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ON THE SPEED OF CONVERGENCE TO AN EXPONENTIAL DISTRIBUTION FOR THE TIME OF THE FIRST OCCURRENCE OF A RARE EVENT IN A REGENERATING PROCESS

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In this article, the time of the first occurrence of a rare event in a regenerating process is investigated. We obtain the bound of deviation from the distribution of the time of the first occurrence of a rare event in a regenerating process to an exponential distribution.

1. INTRODUCTION

Probability analysis for rare events of regenerating processes has been widely used in reliability theory, queuing theory, and risk theory and has drawn considerable attention in the past few years; see, for example, Asmussen [2], Brown [3], and Resnick [11]. Some important quantities in reliability systems, queuing networks, and risk models (e.g., the first failure rate of the system, the first loss of a customer of the queues with limited waiting room, the bankrupt time, etc.) can be described by the time of the first occurrence of a rare event in a regenerating process; see Keilson [10]. However, it is very difficult to give an exact probability evaluation of the first occurrence time of the rare events in a general regenerating process. This stimulates interest in asymptotic methods in assessing the first occurrence time's probabilities. Moreover, in many regenerating random processes of great practical interest, "small

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parameters" are usually present (e.g., the rare event of the regenerating processes occurs with a smaller probability). This feature makes it more possible to use powerful approximation methods for the estimates of the occurrence probability. The goal of this article is to establish the error bound for the asymptotic probability estimate of the first occurrence time of a rare event in a general regenerating random process.

First, we give a mathematical description of a rare event in a general regenerating random process. Let $\{(K_n(t): t \geq 0): n \geq 1\}$ be a sequence of regenerating processes. For the nth regenerating process $(K_n(t): t \geq 0)$, let

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_k^{(n)} < \cdots$$

be the successive regeneration instants. Set

$$\xi_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}, \qquad k = 1,2,\ldots.$$

In general, we assume that in each regeneration period of the nth regenerating process, an event $A_k^{(n)}$ can occur at some instant $t_{k+1}^{(n)} + \eta_k^{(n)}$, where $0 \leq \eta_k^{(n)} < \xi_k^{(n)}$. Here, we assume that $I_{\{A_k^{(n)}\}}$ and $\eta_k^{(n)}$ are measurable relative to the σ field generated by $\{K_n(t): t_{k-1}^{(n)} \leq t < t_k^{(n)}\}$ and that $\{I_{\{A_k^{(n)}\}}: k \geq 1\}$ and $\{\eta_k^{(n)}: k \geq 1\}$ are sequences of independent and identically distributed random variables, where $I_{\{A_k^{(n)}\}}$ is the indicator function of the set $A_k^{(n)}$. Let $A^{(n)} = \bigcup_{k=1}^{\infty} A_k^{(n)}$ and $\theta^{(n)}$ be the time of the first occurrence of the event $A^{(n)}$ in the nth regenerating process. In reliability theory, $\theta^{(n)}$ is called the time of the first system failure. Moreover, we introduce the following notation first used by Solovyev [13]:

$$\zeta_k^{(n)} = \xi_k^{(n)} I_{\{(A_k^{(n)})^c\}} + \eta_k^{(n)} I_{\{A_k^{(n)}\}},$$

$$p^{(n)} = P(I_{\{(A_k^{(n)})^c\}} = 1) \quad \text{and} \quad \alpha^{(n)} = E\zeta_1^{(n)}.$$

Solovyev [13] obtained that if $E(\zeta_1^{(n)})^q < \infty$ for some $q > 1$, then

$$\lim_{p^{(n)} \to 1} P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x\right) = e^{-x}. \tag{1}$$

Furthermore, Solovyev [13] established that if $E(\zeta_1^{(n)})^q < \infty$ for some $2 < q \leq 3$, then there is a constant C such that for all $n \geq 1$,

$$\sup_{0 \leq x < \infty} \left| P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x\right) - e^{-x}\right| \leq C(1-p^{(n)}). \tag{2}$$

