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# NONLINEAR MODULATION OF RANDOM WAVE SPECTRA FOR SURFACE-GRAVITY WAVES WITH LINEAR SHEAR CURRENTS

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**Abstract**

We first derive Alber's equation for the Wigner distribution function using the fourth-order nonlinear Schrödinger equation, and on the basis of this equation we next analyse the stability of the narrowband approximation of the Joint North Sea Wave Project spectrum. Therefore, one interesting result of this study concerns the effect of modulational instability obtained from the fourth-order nonlinear Schrödinger equation. The analysis is restricted to one horizontal direction, parallel to the direction of wave motion, to take advantage of potential flow theory. We find that shear currents considerably modify the instability behaviours of weakly nonlinear waves. The key point of this study is that the present fourth-order analysis shows considerable deviations in the modulational instability properties from the third-order analysis and reduces the growth rate of instability. Moreover, we present here a connection between the random and deterministic properties of a random wavetrain for vanishing spectrum bandwidth.

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## 1. Introduction

Nonlinear effects such as sideband instability have played a significant role in numerous nonlinear analysis domains [19]. In appropriate physical conditions, the nonlinear Schrödinger equation (NLSE) can be utilized to investigate nonlinear evolution of water waves, and this equation can be used to characterize sideband instability [32]. For perturbations with small amplitude and long wavelength, nonlinear analysis is appropriate. However, predictions using the cubic NLSE do not match the exact results of Longuet–Higgins [20, 21] for wave steepness greater than 0.15. As a result,

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Dysthe [7] developed an improved version of the NLSE by adding fourth-order effects, and he claimed that the stability analysis derived from the fourth-order NLSE for deep water produces results that are compatible with both the results of Longuet–Higgins and Benjamin and Feir [3]. The importance of a wave-induced mean flow response is the new effect added to the fourth-order which produces a significant deviation in the stability characteristics. A few of the features that are significantly improved by the fourth-order effects have been explored by Janssen [14]. In the presence of wind blowing over water, Dhar and Das [5] developed a fourth-order modified NLSE for deep-water surface gravity waves and investigated the impact of wind on the Benjamin–Feir instability. Dhar and Kirby [6] currently derived a fourth-order NLSE for capillary-gravity waves on finite depth with constant vorticity. They remarked that vorticity significantly alters the modulational instability properties and that, when vorticity and capillarity are combined, the growth rate of instability (GRI) influenced by capillarity on finite depth is increased when vorticity is negative. Therefore, we conclude that a fourth-order NLSE is a good starting point for the investigation of nonlinear effects of surface waves on deep water.

Recently, the study of surface waves when vorticity is present has received much attention. Nonlinear wave–current interactions are of interest to ocean engineers and scientists, because waves and currents typically coexist in an ocean. The wave–current interactions depend on the propagation direction and also on the vertical distribution of the currents. So, it is important to study the various characteristics of water waves travelling in a shearing flow. In fact, vorticity coexists with the underlying currents. Therefore, both the influence of depth uniform currents and vorticity must be considered when deriving a NLSE. Because of this, Liao et al. [18] used the multiple scale technique to develop a linear-shear-current modified cubic Schrödinger equation for gravity waves in finite water depth.

Starting with the studies of Phillips [28] and Hasselmann [10, 11], attention has been paid to the energy transfer due to four-wave interactions in an ocean [12, 29, 31]. Lower-order corrections to Hasselmann's spectral transport equation (STE) were found by Willebrand [31], and by Watson and West [29] for an inhomogeneous random ocean. Additionally, an STE that explains the development of narrowband Gaussian random surface wavetrains has been established by Alber [1], and by Alber and Saffman [2]. They derived the STE from the weakly nonlinear equations of Davey and Stewartson. Eventually, Crawford et al. [4] obtained a unified equation for the development of a random field of gravity waves on infinite depth of water, starting with the complete equations of motion. This evolution equation describes the impacts of inhomogeneity and also the energy transfer process associated with a homogeneous spectrum. Again, Janssen [13] investigated the nonlinear interactions of narrowband Gaussian random inhomogeneous wavetrains, and, using the multiple scale technique, he described the long-term behaviour of an unstable modulation. Starting from the NLSE and employing the Wingner–Moyal transform [22, 30], Onorato et al. [23] discussed the modulational instability of the narrowband approximation of the Joint North Sea Wave Project (JONSWAP) spectrum. Recently, Halder and Dhar [8] described the random effect on the modulational instability of two Stokes waves on deep water. In a subsequent paper [9], they studied the same for interfacial gravity waves on deep water in the presence of air flowing over water, and they stated that the random effect is to decrease the instability growth rate and the extent of the instability region.

This paper is devoted to studying the theory of weakly nonlinear periodic gravity waves on linear shear currents on deep water. Considering the significance of the fourth-order NLSE, as stated by Dysthe [7], the purpose of this paper is to develop the Alber's equation for the Wigner distribution function, starting from a fourth-order NLSE, and to describe the modulational instability for random wave spectra and finally to discuss the influence of the BFI on vorticity. The key point is that the addition of fourth-order effects significantly affects estimates of the geometry of unstable regions in parameter space. Further, in the present paper we provide a bridge between the deterministic and random schools by investigating the stability properties of random wavetrains.

The presentation of the paper is as follows. In Section 2, we derive Alber's equation for the wave envelope. We discuss the stability analysis for the random wave spectra in Section 3. Section 4 deals with the limit of vanishing bandwidth. The BFI in the context of freak waves is presented in Section 5, and Section 6 concludes the paper.

