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ON 2-GENERATOR 2-RELATION SOLUBLE GROUPS

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The class of non-metacyclic finite soluble groups known to have 2-generator 2-relation presentations is small. Classes of such groups are given in (**3**), (**4**), (**8**) and (**9**). Some subclasses of the groups discussed in (**1**) and (**2**) also provide examples, while a class of finite nilpotent 2-generator 2-relation groups is given by Macdonald in (**7**).

Any finite deficiency zero group G must have G/G' at most 3-generated (**6**). The question arises as to whether there is a bound on the rank of other factors of the derived series. For the groups given in (**4**), (**8**), and (**9**) these factors are at most 2-generated (**5**). We shall show that examples with G'/G'' of unbounded rank can be found in the class investigated in (**2**), that is the class

$$F^{a,b,c} = \langle x, y \mid x^2 = 1, xy^a xy^b xy^c = 1\rangle.$$

For this purpose it is convenient to define the class

$$F_\lambda^{a,b,c} = \langle x, y \mid x^2 = y^{\lambda(a+b+c)} = xy^a xy^b xy^c\rangle,$$

noting that $F_0^{a,b,c} = F^{a,b,c}$, while $F_2^{a,b,c}$ is studied in (**3**).

We first prove some general results concerning $F_\lambda^{a,b,c}$. Introduce the notation $n = a+b+c$ and $d = (a-b, b-c, \lambda n)$. The group $F_\lambda^{a,b,c}$ has an infinite abelian factor group if $n = 0$ or $\lambda = -2$ and, since $F_\lambda^{a,b,c}$ is isomorphic to $F_\lambda^{-a,-b,-c}$, we assume that $n > 0$ and $\lambda \neq -2$. Once the case $(a, b, c) = 1$ is solved, arguments similar to those in (**3**) enable the case $(a, b, c) \neq 1$ to be dealt with, so from now on assume $(a, b, c) = 1$.

For any non-zero integer r let C_r denote the cyclic group of order $|r|$. Putting $G = F_\lambda^{a,b,c}$ we have $G/G' \cong C_{(\lambda+2)n}$. Define $K_\lambda^{a,b,c}$ to be G'/G''.

Lemma 1. *G'' contains $\langle y^{(\lambda+2)n}\rangle^G$, the normal closure of $y^{(\lambda+2)n}$ in $F_\lambda^{a,b,c}$.*

Proof. In $F_\lambda^{a,b,c}$ we have

$$x^{-1}y^{-a}x = y^b xy^c \tag{1}$$

and, since x^2 is central, two additional relations obtained from (1) by permuting a, b, c cyclically. Now xy^n and $xy^{n-a}xy^a x^{-1}$ belong to G' so G'' contains

$$\begin{aligned}[xy^{n-a}xy^a x^{-1}, xy^n] &= xy^{-a}x^{-1}y^{-n+a}(x^{-1}y^{-a}x)y^{a+n}\\ &= xy^{-a}(x^{-1}y^{-c}x)y^{2n-b}\\ &= x^2y^{2n}\\ &= y^{(\lambda+2)n}.\end{aligned}$$

The Reidemeister-Schreier algorithm may be used to find a presentation for the derived group of $F_\lambda^{a,b,c}/\langle y^{(\lambda+2)n}\rangle^G$ and this derived group shown to be abelian using a technique similar to that in Section 3 of (**2**). This proves the first part of the next theorem.

Theorem 2.

(i) $F_\lambda^{a,b,c}/\langle y^{(\lambda+2)n}\rangle^G$ *is metabelian.*

(ii) $\langle y^{(\lambda+2)n}\rangle^G = G''$.

(iii) *If* λ *is even,* $K_\lambda^{a,b,c} \cong K_0^{a,b,c}$.

(iv) *If* λ *is odd,* $K_\lambda^{a,b,c} \cong K_{-1}^{a,b,c}$ *and* $K_{-1}^{a,b,c}$ *is the maximal elementary abelian* 2-*factor of* $K_0^{a,b,c}$.

Proof. (ii) This is a consequence of (i) and Lemma 1.