Afterward, Brown [3] considerably weakened Solovyev's assumption on the moment of $\zeta_1^{(n)}$. Brown proved that (2) is still true if only the second moment of $\zeta_1^{(n)}$ is finite. *Naturally, we would expect to know whether the bound like to* (2) *holds true under the case* $E(\zeta_1^{(n)})^q < \infty$ *for some* $1 < q < 2$. This is very interesting for both theoretical research and application research; see, for example, Gertsbakh [6] and Cao and Chen [4]. Solovyev's proof for (2) makes essential use of the assumption

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that $E(\zeta_1^{(n)})^q < \infty$ for some $2 < q \le 3$ in having the aid of Esseen's inequality. For $E(\zeta_1^{(n)})^2 < \infty$, Brown [3] established (2) with the help of renewal theory and NWU (new worse than used) theory. In this article, we establish an effective estimation of the rate of convergence of the distribution of $(1 - p^{(n)})\theta^{(n)}/\alpha^{(n)}$ to an exponential distribution under the condition of $E(\zeta_1^{(n)})^q < \infty$ for some $1 < q < 2$ by using the basic probability method.

Note that as $p^{(n)}$ is close to 1, the event $A^{(n)}$ is a rare event. Thus, (2) deals just with the approximation for a rare event. Hüsler [8] used the point process to study the rare events of nonstationary sequences, and Shwartz and Weiss [12] investigated the rare events of queuing networks via large deviation and time reversal. Glynn and Iglehart [7] established the importance of sampling approach to give the rare event simulation. Asmussen [1, 2] considered rare events in the presence of heavy tails in stochastic networks and in risk models. In reliability theory, many systems have low element failure rates and/or the element failure rates are much smaller than repair rates. In these cases, we are interested in the time of the first system failure. It is just described by Solovyev [13] (see also Kalashnikov [9]) that the operation of the so-called "fast repaired" model until the appearance of the first system failure can be equivalent to the occurrence of the event $A^{(n)}$ in the regenerating process $\{(K_n(t) : t \ge 0) : n \ge 1\}$. Therefore, (2), in fact, gives the deviation from the distribution of the time of the first system failure to an exponential distribution.

2. THE MAIN RESULT

Our main result is presented in this section.

THEOREM 1: *If* $\inf_{n\ge1}\alpha^{(n)} > 0$ *and* $\sup_{n\ge1}(\zeta_1^{(n)})^q < \infty$ *for some* $1 < q \le 2$, *then there exists a constant* $D(q)$ *depending on q such that for all* $n \ge 1$ *and each* $\delta > 0$ $(0 < \delta < (q-1)/(q+1))$,

$$\sup_{(1-p^{(n)})^\delta \le x < \infty} \left| P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x\right) - e^{-x}\right| \le D(q)(1-p^{(n)})^\delta.$$

PROOF: The proof of the theorem is divided into several steps.

Step 1. Set

$$X^{(n)} = \alpha^{(n)} + \alpha^{(n)} I_{\{(A_1^{(n)})^c\}} + \alpha^{(n)} I_{\{(A_1^{(n)})^c \cap (A_2^{(n)})^c\}} + \cdots.$$

Then, we have

$$P\left(\frac{(1-p^{(n)})X^{(n)}}{\alpha^{(n)}} > x\right) = P\left(1 + \sum_{k=1}^{\infty} I_{\{\cap_{i=1}^k \{A_i^{(n)}\}^c\}} > \frac{x}{1-p^{(n)}}\right)$$

$$= (p^{(n)})^{[x/(1-p^{(n)})]}, \tag{3}$$

where $[x/(1-p^{(n)})]$ is the greatest integer equal to or less than $x/(1-p^{(n)})$. Furthermore,

