The Journal of Symbolic Logic
Volume 83, Number 2, June 2018

ABSTRACT INDUCTIVE AND CO-INDUCTIVE DEFINITIONS

GIOVANNI CURI

Abstract. In [G. Curi, *On Tarski's fixed point theorem*. Proc. Amer. Math. Soc., 143 (2015), pp. 4439–4455], a notion of *abstract inductive definition* is formulated to extend Aczel's theory of inductive definitions to the setting of complete lattices. In this article, after discussing a further extension of the theory to structures of much larger size than complete lattices, as the class of all sets or the class of ordinals, a similar generalization is carried out for the theory of *co-inductive definitions* on a set. As a corollary, a constructive version of the general form of Tarski's fixed point theorem is derived.

Introduction. In its best known and most used form, Tarski's fixed point theorem states that every monotone function $f : L \to L$ on a complete lattice L has a least fixed point p. Albeit intuitionistically valid, Tarski's proof of this result is circular or *impredicative*: the fixed point is indeed defined as $p = \bigwedge\{x \in L \mid f(x) \leq x\}$, and since $f(p) \leq p$, p occurs in its own definition. The circularity in the definition of p may be taken to account for the fact that, despite its intuitionistic validity, Tarski's proof has often been perceived as nonconstructive, even outside the constructivists' community (e.g., [7]).

An alternative standard construction is sometimes considered more satisfactory. It consists in finding p via transfinite iterations of f, starting from the bottom element in the lattice. This alternative approach involves however the application of highly nonconstructive principles, and so, although in a way more explicit than Tarski's approach, it does not provide an effective method for finding the fixed point.

In [1], in the context of classical set theory, P. Aczel described the natural relationship between monotone set-operators on power-set lattices and inductive definitions. In particular, he showed that a monotone function f on $\mathsf{Pow}(X)$, X a set, can be *generated* by an inductive definition Φ, and that the set $I(\Phi)$ inductively defined by Φ is the least fixed point of f. In [1] it is also shown that if Φ is bounded by a regular cardinal k, k iterations of f will suffice to reach the fixed point (see Section 2 below). This result refines the above mentioned construction of p via transfinite iterations of f, albeit still in a classical (noneffective) setting.

The constructive analysis of set-theoretic inductive definitions came in [2]. It had to be developed in a system for set theory in which neither impredicative nor nonintuitionistic principles were available. This system, described later on, is

Received December 22, 2016.
2010 *Mathematics Subject Classification.* Primary 03G10, 03E70, Secondary 03F65, 18B35.
Key words and phrases. inductive definitions, co-inductive definitions, complete lattices, Tarski's fixed point theorem.
Dedicated to Peter Aczel, on the occasion of his 75th birthday.

0022-4812/18/8302-0010
DOI:10.1017/jsl.2018.13

the so-called Constructive Zermelo–Fraenkel set theory, CZF [5]. CZF is a weak intuitionistic subsystem of classical ZF, so its theorems are also theorems of ZF.

In this context, Aczel formulated a constructive analog of the operation of taking the transfinite iteration of the monotone operator generated by an inductive definition Φ. Introducing a concept of regular set as a substitute of the notion of regular cardinal, Aczel then showed that when Φ has a regular bound, CZF proves that $I(\Phi)$ exists and is a set, and thus the least fixed point of the associated monotone set-operator. In a standard extension of CZF, the system CZF+REA, every bounded inductive definition is also regular bounded. So a consequence of this development is a fully constructive proof of the existence of least fixed points of monotone maps $f : \mathsf{Pow}(X) \to \mathsf{Pow}(X)$ generated by bounded inductive definitions.

In [9] I formulate a notion of *abstract inductive definition* to extend the theory of inductive definitions from lattices of the form $\mathsf{Pow}(X)$ to general complete lattices. As a direct consequence of this extension, a constructive version of Tarski's theorem on the existence of the least fixed point of a monotone mapping on a complete lattice is there obtained.

In this article, a further extension is considered, to include structures of a much larger 'size' than complete lattices, as the class of all sets. To this purpose, I formulate notions of *locally set-generated large* $\bigvee$*-semilattices* and of $\bigvee$*-semilattices with a bounded presentation*. Abstract inductive definitions on locally set-generated large $\bigvee$-semilattices are shown to determine generalized elements of such lattices. In case the lattice under analysis has also a bounded presentation, and the abstract inductive definition is bounded, this generalized element is proven to be a standard element of the lattice, and a fixed point of the associated monotone operator.

Even when considered in ZFC, the notions of locally set-generated and boundedly presented large $\bigvee$-semilattice do not collapse to standard known concepts, so that the results derived about them might be of interest also in the realm of classical set theory; in particular one has a result asserting the existence of a least fixed point for certain monotone mappings on the class of all ordinals, as such a class is an example of a large $\bigvee$-semilattice with bounded presentation. These notions seem also to be of potential interest in connection with the constructive semantics of constructive set theories [10].

The general version of Tarski's fixed point theorem states that a monotone mapping f on a complete lattice has a complete lattice of fixed points, so that in particular every such f has a greatest fixed point. In CZF, this result is not directly derived from the existence of the least fixed point. A constructive proof of the existence of the greatest fixed point of a monotone mapping on a complete lattice again of the form $L = \mathsf{Pow}(X)$ for X a set, follows by work of Aczel on co-inductive definitions. In a suitable extension of CZF, an inductive definition Φ indeed also determines a *co-inductively defined* class $C(\Phi)$, and whenever Φ is a set, also $C(\Phi)$ is a set, and the greatest fixed point of the associated monotone set-operator ([4], [6, Chapter 13], and Section 6). In the last and main section of this article, I develop abstract lattice-theoretic versions of Aczel's results on the existence of co-inductively defined classes and sets. Specifically, I show that every abstract inductive definition Φ on a locally set-generated large $\bigvee$-semilattice L, in addition to an inductively defined generalized element of L, also gives rise to a co-inductively defined generalized element. When a set-generated L has a bounded presentation, and Φ is a set, such a generalized element is an ordinary element of L, and the greatest fixed-point of the associated monotone operator. As a corollary of the proofs of the existence of greatest and least fixed point, a constructive version of the general form of Tarski's fixed point theorem is finally derived.

We begin by recapitulating the main facts concerning the set-theoretical framework to be adopted, inductive definitions, and their extension to (constructively presented) complete lattices.

§1. Constructive set theories. The constructive analysis of the theory of inductive definitions is carried out in certain subsystems of classical Zermelo–Fraenkel set theory, collectively called *constructive set theories* [5, 11]. These theories are based on intuitionistic logic, and lack the set-theoretic principles of Powerset and (unrestricted) Separation. The results in the following sections are derived in some of these systems.

More specifically, we shall be working in constructive Zermelo–Fraenkel set theory (CZF, [2, 5]) and in some of its variants. CZF is a formal system in the same language of classical set theory, with $\in$ as the only nonlogical symbol. It uses intuitionistic first-order predicate logic with equality [13], and is axiomatized by the following axioms and axioms schemes:

1. Extensionality: $\forall a \forall b(\forall y(y \in a \leftrightarrow y \in b) \rightarrow a = b)$.
2. Pair: $\forall a \forall b \exists x \forall y(y \in x \leftrightarrow y = a \vee y = b)$.
3. Union: $\forall a \exists x \forall y(y \in x \leftrightarrow (\exists z \in a)(y \in z))$.
4. Restricted Separation scheme:
$$\forall a \exists x \forall y(y \in x \leftrightarrow y \in a \wedge \phi(y)),$$
for ϕ a restricted formula. A formula ϕ is *restricted* if the quantifiers that occur in it are of the form $\forall x \in b$, $\exists x \in c$.
5. Subset Collection scheme: for all formulae ϕ,
$$\forall a \forall b \exists c \forall u((\forall x \in a)(\exists y \in b)\phi(x, y, u) \rightarrow$$
$$(\exists d \in c)((\forall x \in a)(\exists y \in d)\phi(x, y, u) \wedge (\forall y \in d)(\exists x \in a)\phi(x, y, u))).$$
6. Strong Collection scheme: for all formulae ϕ,
$$\forall a((\forall x \in a)\exists y \phi(x, y) \rightarrow$$
$$\exists b((\forall x \in a)(\exists y \in b)\phi(x, y) \wedge (\forall y \in b)(\exists x \in a)\phi(x, y))).$$
7. Infinity: $\exists a(\exists x \in a \wedge (\forall x \in a)(\exists y \in a)x \in y)$.
8. Set Induction scheme: for all formulae ϕ,
$$\forall a((\forall x \in a)\phi(x) \rightarrow \phi(a)) \rightarrow \forall a \phi(a).$$

We use CZF^- for the system obtained from CZF by leaving out the Subset Collection scheme. Subset Collection is perhaps the most unusual of the CZF axioms and schemes; it can be seen as a strengthening of Myhill's axiom Exponentiation Axiom, asserting that the class b^a of functions from a set a to a set b is a set.

Certain consequences of the fully impredicative Powerset Axiom follow in CZF by the Subset Collection scheme; this is often presented in the equivalent form (over the remaining axioms of CZF) of the Fullness Axiom. For sets a, b, let $mv(b^a)$ be the class of subsets r of $a \times b$ such that $(\forall x \in a)(\exists y \in b)\ (x, y) \in r$.

Fullness: Given sets a, b there is a subset c of $mv(b^a)$ such that for every $r \in mv(b^a)$ there is $r_0 \in c$ with $r_0 \subseteq r$.

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

An intuitionistic, but fully impredicative, version of classical ZF is H. Friedman's Intuitionistic Zermelo–Fraenkel set theory based on collection, IZF. It has the same theorems as CZF extended by the unrestricted Separation Scheme and the Powerset Axiom. Moreover, the theory obtained from CZF, or from IZF, by adding the Law of Excluded Middle has the same theorems as ZF.

