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Gaussian Distribution of Trie Depth for Strongly Tame Sources

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The depth of a trie has been deeply studied when the source which produces the words is a simple source (a memoryless source or a Markov chain). When a source is simple but not an unbiased memoryless source, the expectation and the variance are both of logarithmic order and their dominant terms involve characteristic objects of the source, for instance the entropy. Moreover, there is an asymptotic Gaussian law, even though the speed of convergence towards the Gaussian law has not yet been precisely estimated. The present paper describes a 'natural' class of general sources, which does not contain any simple source, where the depth of a random trie, built on a set of words independently drawn from the source, has the same type of probabilistic behaviour as for simple sources: the expectation and the variance are both of logarithmic order and there is an asymptotic Gaussian law. There are precise asymptotic expansions for the expectation and the variance, and the speed of convergence toward the Gaussian law is optimal. The paper first provides analytical conditions on the Dirichlet series of probabilities of a general source under which this Gaussian law can be derived: a pole-free region where the series is of polynomial growth. In a second step, the paper focuses on sources associated with dynamical systems, called dynamical sources, where the Dirichlet series of probabilities is expressed with the transfer operator of the dynamical system. Then, the paper extends results due to Dolgopyat, already generalized by Baladi and Vallée, and shows that the previous analytical conditions are fulfilled for 'most' dynamical sources, provided that they 'strongly differ' from simple sources. Finally, the present paper describes a class of sources not containing any simple source, where the trie depth has the same type of probabilistic behaviour as for simple sources, even with more precise estimates.

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1. Introduction

1.1. Tries

A *trie* is a tree structure which is used as a dictionary in various applications, such as partial match queries, text processing tasks or compression. As Flajolet wrote in [13], this justifies considering the trie structure as one of the central general-purpose data structures of computer science.

The trie structure is based on a splitting according to symbols encountered in strings. If $\mathcal{X}$ is a set of strings over the alphabet $\Sigma = \{m_1, m_2, \ldots\}$ (finite or countably infinite), then the trie associated with $\mathcal{X}$ is defined recursively by the following rules: if $\mathcal{X}$ is empty, then Trie($\mathcal{X}$) is empty; if $\mathcal{X}$ has only one element X, then Trie($\mathcal{X}$) is a leaf labelled with X. For $|\mathcal{X}| \geqslant 2$, the trie Trie($\mathcal{X}$) is an internal node to which the sequence

$$(\mathrm{Trie}(\mathcal{X}_{[m_1]}), \mathrm{Trie}(\mathcal{X}_{[m_2]}), \ldots, \mathrm{Trie}(\mathcal{X}_{[m_r]}), \ldots)$$

is attached. Here, the set $\mathcal{X}_{[m]}$ gathers the words of $\mathcal{X}$ that start with the symbol m and are stripped of their initial symbol m.

As was recognized largely by Jacquet, Louchard and Szpankowski (see, *e.g.*, [24, 29, 30]), digital tree analyses can serve as the basis of a remarkably precise understanding of the Lempel and Ziv schemes for data compression. The complexity of many algorithms that use the trie as their main underlying data structure can be expressed with various parameters of tries, for instance the *path length*, the *size*, the *height*, or the *depth*. The size is the total number of internal nodes; the length of a branch is the number of internal nodes it contains; the path length is the sum of the lengths of all the branches; the depth is the length of a (uniformly randomly selected) branch; the height is the maximum of the lengths of all the branches.

1.2. Tries built on simple sources

The probabilistic behaviour of these trie parameters strongly depends on the process which emits the words contained on the trie. In the context of information theory, a *source* is a probabilistic process, with discrete time, that emits symbols from the alphabet Σ one by one. If Y_i is the symbol emitted at time $t = i$, the source, described by the sequence $(Y_1, Y_2, \ldots, Y_i, \ldots)$ of random variables, emits infinite words of $\Sigma^{\mathbb{N}}$ and defines a probability distribution on the set $\Sigma^{\mathbb{N}}$. The sources for which the correlations between successive symbols are weak are called *simple* sources: there are *memoryless* sources, where the symbols Y_i are drawn independently with the same distribution, or *Markov chains* (of order 1), where the random variable Y_{i+1} depends only on the previously emitted symbol Y_i.

When the trie is built on a simple source, the probabilistic behaviour of all the trie parameters is now well understood, and the book by Szpankowski [37] provides a complete review of the main results. The first work in the average-case analysis of tries is due to Knuth [27], followed by the seminal paper by Flajolet and Sedgewick [15]. Over time, in work by Jacquet, Louchard, Régnier, Szpankowski [23, 24, 22, 29, 30] and many others, all the main trie parameters have been analysed, in the case of simple sources. For instance, the trie depth has a mean value of order $\log n$, and its distribution is known

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to be asymptotically Gaussian, except in the case when the simple source is an *unbiased* memoryless source (all the symbols are independently emitted with the same probability). However, even for these simple sources, the existing results are not as precise as they could be: neither the speed of convergence towards the limit law nor the complete asymptotic expansions of the mean (and the variance) are precisely described in the literature. The recent work of Flajolet, Roux and Vallée [14] is a first step towards making the asymptotic behaviour of tries for simple sources more precise.