(iii) and (iv). By (ii) $K_\lambda^{a,b,c}$ is the derived group of $F_\lambda^{a,b,c}/\langle y^{(\lambda+2)n}\rangle^G$. The Reidemeister-Schreier method referred to above then shows that $K_\lambda^{a,b,c}$ depends on λ only to the extent of its parity. For, putting $x_i = y^{i-1}xy^{n-i+1}$, $1 \leqq i \leqq n$, we obtain the presentation

$$\langle x_1, x_2, \ldots, x_n \mid x_{i+a}^{\varepsilon_{i+a}} = x_i^{\varepsilon_i} x_{i+a+b}^{\varepsilon_{i+a+b}}, 1 \leqq i \leqq 2n\rangle$$

for $K_\lambda^{a,b,c}$ when λ is even, and

$$\langle x_1, x_2, \ldots, x_n \mid x_{i+a} = x_i x_{i+a+b}, x_i^2 = 1, 1 \leqq i \leqq n\rangle$$

when λ is odd. In these two presentations the subscripts on the x's are reduced modulo n and ε_i is $+1$ or -1 according to whether i reduced modulo $2n$ lies in the range 1 to n or $n+1$ to $2n$ respectively.

To find the rank of $K_{-1}^{a,b,c}$ associate with $F_{-1}^{a,b,c}$ the $n\times n$ matrix $A = (a_{ij})$ over the field $\boldsymbol{F}_2$ defined by

$$a_{ij} = \delta_{i,j} + \delta_{i+a,j} + \delta_{i+a+b,j}$$

where

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i \equiv j \pmod{n}, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 3. *The elementary abelian* 2-*group* $K_{-1}^{a,b,c}$ *has rank* $n-t$ *where* t *is the rank of the matrix* A.

Proof. This is immediate from the presentation for $K_\lambda^{a,b,c}$, λ odd, given in the proof of Theorem 2.

We can now show that the rank of $K_{-1}^{a,b,c}$ is unbounded.

Theorem 4. *Given any* $k \geqq 2$ *there exists a positive integer* $m \leqq 2^k - k - 1$ *such that* $K_{-1}^{1,k-1,m}$ *has rank* k.

Proof. Let $\boldsymbol{F}_2^k$ be the vector space of k-tuples over $\boldsymbol{F}_2$ with basis $\{e_i : 1 \leqq i \leqq k\}$ where, as usual, e_i denotes the k-tuple whose only non-zero entry is in the i-th position. Define $\theta : \boldsymbol{F}_2^k \to \boldsymbol{F}_2^k$ by

$$e_i\theta = e_{i+1}, 1 \leqq i \leqq k-1;\; e_k\theta = e_1 + e_2.$$

Clearly θ is an isomorphism so θ belongs to $GL(k, 2)$ and has order $|\theta|$, say, where $|\theta| \leqq 2^k - 1$.

Consider k to be fixed. We shall find necessary and sufficient conditions on the positive integer m so that the $(k+m)\times(k+m)$ matrix A associated with $F_{-1}^{1, k-1, m}$ has rank m. The rank of A is the same as the rank of the circulant matrix B over $\mathbf{F}_2$ with first row

$$(\underbrace{1\ \ 0\ \ 0\ \ \ldots\ \ 0}_{m}\ \ \underbrace{1\ \ 1\ \ 0\ \ 0\ \ \ldots\ \ 0}_{k}).$$

Reduce B to upper triangular form using elementary row operations. It is straightforward to check that after this reduction the final k-tuple of entries in row i is $(e_1 + e_2)\theta^{i-1}$ for $1 \leqq i \leqq m$. The last k rows of the reduced matrix consist entirely of zeros if, and only if, $(e_1 + e_2)\theta^m = e_1$. But $e_1\theta^k = e_1 + e_2$ so the conditions on m are that $m = t - k$ where t is divisible by $|\theta|$. In particular we can choose $m = |\theta| - k$ and the result follows.

Corollary 5. *$K_0^{a, b, c}$ has unbounded rank.*

Proof. This follows from Theorem 2(iv) and Theorem 4.

To show that the class $F_\lambda^{a, b, c}$ contains examples of finite soluble groups with derived groups of unbounded rank it remains to study G''.