We shall make use of class notation and terminology familiar from classical set theory [5]. For example, given any set or class X, one has the class $\mathsf{Pow}(X) = \{x \mid x \subseteq X\}$ of subsets of X. The class of all sets is defined as $\mathrm{V} = \{x \mid x = x\}$.

If not indicated otherwise, in the following we shall be working in the system CZF^-.

§2. Inductive definitions. An *inductive definition* in the sense of Aczel is any class Φ of pairs of sets. One may regard a pair $(x, X) \in \Phi$ as an instance of a rule of inference of a formal system, with X as the set of premisses of an inference step, and x as the conclusion [1]. A class A is Φ-*closed* if it is closed for deduction in this system, i.e., if

$$(x, X) \in \Phi, \text{ and } X \subseteq A, \text{ then } x \in A.$$

The least Φ-closed class is denoted by $I(\Phi)$, when it exists. $I(\Phi)$ is the class inductively defined by Φ.

Given an inductive definition Φ, one has a monotone class-operator Γ_Φ defined, for every class Y, by $\Gamma_\Phi(Y) = \{x \mid (\exists X)\,(x, X) \in \Phi \;\&\; X \subseteq Y\}$. We shall say that Φ *generates* Γ_Φ. Note that a class A is Φ-closed iff $\Gamma_\Phi(A) \subseteq A$.

If the class $\{x \mid (\exists X)\,(x, X) \in \Phi \;\&\; X \subseteq Y\}$ is a set for every set Y, Φ is said to be local. If $\Phi \subseteq S \times \mathsf{Pow}(S)$ for S a set, we say that Φ is an inductive definition on S. For Φ a local inductive definition on S, $\Gamma_\Phi : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$ is a monotone set-operator on $\mathsf{Pow}(S)$. Conversely, with a monotone set-operator $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$ one can associate the local inductive definition $\Phi_\Gamma = \{(x, X) \mid x \in \Gamma(X)\}$.

Observe that Φ_Γ is the largest inductive definition among those that generate Γ. For instance, for S a set, the identity $Id : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$ can be generated by the set $\Phi = \{(x, \{x\}) \mid x \in S\}$, while Φ_{Id} is the class $\{(x, X) \mid x \in X,\ X \in \mathsf{Pow}(S)\}$. For $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$, if $\Gamma = \Gamma_\Phi$, with Φ an inductive definition on S, and if $I(\Phi)$ exists and is a set, then $I(\Phi)$ is the least fixed-point of Γ.

In ZFC, the least fixed-point of a monotone operator $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$, for S a set, can be obtained via transfinite iterations of Γ along all ordinals, starting from the empty set [1]. Again in the context of classical set theory ZFC, Aczel [1] shows that if $\Gamma = \Gamma_\Phi$ for an inductive definition $\Phi \subseteq S \times \mathsf{Pow}(S)$ with the property that $card(X) < \kappa$ for all $(x, X) \in \Phi$, with κ a regular cardinal, then κ iterations suffice to get the least fixed point of Γ.

The constructive analysis of inductive definitions is carried out in Aczel [2], assuming the axioms of CZF^-. The following theorem is called *the class inductive definition theorem* [5].

THEOREM 2.1 (CZF^-). *Given any class Φ of ordered pairs, there exists a least Φ-closed class $I(\Phi)$, the class inductively defined by Φ.*

Despite asserting just the existence of an inductively defined class, rather than of a set, this result can already be profitably exploited in the development of constructive mathematics, e.g., [8].

In constructive systems for set theory no completely satisfactory notion of ordinal and cardinal number is available (see however [6, Section 9.4]). Theorem 2.1 is thus proved by formulating a constructive version of the iteration of the monotone operator associated with the given inductive definition. The class of all sets plays the role of the class of ordinals, and set-induction is used in place of transfinite induction on ordinals.

Regular cardinals are replaced in this context by regular *sets*. A set κ is *regular* if it is transitive, inhabited, and for any $u \in \kappa$ and any set $R \subseteq u \times \kappa$, if $(\forall x \in u)(\exists y)\langle x, y\rangle \in R$, then there is a set $v \in \kappa$ such that

$$(\forall x \in u)(\exists y \in v)((x, y) \in R) \wedge (\forall y \in v)(\exists x \in u)((x, y) \in R). \quad (1)$$

κ is said to be *weakly regular* if in the above definition of regularity the second conjunct in (1) is omitted.

A class K is a *bound* for Φ if, for every $(x, X) \in \Phi$, there is a set $\lambda \in K$ and an onto mapping $f : \lambda \to X$. The inductive definition Φ is defined to be *bounded* (resp. *weakly regular bounded*) if:
1. $\{x \mid (x, X) \in \Phi\}$ is a set for every set X;
2. Φ is bounded by a set (resp. by a weakly regular set).

Theorem 2.2 (CZF). *If Φ is weakly regular bounded, then $I(\Phi)$ is a set.*

The proof shows that, if κ is a weakly regular bound for Φ, then 'κ iterations' of the monotone operator associated with Φ suffice to get the least fixed point. As one can also prove that set-iterations of the operator yields sets, the conclusion follows. The *Regular Extension Axiom*, REA, states:

REA: every set is a subset of a regular set.

The *weak regular extension axiom*, wREA, is the statement that every set is the subset of a weakly regular set. In the context of CZF + wREA, if Φ is bounded by a set, then it is bounded by a weakly regular set. The following result therefore holds in CZF + wREA.

Corollary 2.3 (CZF + wREA). *If Φ is bounded, then $I(\Phi)$ is a set.*

The prime example of an inductive definition is the one yielding the class $\mathbb{N}$ of the nonnegative integers: $\Phi_{\mathbb{N}} = \{(\emptyset, \emptyset)\} \cup \{(x^+, \{x\}) : x \in \mathrm{V}\}$, where $\mathrm{V} = \{x \mid x = x\}$ is the class of all sets, and $x^+ = x \cup \{x\}$. Note that $\Phi_{\mathbb{N}}$ is not a set, but has the set $\{\emptyset, \{\emptyset\}\}$ as bound.

§3. Abstract inductive definitions. An *inductive definition on a set* is an inductive definition Φ that is a subclass of the cartesian product $S \times \mathsf{Pow}(S)$, for S a set. An inductive definition Φ is local if $\Gamma_\Phi(Y)$ is a set for every set Y. As recalled in the previous section, inductive definitions on a set S that are local generate all the monotone operators of type $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$, and the smallest Φ-closed classes that are sets are least fixed points of these operators (see also Proposition 3.2 below).

The class $\mathsf{Pow}(S)$ can be seen as a complete lattice under the inclusion ordering. A natural question is then if the theory of inductive definitions on a set can be extended to compute least fixed points of monotone operators on general complete lattices. In [9] I showed that this is indeed the case, thereby obtaining a constructive version of Tarski's fixed point theorem on the existence of the least fixed point of a monotone mapping on a complete lattice. In this section I recapitulate the main definitions and results involved in that extension.

In constructive set theories, no nontrivial complete lattice can be assumed to be carried by a set [9, Theorem 2.5]. In such systems, the adopted notion is that of *set-generated* $\bigvee$*-semilattice* [5]: a partially ordered class, or *poclass*, $(X, \leq)$ is a class X together with a class-relation $\leq$ that is reflexive, transitive, and antisymmetric. A partially ordered class $(X, \leq)$ is a large $\bigvee$*-semilattice* if every subset of X has a supremum. A large $\bigvee$-semilattice L is said to be set-generated if it has a generating set B, i.e., a subset B of L such that, for all $x \in L$,

i. $\downarrow^B x \equiv \{b \in B \mid b \leq x\}$ is a set,
ii. $x = \bigvee \downarrow^B x$.

Considered in classical ZF, a large $\bigvee$-semilattice is set-generated if and only if it is a (small) $\bigvee$-semilattice in the usual sense.

Note that a set-generated $\bigvee$-semilattice is also a complete lattice. The same does *not* hold for a general large $\bigvee$-semilattice (not even in classical systems as ZFC: consider the universal class V ordered by inclusion).

The powerclass $\mathsf{Pow}(S)$ of a set S, ordered by inclusion, is the prime example of a set-generated $\bigvee$-semilattice, with generating set $B_S = \{\{x\} : x \in S\}$. We can regard an inductive definition on a set $\Phi \subseteq S \times \mathsf{Pow}(S)$ as a subclass of $B_S \times \mathsf{Pow}(S)$, by identifying the elements of S with the corresponding singletons in B_S.

Abstracting from this example, we define the notion of abstract inductive definition [9]. Let L be a set-generated $\bigvee$-semilattice L with generating set B. An *abstract inductive definition on* L is any class of ordered pairs

$$\Phi \subseteq B \times L.$$

A subclass $Y \subseteq B$ will be called c_L*-closed*, or a *generalized element of* L, if, for every subset U of Y, the set $\downarrow^B \bigvee U$ is contained in Y; that is, Y is c_L-closed if

$$\bigcup_{U \in \mathsf{Pow}(Y)} \downarrow^B \bigvee U = Y.$$

If Y is a set, Y is c_L-closed iff $Y = \downarrow^B \bigvee Y$.

Note that, if $L = \mathsf{Pow}(S)$, every subclass Y of B_S is c_L-closed. A c_L-closed class Y can be thought of as denoting a generalized element of L for, if Y were a set, we could consider its join that is an actual element of L; on the other hand, every element a of L is the join of a unique c_L-closed subset of B, the set $\downarrow^B a$.

A class $Y \subseteq B$ will be said Φ-*closed* if it is c_L-closed and if, whenever $(b, a) \in \Phi$,

$$\downarrow^B a \subseteq Y \implies b \in Y.$$

We shall denote by $\mathcal{I}(\Phi)$ the least Φ-closed class, if it exists.