1.3. General sources, Dirichlet generating functions and dynamical sources

We are interested in the case when the words contained in the trie are emitted by a general source. A general source for the alphabet Σ is completely defined by the set (p_w) of its *fundamental probabilities*: for $w \in \Sigma^\star$, the fundamental probability p_w is the probability that a word emitted by the source begins with the (finite) prefix w. As noted early on for simple sources [12, 27], and further extended to the case of a general source [39, 8, 40], a central object of the analysis involves the *Dirichlet generating functions of probabilities* – the plain generating function $\Lambda(s)$, or the bivariate generating function $\Lambda(s,v)$ – which are defined in terms of the series

$$\Lambda_k(s) := \sum_{w\in\Sigma^k} p_w^s, \tag{1.1}$$

as

$$\Lambda(s,v) := \sum_{k\geqslant 0} v^k \Lambda_k(s) = \sum_{k\geqslant 0} v^k \sum_{w\in\Sigma^k} p_w^s, \quad \Lambda(s) := \Lambda(s,1) = \sum_{w\in\Sigma^\star} p_w^s. \tag{1.2}$$

In the past decade, Vallée [39] has introduced and studied the class of *dynamical sources*. This model of sources, built from dynamical systems, first encompasses all the simple sources (memoryless sources and Markov chains), and unifies their treatment. It also provides a natural and general framework where the dependency between symbols may depend on the whole history. Moreover, the Dirichlet generating function defined in (1.2) may be expressed via generalized *transfer operators*, namely *secant* transfer operators, which are introduced in [39]. This explains why this model can be studied precisely and analysed with mixed tools, originating in both dynamical systems theory and analytic combinatorics.

1.4. Tameness of a source

Observe that, for any $k \geqslant 0$, the series $\Lambda_k(s)$ satisfy $\Lambda_k(1) = 1$, so that the equality $\Lambda(1,v) = 1/(1-v)$ holds and proves that $(s,v) \mapsto \Lambda(s,v)$ is always singular at $(1,1)$. The behaviour of $\Lambda(s,v)$, when $\Re s$ is close to 1 and v close to 1, summarizes the main probabilistic properties of the source, and is central to Rice's methodology, which is one of the main tools for analysing trie parameters. We first consider the case when v equals 1, and we are interested in *tameness properties* of the source. The word *tame* was proposed by Philippe Flajolet and used for the first time in [40]. Subsequently, most papers that deal with probabilistic sources have used similar notions, and the word 'tame' is now widely used, for instance in the paper [9], in this issue of *Combinatorics, Probability and Computing*. A *tameness region* for the source is a region which strictly contains the

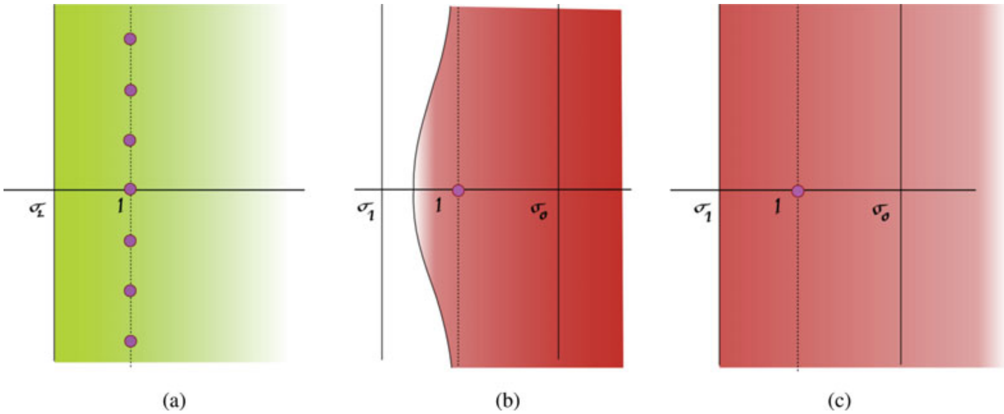

Figure 1. (Colour online) Three situations for the pole-free region $\mathcal{R}$: the periodic case (a), and the aperiodic case, which gives rise to two main subcases: H-tameness (b) and S-tameness (c).

half-plane $\Re s \geqslant 1$, where $\Lambda(s)$ is *analytic* and of *polynomial growth* for $|s| \to \infty$. Figure 1 describes three possible shapes for tameness regions, which will be made precise later on in the paper, and are now summarized briefly.

Periodic sources. If $\Lambda(s)$ admits a pole on the punctured line $\{\Re s = 1, s \neq 1\}$, it admits an infinite set of poles s_k regularly spaced on this line, of the form $s_k = 1 + 2i\pi k\tau$ (for some real $\tau > 0$, and k varying in $\mathbb{Z}$), and the source is thus called *periodic*. In this case, there is a vertical strip $\{1 - \alpha < \Re s < 1\}$ (for some $\alpha > 0$) which is pole-free, and the tameness region is a punctured half-plane $\{\Re s > 1 - \alpha, s \neq s_k\}$: the source is said to be P-tame (see Figure 1(a)).