## 2. Alber's equation for the wave envelope

In a recent paper, Pal and Dhar [27] derived a fourth-order linear shear currents modified NLSE for two-dimensional periodic gravity waves in a finite depth of water using multiscale expansion. In a frame of reference travelling with the group velocity, the fourth-order NLSE in dimensional form on deep water for the wave envelope $B(\xi, \tau)$ can be written as

$$i\frac{\partial B}{\partial \tau} + \beta_1 \frac{\partial^2 B}{\partial \xi^2} + i\beta_2 \frac{\partial^3 B}{\partial \xi^3} = \mu_1 |B|^2 B + i\mu_2 |B|^2 \frac{\partial B}{\partial \xi} + i\mu_3 B^2 \frac{\partial B^*}{\partial \xi} + \mu_4 BH\Big[\frac{\partial}{\partial \xi}|B|^2\Big], \quad (2.1)$$

where $\xi = x - c_g t$, $c_g$ is the group velocity of the carrier wave in dimensional form, $\tau = t$, $B^*$ is the complex conjugate of $B$ and $H$ is the Hilbert transform operator. The coefficients of equation (2.1) are given in Appendix A.

The linear dispersion relationship connecting the frequency $\Sigma$ and the wave number $k$ is

$$\Sigma^2(1 - \overline{v})(1 - \overline{v} + \overline{\Omega}) = gk, \quad (2.2)$$

where $\overline{v} = v/c$, is the nondimensional depth uniform current, $c = \Sigma/k$ is the wave velocity, $\overline{\Omega} = \Omega/\Sigma$ is the nondimensional uniform vorticity and $g$ is the gravitational acceleration.

Starting from equation (2.1), we next obtain Alber's equation. Following Alber [1] and Wigner–Moyal [22, 30], we introduce the Wigner distribution function $n(\xi, k, \tau)$ corresponding to the wave amplitude $B(\xi, \tau)$ given by

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$$n(\xi, k, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle B(\xi + y/2, \tau) B^*(\xi - y/2, \tau) \rangle e^{-iky}\, dy,$$

where $\xi = (\xi_1 + \xi_2)/2$ is the average coordinate, $y = \xi_1 - \xi_2$ is the spatial separation coordinate and the angle brackets represent an ensemble average.

The wave intensity corresponding to $B(\xi, \tau)$ can be expressed as

$$\langle |B(\xi, \tau)|^2 \rangle = \int_{-\infty}^{+\infty} n(\xi, k, \tau)\, dk. \tag{2.3}$$

To derive the kinetic equation of $n(\xi, k, \tau)$, namely, the Wigner–Moyal equation, we take the time derivative of equation (2.1). The nonlinear terms in NLSE will generate the fourth-order correlators

$$\langle B_1 B_1^* B_1 B_2^* \rangle, \quad \left\langle B_1 B_1^* \frac{\partial B_1}{\partial \xi_1} B_2^* \right\rangle, \quad \left\langle B_1 B_1 \frac{\partial B_1^*}{\partial \xi_1} B_2^* \right\rangle, \quad \left\langle B_1 H \frac{\partial}{\partial \xi_1} [B_1 B_1^*] B_2^* \right\rangle,$$

respectively, where $B_1 = B(\xi + y/2)$ and $B_2 = B(\xi - y/2)$. For proceeding with the calculation, a closure that connects second- and fourth-order correlators must be considered (see [1, 9]). This closure is obtained by considering the quasi-Gaussian approximation

$$\begin{aligned}
&\langle B_1 B_1^* B_1 B_2^* \rangle = 2 \langle B_1 B_2^* \rangle \langle |B_1|^2 \rangle, \\
&\left\langle B_1 B_1^* \frac{\partial B_1}{\partial \xi_1} B_2^* \right\rangle = \left\langle \frac{\partial B_1}{\partial \xi_1} B_2^* \right\rangle \langle |B_1|^2 \rangle + \left\langle B_1^* \frac{\partial B_1}{\partial \xi_1} \right\rangle \langle B_1 B_2^* \rangle, \\
&\left\langle B_1 B_1 \frac{\partial B_1^*}{\partial \xi_1} B_2^* \right\rangle = 2 \langle B_1 B_2^* \rangle \left\langle B_1 \frac{\partial B_1^*}{\partial \xi_1} \right\rangle, \\
&\left\langle B_1 H \frac{\partial}{\partial \xi_1} [B_1 B_1^*] B_2^* \right\rangle = \langle B_1 B_2^* \rangle \left\langle H \frac{\partial}{\partial \xi_1} |B_1|^2 \right\rangle,
\end{aligned}$$

so that the fourth-order correlators can be written as a sum of the products of pairs of second-order correlators.

This practice is familiar for the statistical description of progressive water waves [33] and of several other fields, for example plasma physics [34]. By a standard procedure as described in [1, 23] the resulting Alber's equation takes the form

$$\begin{aligned}
&\frac{\partial n}{\partial \tau} + 2\beta_1 k \frac{\partial n}{\partial \xi} + \beta_2 \left( \frac{1}{4} \frac{\partial^3 n}{\partial \xi^3} - 3k^2 \frac{\partial n}{\partial \xi} \right) = 4\mu_1 \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1} \langle |B|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m+1} n}{\partial k^{2m+1}} \\
&\quad + \mu_2 \left[ \sum_{m=0}^{\infty} b_m \left( \frac{\partial^{2m} \langle |B|^2 \rangle}{\partial \xi^{2m}} \frac{\partial^{2m+1} n}{\partial \xi \partial k^{2m}} + \frac{\partial^{2m+1} \langle |B|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m} n}{\partial k^{2m}} \right) \right. \\
&\quad \left. - 2 \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1} \langle |B|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m+1} (kn)}{\partial k^{2m+1}} \right] + 2\mu_3 \sum_{m=0}^{\infty} b_m \frac{\partial^{2m+1} \langle |B|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m} n}{\partial k^{2m}} \\
&\quad + \frac{2\mu_4}{\pi} \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1}}{\partial \xi^{2m+1}} \left( \int_{-\infty}^{+\infty} \frac{d\xi'}{\xi' - \xi} \left[ \frac{\partial}{\partial \xi'} \langle |B(\xi')|^2 \rangle \right] \right) \frac{\partial^{2m+1} n}{\partial k^{2m+1}},
\end{aligned} \tag{2.4}$$

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where

$$a_m = \frac{(-1)^m}{(2m+1)!\ 2^{2m+1}} \quad \text{and} \quad b_m = \frac{(-1)^m}{(2m)!\ 2^{2m}}.$$

## 3. Stability of random wave spectra

Alber [1] stated that a homogeneous spectrum is unstable subject to long-wavelength perturbations if the bandwidth is sufficiently small. So, to emphasize inhomogeneous ensemble wavetrains, let the distribution function $n(\xi, k, \tau)$ be written as sum of homogeneous envelope spectra and an infinitesimal perturbation given by

$$n(\xi, k, \tau) = n_0(k) + \epsilon n_1(\xi, k, \tau), \tag{3.1}$$

with $n_1(\xi, k, \tau) \ll n_0(k)$, and where $\epsilon$ is a slow ordering parameter.