Given an abstract inductive definition Φ on L, and an element a in L, the class

$$\{b \in B \mid (\exists a')\,(b, a') \in \Phi \;\&\; a' \leq a\}$$

may not be a set in general. If for every $a \in L$, this class is a set we say that Φ is *local*. A local abstract inductive definition Φ determines a mapping $\Gamma_\Phi : L \to L$, given by, for $a \in L$,

$$\Gamma_\Phi(a) \equiv \bigvee\{b \in B \mid (\exists a')\,(b, a') \in \Phi \;\&\; a' \leq a\}.$$

If $a_1 \leq a_2$, then $\Gamma_\Phi(a_1) \leq \Gamma_\Phi(a_2)$, i.e., Γ_Φ is *monotone*.

The following two propositions are proved in [9]. We reproduce their proofs here as they will be needed in the following.

Proposition 3.1. *Let $\Gamma : L \to L$ be a monotone operator on a set-generated $\bigvee$-semilattice L. Then, the abstract inductive definition $\Phi_\Gamma = \{(b, a) \in B \times L \mid b \leq \Gamma(a)\}$ is local, and for every $a \in L$, $\Gamma(a) = \Gamma_{\Phi_\Gamma}(a)$.*

Proof. Define $\Phi_\Gamma \subseteq B \times L$ by

$$(b, a) \in \Phi_\Gamma \iff b \leq \Gamma(a).$$

Φ_Γ is local as, for $a \in L$, $\{b \in B \mid (\exists a')\ (b, a') \in \Phi_\Gamma \ \&\ a' \leq a\} = \{b \in B \mid (\exists a')\ b \leq \Gamma(a') \ \&\ a' \leq a\}$. By monotonicity of Γ, this class is the same as the class $\{b \in B \mid b \leq \Gamma(a)\}$, that is a set by the assumption that L is set-generated. The join of this set therefore exists, and again as L is set-generated, is equal to $\Gamma(a)$. ⊣

Proposition 3.2. *Given a local abstract inductive definition Φ on a $\bigvee$-semilattice L with generating set B, a one-to-one correspondence exists between the Φ-closed subclasses Y of B that are sets, and the elements a of L such that $\Gamma_\Phi(a) \leq a$. The correspondence associates with Y its supremum, and with a the c_L-closed set $\downarrow^B a$.*

Moreover, whenever the class $\mathcal{I}(\Phi)$ exists and is a set, Γ_Φ has $\bigvee \mathcal{I}(\Phi)$ as least fixed point.

Proof. Assume $Y \subseteq B$ is Φ-closed and that it is a set. Then, $\bigvee Y$ exists in L and we have

$$\Gamma_\Phi(\bigvee Y) = \bigvee\{b \in B \mid (\exists a)\ (b, a) \in \Phi \ \&\ a \leq \bigvee Y\}.$$

To conclude that $\Gamma_\Phi(\bigvee Y) \leq \bigvee Y$, let $b \in B$ be such that there is $a \in L$ with $(b, a) \in \Phi$ and $a \leq \bigvee Y$. To show that $b \leq \bigvee Y$ it suffices to prove that $\downarrow^B a \subseteq Y$, since Y is Φ-closed. But this follows, by $a \leq \bigvee Y$, from the assumption that Y is a set and that it is c_L-closed (take $U = Y$ in the definition of c_L-closed). Conversely, to $a \in L$ such that $\Gamma_\Phi(a) \leq a$, we associate the c_L-closed class $\downarrow^B a$. As L is set-generated, $\downarrow^B a$ is a set; using the assumption that $\Gamma_\Phi(a) \leq a$, one immediately sees that $\downarrow^B a$ is also Φ-closed. Finally, one has $Y = \downarrow^B \bigvee Y$, as Y is c_L-closed, and $a = \bigvee \downarrow^B a$, since L is set-generated.

Now assume $\mathcal{I}(\Phi)$ exists and is a set. As $\mathcal{I}(\Phi)$ is Φ-closed, by what has just been shown, $\Gamma_\Phi(\bigvee \mathcal{I}(\Phi)) \leq \bigvee \mathcal{I}(\Phi)$. To prove the converse, note that by monotonicity of Γ_Φ, $\Gamma_\Phi(\Gamma_\Phi(\bigvee \mathcal{I}(\Phi))) \leq \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$. Then, $\downarrow^B \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$ is Φ-closed, again by the correspondence just proved. Then, as $\mathcal{I}(\Phi)$ is the least Φ-closed class, $\mathcal{I}(\Phi) \subseteq \downarrow^B \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$, so that $\bigvee \mathcal{I}(\Phi) \leq \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$. Thus $\bigvee \mathcal{I}(\Phi)$ is a fixed point for Γ_Φ. If $a \in L$ is another fixed point, then in particular $\Gamma_\Phi(a) \leq a$. Therefore $\downarrow^B a$ is Φ-closed, and $\mathcal{I}(\Phi) \subseteq \downarrow^B a$, which gives $\bigvee \mathcal{I}(\Phi) \leq a$. ⊣

Theorem 3.3. *Let Φ be an abstract inductive definition on a $\bigvee$-semilattice L set-generated by a set B. Then, the smallest Φ-closed class $\mathcal{I}(\Phi)$ exists.*

This result is proved in [9] generalizing the argument given in [2] for Theorem 2.1, in particular showing how to define constructively the transfinite iterations of a monotone operator on a general set-generated $\bigvee$-semilattice. It will also be a corollary of Theorem 5.3.

By this theorem, then, if $\Gamma : L \to L$ is a monotone operator on L, the smallest Φ-closed class $\mathcal{I}(\Phi_\Gamma)$ exists. By Propositions 3.1 and 3.2, if $\mathcal{I}(\Phi_\Gamma)$ is a set, $\bigvee \mathcal{I}(\Phi_\Gamma)$ is the least fixed point of Γ. Observe that in the presence of the unbounded Separation scheme, e.g., in the system ZF or IZF, $\mathcal{I}(\Phi_\Gamma)$ is always a set, as it is a subclass of the set B. In such impredicative systems then Theorem 3.3 yields Tarski fixed point theorem.

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

In the weaker setting of constructive set theory, more demanding conditions have to be imposed. A $\bigvee$-semilattice L set-generated by a set B is said to be *set-presented* [5] if a mapping $D : B \to \mathsf{Pow}(\mathsf{Pow}(B))$ is given with the property that

$$b \leq \bigvee U \iff (\exists W \in D(b)) W \subseteq U,$$

for every $b \in B, U \in \mathsf{Pow}(B)$. For instance, the set-generated $\bigvee$-semilattice $(\mathsf{Pow}(S), B_S)$ is set-presented by the mapping $D(\{x\}) = \{\{\{x\}\}\}$. Note that in a set-presented $\bigvee$-semilattice one in particular has, for all $b \in B$ and $W \in D(b)$, $b \leq \bigvee W$.

A *bound* for an abstract inductive definition Φ is a set α such that, whenever $(b, a) \in \Phi$ there is $x \in \alpha$ such that the set $\downarrow^B a$ is an image of x under a function f. An abstract inductive definition Φ is *bounded* if

1. $\{b \in B \mid (b, a) \in \Phi\}$ is a set for every $a \in L$.
2. Φ has a bound.

The following result is also proved in [9], by showing that, when Φ is bounded by a set, a set of iterations of the associated monotone operator suffices to yield the least fixed point. It will also arise as a corollary of Theorem 5.4.

THEOREM 3.4 (CZF + wREA). *Let Φ be a bounded abstract inductive definition on a set-presented $\bigvee$-semilattice L. Then, the smallest Φ-closed class $\mathcal{I}(\Phi)$ is a set.*

As a corollary one directly gets the constructive Tarski's fixed point theorem.

COROLLARY 3.5 (CZF + wREA). *Let $\Gamma : L \to L$ be a monotone operator on a set-presented $\bigvee$-semilattice L. If $\Gamma = \Gamma_\Phi$ for Φ a bounded abstract inductive definition on L, then Γ has the least fixed point $p = \bigvee \mathcal{I}(\Phi)$.*

§4. Inductive definitions on a set as abstract inductive definitions and conversely. As hinted before, a standard inductive definition on a set S can be regarded as an abstract inductive definition. With $\Phi \subseteq S \times \mathsf{Pow}(S)$ one associates the class $\Phi_a \subseteq B_S \times \mathsf{Pow}(S)$, where $B_S = \{\{x\} : x \in X\}$ is the base of singletons for the lattice $\mathsf{Pow}(S)$, and Φ_a is given by $\{(\{x\}, X) \mid (x, X) \in \Phi\}$. One then has that $I(\Phi)$ exists if and only if $\mathcal{I}(\Phi_a)$ exists, and $I(\Phi) = \{x \mid \{x\} \in \mathcal{I}(\Phi_a)\}$.

On the other hand, with an abstract inductive definition Φ on a $\bigvee$-semilattice L with generating set B, one can associate the standard inductive definition $\Phi_{st} = \Phi_{st,1} \cup \{(b, \downarrow^B a) \mid (b, a) \in \Phi\}$, where $\Phi_{st,1} = \{(b, U) \in B \times \mathsf{Pow}(B) \mid b \leq \bigvee U\}$. Observe first that a class $Y \subseteq B$ is c_L-closed if and only if it is $\Phi_{st,1}$-closed in the standard sense. Then a class $Y \subseteq B$ is Φ-closed in the abstract sense iff it is Φ_{st}-closed in the standard sense, $\mathcal{I}(\Phi)$ exists if and only if $I(\Phi_{st})$ exists, and in that case they coincide. Theorem 3.3 can thus be derived via this identification, applying Theorem 2.1.[1] As mentioned before, the direct proof of Theorem 3.3 shows however how to define, in the setting of constructive set theory, the infinitary iterations of a monotone operator on a complete lattice.