$$\begin{aligned}(p^{(n)})^{[x/(1-p^{(n)})]} &= e^{[x/(1-p^{(n)})]\log p^{(n)}} \\ &= \exp\left\{-\left[\frac{x}{1-p^{(n)}}\right]\left(\sum_{k=1}^{\infty}\frac{(1-p^{(n)})^k}{k}\right)\right\}. \qquad (4)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\exp\left\{-\left[\frac{x}{1-p^{(n)}}\right]\left(\sum_{k=1}^{\infty}\frac{(1-p^{(n)})^k}{k}\right)\right\} \\ &\quad\geq \exp\left\{-\sum_{k=1}^{\infty}\frac{x(1-p^{(n)})^{k-1}}{k}\right\} \qquad (5)\end{aligned}$$

and

$$\begin{aligned}&\exp\left\{-\left[\frac{x}{1-p^{(n)}}\right]\left(\sum_{k=1}^{\infty}\frac{(1-p^{(n)})^k}{k}\right)\right\} \\ &\quad\leq \exp\left\{\sum_{k=1}^{\infty}\frac{(1-p^{(n)})^k}{k}-\sum_{k=1}^{\infty}\frac{x(1-p^{(n)})^{k-1}}{k}\right\}. \qquad (6)\end{aligned}$$

Moreover, it is easy to prove that there exists a constant D_1 such that for $n \geq 1$,

$$\sup_{0\leq x<\infty}\left|\exp\left\{-\sum_{k=1}^{\infty}\frac{x(1-p^{(n)})^{k-1}}{k}\right\}-e^{-x}\right| \leq D_1(1-p^{(n)}),$$

and

$$\begin{aligned}&\sup_{0\leq x<\infty}\left|\exp\left\{\sum_{k=1}^{\infty}\frac{(1-p^{(n)})^k}{k}-\sum_{k=1}^{\infty}\frac{x(1-p^{(n)})^{k-1}}{k}\right\}-e^{-x}\right| \\ &\quad\leq D_1(1-p^{(n)}).\end{aligned}$$

Hence, by (3)–(6), we have

$$\sup_{0\leq x<\infty}\left|P\left(\frac{(1-p^{(n)})X^{(n)}}{\alpha^{(n)}}>x\right)-e^{-x}\right| \leq D_1(1-p^{(n)}). \qquad (7)$$

Step 2. Let $\mathcal{F}_0^{(n)}$ be the trivial σ field, and

$$\mathcal{F}_k^{(n)} = \sigma(\xi_1^{(n)},\ldots,\xi_k^{(n)};\eta_1^{(n)},\ldots,\eta_k^{(n)};I_{\{A_1^{(n)}\}},\ldots,I_{\{A_k^{(n)}\}}),$$

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$k = 1, 2, \ldots$. We know that

$$\{(\zeta_k^{(n)} - \alpha^{(n)}) I_{\{\cap_{i=0}^{k-1} (A_i^{(n)})^c\}} : k \geq 1\}$$

is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_k^{(n)} : k \geq 1\}$, where $(A_0^{(n)})^c = \Omega$. Now, define $S_0^{(n)} = 0$ and

$$S_k^{(n)} = \sum_{i=1}^{k} (\zeta_i^{(n)} - \alpha^{(n)}) I_{\{\cap_{j=0}^{i-1} (A_j^{(n)})^c\}}, \qquad k \geq 1.$$

Then, $\{S_k^{(n)}, \mathcal{F}_{k-1}^{(n)} : k \geq 1\}$ is a martingale. Using the Burkholder inequality for martingale (see Chow and Teicher [5, Cor. 1, p. 397]) and the C_r inequality, we obtain for $x > 0$,