In view of (2.3) and (3.1),

$$\langle |B(\xi, \tau)|^2 \rangle = \langle |B_0|^2 \rangle + \epsilon \langle |B_1(\xi, \tau)|^2 \rangle, \tag{3.2}$$

where

$$\langle |B_0|^2 \rangle = \int_{-\infty}^{+\infty} n_0(k)\, dk, \quad \langle |B_1(\xi, \tau)|^2 \rangle = \int_{-\infty}^{+\infty} n_1(\xi, k, \tau)\, dk. \tag{3.3}$$

Substituting equations (3.1) and (3.2) into equation (2.4) and linearizing, we get the following equation for the perturbation, that is,

$$\begin{aligned}
\frac{\partial n_1}{\partial \tau} + 2\beta_1 k \frac{\partial n_1}{\partial \xi} + \beta_2 \Big( \frac{1}{4} \frac{\partial^3 n_1}{\partial \xi^3} - 3k^2 \frac{\partial n_1}{\partial \xi} \Big) &= 4\mu_1 \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1} \langle |B_1|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m+1} n_0}{\partial k^{2m+1}} \\
&+ \mu_2 \Big[ \langle |B_0|^2 \rangle \frac{\partial n_1}{\partial \xi} + \sum_{m=0}^{\infty} b_m \frac{\partial^{2m+1} \langle |B_1|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m} n_0}{\partial k^{2m}} - 2 \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1} \langle |B_1|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m+1} (k n_0)}{\partial k^{2m+1}} \Big] \\
&+ 2\mu_3 \sum_{m=0}^{\infty} b_m \frac{\partial^{2m+1} \langle |B_1|^2 \rangle}{\partial \xi^{2m+1}} \frac{\partial^{2m} n_0}{\partial k^{2m}} \\
&+ \frac{2\mu_4}{\pi} \sum_{m=0}^{\infty} a_m \frac{\partial^{2m+1}}{\partial \xi^{2m+1}} \Big( \int_{-\infty}^{+\infty} \frac{d\xi'}{\xi' - \xi} \Big[ \frac{\partial}{\partial \xi'} \langle |B_1(\xi')|^2 \rangle \Big] \Big) \frac{\partial^{2m+1} n_0}{\partial k^{2m+1}}.
\end{aligned} \tag{3.4}$$

Considering the Fourier transform of (3.4) defined by

$$\overline{n}_1(k, \tau) = \int_{-\infty}^{+\infty} n_1(\xi, k, \tau) e^{-ip\xi}\, d\xi, \quad \langle |\overline{B}_1(\tau)|^2 \rangle = \int_{-\infty}^{+\infty} \langle |B_1(\xi, \tau)|^2 \rangle e^{-ip\xi}\, d\xi,$$

where $p$ is the wave number of perturbation, and taking $\tau$ dependence of $n_1(k, \tau)$ to be of the form $e^{-i\nu\tau}$, we get

$$[-\nu + 2\beta_1 kp - \beta_2 (p^2/4 + 3k^2) p)] \overline{n}_1 = 4\mu_1 \langle |\overline{B}_1|^2 \rangle \sum_{m=0}^{\infty} \frac{p^{2m+1}}{(2m+1)!\ 2^{2m+1}} \frac{\partial^{2m+1} n_0}{\partial k^{2m+1}}$$

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$$+\mu_2\Big[p\overline{n}_1\langle|B_0|^2\rangle+p\langle|\overline{B}_1|^2\rangle\sum_{m=0}^{\infty}\frac{p^{2m}}{(2m)!\ 2^{2m}}\frac{\partial^{2m}n_0}{\partial k^{2m}}$$

$$-2\langle|\overline{B}_1|^2\rangle\sum_{m=0}^{\infty}\frac{p^{2m+1}}{(2m+1)!\ 2^{2m+1}}\frac{\partial^{2m+1}(kn_0)}{\partial k^{2m+1}}\Big]+2\mu_3p\langle|\overline{B}_1|^2\rangle\sum_{m=0}^{\infty}\frac{p^{2m}}{(2m)!\ 2^{2m}}\frac{\partial^{2m}n_0}{\partial k^{2m}}$$

$$-4\mu_4|p|\langle|\overline{B}_1|^2\rangle\sum_{m=0}^{\infty}\frac{p^{2m+1}}{(2m+1)!\ 2^{2m+1}}\frac{\partial^{2m+1}n_0}{\partial k^{2m+1}}. \tag{3.5}$$