Aperiodic sources. On the other hand, if the only pole located on the line $\{\Re s = 1\}$ is $s = 1$, the source is said to be *aperiodic*. In this case, the poles of Λ may come close to the left of the vertical line $\Re s = 1$ when $|\Im s|$ becomes large, and an aperiodic source is tame if the poles of $\Lambda(s)$ come close to the vertical line $\Re s = 1$ *but not too fast*, namely with polynomial speed: this means that the points $s = \sigma + it$ of the frontier of $\mathcal{R}$ satisfy $1 - \sigma = \Omega(|t|^{-\beta})$ for some $\beta \geqslant 0$. The smallest possible exponent β is the hyperbolic exponent.

(i) When $\beta > 0$, the tameness region has a hyperbolic shape (see Figure 1(b)), and the source is *hyperbolic tame* (H-tame for short).

(ii) The case $\beta = 0$ gives rise to the largest possible tameness region, which is now a vertical strip (see Figure 1(c)), and the source is said to be *strip tame* (S-tame for short).

Strongly tame sources. Very often, in this last situation, where there exists a vertical strip as a tameness region, the tameness region is large enough to be 'perturbed'. This gives rise to the notion of strong tameness, which describes 'nice' behaviour for the Dirichlet series $\Lambda(s, v)$: there exists a complex neighbourhood of $v = 1$ and a vertical strip $\mathcal{R}$ in which the

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Dirichlet generating function $\Lambda(s,v)$ admits a unique pole and is of polynomial growth, when $|\Im s| \to \infty$ (uniformly with respect to v).[1]

1.5. Role of source tameness in the analysis of tries

This paper has three main aims.

(a) We study the probabilistic behaviour of trie parameters, when the trie is built on a general source. With the use of Rice's methodology, we make the role of tameness in the analysis of trie parameters more precise, first in the general case in Section 2, and then, in Section 3, in the particular case of simple sources.

(b) We focus on the case when the source is strongly tame (the best situation from the tameness perspective). In this case the analysis of trie parameters can be performed in a transparent way, with the joint use of Rice's methodology and the Quasi-Powers Theorem. This leads to asymptotic Gaussian laws with optimal speed (see Section 2).

(c) We exhibit general sources which arise in a natural way and are strongly tame. Most simple sources are P-tame or H-tame, but a simple source is *never* strongly tame. Thus, strongly tame sources have to be found amongst sources that are not simple. We shall prove that a source is strongly tame as soon as it *strongly differs from a simple source*. We deal with the class of *good* dynamical sources that satisfy the *UNI Condition* (uniform non-integrability). This class was introduced by Dolgopyat [10]. In Sections 5 and 6 we extend results due to Dolgopyat and generalized by Baladi and Vallée [2].

1.6. Comparison with previous results

In the case of simple sources, in Section 3 we study precisely the possible types of tameness and obtain precise remainder terms in the asymptotic estimates of the expectation and the variance of trie depth, correcting 'classical' results of the literature. The type of remainder term is closely related to the type of tameness (P,H) of $\Lambda(s)$. Section 3 is a summary of results that are partially described in [14] but not yet well known.

In the case of a general dynamical source, the probabilistic analysis of three main parameters of a trie built on a dynamical source was achieved by Clément, Flajolet and Vallée [8]: the authors studied the path length, the size, and the height, mainly in the average case, except for the height which was analysed in distribution. Subsequently, Bourdon [3] extended this study to Patricia tries. The study of the size and path length in papers [8, 3] is not completely exact since it cannot be applied to any dynamical source. The proof needs the source to be tame, and the results of [14, 33, 34] are needed to complete the proof of [8]. Our parameter of interest, the depth, was precisely analysed by Flajolet and Vallée [17], for the particular source related to the continued fraction dynamical system. The authors exhibited the mean value of the depth and related it to some classical constants, together with the Riemann hypothesis.

[1] In Section 7 we shall return to the possible perturbation of the notion of H-tameness.

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1.7. Main results of the paper

The present paper is devoted to the *distributional* analysis of the *depth* of a trie built on a general source. It can be viewed as an extension of the three papers [8, 17, 2]. We use the general methodology for analysis of tries described in [8]. We also apply some ideas that come from [17], well adapted to the study of this particular parameter (the depth), and extend them to a general dynamical source. And finally, since we wish to obtain distributional results, we extend results of Dolgopyat [10], already generalized by Baladi and Vallée [2], to the 'secant' transfer operator associated with a dynamical source. The main results of the paper can be described as follows.

(i) Consider a general source which is strongly tame. The depth of a random trie built on n words independently emitted by this source is asymptotically Gaussian, with an expectation and a variance of order $\log n$ and a speed of convergence of order $(\log n)^{-1/2}$.

(ii) Any dynamical source of the *Good Class* which satisfies the *UNI Condition* is strongly tame. Moreover, the constants which appear in the main terms of the mean and the variance of the trie depth are expressed in terms of the spectral objects of transfer operators, and they are computable.