$$\begin{aligned}
P&\left(\max_{1 \leq k \leq m} |S_k^{(n)}| \geq \frac{\alpha^{(n)} x}{1 - p^{(n)}}\right) \\
&\leq \left(\frac{1 - p^{(n)}}{\alpha^{(n)} x}\right)^q E\left(\max_{1 \leq k \leq m} |S_k^{(n)}|\right)^q \\
&\leq A_q \left(\frac{1 - p^{(n)}}{\alpha^{(n)} x}\right)^q E\left(\sum_{k=1}^{m} (S_k^{(n)} - \alpha^{(n)})^2 I_{\{\cap_{i=0}^{k} (A_i^{(n)})^c\}}\right)^{q/2} \\
&\leq A_q \left(\frac{1 - p^{(n)}}{\alpha^{(n)} x}\right)^q \sum_{k=1}^{m} E((S_k^{(n)} - \alpha^{(n)})^q I_{\{\cap_{i=0}^{k} (A_i^{(n)})^c\}}),
\end{aligned}$$

where A_q is a constant only depending on q. Consequently, there exists a constant B_q only depending on q such that, for $x > 0$,

$$P\left(\max_{1 \leq k < \infty} |S_k^{(n)}| \geq \frac{\alpha^{(n)} x}{1 - p^{(n)}}\right) \leq \frac{B_q}{(\alpha^{(n)} x)^q} (1 - p^{(n)})^{q-1}. \tag{8}$$

Step 3. First, we have

$$\begin{aligned}
\theta^{(n)} &= \zeta_1^{(n)} + \sum_{k=2}^{\infty} \zeta_k^{(n)} I_{\{\cap_{i=0}^{k-1} (A_i^{(n)})^c\}} \\
&= \sum_{k=1}^{\infty} (\zeta_k^{(n)} - \alpha^{(n)}) I_{\{\cap_{i=0}^{k-1} (A_i^{(n)})^c\}} + \sum_{k=1}^{\infty} \alpha^{(n)} I_{\{\cap_{i=0}^{k-1} (A_i^{(n)})^c\}}.
\end{aligned}$$

Set

$$Y^{(n)} = \sum_{k=1}^{\infty} (\zeta_k^{(n)} - \alpha^{(n)}) I_{\{\cap_{i=0}^{k-1} (A_i^{(n)})^c\}}.$$

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Then,

$$
\begin{aligned}
P&\left(\frac{(1-p^{(n)})X^{(n)}}{\alpha^{(n)}} > x\right)\\
&= P\left(X^{(n)} + Y^{(n)} > \frac{\alpha^{(n)}x}{1-p^{(n)}}\right)\\
&= P\left(X^{(n)} + Y^{(n)} > \frac{\alpha^{(n)}x}{1-p^{(n)}}, |Y^{(n)}| \geq \frac{\alpha^{(n)}(1-p^{(n)})^{\delta}}{1-p^{(n)}}\right)\\
&\quad + P\left(X^{(n)} + Y^{(n)} > \frac{\alpha^{(n)}x}{1-p^{(n)}}, |Y^{(n)}| < \frac{\alpha^{(n)}(1-p^{(n)})^{\delta}}{1-p^{(n)}}\right).
\end{aligned} \tag{9}
$$

On the other hand,

$$
\begin{aligned}
P&\left(X^{(n)} + Y^{(n)} > \frac{\alpha^{(n)}x}{1-p^{(n)}}, |Y^{(n)}| < \frac{\alpha^{(n)}(1-p^{(n)})^{\delta}}{1-p^{(n)}}\right)\\
&\leq P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x-(1-p^{(n)})^{\delta})\right),
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
P&\left(X^{(n)} + Y^{(n)} > \frac{\alpha^{(n)}x}{1-p^{(n)}}, |Y^{(n)}| < \frac{\alpha^{(n)}(1-p^{(n)})^{\delta}}{1-p^{(n)}}\right)\\
&\geq P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x+(1-p^{(n)})^{\delta})\right)\\
&\quad - P\left(|Y^{(n)}| \geq \frac{\alpha^{(n)}(1-p^{(n)})^{\delta}}{1-p^{(n)}}\right).
\end{aligned} \tag{11}
$$

