dimension (see [15, Fact 10.2]) and hence contains commutative Noether rings with a dualizing complex or with finite Krull dimension.

(C2) rings such that any projective module has a finite injective dimension (see [31, Theorem 4.2]). Note that such class of rings contains rings with finite left Gorenstein global dimension.

(C3) left coherent rings R such that $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} < \infty$ (see Corollary 3.4). Note that such class of rings contains strictly Ding-Chen rings (see Remark 3.10).

We end this section by pointing out that the classes of rings in (C1) (resp. (C2))and (C3) above are independent to each other as shown by the following remarks.

Remark 3.12. Let $R_2 = F_\alpha$ be the free Boolean ring on $\aleph_\alpha$ generators with α an infinite cardinality, as in Remark 3.11. Then, R_2 is in the class (C3) but not in the class (C1). Furthermore, R_2 admits infinite Gorenstein global dimension. it follows by Theorem 4.4 (in § 3) that there exists a projective left R-module M with $\text{id}_R(M) = \infty$. In other words, R_2 is not in the class (C2).

Remark 3.13. There exists a commutative ring R with finite global dimension (hence R is in both the classes (C1) and (C2)) which is not coherent, see [22, Example in p.128]. Thus, R is not in the class (C3).

Remark 3.14. There exists a commutative Noether ring R with finite Krull dimension (so R is a commutative coherent ring with $\sup\{\text{pd}_R(M)|\ M \text{ is a level } R\text{-module}\} < \infty$, that is, R is in the class (C1)) which is not Gorenstein. Note that a commutative Noether ring R is Gorenstein if and only if $\text{FP-id}_R(R) = \text{id}_R(R) < \infty$ by the definition, equivalently, if and only if all projective R-modules have finite injective dimension (see [11, Theorem 9.1.11]). These facts imply that R is neither in the class (C2) nor in the class (C3).

Remark 3.14 shows that rings in (C1) may not be in (C2). Besides, we know from Lemma 4.3 (in § 3) that any ring in (C2) satisfies that all flat modules have finite projective dimension. So we pose a question below:

Question 3.15. *Does the class* (C1) *strictly contain the class* (C2)*?*

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4. Gorenstein global dimensions of coherent rings

As another application of Theorem 3.3, we will characterize Gorenstein global dimensions for left coherent rings.

For a ring R, the invariant $\mathrm{silffpi}(R)$ is defined as the supremum of the injective dimensions of R-modules with finite FP-injective dimension; the invariant $\mathrm{splfpi}(R)$ is defined as the supremum of the projective dimensions of FP-injective R-modules.

It is proved by Bennis and Mahdou [2, Theorem 1.1] that the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid M \text{ is an } R \text{ -module}\}$$

holds. Bennis and Mahdou call the common value of the equality the *left Gorenstein global dimension* of R and denote it by $l.\mathrm{Ggldim}(R)$.

Proposition 4.1. *The following quantities are equal to each other whenever R is a left coherent ring.*

(1) $l.\mathrm{Ggldim}(R)$.

(2) $\max\{\mathrm{silffpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\}$.

(3) $\max\{\mathrm{splfpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\}$.

Proof. Assume $l.\mathrm{Ggldim}(R) = n < \infty$. Then by [2, Corollary 2.7], the regular module ${}_RR$ has a finite injective dimension at most n, and of course $\text{FP-id}_R(R) \leq n$. Now let E be an FP-injective R-module. Then we have a pure short exact sequence of R-modules

$$0 \longrightarrow E \longrightarrow I \longrightarrow E' \longrightarrow 0$$

with I injective. Since $\mathrm{fd}_R(I) \leq n$ by [2, Corollary 2.7], one has $\mathrm{fd}_R(E) \leq n$. Thus one can deduce easily that any R-module with finite FP-injective dimension has flat dimension at most n. It follows again from [2, Corollary 2.7] that any R-module with finite FP-injective dimension has both injective dimension and projective dimension at most n. This means $\mathrm{silffpi}(R) \leq n$ and $\mathrm{splfpi}(R) \leq n$, and also $\mathrm{FFPID}(R) \leq n$. Thus we have shown that $\max\{\mathrm{silffpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\} \leq l.\mathrm{Ggldim}(R)$, and $\max\{\mathrm{splfpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\} \leq l.\mathrm{Ggldim}(R)$.