1.8. Plan of the paper

Section 2 describes the general framework of sources and tries, and states an initial result which explains how to deal with the asymptotic behaviour of the trie depth when the source is strongly tame. Section 3 introduces tameness of sources more generally, and studies the behaviour of the trie depth in the case of simple sources. In Section 4 we introduce dynamical sources, and describe the subclass of interest, the *Good-UNI Class*, which gathers dynamical sources that can be proved to be strongly tame. We explain the central role that is played by the secant transfer operator, as it transfers geometric properties of the source into analytic properties for the generating function of the trie depth. Finally, Sections 5 and 6 focus on the case when the source belongs to the Good-UNI Class. We describe the main spectral properties of the secant transfer operator, when the parameter s is close to the real axis (Section 5) or far from the real axis (Section 6). The results of this paper have been stated in [5].

2. General framework: sources, tries, the Gaussian law for the depth of a trie

Here, the main objects of interest are introduced: sources, with their fundamental probabilities and their generating functions $\Lambda(s), \Lambda(s,v)$, in Section 2.1, and then tries in Section 2.2. In Section 2.3 we relate the probabilistic behaviour of the trie depth to the generating function of the source. This expression involves a binomial sum, leading us to Rice's methodology, which is recalled in Section 2.4. It is possible to use this method if we have good knowledge about the Dirichlet series $\Lambda(s,v)$ when both $\Re s$ and v are close to 1. This leads us to introduce the notion of strong tameness. Then, Section 2.6 focuses on the case when the source is strongly tame, and provides a simple estimate for the probability generating function of the depth. Finally, Sections 2.7 and 2.8 explain how an asymptotic

Gaussian law can be derived for the trie depth when the words are emitted by a strongly tame source.

2.1. General model for a source

Throughout this paper, an alphabet Σ (finite or denumerable) of symbols is fixed.

Definition 1. A *probabilistic source* over the alphabet Σ is defined by a sequence of random variables $(Y_1, \dots, Y_i, \dots)$. Each Y_i represents the symbol which is emitted by the source at time $t = i$ and the source produces infinite words of $\Sigma^{\mathbb{N}}$. A probabilistic source defines a probability $\mathbb{P}$ on the space $\Sigma^{\mathbb{N}}$ which is specified by the set $\{p_w, w \in \Sigma^\star\}$ of *fundamental probabilities* p_w, where p_w is the probability that an infinite word begins with the finite prefix w. Namely, for $w \in \Sigma^k$, we have $p_w := \mathbb{P}[(Y_1, Y_2, \dots, Y_k) = w]$.

Our analyses mainly deal with the Λ series of Dirichlet type, which involve fundamental probabilities, already defined in (1.2). For instance, the entropy $h(\mathcal{S})$ of a probabilistic source $\mathcal{S}$ is defined in terms of fundamental probabilities, *i.e.*,

$$h(\mathcal{S}) := \lim_{k\to\infty} \frac{-1}{k} \sum_{w\in\Sigma^k} p_w \log p_w = \lim_{k\to\infty} \frac{-1}{k} \frac{d}{ds} \Lambda_k(s)|_{s=1}, \tag{2.1}$$

and thus involves the Λ series.

2.2. Description of a trie

We now describe the second main object of this work, the trie, which is a tree structure, used as a dictionary, that compares words via their prefixes.

Definition 2. Given a finite set $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ formed with n (infinite) words emitted by the source, the tree Trie($\mathcal{X}$) built on the set $\mathcal{X}$ is defined recursively by the following rules.

(i) If $|\mathcal{X}| = 0$, Trie$(\mathcal{X}) = \emptyset$.
(ii) If $|\mathcal{X}| = 1$, $\mathcal{X} = \{X\}$, Trie$(\mathcal{X})$ is a leaf labelled by X.
(iii) If $|\mathcal{X}| \geqslant 2$, then Trie$(\mathcal{X})$ is formed with an internal node and n subtries respectively equal to

$$\text{Trie}(\mathcal{X}_{[m_1]}), \dots, \text{Trie}(\mathcal{X}_{[m_r]}),$$

where $\mathcal{X}_{[m]}$ denotes the subset which gathers the words of $\mathcal{X}$ that begin with the symbol m, stripped of their initial symbol m. If the set $\mathcal{X}_{[m]}$ is non-empty, the edge which links the subtrie Trie$(\mathcal{X}_{[m]})$ to the internal node is labelled with the symbol m.

For a sequence $\mathcal{X} := \{X_1, X_2, \dots, X_n\}$ with $n \geqslant 2$, the trie Trie$(\mathcal{X})$ has exactly n branches, and the length of a branch is the number of (internal) nodes it contains. For $i \in [1..n]$, the length of the ith branch of the trie (corresponding to the word X_i) is denoted by $D_n^{(i)}$. In this paper, the parameter of interest is the depth D_n of a random branch. If $\mathbb{P}$ is the