Note that, even when Φ is a set or is bounded, Φ_{st} is generally a proper unbounded inductive definition. So this association cannot be used to derive Theorem 3.4 by

[1] As observed by the anonymous referee of [9]. This corresponds to the fact that one can (impredicatively) derive Tarski fixed point theorem on the existence of the least fixed point of a monotone mapping $f : L \to L$ by its particular case (known as Knaster–Tarski fixed point theorem) for $L = \mathsf{Pow}(X)$, X a set.

its special case for standard inductive definitions (Theorem 2.3). However, when L is set-presented, one can associate with Φ the standard inductive definition $\Phi'_{st} = \Phi'_{st,1} \cup \{(b, \downarrow^B a) \mid (b, a) \in \Phi\}$, with $\Phi'_{st,1} = \{(b, W) \in B \times \mathsf{Pow}(B) \mid W \in D(b)\}$. Again, $I(\Phi'_{st})$ exists if and only if $\mathcal{I}(\Phi)$ does, and they coincide in that case. Moreover, Φ'_{st} is bounded in the standard sense if and only if Φ is bounded in the abstract sense. Theorem 3.4 can then be obtained by applying Theorem 2.3 to Φ'_{st}.

The reduction of abstract inductive definitions to standard inductive definitions is however not entirely satisfactory. Given a set-generated $\bigvee$-semilattice L, no uniform way is available for replacing an abstract inductive definition Φ with a standard inductive definition Φ_{st}, inductively defining the same class and such that the latter is bounded whenever the former is. Moreover, while the same abstract inductive definition shall determine both a least inductively defined element and a greatest co-inductively defined element, as for the standard case, the replacement of an abstract inductive definition with the standard ones considered above does not define the co-inductively defined element generated by the original abstract inductive definition (see Section 6).

§5. Locally set-generated and boundedly presented $\bigvee$-semilattices. Abstract inductive definitions on set-generated $\bigvee$-semilattices generalize inductive definitions *on a set*, i.e., of the form $\Phi \subseteq S \times \mathsf{Pow}(S)$, for S a set. A natural question is if one has also an abstract analog of arbitrary inductive definitions. In this section we show that this generalization is indeed possible, introducing the notions of locally set-generated $\bigvee$-semilattice and of $\bigvee$-semilattice with a bounded presentation.

A general inductive definition Φ can also be seen as an abstract inductive definition on a $\bigvee$-semilattice. However, the $\bigvee$-semilattice has to be of a much larger 'size' than that of a set-generated $\bigvee$-semilattice.

We shall say that a large $\bigvee$-semilattice L is *locally set-generated* if it is generated via subsets of a possibly proper subclass of L, i.e., if a subclass B of L is given such that, for all $x \in$ L,

i. $\downarrow^B x \equiv \{b \in B \mid b \leq x\}$ is a set,
ii. $x = \bigvee \downarrow^B x$.

So a locally set-generated $\bigvee$-semilattice is defined as a set-generated $\bigvee$-semilattice, but the requirement that the base B is a set is dropped.

Note that, even in classical ZF set theory, a locally set-generated $\bigvee$-semilattice is in general carried by a proper class. Also, by contrast with set-generated $\bigvee$-semilattices, a locally set-generated $\bigvee$-semilattice need not be a complete lattice. Indeed, the prime example of such a structure is the class of all sets $\mathrm{V} = \{x \mid x = x\}$, ordered by inclusion, with $B_{\mathrm{V}} = \{\{x\} \mid x \in \mathrm{V}\}$. Observe that, in the present constructive context, V itself wouldn't work as a base for V. Another example is given by the powerclass $\mathsf{Pow}(X)$ of any class X, locally set-generated by the class $B_{\mathsf{Pow}(X)} = \{\{x\} \mid x \in \mathrm{X}\}$. Thus for instance the double powerclass $\mathsf{PowPow}(S)$ of a set S is locally set-generated, while it cannot be proved to be set-generated in CZF if S is nonempty. For an example in classical set theory, consider the class of all ordinals **Ord**, locally set-generated by $B_{\mathbf{Ord}} = \{\alpha \in \mathbf{Ord} \mid (\exists \beta \in \mathbf{Ord})\alpha = \beta^+\}$ (this example in fact also works in CZF, assuming the class of ordinals is defined as the class of the transitive sets all of whose members are transitive sets [6]; to verify this, note that these ordinals are closed by successor and set-indexed union [6, Lemma 9.4.2], and for ordinals α, β it holds $\alpha^+ \leq \beta$ if and only if $\alpha \in \beta$).

Reading through the proofs of Propositions 3.1 and 3.2, one sees that the assumption that the class B of generators is a set is not used, the features of set-generated $\bigvee$-semilattice exploited being instead conditions i, ii. We can therefore read those two propositions as valid more generally for monotone operators on locally set-generated $\bigvee$-semilattices.

PROPOSITION 5.1. *Let $\Gamma : \mathrm{L} \to \mathrm{L}$ be a monotone operator on a locally set-generated $\bigvee$-semilattice* L*. Then, the abstract inductive definition $\Phi_\Gamma = \{(b, a) \in B \times \mathrm{L} \mid b \leq \Gamma(a)\}$ is local, and for every $a \in \mathrm{L}$, $\Gamma(a) = \Gamma_{\Phi_\Gamma}(a)$.*

PROPOSITION 5.2. *Given a local abstract inductive definition Φ on a locally set-generated $\bigvee$-semilattice* L *with generating class B, a one-to-one correspondence exists between the Φ-closed subclasses Y of B that are sets, and the elements a of* L *such that $\Gamma_\Phi(a) \leq a$. The correspondence associates with Y its supremum, and with a the c_L-closed set $\downarrow^B a$.*

Moreover, whenever the class $\mathcal{I}(\Phi)$ exists and is a set, Γ_Φ has $\bigvee \mathcal{I}(\Phi)$ as least fixed point.

We have the following generalization of Theorem 3.3.

THEOREM 5.3. *Let Φ be an abstract inductive definition on a $\bigvee$-semilattice* L *locally set-generated by a class B. Then, the smallest Φ-closed class $\mathcal{I}(\Phi)$ exists.*

PROOF. The result can be proved either directly, observing that the proof of Theorem 3.3 in [9] does not require the condition that the class B of generators of L is a set, thus showing that the constructive formulation of the transfinite iterations of the monotone operator associated with Φ extends to this more general situation, or using Theorem 2.1, as follows. Recall by Section 4 that one may regard an abstract inductive definition Φ on a $\bigvee$-semilattice L with generating set B as the standard inductive definition $\Phi_{st} = \{(b, U) \in B \times \mathsf{Pow}(B) \mid b \leq \bigvee U\} \cup \{(b, \downarrow^B a) \mid (b, a) \in \Phi\}$. Again no role is played in this identification by the assumption that B is a set. We can therefore apply Theorem 2.1 to get the existence of the smallest Φ-closed class $I(\Phi_{st})$. One then immediately checks that $\mathcal{I}(\Phi) = I(\Phi_{st})$. ⊣

An arbitrary standard inductive definition Φ can then be seen as an abstract inductive definition on $\mathrm{L} = (\mathrm{V}, B_\mathrm{V})$, associating with Φ the class $\Phi_a \equiv \{(\{x\}, X) \mid (x, X) \in \Phi\}$.

A locally set-generated $\bigvee$-semilattice (L, B) will be said *boundedly presented*, or to have a *bounded presentation*, if a set k exists such that, for all $b \in B$ and all $U \in \mathsf{Pow}(B)$,

$$b \leq \bigvee U \Rightarrow (\exists \lambda \in k)(\exists f : \lambda \to U) b \leq \bigvee \mathrm{Range}(f).$$

The universal class $(\mathrm{V}, B_\mathrm{V})$ is boundedly presented by $k = \{\{0\}\}$. Other examples are given by set-presented $\bigvee$-semilattices: a set-presented L is boundedly presented by $k = \{W \mid (\exists b \in B) W \in D(b)\} = \bigcup \mathrm{Range}(D)$.

As already observed, a class $I \subseteq B$ is c_L-closed if and only if it is closed in the standard sense for the standard inductive definition $\Phi_{st,1} = \{(b, U) \in B \times \mathsf{Pow}(B) \mid b \leq \bigvee U\}$; if the $\bigvee$-semilattice is boundedly presented, a class I is c_L-closed if and only if it is $\Phi''_{st,1}$-closed, where $\Phi''_{st,1}$ is the standard inductive

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

definition

$$\{(b,U) \in B \times \mathsf{Pow}(B) \mid (\exists \lambda \in k)(\exists f : \lambda \to \mathrm{V})\, \mathrm{Range}(f) = U \,\&\, b \leq \bigvee \mathrm{Range}(f)\}.$$

Indeed, if $U \subseteq I$, for U a subset of B, and $b \leq \bigvee U$, then there is $\lambda \in k$ and $f : \lambda \to U$ such that $b \leq \bigvee \mathrm{Range}(f)$.

THEOREM 5.4 (CZF + wREA). *Let Φ be a bounded abstract inductive definition on a boundedly presented $\bigvee$-semilattice* L. *Then, the smallest Φ-closed class $\mathcal{I}(\Phi)$ is a set.*

PROOF. Again, one can prove this result either directly or indirectly. Indirectly, one associates with Φ the standard $\Phi''_{st} = \Phi''_{st,1} \cup \{(b, \downarrow^B a) \mid (b,a) \in \Phi\}$, with $\Phi''_{st,1}$ defined as above. This inductive definition is bounded in the standard sense, as k is a bound for $\Phi''_{st,1}$ by the definition of $\Phi''_{st,1}$, and since Φ is bounded; remarkably, condition i on locally set-generated $\bigvee$-semilattices is precisely what is needed to have that $\{x \mid (x,X) \in \Phi''_{st,1}\}$ is a set for every set X, so that $\{x \mid (x,X) \in \Phi''_{st}\}$ is too a set for every set X, as Φ is bounded. In CZF + wREA, then, the smallest Φ-closed class $I(\Phi''_{st})$ exists and is a set, by Corollary 2.3. It is not difficult then to check that $\mathcal{I}(\Phi) = I(\Phi''_{st})$.