Taking into consideration the following relationships, which are obtained by using the Taylor's theorem of $n_0(k\pm p/2)$, that is,

$$2\sum_{m=0}^{\infty}\frac{p^{2m}}{(2m)!\ 2^{2m}}\frac{\partial^{2m}n_0}{\partial k^{2m}}=n_0(k+p/2)+n_0(k-p/2),$$

$$2\sum_{m=0}^{\infty}\frac{p^{2m+1}}{(2m+1)!\ 2^{2m+1}}\frac{\partial^{2m+1}n_0}{\partial k^{2m+1}}=n_0(k+p/2)-n_0(k-p/2),$$

$$4\sum_{m=0}^{\infty}\frac{p^{2m+1}}{(2m+1)!\ 2^{2m+1}}\frac{\partial^{2m+1}(kn_0)}{\partial k^{2m+1}}=2k[n_0(k+p/2)-n_0(k-p/2)]$$
$$+p[n_0(k+p/2)+n_0(k-p/2)], \tag{3.6}$$

equation (3.5) becomes

$$[-\nu+f(k)]\overline{n}_1=[g_+(k)n_0(k+p/2)+g_-(k)n_0(k-p/2)]\langle|\overline{B}_1|^2\rangle, \tag{3.7}$$

where

$$f(k)=2\beta_1kp-\beta_2(p^2/4+3k^2)p-\mu_2p\langle|B_0|^2\rangle,\quad g_\pm(k)=\pm2\mu_1\mp\mu_2k+\mu_3p\mp2\mu_4|p|.$$

The Fourier transform of the second relationship of (3.3) with respect to $\xi$ gives

$$\langle|\overline{B}_1(\tau)|^2\rangle=\int_{-\infty}^{+\infty}\overline{n}_1(k,\tau)\,dk. \tag{3.8}$$

Equations (3.7) and (3.8) reduce to

$$1+\int_{-\infty}^{+\infty}\frac{g_+(k)n_0(k+p/2)+g_-(k)n_0(k-p/2)}{\nu-f(k)}\,dk=0, \tag{3.9}$$

which represents the dispersion relationship for determining the perturbed frequency $\nu$ for the given homogeneous envelope spectrum $n_0(k)$. Set

$$f(k)=2\beta_1kp+\epsilon\tilde{f}(k),\quad g_\pm(k)=\pm2\mu_1+\epsilon\tilde{g}_\pm(k), \tag{3.10}$$

where

$$\tilde{f}(k)=-\beta_2(p^2/4+3k^2)p-\mu_2p\langle|B_0|^2\rangle,\quad \tilde{g}_\pm(k)=\mp\mu_2k+\mu_3p\mp2\mu_4|p|$$

are the fourth-order contributions of the NLSE (2.1).

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Inserting (3.10) in equation (3.9) and keeping terms up to $O(\epsilon)$, the nonlinear dispersion relationship reduces to

$$\Phi(\nu) = \epsilon\, \Psi(\nu), \tag{3.11}$$

with

$$\Phi(\nu) = 1 + 2\mu_1 \int_{-\infty}^{+\infty} \frac{n_0(k+p/2) - n_0(k-p/2)}{\nu - 2\beta_1 kp}\, dk,$$
$$\Psi(\nu) = 2\mu_1 \int_{-\infty}^{+\infty} \frac{[n_0(k-p/2) - n_0(k+p/2)]\tilde{f}(k)}{(\nu - 2\beta_1 kp)^2}\, dk - \int_{-\infty}^{+\infty} \frac{[\tilde{g}_+(k) n_0(k+p/2) + \tilde{g}_-(k) n_0(k-p/2)]}{\nu - 2\beta_1 kp}\, dk.$$

For studying the instability of the homogeneous envelope spectrum, we find that the spectrum for $\zeta(x,t)$ is well approximated by the JONSWAP spectrum [12, 16]

$$P(\Sigma) = \frac{\alpha g^2}{\Sigma^5} e^{-(5/4)(\Sigma/\Sigma_0)^{-4}} \gamma^{\exp[-(\Sigma-\Sigma_0)^2/2\delta^2\Sigma_0^2]},$$

where $\alpha$ is a Philips constant, $\gamma$ is a peak enhancement factor, $\delta$ is the spectral bandwidth, $\Sigma_0 = \Sigma(k_0)$ and, for the present study, the frequency $\Sigma$ satisfies the linear dispersion relationship (2.2). For a narrowband approximation of the JONSWAP spectrum, $(\Sigma - \Sigma_0)/\Sigma_0 = r \ll 1$, and this can be obtained by a second-order Taylor series expansion of $P(r)$ around $r = 0$: that is,

$$P(r) \simeq P(0) + rP^{'}(0) + \frac{r^2}{2} P^{''}(0) \simeq P_0 \Big[1 - \frac{P_0^{''}}{2P_0} r^2\Big]^{-1} \quad \text{if } r \ll 1. \tag{3.12}$$

Since $P^{'}(0) = 0$, it is necessary to retain the term up to the second order of $r$ for considering the approximation of $P(r)$ (see [15]). Equation (3.12) reduces to the Lorentzian spectrum in wavenumber space given by

$$P(k) = \frac{H_s^2}{16\pi} \frac{\sigma}{\sigma^2 + (k-k_0)^2},$$

where

$$H_s = 4\sqrt{\pi \frac{\alpha g^2 \gamma \sigma}{e^{5/4}\Sigma_0^5}} \quad \text{and} \quad \sigma = \sqrt{\frac{2\delta^2}{20\delta^2 + \ln\gamma}} \Big(\frac{\Sigma_0}{\nu + g/\sqrt{\Omega^2 + 4gk_0}}\Big).$$

For a symmetric spectrum $P(k)$ of the free surface elevation, the spectrum for a complex envelope $B$ is given by $n_0(k) = 4P(k+k_0)$ (see [4, 23]).