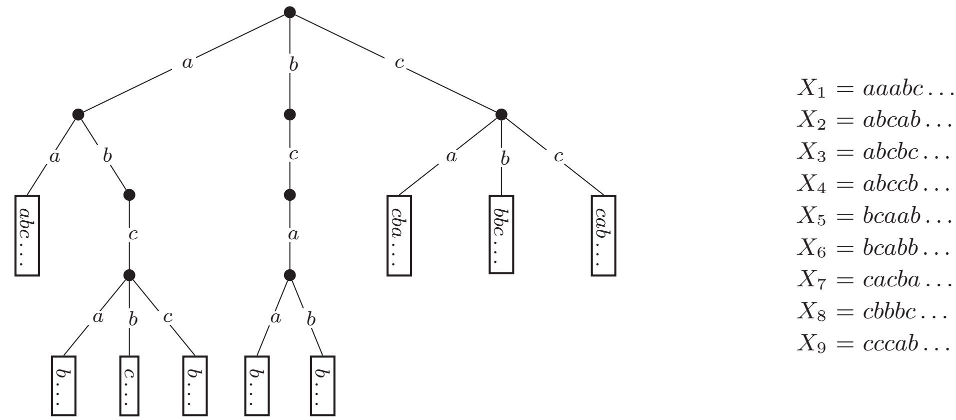

Figure 2. The trie $T(\mathcal{X})$ associated with a set $\mathcal{X}$ of nine (infinite) words on the alphabet $\Sigma := \{a, b, c\}$.

probability associated with the source by Definition 1, the depth D_n satisfies

$$\mathbb{P}[D_n \geqslant k+1] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}[D_n^{(i)} \geqslant k+1]. \tag{2.2}$$

In the following, the parameter D_n will simply be called the depth of the trie. This is a random variable that depends on the set $\mathcal{X}$ of words, and we study its distribution when the source is fixed, when the set $\mathcal{X}$ is formed with words that are independently drawn from the source, and the cardinality n of $\mathcal{X}$ tends to ∞.

2.3. Probability generating function of the trie depth

Let D be a random variable over a probability space $(\Omega, \mathbb{P})$, with positive integer values. Its probability generating function is defined by

$$G(v) := \mathbb{E}[v^D] = \sum_{k \geqslant 0} v^k\, \mathbb{P}[D = k],$$

and its moment generating function $M(u) := \mathbb{E}[\exp(uD)]$ is exactly equal to $G(e^u)$.

This paper deals with the sequence of random variables D_n defined in (2.2). We first provide an expression for the probability generating function G_n of the variable D_n.

Proposition 2.1. *Consider a probabilistic source, and let $\mathbb{P}$ denote the probability associated with the source. Consider a set of n infinite words independently emitted by the source. Then the depth D_n of the trie built on this set satisfies the following.*

(i) *The distribution of D_n involves the fundamental probabilities of the source, in the form*

$$\mathbb{P}[D_n \geqslant k+1] = \sum_{|w|=k} p_w[1-(1-p_w)^{n-1}], \quad \textit{for } k \geqslant 0.$$

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(ii) *The probability generating function* $G_n(v)$ *of* D_n *is expressed via the function* $\Lambda(s,v)$ *defined in* (1.2)*, i.e.,*

$$n\left[\frac{G_n(v)-1}{v-1}\right] = \sum_{\ell=2}^{n}(-1)^{\ell}\binom{n}{\ell}\ell\Lambda(\ell,v). \tag{2.3}$$

(iii) *The mean value of* D_n *is expressed via the function* $\Lambda(s)$ *defined in* (1.2)*, i.e.,*

$$\mathbb{E}[D_n] = \frac{1}{n}\sum_{\ell=2}^{n}(-1)^{\ell}\binom{n}{\ell}\ell\Lambda(\ell). \tag{2.4}$$

Proof. **(i)** Let $D_n^{(i)}$ be the length of the branch whose leaf contains the designated word X_i. The event $[D_n^{(i)} \geqslant k+1]$ means that the word X_i shares its prefix of length k with at least another word X_j. Thus, the independence of the words of a set $\mathcal{X}$ implies the equality

$$\mathbb{P}[D_n^{(i)} \geqslant k+1] = \sum_{w\in\Sigma^k} p_w[1-(1-p_w)^{n-1}],$$

and now the definition of the typical depth, with relation (2.2), implies assertion (i):

$$\mathbb{P}[D_n \geqslant k+1] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}[D_n^{(i)} \geqslant k+1] = \sum_{w\in\Sigma^k} p_w[1-(1-p_w)^{n-1}].$$

(ii) Next, a straight binomial expansion provides an expression for $\mathbb{P}[D_n \geqslant k+1]$ that reduces to a linear combination of the series $\Lambda_k(\ell)$ defined in (1.2) in the form

$$\mathbb{P}[D_n \geqslant k+1] = \sum_{\ell=1}^{n-1}(-1)^{\ell+1}\binom{n-1}{\ell}\sum_{|w|=k}p_w^{\ell+1} = \frac{1}{n}\sum_{\ell=2}^{n}(-1)^{\ell}\binom{n}{\ell}\ell\Lambda_k(\ell).$$

The probability generating function is given by

$$G_n(v) := \sum_{k=0}^{\infty}\mathbb{P}[D_n=k]\,v^k = 1+(v-1)\sum_{k=0}^{\infty}\mathbb{P}[D_n \geqslant k+1]\,v^k,$$

and, with the definition of function $\Lambda(s,v)$ in (1.2), the following equality holds:

$$n\left[\frac{G_n(v)-1}{v-1}\right] = \sum_{k=0}^{\infty}\sum_{\ell=2}^{n}(-1)^{\ell}\binom{n}{\ell}\ell\Lambda_k(\ell)v^k = \sum_{\ell=2}^{n}(-1)^{\ell}\binom{n}{\ell}\ell\Lambda(\ell,v).$$

(iii) This is clear, since $\mathbb{E}[D_n]$ equals the derivative of $v \mapsto G_n(v)$ at $v=1$. □

2.4. Rice's method

An important tool that deals with binomial sums of the form (2.3) is Rice's formula [31, 32]. As recalled in the following proposition, it transforms a binomial sum into an integral in the complex plane, and has been widely used in analytic combinatorics since the seminal paper of Flajolet and Sedgewick [15].