This proof however does not provide us with an iteration stage at which the monotone operator Γ_Φ associated with Φ stabilizes. To directly construct the smallest Φ-closed class $\mathcal{I}(\Phi)$ as a set via iterations, one can proceed along the lines of the proof of [9, Theorem 4.9], by first replacing the set $S = \alpha \cup \{V : (\exists b \in B) V \in D(b)\} = \alpha \cup \bigcup \mathrm{Range}(D)$ with $S = \alpha \cup k$, where α is a bound for Φ. We leave the details of this alternative proof to the reader. ⊣

Even when B is a set, the concept of boundedly presented $\bigvee$-semilattice (L, B) is a generalization of the notion of a set-presented $\bigvee$-semilattice. This notion could thus be useful in connection with the semantics of constructive set theories, where particular types of set-presented $\bigvee$-semilattices (set-presented cHa's) are used as spaces of truth values [10].

As hinted previously, the notions of locally set-generated and boundedly presented $\bigvee$-semilattice may be of interest also in the context of classical set theory; considered in such a context they do not collapse to known standard notions, as is instead the case for the notions of set-generated and set-presented $\bigvee$-semilattices. For example, the locally set-generated $\bigvee$-semilattice $(\mathbf{Ord}, B_{\mathbf{Ord}})$ is easily seen to be boundedly presented (by $k = \{\{0\}\}$; to prove this note that $\beta \in B_{\mathbf{Ord}} \iff (\exists \gamma \in \mathbf{Ord})\beta = \gamma \cup \{\gamma\}$, so that, for $U \subseteq B_{\mathbf{Ord}}$, $\gamma \cup \{\gamma\} \subseteq \bigcup U$ gives $\delta \in U$ such that $\gamma \in \delta$, whence $\beta = \gamma \cup \{\gamma\} \subseteq \delta$). So by Theorem 5.4, every monotone operator on the ordinals that is generated by a bounded abstract inductive definition has a least fixed point. Note that as the successor function $^+$ is monotone but with no least fixed point, we deduce that $^+$ cannot be generated by a bounded abstract inductive definition (in CZF as well as) in ZFC.

§6. Abstract co-inductive definitions.

In [12], Tarski proves that a monotone operator f on a complete lattice also has a greatest fixed point. Using the existence of a least and a greatest fixed point of f, he then proves that f has, more generally, a complete lattice of fixed points. In this section I prove a constructive version of this result. As before, unless otherwise explicitly indicated, we shall be working in CZF^-.

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

Tarski first exhibited the existence of a greatest fixed point of f, then applied this result to the lattice obtained from the given one taking the dual order, to show that f also has a least fixed point. Unfortunately, we cannot use an analogous approach, since the dual of a set-generated $\bigvee$-semilattice need not a priori be set-generated.

As for the case of the least fixed point, the existence of the greatest fixed point in the special case of monotone functions on $\bigvee$-semilattices of the form $(\mathsf{Pow}(S), \subseteq)$, for S a set, has been proved by Aczel (cf. [4, 6]).

Specifically, given an inductive definition Φ on a class S, $\Phi \subseteq S \times \mathsf{Pow}(S)$, a class $C \subseteq S$ is said to be Φ-inclusive if $C \subseteq \Gamma_\Phi(C)$, with Γ_Φ the operator on subclasses associated with Φ: $\Gamma_\Phi(C) = \{x \mid (\exists X)\,(x, X) \in \Phi \,\&\, X \subseteq C\}$. Aczel showed that the class $J = \bigcup\{Y \in \mathsf{Pow}(S) \mid Y \subseteq \Gamma_\Phi(Y)\}$ is the largest Φ-inclusive subclass of S, the class co-inductively defined by Φ, denoted by $C(\Phi)$. This result is proven in the system $\mathrm{CZF}^- + \mathrm{RRS}$, where the Relation Reflection Scheme RRS is the following axiom scheme.

Relation Reflection Scheme, RRS:

> For classes S, R with $R \subseteq S \times S$, if $a \in S$ and $\forall x \in S\ \exists y \in S\ R(x, y)$ then there is a set $S_0 \subseteq S$ such that $a \in S_0$ and $\forall x \in S_0\ \exists y \in S_0\ R(x, y)$.

This scheme can be regarded as a weakening of the Relativized Dependent Choices Axiom, RDC. By contrast with RDC, RRS is valid in all topological models (all cHa-valued models). Note also that RRS is a theorem of ZF (see [4] for a proof of these facts).

The following strengthening of RRS and REA is used to show that J is a set when S, Φ are sets. A regular set A is *strongly regular* if it is closed under the union operation, i.e., if $\forall x \in A\ \cup x \in A$. Let A be a strongly regular set. A is defined to be *RRS-strongly regular* if also, for all sets $A' \subseteq A$ and $R \subseteq A' \times A'$, if $a_0 \in A'$ and $\forall x \in A'\ \exists y \in A'\ xRy$ then there is $A_0 \in A$ such that $a_0 \in A_0 \subseteq A'$ and $\forall x \in A_0\ \exists y \in A_0\ xRy$.

RRS-$\bigcup$REA: Every set is the subset of a RRS-strongly regular set.

In the system CZF+ RRS-$\bigcup$REA, the class $J = \bigcup\{Y \in \mathsf{Pow}(S) \mid Y \subseteq \Gamma_\Phi(Y)\}$ can be proved to be a set when S and Φ are sets [6].

If the largest Φ-inclusive subclass $C(\Phi) = J$ of S exists, then it is easily seen to be the greatest fixed point of Γ_Φ. As recalled before, every monotone operator $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$ can be obtained as Γ_Φ for a suitable Φ. Then, when Γ is obtained from Φ a set and when S is a set, Γ has $C(\Phi)$ as greatest fixed point (note indeed that for X to be a fixed point of $\Gamma : \mathsf{Pow}(S) \to \mathsf{Pow}(S)$, X must be an element of $\mathsf{Pow}(S)$, and so it has to be a set).

I now extend these results to the abstract setting. To this purpose, given a locally set-generated $\bigvee$-semilattice (L, B) and an abstract inductive definition $\Phi \subseteq B \times \mathrm{L}$, we define an operator $\bar{\Gamma}_\Phi$ on the generalized elements of L (i.e., the c_L-closed subclasses of B) as follows: for $Y \subseteq B$ a c_L-closed class, let

$$\bar{\Gamma}_\Phi(Y) \equiv c_L\{b \in B \mid (\exists a)\,(b, a) \in \Phi \,\&\, {\downarrow^B} a \subseteq Y\},$$

where, for Z a subclass of B,

$$c_L Z \equiv \{b \in B \mid (\exists U \in \mathsf{Pow}(Z))\, b \in {\downarrow^B} \bigvee U\} = \bigcup_{U \in \mathsf{Pow}(Z)} {\downarrow^B} \bigvee U.$$

Note that $Z \subseteq c_L\, Z$, and if Z is a set, $c_L\, Z = {\downarrow^B} \bigvee Z$ is a set, too. Note also that $\bar{\Gamma}_\Phi$ is a monotone operator on classes, and, that by contrast with Γ_Φ (Section 3), it

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

is defined even when Φ is not local. Also, for every class Y, $\bar{\Gamma}_\Phi(Y)$ is easily seen to be a c_L-closed class (cf. also [9]).

We shall say that a subclass Y of B is Φ-*inclusive* if Y is c_L-closed and

$$Y \subseteq \bar{\Gamma}_\Phi(Y). \qquad (\star)$$

Condition $\star$ will be referred to as *the characteristic condition of* Φ-*inclusivity.* When it exists, the largest Φ-inclusive subclass of B will be denoted by $\mathcal{C}(\Phi)$.

Observe that for Y a c_L-closed class, the equivalence

$\bar{\Gamma}_\Phi(Y) \subseteq Y$ if and only if, for all $(b,a) \in \Phi, \downarrow^B a \subseteq Y \implies b \in Y$,

justifies the abstract definition of Φ-closed class. By contrast, at the abstract level, it does not hold in general that $Y \subseteq \bar{\Gamma}_\Phi(Y)$ if and only if $(\forall b \in Y)(\exists a)(b,a) \in \Phi \;\&\; \downarrow^B a \subseteq Y$.

If Φ is local, so that $\Gamma_\Phi(a) \equiv \bigvee\{b \in B \mid (\exists a')\,(b,a') \in \Phi \;\&\; a' \leq a\}$ is defined for every $a \in \mathrm{L}$, one has an obvious one-one correspondence between Φ-inclusive classes Y that are sets, and elements a of L such that $a \leq \Gamma_\Phi(a)$; the correspondence associates $\bigvee Y$ to Y and $\downarrow^B a$ to a.

PROPOSITION 6.1. *Let Φ be a local abstract inductive definition on a $\bigvee$-semilattice* L *locally set-generated by a class B. If the class $\mathcal{C}(\Phi)$ exists and is a set, $\Gamma_\Phi : \mathrm{L} \to \mathrm{L}$ has $\bigvee \mathcal{C}(\Phi)$ as greatest fixed point.*

PROOF. Since $\mathcal{C}(\Phi)$ is Φ-inclusive,

$$\bigvee \mathcal{C}(\Phi) \leq \Gamma_\Phi(\bigvee \mathcal{C}(\Phi)).$$

By monotonicity,

$$\Gamma_\Phi(\bigvee \mathcal{C}(\Phi)) \leq \Gamma_\Phi(\Gamma_\Phi(\bigvee \mathcal{C}(\Phi))),$$

therefore $\downarrow^B \Gamma_\Phi(\bigvee \mathcal{C}(\Phi))$ is Φ-inclusive.