In the next, we show that $l.\mathrm{Ggldim}(R) \leq \max\{\mathrm{silffpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\}$. Suppose $\max\{\mathrm{silffpi}(R),\ \text{FP-id}_R(R),\ \mathrm{FFPID}(R)\} = n < \infty$. By Theorem 3.3 and $\max\{\text{FP-id}_R(R),\ \mathrm{FFPID}(R)\} \leq n$ one has a projective cotorsion pair $(\mathcal{GP},\ \mathcal{FI}_n)$. For any R-module M, there is an exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i projective. Then by dimension shifting, one gets

$$\mathrm{Ext}_R^1(K, E) \cong \mathrm{Ext}_R^{n+1}(M, E)$$

for any R-module E with $\text{FP-id}_R(E) \leq n$. But $\mathrm{Ext}_R^{n+1}(M,\ E) = 0$ as $\mathrm{silffpi}(R) \leq n$. Thus, $\mathrm{Ext}_R^1(K,\ E) = 0$ for any R-module E with $\text{FP-id}_R(E) \leq n$. It follows that K is Gorenstein

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projective since $(\mathcal{GP}, \mathcal{FI}_n)$ is a cotorsion pair. Therefore, M has a finite Gorenstein projective dimension at most n, as desired.

In the last, we show that $l.\text{Ggldim}(R) \leq \max\{\text{splfpi}(R), \text{FP-id}_R({}_RR), \text{FFPID}(R)\}$. Now suppose $\max\{\text{splfpi}(R), \text{FP-id}_R(R), \text{FFPID}(R)\} = n < \infty$. Again, by Theorem 3.3 and $\max\{\text{FP-id}_R(R), \text{FFPID}(R)\} \leq n$ one obtains a projective cotorsion pair $(\mathcal{GP}, \mathcal{FI}_n)$. Thus, for any R-module M, there is a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow G \longrightarrow 0$$

of R-modules with G Gorenstein projective and $N \in \mathcal{FI}_n$. Since $\text{splfpi}(R) \leq n$, it is a routine to see that $\text{pd}_R(N) \leq n$, and hence, it is a routine to check that M has Gorenstein projective dimension at most n, as desired. □

Let R be a ring. Then the invariant $\text{silp}(R)$ (respectively, $\text{silf}(R)$) is defined as the supremum of the injective dimensions of projective (respectively, flat) R-modules. The invariant $\text{splf}(R)$ is defined as the supremum of the projective dimensions of flat R-modules.

Corollary 4.2. *Let R be a Ding-Chen ring. Then the following invariants of R coincide with each other.*

(1) $l.\text{Ggldim}(R)$.

(2) $\text{silffpi}(R)$.

(3) $\text{splfpi}(R)$.

(4) $\text{splf}(R)$.

(5) $\text{silf}(R)$.

(6) $\text{silp}(R)$.

Proof. Since R is Ding-Chen, [8, Proposition 3.6] yields $\text{FFPID}(R) < \infty$. Now the quantities (1), (2) and (3) are equal by Proposition 4.1. The quantities (5) and (6) are equal by [7, Execise 8.5.19], which tells us that $\text{silf}(R) = \text{silp}(R)$ for any ring R. Finally, (2)=(5) and (3)=(4) follow by [8, Proposition 3.16]. □

As mentioned in the proof of Corollary 4.2, it is known from [7, Execise 8.5.19] that $\text{silf}(R) = \text{silp}(R)$ for any ring R. In particular, $\text{silf}(R) < \infty$ if and only if $\text{silp}(R) < \infty$. Furthermore, in this case one also gets that $\text{FPD}(R) < \infty$ by [7, Execise 8.5.14], where $\text{FPD}(R)$ is defined as the supremum of the projective dimensions of all R-modules with finite projective dimension.

The next lemma can be viewed as a continuation of these facts.

Lemma 4.3. *Let R be a ring. Then the following are equivalent:*

(1) *Every flat R-module has a finite injective dimension.*

(2) *Every projective R-module has a finite injective dimension.*

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Moreover, $\mathrm{FPD}(R) < \infty$ *holds if any one of the above conditions is satisfied.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Assume that every projective R-module has a finite injective dimension. Then one gets easily that any R-module of finite projective dimension has finite injective dimension. Since $\mathrm{FPD}(R) < \infty$ implies that any flat R-module has a finite projective dimension by [7, Theorem 8.5.17], we need only to show $\mathrm{FPD}(R) < \infty$.