Proposition 2.2 (Rice's integral). *Consider a sequence $S(n)$ defined as a binomial sum of the sequence $T(\ell)$, namely*

$$S(n) = \sum_{\ell=2}^{n} (-1)^{\ell} \binom{n}{\ell} T(\ell).$$

(i) *Assume that there is a lifting $\varpi(s)$ of the sequence $k \mapsto T(k)$ which is analytic in the half-plane $\Re(s) > C$, with $1 < C < 2$, and is of polynomial growth there (i.e., $\varpi(s)$ is $O(|s|^r)$ when $s \to \infty$). Then, for any real d with $C < d < 2$, the sequence $S(n)$ admits an integral representation:*

$$S(n) = -\frac{1}{2i\pi} \int_{d-i\infty}^{d+i\infty} \varpi(s) L_n(s)\, ds \quad \text{with } L_n(s) = \frac{(-1)^n\, n!}{s(s-1)(s-2)\cdots(s-n)}. \tag{2.5}$$

(ii) *Assume now that the lifting $\varpi(s)$ of the sequence $T(k)$ is meromorphic in a region $\mathcal{R}$ that contains the half-plane $\Re s \geqslant 1$ and is of polynomial growth there (for $|\Im s| \to \infty$). Then*

$$S(n) = -\left[\sum_k \operatorname{Res}[\varpi(s) L_n(s); s_k] + \frac{1}{2i\pi} \int_{\mathcal{C}} \varpi(s) L_n(s)\, ds \right], \tag{2.6}$$

where $\mathcal{C}$ is a curve (oriented from the south to the north) of class $\mathcal{C}^1$ included in $\mathcal{R}$ and the sum is extended to all poles s_k of $L_n(s)$ inside the domain $\mathcal{D}$ delimited by the vertical line $\Re s = d$ and the curve $\mathcal{C}$.

The dominant singularities of $\varpi(s)\, L_n(s)$ provide the asymptotic behaviour of $S(n)$, and the remainder integral is estimated using the polynomial growth of $\varpi(s)\, L_n(s)$ when $|\Im(s)| \to \infty$.

We wish to apply Rice's method to the present case described by (2.3) and its particular case (2.4). This introduces the function $\varpi_v(s)$ related to the Dirichlet generating function Λ, via the equality $\varpi_v(s) = s\Lambda(s, v)$, and we thus need 'good behaviour' of the function $s \mapsto \Lambda(s, v)$, first in the half-plane $\Re s > 1$, then near the point $(s, v) = (1, 1)$, and finally on the left of the vertical line $\Re s = 1$.

2.5. Strongly tame sources

Here we describe a situation where the bivariate Dirichlet series $\Lambda(s, v)$ has nice behaviour (in fact the best possible behaviour, as we will see later on). The following definition is not well justified here, and we explain later on, in Section 3.7, why this notion of strong tameness appears in a natural way.

Definition 3 (strongly tame source). A source is strongly tame if there exist a complex neighbourhood $\mathcal{V}$ of $v = 1$, two functions (called the entropic functions) $v \mapsto \sigma(v)$ and $v \mapsto r(v)$ defined on $\mathcal{V}$, and a half-plane $\mathcal{R} := \{s;\ \Re s > 1 - \gamma\}$ determined by its width $\gamma > 0$, such that the following hold.

(a) For any $v \in \mathcal{V}$, the unique singularity of $s \mapsto \Lambda(s, v)$ in $\mathcal{R}$ is the (simple) pole located at $1 + \sigma(v)$, with residue $r(v)$.

(b) The functions σ and r satisfy

$$\sigma(1) = 0, \quad r(1) = \sigma'(1) = 1/h(\mathcal{S}), \quad \text{and} \quad \sigma''(1) + \sigma'(1) \neq 0.$$

(c) The function $(s, v) \mapsto \Lambda(s, v)$ is of polynomial growth in $\mathcal{R} \times \mathcal{V}$: there exist $\nu > 0$ and $C, D > 0$ such that, for any $s = \sigma + it \in \mathcal{R}$ with $|t| \geqslant C$, and any $v \in \mathcal{V}$, we have

$$|\Lambda(s, v)| \leqslant D|t|^{\nu}.$$

A source that satisfies $\sigma''(1) + \sigma'(1) \neq 0$ is said to be log-convex.

2.6. The probability generating function $G_n(v)$ for a strongly tame source

We will now focus on strongly tame sources, and the following result provides in this case a simple expression for the moment generating function of the trie depth.