So, $\downarrow^B \Gamma_\Phi(\bigvee \mathcal{C}(\Phi)) \subseteq \mathcal{C}(\Phi)$, which gives $\Gamma_\Phi(\bigvee \mathcal{C}(\Phi)) \leq \bigvee \mathcal{C}(\Phi)$. Thus $\bigvee \mathcal{C}(\Phi)$ is a fixed point. To see that it is the greatest, assume a is a fixed point. Then $\downarrow^B a$ is Φ-inclusive, so that $\downarrow^B a \subseteq \mathcal{C}(\Phi)$, which gives $a \leq \bigvee \mathcal{C}(\Phi)$. ⊣

Now let (L, B) be a locally set-generated $\bigvee$-semilattice and $\Phi \subseteq B \times \mathrm{L}$ an abstract inductive definition on L. Let

$$J = \bigcup\{Y \in \mathsf{Pow}(B) \mid Y\, c_L\text{-closed} \;\&\; Y \subseteq \bar{\Gamma}_\Phi(Y)\}$$

the union of all Φ-inclusive sets. In general, the union of c_L-closed sets is not c_L-closed. The following lemma shows, using the Strong Collection scheme, that this is instead the case for J.

LEMMA 6.2. *The class J is c_L-closed.*

PROOF. Let $b \leq \bigvee U$, $U \in \mathsf{Pow}(B) \;\&\; U \subseteq J$. Then, for all $c \in U$ there is a Φ-inclusive Y such that $c \in Y$. By Strong Collection, there is a set K such that $(\forall c \in U)(\exists Y \in K)(Y\ \Phi\text{-inclusive} \;\&\; c \in Y) \;\&\; (\forall Y \in K)(\exists c \in U)(Y\ \Phi\text{-inclusive} \;\&\; c \in Y)$.

Let $\bar{Y} = \bigcup K$. By monotonicity of $\bar{\Gamma}_\Phi$, the union of any set of sets satisfying the characteristic property of Φ-inclusivity also satisfies the characteristic property, therefore $U \subseteq \bar{Y} \subseteq \bar{\Gamma}_\Phi(\bar{Y})$. Furthermore, again by monotonicity, $\bar{Y} \subseteq \bar{\Gamma}_\Phi(\downarrow^B \bigvee \bar{Y})$. Since $\bar{\Gamma}_\Phi(\downarrow^B \bigvee \bar{Y})$ is c_L-closed, it also holds $b \in \downarrow^B \bigvee U \subseteq \downarrow^B \bigvee \bar{Y} \subseteq \bar{\Gamma}_\Phi(\downarrow^B \bigvee \bar{Y})$. Thus, $\downarrow^B \bigvee \bar{Y}$ is Φ-inclusive, so that $b \in J$, as wished. ⊣

The following theorem is an extension of [4, Theorem 3.1] (or [6, Theorem 13.1.3]) to the abstract setting. As for the case of inductive definitions, the main difficulty in carrying out this extension is due to the nonpoint set nature of $\bigvee$-semilattices. The same will hold for Theorem 6.5 below.

THEOREM 6.3 (CZF$^-$ + RRS). *Let Φ be an abstract inductive definition on a $\bigvee$-semilattice* L *locally set-generated by the class B. The class $J = \bigcup\{Y \in \mathsf{Pow}(B) \mid Y\, c_L\text{-closed} \;\&\; Y \subseteq \bar{\Gamma}_\Phi(Y)\}$ is the greatest Φ-inclusive sub-class of B. I.e., $J = \mathcal{C}(\Phi)$.*

PROOF. By the previous lemma, J is c_L-closed. To prove that J satisfies the characteristic property of Φ-inclusivity, i.e., that $J \subseteq \bar{\Gamma}_\Phi(J)$, let $b \in J$. Then, $b \in Y$ for Y such that $Y \subseteq \bar{\Gamma}_\Phi(Y)$. By monotonicity, $b \in \bar{\Gamma}_\Phi(J)$. So J is Φ-inclusive. It remains to show that it is the greatest Φ-inclusive class. Assume Z is a Φ-inclusive class, and $b_0 \in Z$. We want to prove $b_0 \in J$. To this purpose, it suffices to construct a Φ-inclusive set W such that $b_0 \in W$. Let $A = \mathsf{Pow}(Z)$, and let $X \in A$. Thus $X \subseteq Z$, and

$$(\forall x \in X)(\exists U)[U \in \mathsf{Pow}(\{b \in B \mid (\exists a)\ (b,a) \in \Phi \;\&\; {\downarrow}^B a \subseteq Z\}) \;\&\; x \leq \bigvee U].$$

In particular, then,

$$\text{for all } b \in U \text{ there is } a \in \mathrm{L} \text{ such that } (b,a) \in \Phi \text{ and } {\downarrow}^B a \subseteq Z.$$

By Strong Collection, a subset T_1 of L exists such that

$$(\forall b \in U)(\exists a \in T_1)((b,a) \in \Phi \;\&\; {\downarrow}^B a \subseteq Z)$$

and

$$(\forall a \in T_1)(\exists b \in U)((b,a) \in \Phi \;\&\; {\downarrow}^B a \subseteq Z).$$

Let $K = \bigcup_{a \in T_1} {\downarrow}^B a$. So $K \in A = \mathsf{Pow}(Z)$. It follows that

$$(\forall x \in X)(\exists K \in A) x \in \bar{\Gamma}_\Phi(K) \;\&\; K \subseteq Z.$$

Applying again the Strong Collection scheme, we obtain a set T_2 such that

$$(\forall x \in X)(\exists K \in T_2)[x \in \bar{\Gamma}_\Phi(K) \;\&\; K \subseteq Z]$$

and

$$(\forall K \in T_2)(\exists x \in X)[x \in \bar{\Gamma}_\Phi(K) \;\&\; K \subseteq Z].$$

Defining $Z' = \bigcup T_2$, we have then shown that

$$(\forall X \in A)(\exists Z' \in A)[X \subseteq \bar{\Gamma}_\Phi(Z')].$$

By the Relation Reflection Scheme, we get a set $A_0 \subseteq A$ such that $\{b_0\} \in A_0$ and

$$(\forall X \in A_0)(\exists Z' \in A_0)[X \subseteq \bar{\Gamma}_\Phi(Z')].$$

Letting $W' = \bigcup A_0$, we have $W' \subseteq \bar{\Gamma}_\Phi(W')$, so that W' satisfies the characteristic property of Φ-inclusivity. By monotonicity, $W' \subseteq \bar{\Gamma}_\Phi({\downarrow}^B \bigvee W')$, and ${\downarrow}^B \bigvee W' \subseteq \bar{\Gamma}_\Phi({\downarrow}^B \bigvee W')$, as $\bar{\Gamma}_\Phi(Y)$ is c_L-closed for all Y. $W = {\downarrow}^B \bigvee W'$ is then the Φ-inclusive set we were looking for. ⊣

If Φ is bounded, then Φ can be proved to be local in CZF, using the Subset Collection scheme (or the weaker Exponentiation axiom). When Φ is a set (and so in particular it is bounded) this can be proved already assuming CZF$^-$.

LEMMA 6.4. *Let Φ be an abstract inductive definition on a locally set-generated $\bigvee$-semilattice* L. *If Φ is a set, then it is local.*

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

PROOF. We have to show that the class $\{b \in B \mid (\exists a')\,(b,a') \in \Phi \ \&\ a' \leq a\}$ is a set for every $a \in \mathrm{L}$. Note that we have $\{b \in B \mid (\exists a')\,(b,a') \in \Phi \ \&\ a' \leq a\} = \{b \in \Phi_1 \mid (\exists a' \in \Phi_2)\,(b,a') \in \Phi \ \&\ {\downarrow^B} a' \subseteq {\downarrow^B} a\}$, where $\Phi_i = \mathrm{Range}(\pi_i)$, π_i for $i = 1, 2$ the first and second projections on Φ. As, by (Replacement that is a consequence of) Strong Collection, $\mathrm{Range}(\pi_i)$ are sets, by Restricted Separation, the latter class is a set for every $a \in \mathrm{L}$. ⊣

The following result extends [6, Theorem 13.2.3] to the abstract context of set-generated $\bigvee$-semilattices with a bounded presentation, thus in particular to set-presented $\bigvee$-semilattices.

THEOREM 6.5 (CZF+RRS-$\bigcup$REA). *Let Φ be an abstract inductive definition on a set-generated $\bigvee$-semilattice (L, B) boundedly presented by a set k. If Φ is a set, the largest Φ-inclusive class $\mathcal{C}(\Phi)$ is a set.*

PROOF. Since Φ is local, we have

$J = \bigcup\{Y \in \mathsf{Pow}(B) \mid Y\, c_L\text{-closed} \ \&\ Y \subseteq \bar{\Gamma}_\Phi(Y)\} = \bigcup\{{\downarrow^B} a \mid a \in L, a \leq \Gamma_\Phi(a)\}$.