Neglecting $O(\epsilon)$ terms of equation (3.11) and using the expression for $n_0(k)$ given by (3.6), the reduced equation $\Phi(\nu) = 0$ becomes

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$$1+\frac{\mu_1\sigma {H_s}^2}{2\pi}\left[\int_{-\infty}^{+\infty}\frac{dp}{(\nu+2\kappa kp)\{\sigma^2+(k+p/2)^2)\}}\right.$$
$$\left.-\int_{-\infty}^{+\infty}\frac{dp}{(\nu+2\kappa kp)\{\sigma^2+(k-p/2)^2)\}}\right]=0. \quad (3.13)$$

Now, setting $k\pm p/2=z$ in (3.13) and integrating the resulting integral by residues at $z=i\sigma$, we have the nonlinear dispersion relationship

$$\nu=\Big(-2i\kappa\sigma+\sqrt{\kappa^2p^2-H_s^2\mu_1\kappa}\Big)p, \quad (3.14)$$

where $\kappa=-\beta_1$. The GRI $\nu_i$, expressed by the imaginary part of $\nu$ of the perturbation corresponding to the positive complex root of equation (3.14), is the following, and can be obtained when $p^2<H_s^2\mu_1/\kappa$: that is,

$$\nu_i=\Big(\sqrt{H_s^2\mu_1\kappa-\kappa^2p^2}-2\kappa\sigma\Big)p. \quad (3.15)$$

Note that, for $\sigma\to 0$, equation (3.15) represents the BFI.

It is important to mention that the last term of $\nu_i$ in equation (3.15) has a stabilizing (defocusing) effect and plays the same role as the Landau damping in plasma physics [17], that is, a damping of the perturbation. There is a contest between exponential growth and damping of the perturbation which depends on the two parameters, $\alpha$ and $\gamma$, of the Lorentzian spectrum. For $\sigma>(1/2)\sqrt{H_s^2\mu_1/\kappa-p^2}$, the damping dominates the modulational instability, and the reverse effect to this will occur if $\sigma<(1/2)\sqrt{H_s^2\mu_1\kappa-p^2}$.

Now, we take $\nu=i\nu_i+\epsilon\nu_1$ as the root of equation (3.11). Putting this value of $\nu$ into equation (3.11), we readily obtain $\nu_1$ in the lowest order, that is,

$$\nu_1=\frac{\Psi(i\nu_i)}{\Phi'(i\nu_i)},$$

where

$$\Phi'(i\nu_i)=-2\mu_1(J_1-J_2),\quad \Psi(i\nu_i)=2\mu_1(J_3-J_4)-(J_5+J_6),$$

and the expressions for the integrals $J_n(n=1,\dots,6)$ are given in Appendix B. Following the same procedure, we evaluate the integrals to obtain the expression of the GRI $\Gamma_i$ corresponding to the fourth-order results,

$$\Gamma_i=\nu_i+\Im(\nu_1)=\nu_i-\frac{\mu_4|p|}{2\mu_1}\left[\frac{(\nu_i+2\kappa p\sigma)^2+\kappa^2p^4}{(\nu_i+2\kappa p\sigma)}\right]. \quad (3.16)$$

Herein, the last term in the big brackets of (3.16) is obtained from the fourth-order nonlinear term of the right-hand side of equation (2.1) involving the Hilbert transform, and $\nu_i$ is given by (3.15).

Figures 1 and 2 show the marginal stability curves in the $(\gamma,\alpha)$ plane for $p\to 0$ and $p=0.5$, respectively. Herein, we wish to study the modulational instability of surface gravity waves with random phase spectra. Therefore, in Figures 1 and 2,

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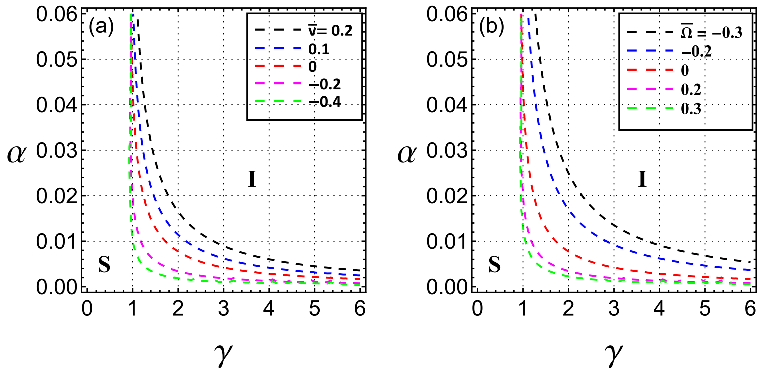


FIGURE 1. Instability region in the $(\gamma, \alpha)$ plane for $p \to 0$: (a) $\overline{\Omega} = 0$; and (b) $\overline{v} = 0$. **I** and **S** indicate the instability and stability regions, respectively.

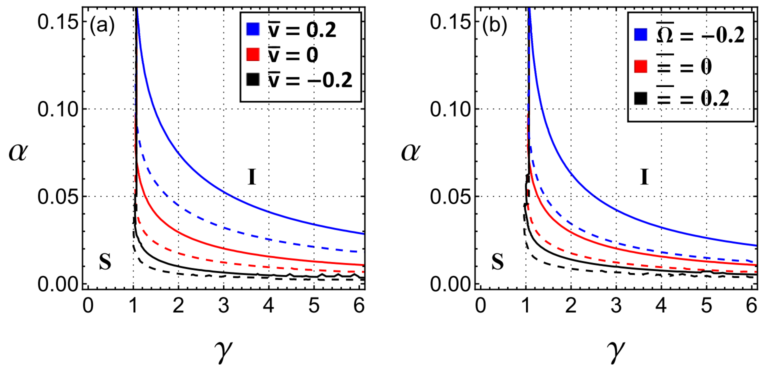


FIGURE 2. Instability diagram in the $(\gamma, \alpha)$ plane for $p = 0.5$: (a) $\overline{\Omega} = 0$; and (b) $\overline{v} = 0$. Dashed line: third-order results; solid line: fourth-order results. **I** and **S** indicate the instability and stability regions, respectively.

*I* indicates the modulational instability region. One can observe from these figures that spectra with larger values of $\gamma$ and $\alpha$ are more likely to show the modulational instability (see [23]). The depth uniform reverse currents can expand the instability region, whereas following currents have the opposite effect. Again, positive vorticity ($\overline{\Omega} < 0$) reduces the instability region, whereas negative vorticity ($\overline{\Omega} > 0$) increases the instability region. Figure 2 suggests that fourth-order results reduce the instability region compared with third-order results for fixed values of $\overline{\Omega}$ and $\overline{v}$. As a check, the modulational instability diagram that we obtain is compared in Figure 1 for $\overline{\Omega} = 0$, $\overline{v} = 0$ with that found by Onorato et al. [23]. Thus, we can verify that this diagram reproduced exactly. It is to be noted that, for the purpose of finding the effects of modulational instability, we have taken values in the $(\gamma, \alpha)$ plane that lie far away from the marginal stability curve (see Figure 1).