Lemma 6. *In $F_\lambda^{a, b, c}$ y^{2n} commutes with $x^{-1}y^dx$.*

Proof. From (1) $x^{-1}y^ax = y^{-c}x^{-1}y^{-b}$ and $x^{-1}y^bx = y^{-a}x^{-1}y^{-c}$ so

$$x^{-1}y^{a-b}x = y^{-c}x^{-1}y^{c-b}xy^a. \tag{2}$$

By symmetry,

$$x^{-1}y^{b-c}x = y^{-a}x^{-1}y^{a-c}xy^b \tag{3}$$

and

$$x^{-1}y^{c-a}x = y^{-b}x^{-1}y^{b-a}xy^c. \tag{4}$$

Now from (2), (3) and (4)

$$x^{-1}y^{a-b}x = y^{-c}(y^{-b}x^{-1}y^{c-a}xy^a)y^a = y^{-c-b}(y^{-b}x^{-1}y^{b-a}xy^c)y^{2a}.$$

Hence

$$\begin{aligned} x^{-1}y^{a-b}x &= y^{-c-2b}(x^{-1}y^{b-a}x)y^{c+2a} \\ &= y^{-c-2b}(y^{-2a-c}x^{-1}y^{a-b}xy^{c+2b})y^{c+2a} \end{aligned}$$

and so $[y^{2n}, x^{-1}y^{a-b}x] = 1$. By symmetry $[y^{2n}, x^{-1}y^{b-c}x] = 1$ and, since $y^{\lambda n}$ is central, $[y^{2n}, x^{-1}y^{\lambda n}x] = 1$. Therefore, since $d = (a-b, b-c, \lambda n)$, $[y^{2n}, x^{-1}y^dx] = 1$.

Corollary 7. *In $G = F_\lambda^{a, b, c}$, if either $d = 1$ or $\lambda = \pm 1$ then G'' is a central cyclic subgroup.*

Proof. If $d = 1$ then $[y^{2n}, x^{-1}yx] = 1$ so $[y^{2n}, x^{-1}y^ax] = 1$. Therefore, using (1), $[y^{2n}, x] = 1$ and y^{2n} is central. By Theorem 2(ii) $G'' = \langle y^{(\lambda+2)n}\rangle^G$ so G'' is a central cyclic subgroup as required. If $\lambda = \pm 1$ then y^n is central giving the same result in this case.

Notice that when $\lambda = \pm 1$ the only possible values of d are 1 and 3. For, since $(a, b, c) = 1$,
$d = (a-b, b-c, a+b+c) = (a-b, b-c, 3a) = (a-b, b-c, 3a, 3b, 3c) = (a-b, b-c, 3)$.

Corollary 8. *$F_\lambda^{a,b,c}$ is a finite soluble group of derived length $\leqq 3$ if either $d = 1$ or $\lambda = \pm 1$.*

Proof. In both cases the centre of $F_\lambda^{a,b,c}$ has finite index and the commutator quotient of $F_\lambda^{a,b,c}$ is finite. Hence the result follows by a well-known theorem due to Schur.

The order of G'' in the cases $d = 1$ and $\lambda = \pm 1$ may be calculated using the Reidemeister-Schreier method. We omit the details of this argument which, now that we have Corollary 7, is similar to that of Theorem 3.3 and Lemma 4.1 of (**3**). The method shows that G'' is trivial when $d = 1$ and is C_2 when $d = 3$.

The case $d = 1$ verifies part of the conjecture given in §12 of (**2**).

The following example shows that the rank of the third factor of the derived series of a finite soluble 2-generator 2-relation group can also exceed 3. Let H be defined by

$$\langle x, y \mid x^3 = 1, yxy^3xy^{-3}x = 1 \rangle.$$

Then $H/H' \cong C_3$ and the Reidemeister-Schreier method shows that

$$H' = \langle x, y, z \mid xy^3 = z^3, yz^3 = x^3, zx^3 = y^3 \rangle$$

so that $H'/H'' \cong C_2 \times C_{14}$. A machine implementation of the Todd-Coxeter coset enumeration algorithm now shows that H'' is the direct product of seven copies of C_2.

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