By Theorem 6.3, J is the largest Φ-inclusive class, we now prove that it is a set. By RRS-$\bigcup$REA there is a RRS-strongly regular set A such that

$$\{\{b\} \mid b \in B\} \cup \{{\downarrow^B} a \mid (\exists b \in B)(b,a) \in \Phi\} \cup k \subseteq A.$$

Let

$$J_A = \bigcup\{X \in A \cap \mathsf{Pow}(B) \mid \bigvee X \leq \Gamma_\Phi(\bigvee X)\}.$$

Since $\{X \in A \cap \mathsf{Pow}(B) \mid \bigvee X \leq \Gamma_\Phi(\bigvee X)\} = \{X \in A \mid (\forall b \in X) b \in B \ \&\ b \in {\downarrow^B}\Gamma_\Phi(\bigvee X)\}$, and since ${\downarrow^B}\Gamma_\Phi(\bigvee X)$ is a set because L is set-generated, J_A is a set by Restricted Separation and Union.

We can therefore conclude if we prove that $J = J_A$. Clearly, $J_A \subseteq J$: if $b \in J_A$, then $b \in X$, with $\bigvee X \leq \Gamma_\Phi(\bigvee X)$. Thus, $b \in {\downarrow^B} \bigvee X$ and $b \in J$. For the converse, let $b_0 \in J$, i.e., $b_0 \in {\downarrow^B} a_0$ with $a_0 \leq \Gamma_\Phi(a_0)$. We have to find $X \in A$ such that $\bigvee X \leq \Gamma_\Phi(\bigvee X)$, and $b_0 \in X$.

Let $A' = \mathsf{Pow}({\downarrow^B} a_0) \cap A$. For $Z \in A'$, one then has

$$(\forall b \in Z) b \leq \Gamma_\Phi(a_0).$$

Since L is boundedly presented, this gives

$$(\forall b \in Z)(\exists \lambda \in k)(\exists f : \lambda \to \{b' \in B \mid (\exists a)\,(b', a) \in \Phi \,\&\, a \leq a_0\}) b \leq \bigvee \mathrm{Range}(f).$$

Given $b \in Z$ there is thus $\lambda \in k$ and $f : \lambda \to \{b' \in B \mid (\exists a)\,(b', a) \in \Phi \,\&\, a \leq a_0\}$ such that for all $c \in \lambda$ there is a such that $(f(c), a) \in \Phi$ and $a \leq a_0$. Letting, for $c \in \lambda$, $\Phi_c = \{a \in \Phi_2 \mid (f(c), a) \in \Phi\}$, with Φ_2 the range of the second projection on Φ, it follows that

$$(\forall c \in \lambda)(\exists X \in A)[(\exists a \in \Phi_c) X = {\downarrow^B} a \ \&\ a \leq a_0],$$

as A contains ${\downarrow^B} a$ for each $a \in \Phi_2$. Then, since A is regular, and $\lambda \in A$, we have a set $Z_0 \in A$ such that

$$(\forall c \in \lambda)(\exists X \in Z_0)[(\exists a \in \Phi_c) X = {\downarrow^B} a \ \&\ a \leq a_0]$$

and

$$(\forall X \in Z_0)(\exists c \in \lambda)[(\exists a \in \Phi_c) X = {\downarrow^B} a \ \&\ a \leq a_0].$$

Thus $Z_0 \subseteq \mathsf{Pow}(\downarrow^B a_0)$, so that $Z_1 \equiv \bigcup Z_0 \in \mathsf{Pow}(\downarrow^B a_0)$. As A is union-closed, $Z_1 \in A$, so that $Z_1 \in A'$. Also, we have $(\forall c \in \lambda)(\exists a)(f(c), a) \in \Phi$ and $\downarrow^B a \subseteq Z_1$, by which $\bigvee \mathrm{Range}(f) \leq \Gamma_\Phi(\bigvee Z_1)$, yielding $b \leq \Gamma_\Phi(\bigvee Z_1)$.
We have therefore proved

$$(\forall b \in Z)(\exists Z_1)[Z_1 \in A' \,\&\, b \leq \Gamma_\Phi(\bigvee Z_1)].$$

Again as A is regular, and since $Z \in A$, we get a set $Z_2 \in A$ such that

$$(\forall b \in Z)(\exists Z_1 \in Z_2)[Z_1 \in A' \,\&\, b \leq \Gamma_\Phi(\bigvee Z_1)]$$

and

$$(\forall Z_1 \in Z_2)(\exists b \in Z)[Z_1 \in A' \,\&\, b \leq \Gamma_\Phi(\bigvee Z_1)].$$

Letting $Z' = \bigcup Z_2$, since A is union-closed, we have $Z' \in A' = \mathsf{Pow}(\downarrow^B a_0) \cap A$ and $(\forall b \in Z) b \leq \Gamma_\Phi(\bigvee Z')$, so that $\bigvee Z \leq \Gamma_\Phi(\bigvee Z')$.

In conclusion, given $Z \in A'$ we have found $Z' \in A'$ such that $\bigvee Z \leq \Gamma_\Phi(\bigvee Z')$. As A is RRS-strongly regular, and $\{b_0\} \in A' \subseteq A$, there is $A_0 \in A$ such that $\{b_0\} \in A_0 \subseteq A'$ and

$$(\forall Z \in A_0)(\exists Z' \in A_0) \bigvee Z \leq \Gamma_\Phi(\bigvee Z'). \tag{2}$$

Let $Y' = \bigcup A_0 \in A$. We have $b_0 \in Y'$ and

$$\bigvee Y' \leq \Gamma_\Phi(\bigvee Y')$$

indeed, if $b \in Y'$, then $b \in Z$ for some $Z \in A_0$. So $b \leq \Gamma_\Phi(\bigvee Z')$ for a set $Z' \in A_0$. As $Z' \subseteq Y'$, $\Gamma_\Phi(\bigvee Z') \leq \Gamma_\Phi(\bigvee Y')$, which gives $b \leq \Gamma_\Phi(\bigvee Y')$.

We have shown $Y' \in A, b_0 \in Y'$, and $\bigvee Y' \leq \Gamma_\Phi(\bigvee Y')$, so that $b_0 \in J_A$ as wished. ⊣

The following corollary is our constructive analog of Tarski's theorem on the existence of greatest fixed points of monotone operators on complete lattices.

Corollary 6.6 (CZF+RRS-$\bigcup$REA). *Let $\Gamma : L \to L$ be a monotone operator on a set-generated $\bigvee$-semilattice L with a bounded presentation, and let Φ be an abstract inductive definition on L. If Φ is a set, and generates Γ, i.e., $\Gamma = \Gamma_\Phi$, then Γ has the greatest fixed point $p = \bigvee \mathcal{C}(\Phi)$.*

Proof. Recall Proposition 6.1 and Lemma 6.4. ⊣

Remark 6.7. By contrast with the case of least fixed points, this result cannot be extended to monotone operators generated by bounded inductive definitions on boundedly presented $\bigvee$-semilattices, not even classically: for a counterexample, consider the identity on the universal class V.

So, over the system CZF+RRS-$\bigcup$REA, a monotone operator on a boundedly presented set-generated $\bigvee$-semilattice L generated by an abstract inductive definition that is a set, has both a greatest and a least fixed point. This holds therefore in particular for such monotone operators on set-presented $\bigvee$-semilattices.

A natural question is whether it is possible to deduce Theorems 6.3 and 6.5 from the corresponding results for (the particular case of) standard inductive definitions. Note first that the standard inductive definition $\Phi_{st} = \{(b, U) \in B \times \mathsf{Pow}(B) \mid b \leq \bigvee U\} \cup \{(b, \downarrow^B a) \mid (b, a) \in \Phi\}$ we associated to an abstract Φ (or its version Φ'_{st}

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

for set-presented $\bigvee$-semilattices) in Section 4 will not do: any subclass of B, and in particular B, is Φ-inclusive in the standard sense for such inductive definitions, so we have to look elsewhere.

An (unsatisfactory) option is to associate with the given Φ, the standard inductive definition corresponding to the maximal abstract inductive definition inducing $\bar{\Gamma}_\Phi$, i.e., we let

$$\Phi^*_{st} = \{(b, \downarrow^B a) \mid b \in \bar{\Gamma}_\Phi(\downarrow^B a)\}.$$

It is an easy exercise to prove that, if Y is a c_L-closed class, Y is Φ-inclusive in the abstract sense if and only if it is Φ^*_{st}-inclusive in the standard sense. Moreover, if Z is Φ^*_{st}-inclusive, also $c_L Z$ is Φ^*_{st}-inclusive, so $C(\Phi^*_{st})$ is c_L-closed, $\mathcal{C}(\Phi)$ exists if and only if $C(\Phi^*_{st})$ does, and in this case they coincide. However, the association of Φ^*_{st} to Φ is not adequate to derive Theorem 6.5, since, even when Φ is a set and the underlying $\bigvee$-semilattice L is set-presented, Φ^*_{st} is in general only a class.

After proving that every monotone mapping $f : L \to L$ on a complete lattice has both a least and greatest fixed point, in [12] Tarski showed that f has, more generally, a complete lattice of fixed points. To prove this, Tarski argued as follows. Let P be the class of fixed points of $f : L \to L$, and let $W \subseteq P$ be a set. The class $[w, \top] = \{x \in L \mid w \leq x \leq \top\}$, with $w = \bigvee_L W$ the join in L of W, is a complete lattice with the partial order inherited by L. Moreover, the restriction of f to $[w, \top]$ is a monotone mapping. Then, f has a least fixed point p, and one has $p = \bigvee_P W$. Dually, one proves that the infimum $\bigvee_P W$ exists.

For a set-presented $\bigvee$-semilattice L, we can argue similarly in the present context, using Corollaries 3.5 and 6.6, provided it holds that intervals $[a, \top]$ and $[0, a]$, for $a \in L$, have a bounded presentation, and so in particular if they are set-presented. For $[a, \top]$, this follows directly from the following lemma, for which we exploit, for the first time in this article, the Subset Collection scheme (in the form of Fullness, cf. Section 1). Note first that, if B_L is a set of generators for L, then, given $a \in L$, $B^a_L = B_L \cup \{a \vee b \mid b \in B_L\}$ is also a set of generators for L.