Proposition 2.3. *If the source $\mathcal{S}$ is strongly tame, with neighbourhood $\mathcal{V}$, entropic functions $\sigma(v), r(v)$, and width γ, then, for any $\delta \in]0, \gamma[$, there exists a complex neighbourhood $\mathcal{W} \subset \mathcal{V}$ such that, for any $v \in \mathcal{W}$, we have*

$$G_n(v) = (1 - v)\, r(v)\, \Gamma(-\sigma(v))\; n^{\sigma(v)}\, [1 + O(n^{-\delta})], \tag{2.7}$$

where the constant hidden in the O-term is uniform in $\mathcal{W}$.

Proof. If the source is strongly tame, then $s \mapsto \Lambda(s, v)$ is of polynomial growth in the half-plane $\Re s > 1 - \gamma$. Then the line of integration $\Re(s) = d$ can be moved to the left in (2.5), until we reach a vertical line ρ of equation $\Re s = \alpha$, with $\alpha > 1 - \gamma$, with residues $s = 1 + \sigma(v)$ and $s = 1$ taken into account. Then

$$\begin{aligned} n[G_n(v) - 1] = &- \operatorname{Res}[(v-1)s\Lambda(s, v)\, L_n(s); s = 1 + \sigma(v)] \\ &- \operatorname{Res}[(v-1)s\Lambda(s, v)\, L_n(s); s = 1] \\ &- \frac{1}{2i\pi} \int_{\rho} (v-1)s\Lambda(s, v)\, L_n(s)\, ds. \end{aligned} \tag{2.8}$$

The second residue in (2.8) at $s = 1$ is equal to $-n$, and we obtain

$$G_n(v) = \operatorname{Res}\left[(1 - v)\Lambda(s, v)\frac{sL_n(s)}{n}; s = 1 + \sigma(v)\right] + \frac{1}{n}\frac{1}{2i\pi}\int_{\rho}(1 - v)s\Lambda(s, v)\; L_n(s)\, ds. \tag{2.9}$$

The remainder of the proof provides estimates for each term in (2.9). It is based on the following proposition, whose proof (given in the Appendix) is mainly due to Flajolet and Sedgewick [16]. In the present proof we only use the first two assertions (i) and (ii), but assertion (iii) will be used in the case of H-tameness (see Section 3.6).

Proposition 2.4.

(i) *For any fixed s with $s \notin \mathbb{Z}_{\geqslant 0}$, we have*

$$L_n(s) := \frac{n!(-1)^n}{s(s-1)\cdots(s-n)} = -n^s\Gamma(-s)\left[1 + O\left(\frac{1}{n}\right)\right].$$

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The O-term is uniform for s in a bounded set.

(ii) *Consider a vertical line $\Re(s) = \alpha$ with $\alpha \notin \mathbb{Z}_{\leqslant 0}$, and assume that $\varpi(s)$ is continuous on $\Re(s) = \alpha$ and of at most polynomial growth there, i.e., $\varpi(s) = O(|s|^r)$ as $|s| \to \infty$ on $\Re(s) = \alpha$. Then, the integral admits the following estimate, as $n \to \infty$:*

$$\int_{\Re s = \alpha} \varpi(s)\, L_n(s)\, ds = O(n^\alpha).$$

(iii) *Consider a curve ρ of hyperbolic type, namely of the form*

$$\rho := \left\{ s = \sigma + it, |t| \geqslant B, \sigma = \sigma_0 - \frac{A}{|t|^{\beta_0}} \right\} \cup \left\{ s = \sigma + it, \sigma = \sigma_0 - \frac{A}{B^{\beta_0}}, |t| \leqslant B \} \right\},$$

for some strictly positive constants (A, B, β_0) and some real σ_0. Assume further, and assume that $\varpi(s)$ is continuous on ρ and of at most polynomial growth there, i.e., $\varpi(s) = O(|s|^r)$ as $|s| \to \infty$. Then the integral of $\varpi(s) L_n(s)$ on the curve ρ admits the following estimate, as $n \to \infty$:

$$\int_\rho \varpi(s) L_n(s) ds = n^{\sigma_0} \cdot\ O(\exp[-(\log n)^\beta]), \quad \textit{with } \beta < \frac{1}{1+\beta_0}.$$

We now apply Proposition 2.4 to the present situation, where it provides estimates for each term in (2.9). For the first term in (2.9), we apply assertion (i) at $s = 1 + \sigma(v)$, together with the equality $(1/n)\, sL_n(s) = -L_{n-1}(s-1)$,

$$\frac{1+\sigma(v)}{n} L_n(1+\sigma(v)) = n^{\sigma(v)} \Gamma(-\sigma(v)) \left[1 + O\left(\frac{1}{n}\right) \right],$$

and the residue in (2.9) relative to the simple pole at $1 + \sigma(v)$ is

$$\text{Res}\left[(1-v)\Lambda(s,v) \frac{sL_n(s)}{n}; 1 + \sigma(v) \right] = (1-v) r(v) n^{\sigma(v)} \Gamma(-\sigma(v)) \left[1 + O\left(\frac{1}{n}\right) \right],$$

where $r(v)$ is the residue of $\Lambda(s, v)$ at $1 + \sigma(v)$. At $v = 1$, the function σ satisfies $\sigma(1) = 0$, with $\sigma'(1) \neq 0$. Since the Γ function has a simple pole at $s = 0$ with residue equal to 1, this implies that $(1-v)\Gamma(-\sigma(v))$ equals $1/\sigma'(1)$ at $v = 1$ and is also analytic there. With properties of entropic functions $\sigma(v), r(v)$, the expression $(1-v)r(v)\Gamma(-\sigma(v))$ tends to 1 for $v \to 1$, and the first term in (2.9) is $\Theta(n^{\sigma(v)})$.