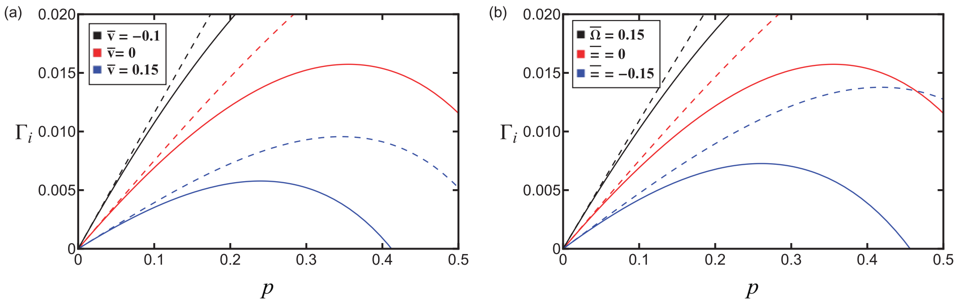


FIGURE 3. Plot of $\Gamma_i$ versus $p$ for $\gamma = 3$ and $\alpha = 0.03$: (a) $\overline{\Omega} = 0$; and (b) $\overline{\nu} = 0$. Dashed line: third-order results; solid line: fourth-order results.

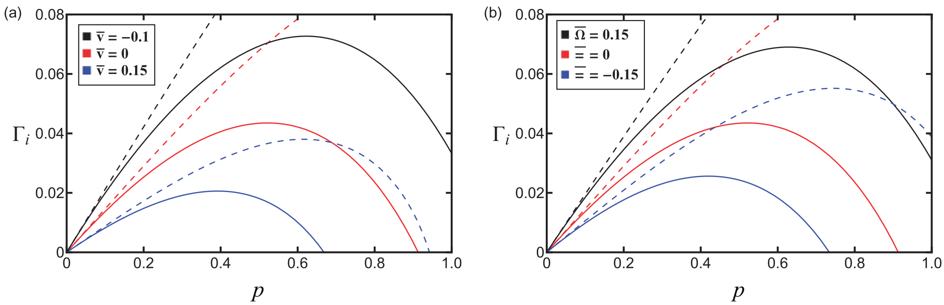


FIGURE 4. Plot of $\Gamma_i$ versus $p$ for $\gamma = 5$ and $\alpha = 0.05$: (a) $\overline{\Omega} = 0$; and (b) $\overline{\nu} = 0$. Dashed line: third-order results; solid line: fourth-order results.

In Figures 3 and 4 we have drawn the GRI as a function of $p$, the wave number of perturbation, for different values of $\overline{\Omega}$ and $\overline{\nu}$. These figures suggest that $\Gamma_i$ increases when both $\alpha$ and $\gamma$ increase. These figures also suggest that the fourth-order results produce a diminished GRI. The depth uniform opposing currents considerably increase the growth rate, whereas following currents diminish the modulational instability. Further, the effect of negative vorticity ($\overline{\Omega} > 0$) is to enhance the growth rate, whereas for $\overline{\Omega} < 0$ we observe a decrease in the growth rate.

## 4. The limit of vanishing bandwidth

For vanishing spectral bandwidth $\sigma \to 0$ and, in this case, we can rediscover the BFI for deterministic wavetrains. For $\sigma \to 0$, equation (3.16) becomes

$$\Gamma_i = \nu_i - \frac{\kappa\mu_4}{2\nu_i} H_s^2 p^2 |p| \quad \text{with} \quad \nu_i = \left(\sqrt{H_s^2 \mu_1 \kappa - \kappa^2 p^2}\right) p. \tag{4.1}$$

There is a relationship between the wave amplitude $\overline{a}$ and the wave significant height $H_s$ given by $\overline{a} = H_s/2$. Employing this relationship in equation (4.1) and replacing

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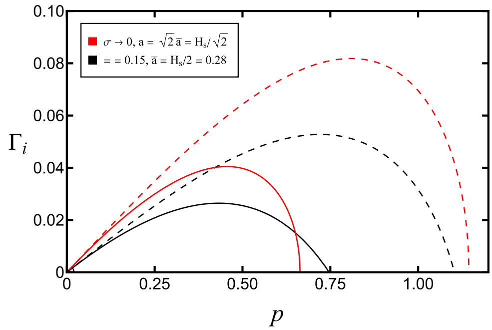


FIGURE 5. Plot of $\Gamma_i$ versus $p$ for $\overline{v} = 0.4$, $\overline{\Omega} = 0.5$, $\gamma = 3$ and $\alpha = 0.03$. Comparison with the deterministic growth rate. Dashed line: third-order results; solid line: fourth-order results.

$2\overline{a}^2$ by $a^2$, where $\overline{a}$ and $a$ represent, respectively, the amplitudes of the random and deterministic wavetrains, we get the deterministic growth rate given by

$$\Gamma_i = [2\kappa a^2(\mu_1 - \mu_4|p|) - \kappa^2 p^2]^{1/2} p. \tag{4.2}$$

For $\overline{\Omega} = 0$ and $\overline{v} = 0$, this expression for $\Gamma_i$ is the same as that of (3.13) of Dysthe [7] for the one-dimensional case. In Figure 5, the growth rate $\Gamma_i$ given by (3.16) is compared with the corresponding deterministic growth rate given by (4.2). This figure suggests that the instability growth rate diminishes due to the effect of randomness, which is consistent with the preceding results of Alber [1] and Halder and Dhar [8, 9]. Further, it is observed that the growth rate obtained from fourth-order results is reduced significantly compared with that obtained from third-order results, which is in agreement with the results of Halder and Dhar [9]. The physical significance related to fourth-order results can be stated as follows. Dysthe [7] stated that a significant improvement in the stability properties can be achieved by considering the effects of fourth-order perturbation. The dominant new effect introduced to the fourth order is the mean flow response to nonuniformities in the radiation stress caused by modulation of a finite amplitude wave.