Using (7), there exists a constant D_2 such that, for all $n \geq 1$,

$$
\begin{aligned}
&\sup_{0\leq x<\infty}\left|P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x-(1-p^{(n)})^{\delta})\right) - e^{-x}\right|\\
&\leq \left(\sup_{0\leq x\leq t_n}\left|P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x-(1-p^{(n)})^{\delta})\right) - e^{-x}\right|\right)\\
&\vee \left(\sup_{t_n\leq x<\infty}\left|P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x-(1-p^{(n)})^{\delta})\right) - e^{-x}\right|\right)\\
&\leq \left(D_1(1-p^{(n)}) + \sup_{t_n\leq t<\infty}|e^{-x} - \exp\{-x+(1-p^{(n)})^{\delta}\}|\right) \vee t_n\\
&< D_2\, t_n,
\end{aligned} \tag{12}
$$

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where $t_n = (1-p^{(n)})^\delta$. An estimation, similar to that of (12), now gives that there is a constant D_3 such that, for $n \geq 1$,

$$\sup_{0\leq x<\infty}\left|P\left(X^{(n)} > \frac{\alpha^{(n)}}{1-p^{(n)}}\,(x+(1-p^{(n)})^\delta)\right) - e^{-x}\right| \\ \leq D_3(1-p^{(n)})^\delta. \qquad \textbf{(13)}$$

On combining (8)–(13), we know that there is a constant $D(q)$ only depending on q such that, for all $n \geq 1$,

$$\sup_{(1-p^{(n)})^\delta\leq x<\infty}\left|P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x\right) - e^{-x}\right| \leq D(q)(1-p^{(n)})^\delta.$$

This completes the proof of the theorem. ■

From Theorem 1, we immediately obtain the following result proved by Solovyev [13].

COROLLARY: *If* $E(\zeta_1^{(n)})^q < \infty$ *for some* $q > 1$, *then*

$$\lim_{p^{(n)}\to 1} P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x\right) = e^{-x}.$$

Finally, we consider an example that satisfies the condition in Theorem 1 but does not satisfy the condition required by Solovyev [13].

Example: As in the notation given in Section 1, we let

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_k^{(n)} < \cdots$$

be the successive regeneration instants of the nth regenerating process $(K_n(t): t \geq 0)$, $n = 1,2,\ldots$. Set

$$\xi_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}, \qquad k = 1,2,\ldots.$$

We assume that

$$\eta_k^{(n)} = \min\left\{\left(\frac{\xi_k^{(n)}}{2}\right), n\right\}$$

and $\xi_k^{(n)}$ has the following density function:

$$\frac{a_n}{2+t^{1+\epsilon+1/n}}, \qquad t \geq 0,$$

where $0 < \epsilon \leq 1$ and

$$a_n^{-1} = \int_0^\infty \frac{dt}{2+t^{1+\epsilon+1/n}}.$$

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$\{I_{\{A_k^{(n)}\}} : k \geq 1\}$, $\{\zeta_k^{(n)} : k \geq 1\}$, $\alpha^{(n)}$, $p^{(n)}$, and $\theta^{(n)}$ are the same as in Section 1. Then, after some calculations, we know that for any n, the $(1+\epsilon/2)$th moment of $\zeta_k^{(n)} (= \xi_k^{(n)} I_{\{(A_k^{(n)})^c\}} + \eta_k^{(n)} I_{\{A_k^{(n)}\}})$ is finite and uniformly bounded in n, *but* the $(1+\epsilon)$th moment of $\zeta_k^{(n)}$ is infinite. Therefore, this model satisfies the condition in Theorem 1 but does not satisfy the condition required by Solovyev [13]. Theorem 1 implies that there exists a constant $D(1+\epsilon/2)$ such that for all $n \geq 1$ and δ $(0 < \delta < \epsilon/(4+\epsilon))$,

$$\sup_{(1-p^{(n)})^{\delta} \leq x < \infty} \left| P\left(\frac{(1-p^{(n)})\theta^{(n)}}{\alpha^{(n)}} > x \right) - e^{-x} \right| \leq D(1+\epsilon/2)(1-p^{(n)})^{\delta}.$$

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