Firstly, we claim that for any R-module with $\mathrm{pd}_R(M) = m < \infty$ and any R-module N with a projective presentation $0 \to K_N \to P_N \to N \to 0$, there is an inequality $m \leq \mathrm{id}_R(P_N)$. Otherwise, let $m > \mathrm{id}_R(P_N)$. Consider the following exact sequence of Abel groups

$$\mathrm{Ext}_R^m(M, P_N) \longrightarrow \mathrm{Ext}_R^m(M, N) \longrightarrow \mathrm{Ext}_R^{m+1}(M, K_N) \ .$$

Note that $\mathrm{Ext}_R^m(M, P_N) = 0$ as $\mathrm{id}_R(P_N) < m$, and $\mathrm{Ext}_R^{m+1}(M, K_N) = 0$ as $\mathrm{pd}_R(M) = m$. Hence, one has $\mathrm{Ext}_R^m(M, N) = 0$, which shows that $\mathrm{pd}_R(M) < m$, it is a contradiction with $\mathrm{pd}_R(M) = m$.

Now we prove that $\mathrm{FPD}(R) < \infty$. If $\mathrm{FPD}(R) = \infty$, then for each $n \in \mathbb{N}$, there exists an R-module M_n with $\mathrm{pd}_R(M_n) < \infty$ but $\mathrm{pd}_R(M_n) \geq n$. Then $\mathrm{pd}_R(\coprod_{n\in\mathbb{N}} M_n) = \sup\{\mathrm{pd}_R(M_n) \mid n \in \mathbb{N}\} = \infty$. For each $n \in \mathbb{N}$, there is a short exact sequence of R-modules $0 \to K_n \to P_n \to M_n \to 0$ with P_n projective. This induces a projective presentation of $\coprod_{n\in\mathbb{N}} M_n$ of the form

$$0 \longrightarrow \coprod_{n\in\mathbb{N}} K_n \longrightarrow \coprod_{n\in\mathbb{N}} P_n \longrightarrow \coprod_{n\in\mathbb{N}} M_n \longrightarrow 0 \ .$$

So, by the above claim, we conclude that $\mathrm{id}_R(\coprod_{n\in\mathbb{N}} P_n) \geq \mathrm{pd}_R(\coprod_{n\in\mathbb{N}} M_n) = \infty$. This is a contradiction with the assumption as $\coprod_{n\in\mathbb{N}} P_n$ is projective. Therefore, $\mathrm{FPD}(R) < \infty$, and the proof is completed. □

Now we can give some characterizations of when the left Gorenstein global dimension of a Ding-Chen ring is finite.

Theorem 4.4. *Let R be a Ding-Chen ring. Then the following are equivalent:*

(1) *$l.\mathrm{Ggldim}(R) < \infty$.*

(2) $\mathrm{silffpi}(R) < \infty$.

(3) $\mathrm{splfpi}(R) < \infty$.

(4) $\mathrm{splf}(R) < \infty$.

(5) $\mathrm{silf}(R) < \infty$.

(6) $\mathrm{silp}(R) < \infty$.

(7) *Every FP-injective R-module has finite injective dimension.*

(8) *Every FP-injective R-module has a finite projective dimension.*

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(9) *Every flat R-module has a finite projective dimension.*

(10) *Every flat R-module has finite injective dimension.*

(11) *Every projective R-module has finite injective dimension.*

(12) $\mathrm{FPD}(R) < \infty$.

Proof. (1)–(6) are equivalent by Corollary 4.2. (2) $\Rightarrow$ (7), (3) $\Rightarrow$ (8), (4) $\Rightarrow$ (9), (5) $\Rightarrow$ (10), and (6) $\Rightarrow$ (11) are trivial.

(7) $\Rightarrow$ (2). Assume that every FP-injective R-module has finite injective dimension. Then there must exist an integer m such that $\mathrm{id}_R(M) \leq m$ for any FP-injective module M. Otherwise, for each positive integer i there is an FP-injective module M_i with $i < \mathrm{id}_R(M_i) < \infty$, then $\prod M_i$ is an FP-injective module of infinite injective dimension. This is a contradiction. Now one gets easily that $\mathrm{silffpi}(R) < \infty$, since R is Ding-Chen.

Similar arguments applied, one can show that (8) $\Rightarrow$ (3), (9) $\Rightarrow$ (4), and (10) $\Rightarrow$ (5) hold. Now we obtain that 1)-(10) are equivalent.

Finally, (10)$\Leftrightarrow$(11)$\Rightarrow$(12) follows from Lemma 4.3 and (12)$\Rightarrow$(9) holds by [7, Theorem 8.5.17]. □

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