## 5. The BFI

The concept of the BFI in the context of freak waves has been presented for random waves by Janssen [14] (see also [24]). For the definition of BFI, he suggested the ratio of the mean square slope to the normalized width of the frequency spectrum. When this parameter is greater than one, the random wave field is unstable, whereas in the opposite case it is stable. From the NLSE, Onorato et al. [25] also defined the BFI in the context of freak waves.

It is noteworthy that, among the new fourth-order terms of the NLSE (2.1), only the last term involving Hilbert transform contributes to the instability results given by equations (3.16) and (4.1). Therefore, as far as stability properties are considered, it is

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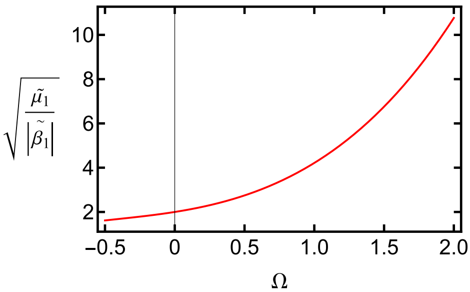


FIGURE 6. Effect of vorticity on the BFI.

enough to use the simplified form of the dimensional NLSE (2.1) on deep water [7] given by

$$i\frac{\partial B}{\partial \tau} + \frac{\Sigma}{k^2}\tilde{\beta}_1\frac{\partial^2 B}{\partial \xi^2} = \Sigma k^2\tilde{\mu}_1|B|^2B + \Sigma k\tilde{\mu}_4 BH\Big[\frac{\partial}{\partial \xi}|B|^2\Big], \tag{5.1}$$

where $B$ is the complex wave envelope and $(\Sigma, k)$ represent, respectively, the carrier frequency and wave number. Here the coefficients appearing in equation (5.1) depend on the parameters $\overline{\nu}$ and $\overline{\Omega}$. Following Onorato et al. [25], we rewrite equation (5.1) by transforming $B^{'} = B/a,\ \xi^{'} = \Delta k\xi,\ \tau^{'} = \Sigma\tilde{\beta}_1(\Delta k/k)^2\tau$ given by

$$i\frac{\partial B}{\partial \tau} + \frac{\partial^2 B}{\partial \xi^2} = \Big(\frac{ak}{\Delta k/k}\Big)^2\Big(\frac{\tilde{\mu}_1}{\tilde{\beta}_1}\Big)|B|^2B + \frac{(ak)^2}{\Delta k/k}\Big(\frac{\tilde{\mu}_4}{\tilde{\beta}_1}\Big)BH\Big[\frac{\partial}{\partial \xi}|B|^2\Big], \tag{5.2}$$

where primes of $B$, $\xi$ and $\tau$ have been omitted and $\Delta k, a$ denote typical bandwidth and wave amplitude, respectively. According to Onorato et al. [25], we define the BFI as the square root of the coefficient that multiplies the cubic nonlinear term of equation (5.2). Therefore,

$$\text{BFI} = \frac{ak}{\Delta k/k}\sqrt{\frac{\tilde{\mu}_1}{|\tilde{\beta}_1|}}. \tag{5.3}$$

The term $\sqrt{\tilde{\mu}_1/|\tilde{\beta}_1|}$ on the right-hand side of the BFI given by (5.3) depends on the parameter $\overline{\Omega}$, the magnitude of the shear. The effect of this term on the BFI against $\overline{\Omega}$ is presented in Figure 6. We observe that, for $\overline{\Omega} > 0$, the BFI increases with an increase of vorticity. Again, as the BFI increases, the nonlinearity also increases. So, we may hope that the number of freak waves is enhanced in the presence of shear currents co-flowing with the waves. It is to be mentioned that this result was first found numerically with the help of numerical simulations of the cubic NLSE in the paper by Onorato et al. [26]. Again, for $\overline{\Omega} < 0$, the presence of vorticity diminishes the BFI.

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## 6. Discussion and conclusions

Dysthe [7] first stated that a fourth-order NLSE is a good starting point for analysing the stability of surface periodic waves on deep water. Therefore, starting from fourth-order NLSE for surface gravity waves in the presence of linear shear currents, in this paper, we first derive Alber's equation, and, based on this equation, we then describe the instability of weakly nonlinear waves with random phase spectra. The key outcomes of our study are as follows. The present fourth-order analysis exhibits significant deviations in the instability growth rate compared with the third-order analysis. Also, we find the effect of linear shear currents on the modulational instability properties of weakly nonlinear waves. It is observed that the fourth-order results produce a decrease in the GRI. Further, the effect of randomness is to decrease the GRI, which is consistent with the previous results [1]. For waves on shearing currents, negative vorticity tends to enhance the modulational instability, whereas positive vorticity decreases the instability. Further, for waves moving in the same direction as the uniform current, the current was observed to have a stabilizing effect on the wavetrains and diminishes the instability growth rate. We also recovered the deterministic GRI for vanishing spectral bandwidth.