LEMMA 6.8 (CZF). *Let (L, B_L) be set-presented by $D : B_L \to \mathsf{Pow}(\mathsf{Pow}(B_L))$, and let $a \in L$. Then, (L, B^a_L), with $B^a_L = B_L \cup \{a \vee b \mid b \in B_L\}$, is also set-presented.*

PROOF. Let $\bar{D} = \{(b, U) \mid U \in D(b)\}$. For $c \in B^a_L$, we shall write $\downarrow^B c$ for $\downarrow^{B_L} c$. For each $c \in B^a_L$, $\downarrow^B c$ is a set, so by Fullness, there is a set C with C full in $mv({}^{\downarrow^B c}\bar{D})$. We then have, by Strong Collection, that a set K exists such that

$$(\forall c \in B^a_L)(\exists C \in K)[C \text{ full in } mv({}^{\downarrow^B c}\bar{D})]$$

and

$$(\forall C \in K)(\exists c \in B^a_L)[C \text{ full in } mv({}^{\downarrow^B c}\bar{D})].$$

Let $\bar{C} = \bigcup K$. For $F \in \bar{C}$, let

$$\bar{F} = \{b' \in B_L \mid (\exists b \in \downarrow B_L)(\exists (b'', U) \in \bar{D})(b, (b'', U)) \in F \;\&\; b' \in U\}.$$

Since $\bar{F}$ and B^a_L are sets, again by Fullness, there is a set C' full in $mv({}^{\bar{F}}B^a_L)$. So, applying again Strong Collection, we get a set K' such that

$$(\forall F \in \bar{C})(\exists C' \in K')[C' \text{ full in } mv({}^{\bar{F}}B^a_L)]$$

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

and

$$(\forall C' \in K')(\exists F \in \bar{C})[C' \text{ full in } mv(^{\bar{F}} B_L^a)].$$

Define $\bar{C}' = \bigcup K'$. Then, (L, B_L^a) is set-presented by $D' : B_L^a \to \mathsf{Pow}(\mathsf{Pow}(B_L^a))$, where for $c \in B_L^a$,

$$D'(c) = \{\text{Range}(G) \mid G \in \bar{C}' \,\&\, c \leq \bigvee \text{Range}(G)\}$$

with $\text{Range}(G) = \{c \in B_L^a \mid (\exists b \in B_L)(b, c) \in G\}$. Indeed, assume $c \in B_L^a$ and $W \in \mathsf{Pow}(B_L^a)$ are such that $c \leq \bigvee W$. Then, for all $b \in \downarrow^B c$, $b \leq \bigvee \bigcup_{d \in W} \downarrow^B d$. Thus, for all $b \in \downarrow^B c$ there is $(b', U) \in \bar{D}$ such that $b' = b$ and $U \subseteq \bigcup_{d \in W} \downarrow^B d$. Therefore, $R(b, (b', U)) = [b' = b \,\&\, U \subseteq \bigcup_{d \in W} \downarrow^B d]$ is a total relation from $\downarrow^B c$ to $\bar{D}$, so that there is $F \in \bar{C}$, $F \subseteq R$. For every $b' \in \bar{F}$ one has $d \in B_L^a$ such that $b' \in \downarrow^B d$ and $d \in W$. So we have a total relation $G \in \bar{C}'$ with $G \subseteq R'$, where $R'(b', d) \equiv [b' \in \downarrow^B d \,\&\, d \in W]$. As for every $b \in \downarrow^B c$, $b \leq \bigvee \bar{F}$, we may conclude observing that $c \leq \bigvee \bar{F} \leq \bigvee \text{Range}(G)$ and $\text{Range}(G) \subseteq W$.

The converse, i.e., that from $\text{Range}(G) \subseteq W$, where $\text{Range}(G) \in D'(c)$, one derives $c \leq \bigvee W$, is obvious. ⊣

We then have the following constructive version of Tarski's general result.

THEOREM 6.9 (CZF+RRS-$\bigcup$REA). *Let $\Gamma : L \to L$ be a monotone operator on a $\bigvee$-semilattice L set-presented by $D : B_L \to \mathsf{Pow}(\mathsf{Pow}(B_L))$, and let Φ be an abstract inductive definition on L that is a set, and that generates Γ, i.e., $\Gamma = \Gamma_\Phi$. Then, Γ has a complete lattice P of fixed points.*

PROOF. Let $P \subseteq L$ be the class of fixed points of Γ, partially ordered by the restriction of the order relation of L. Let Y be any subset of P. We want to show that Y has a join and a meet in P. To prove this, we may reproduce the argument in [12], applying Corollaries 3.5 and 6.6 to the restrictions of Γ to the intervals $[\bigvee_L Y, \top_L]$ and $[0_L, \bigwedge_L Y]$, respectively, if they are themselves set-presented ($0_L, \top_L$ denote the bottom and top elements in L). Clearly, $B_{[\bigvee_L Y, \top_L]} = \{\bigvee_L Y \vee b \mid b \in B_L\}$ is a set of generators for $[\bigvee_L Y, \top_L]$. By Lemma 6.8, it immediately follows that $([\bigvee_L Y, \top_L], B_{[\bigvee_L Y, \top_L]})$ is a set-presented $\bigvee$-semilattice (restrict D' from Lemma 6.8 to elements and subsets of $B_{[\bigvee_L Y, \top_L]}$: for $c \in B_{[\bigvee_L Y, \top_L]}$, $D''(c) = \{U \in D'(c) \mid U \subseteq B_{[\bigvee_L Y, \top_L]}\}$). On the other hand, the interval $[0_L, \bigwedge_L Y]$ has $B_{[0_L, \bigwedge_L Y]} = \{b \in B_L \mid b \leq \bigwedge_L Y\}$ as set of generators, and is set-presented by the restriction D^* of D: for $b \in B_{[0_L, \bigwedge_L Y]}$, $D^*(b) = \{U \in D(b) \mid U \subseteq B_{[0_L, \bigwedge_L Y]}\}$.

To prove then that Y has a meet in P, one observes that for every $y \in Y$, $\bigwedge_L Y \leq y$, so that $\Gamma(\bigwedge_L Y) \leq \Gamma(y) = y$. Therefore, $\Gamma(\bigwedge_L Y) \leq \bigwedge_L Y$, which gives $\Gamma(z) \leq \Gamma(\bigwedge_L Y) \leq \bigwedge_L Y$ when $z \leq \bigwedge_L Y$. Hence, $\Gamma : [0_L, \bigwedge_L Y] \to [0_L, \bigwedge_L Y]$ is a monotone mapping on a set-presented $\bigvee$-semilattice. By Corollary 6.6 it thus has a greatest fixed point p in $[0_L, \bigwedge_L Y]$. We have $p \in P$, $p \leq y$ for every $y \in Y$, and if z is any other element with these properties, $z \leq p$, since p is the greatest fixed point of Γ restricted to $[0_L, \bigwedge_L Y]$. Therefore, $p = \bigwedge_P Y$. We leave to the reader the proof that Y has a join in P. ⊣

An interesting problem is to determine under which conditions on the $\bigvee$-semilattice L and/or the monotone mapping Γ, the associated complete lattice of fixed points is itself (set-generated and) set-presented.

https://doi.org/10.1017/jsl.2018.13 Published online by Cambridge University Press

§7. **Acknowledgements.** I am grateful to the anonymous referees for their useful comments on a previous version of this article.

REFERENCES

[1] P. Aczel, *An introduction to inductive definitions*, ***Handbook of Mathematical Logic*** (J. Barwise, editor), Studies in Logic and the Foundations of Mathematics, Vol. 90, North-Holland, Amsterdam, 1977, pp. 739–782.

[2] ———, *The type theoretic interpretation of constructive set theory: Inductive definitions*, ***Logic, Methodology, and Philosophy of Science VII*** (R. B. Marcus, G. J. W. Dorn, and P. Weingartner, editors), Studies in Logic and the Foundations of Mathematics, Vol. 114, North-Holland, Amsterdam, 1986, pp. 17–49.

[3] ———, *Aspects of general topology in constructive set theory*. ***Annals of Pure and Applied Logic***, vol. 137 (2006), no. 1–3, pp. 3–29.

[4] ———, *The relation reflection scheme*. ***Mathematical Logic Quarterly***, vol. 54 (2008), no. 1, 511.

[5] P. Aczel and M. Rathjen, *Notes on constructive set theory*, Mittag-Leffler, Technical report no. 40, 2000/2001.

[6] ———, *Constructive Set Theory*, Book Draft, 2010.

[7] P. Cousot and R. Cousot, *Constructive versions of Tarski's fixed point theorems*. ***Pacific Journal of Mathematics***, vol. 82 (1979), no. 1, pp. 43–57.

[8] G. Curi, *Topological inductive definitions*. ***Annals of Pure and Applied Logic***, vol. 163 (2012), no. 11, pp. 1471–1483. Kurt Gödel Research Prize Fellowships 2010.

[9] ———, *On Tarski's fixed point theorem*. ***Proceedings of the American Mathematical Society***, vol. 143 (2015), pp. 4439–4455.

[10] N. Gambino, *Heyting-valued interpretations for constructive set theory*. ***Annals of Pure and Applied Logic***, vol. 137 (2006), no. 1–3, pp. 164–188.

[11] J. Myhill, *Constructive set theory*, this Journal, vol. 40 (1975), no. 3, pp. 347–382.

[12] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*. ***Pacific Journal of Mathematics***, vol. 5 (1955), pp. 285–309.

[13] A. Troelstra and D. van Dalen, ***Constructivism in Mathematics, an Introduction***, Vol. I, North-Holland, Amsterdam, 1988.

VIA BOITO, 23 - 35122 PADOVA, ITALY
E-mail: giovanni.curi@email.it