## Appendix A. The coefficients of equation (2.1)

$$\beta_1 = \frac{\Sigma}{k^2}\tilde{\beta}_1 = -\frac{\Sigma}{k^2}\frac{(\overline{c}_g - \overline{v})^2}{(2 - 2\overline{v} + \overline{\Omega})},$$

$$\beta_2 = \frac{\Sigma}{k^3}\tilde{\beta}_2 = \frac{\Sigma}{k^3}\frac{2(\overline{c}_g - \overline{v})^3}{(2 - 2\overline{v} + \overline{\Omega})^2},$$

$$\overline{c}_g = \frac{c_g}{c} = \frac{1 + \overline{\Omega} - \overline{v}^2}{(2 - 2\overline{v} + \overline{\Omega})}, \qquad \overline{\Omega} = \frac{\Omega}{\Sigma}, \qquad c = \frac{\Sigma}{k},$$

$$\mu_1 = \Sigma k^2 \tilde{\mu}_1 = \frac{\Sigma k^2 (M + LN)}{8(1 - \overline{v})^2 (2 - 2\overline{v} + \overline{\Omega})(1 - \overline{v} + \overline{\Omega})},$$

$$\mu_2 = \Sigma k \tilde{\mu}_2 = \frac{\Sigma k\{\delta_4 - \overline{c}_g \delta_2 + 4\overline{c}_g \delta_1 (1 - \overline{v})/(2 - 2\overline{v} + \overline{\Omega})\}}{4(2 - 2\overline{v} + \overline{\Omega})},$$

$$\mu_3 = \Sigma k \tilde{\mu}_3 = \frac{\Sigma k\{\delta_5 - \overline{c}_g \delta_3 + 2\overline{c}_g \delta_1 (1 - \overline{v})/(2 - 2\overline{v} + \overline{\Omega})\}}{4(2 - 2\overline{v} + \overline{\Omega})},$$

$$\mu_4 = \Sigma k \tilde{\mu}_4 = \frac{\Sigma k (2 - 2\overline{v} + \overline{\Omega})}{4\{1 - (\overline{c}_g - \overline{v})\overline{\Omega}\}},$$

where

$$M = 8(1 - \overline{v})^5 + 24(1 - \overline{v})^4\overline{\Omega} + 34(1 - \overline{v})^3\overline{\Omega}^2 + 26(1 - \overline{v})^2\overline{\Omega}^3 + 9(1 - \overline{v})\overline{\Omega}^4 + \overline{\Omega}^5,$$

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$$L = (2 - 2\overline{v} + \overline{\Omega})\overline{\Omega}^2, \quad N = \frac{2(1-\overline{v})^2(1-\overline{v}+\overline{\Omega})(2-2\overline{v}+\overline{\Omega})}{\overline{\Omega}(\overline{c}_g - \overline{v}) - (1-\overline{v})(1-\overline{v}+\overline{\Omega})},$$

$$\delta_1 = \frac{2(1-\overline{v})^2 - \overline{\Omega}^2 + A_1^2/2(1-\overline{v})^2 - \overline{\Omega}^2(2-2\overline{v}+\overline{\Omega})^2}{1-(\overline{c}_g - \overline{v})\overline{\Omega}},$$

$$\delta_2 = \frac{A_1}{(1-\overline{v})^2}\left[4(1-\overline{v}+\overline{\Omega}) - \frac{\{4(1-\overline{v})+\overline{\Omega}\}A_1}{2(1-\overline{v})^2}\right] - \frac{3\overline{\Omega}(1-\overline{v}-\overline{\Omega})(2-2\overline{v}+\overline{\Omega})}{1-(\overline{c}_g-\overline{v})\overline{\Omega}},$$

$$\delta_3 = \frac{(1-\overline{v}+\overline{\Omega})(2-2\overline{v}+\overline{\Omega})\overline{\Omega}}{1-(\overline{c}_g-\overline{v})\overline{\Omega}},$$

$$\delta_4 = -\frac{A_1^2}{(1-\overline{v})^2}\left[1+\frac{1}{2(1-\overline{v})^2}\right] - 4(1-\overline{v})^2 + (1-\overline{v})\overline{\Omega} + 3\overline{\Omega}^2 + A_2,$$

$$\delta_5 = \overline{\Omega}(1-\overline{v}+\overline{\Omega}) - \frac{A_1^2}{2(1-\overline{v})^2} + A_2,$$

with

$$A_1 = 2(1-\overline{v})^2 + 4(1-\overline{v})\overline{\Omega} + \overline{\Omega}^2,$$

$$A_2 = \frac{(2-2\overline{v}+\overline{\Omega})}{1-(\overline{c}_g-\overline{v})\overline{\Omega}}\left[\frac{(1-\overline{v})\overline{c}_g\overline{\Omega}}{1-(\overline{c}_g-\overline{v})\overline{\Omega}} - (1-\overline{v}) - \overline{\Omega}^2(2-2\overline{v}+\overline{\Omega})\right].$$

## Appendix B. Expressions for $J_n$ ($n = 1, \dots, 6$)

$$J_1 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{dz}{(z^2+\sigma^2)(i\nu_i - \kappa p^2 + 2\kappa p z)^2},$$

$$J_2 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{dz}{(z^2+\sigma^2)(i\nu_i + \kappa p^2 + 2\kappa p z)^2},$$

$$J_3 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{(b_1 - b_2 z + b_3 z^2)\,dz}{(z^2+\sigma^2)(i\nu_i - \kappa p^2 + 2\kappa p z)^2},$$

$$J_4 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{(b_1 + b_2 z + b_3 z^2)\,dz}{(z^2+\sigma^2)(i\nu_i + \kappa p^2 + 2\kappa p z)^2},$$

$$J_5 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{(c_2 - \mu_2 z)\,dz}{(z^2+\sigma^2)(i\nu_i - \kappa p^2 + 2\kappa p z)},$$

$$J_6 = \frac{H_s^2\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{(c_1 + \mu_2 z)\,dz}{(z^2+\sigma^2)(i\nu_i + \kappa p^2 + 2\kappa p z)},$$

where

$$b_1 = \beta_2 p^3 + \mu_2 p \langle |B_0|^2 \rangle, \quad b_2 = 3\beta_2 p^2, \quad b_3 = 3\beta_2 p,$$
$$c_1 = \frac{\mu_2}{2} p + \mu_3 p + 2\mu |p|, \quad c_2 = \frac{\mu_2}{2} p + \mu_3 p - 2\mu |p|.$$

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