For the second term in (2.9), assertion (ii) applied to $\varpi_v(s) := s\Lambda(s, v)$ implies that the integral in (2.9) along the vertical line $\Re s = \alpha$ is of order n^α; with the division by n, the second term in (2.9) is of order $n^{\alpha-1} = O(n^{\Re\sigma(v)-\delta})$, when v is close enough to 1, and $\delta < \gamma$ small enough. This completes the proof of Proposition 2.3. □

2.7. Towards asymptotic Gaussian laws

Our main tool for proving an asymptotic Gaussian law for the trie depth D_n is the sequence of moment generating functions $u \mapsto M_n(u) := \mathbb{E}_n[\exp(uD_n)]$, related to the probabilistic generating functions $G_n(v)$ via the equality $M_n(u) = G_n(e^u)$. The following result, known as the Quasi-Powers Theorem, and due to Hwang [21], provides sufficient conditions on $M_n(u)$ under which the asymptotic law of D_n is proved to be Gaussian.

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Theorem 1 (Hwang). *Consider a sequence of variables D_n, defined on probability spaces $(\Omega_n, \mathbb{P}_n)$, and their moment generating functions $M_n(u) := \mathbb{E}_n[\exp(uD_n)]$. Suppose the functions $M_n(u)$ are analytic in a complex neighbourhood $\mathcal{U}$ of zero, and satisfy*

$$M_n(u) = \exp[\beta_n U(u) + V(u)]\big(1 + O(\kappa_n^{-1})\big), \tag{2.10}$$

where the O-term is uniform on $\mathcal{U}$. Moreover, $\beta_n \to \infty$, $\kappa_n \to \infty$ as $n \to \infty$, and $U(u)$, $V(u)$ are analytic on $\mathcal{U}$.
Then the mean and the variance satisfy

$$\mathbb{E}_n[D_n] = \beta_n U'(0) + V'(0) + O(\kappa_n^{-1}), \quad \mathbb{V}_n[D_n] = \beta_n U''(0) + V''(0) + O(\kappa_n^{-1}).$$

Furthermore, if $U''(0) \neq 0$, then the distribution of D_n on Ω_n is asymptotically Gaussian, with speed of convergence $O(\kappa_n^{-1} + \beta_n^{-1/2})$.

2.8. Statement of the main result

The Quasi-Powers Theorem is now applied to the present object of our study, and provides the first main result of the paper.

Theorem 2.5. *Consider a strongly tame source, defined in Definition 3 with entropic functions $(\sigma(v), r(v))$ and width γ, and a random trie with n keys built on the source. Then, the mean and the variance of the trie depth D_n admit the following estimates, for any $\delta \in]0, \gamma[$:*

$$\mathbb{E}[D_n] = \sigma'(1)\log n + c + O(n^{-\delta}),$$
$$\mathbb{V}[D_n] = [\sigma''(1) + \sigma'(1)]\log n + d + O(n^{-\delta}).$$

The constants c and d are expressed with derivatives of functions σ and r at $v = 1$. Moreover, the constant $\sigma''(1) + \sigma'(1)$ is not zero, and the depth D_n asymptotically follows a Gaussian law with speed of convergence $O((\log n)^{-1/2})$.

Proof. The moment generating functions $M_n(u)$ are expressed as in the Quasi-Powers Theorem, with $\beta_n := \log n$, $\kappa_n := n^\beta$, and

$$M_n(u) := \mathbb{E}_n[\exp(uD_n)] = G_n(e^u) = \exp(U(u)\log n + V(u))\big(1 + O(\kappa_n^{-1})\big)$$
$$\text{with} \quad U(u) := \sigma(e^u), \quad V(u) := \log[r(e^u)(1 - e^u)\Gamma(-\sigma(e^u))],$$

Since, in a neighbourhood of $v = 1$, the function $v \mapsto \sigma(v)$ is analytic and $v \mapsto r(v)$ is analytic and bounded away from zero, the functions U and V are analytic in a neighbourhood of $u = 0$. In particular, notice that $(1 - e^u)\Gamma(-\sigma(e^u))$ is analytic, even at $u = 0$. Moreover, the first two derivatives of U at $u = 0$ satisfy

$$U'(0) = \sigma'(1), \quad U''(0) = \sigma''(1) + \sigma'(1),$$

and $U''(0)$ is strictly positive. □

The rest of the paper is devoted to exhibiting a natural class of strongly tame sources to which Theorem 2.5 can be applied. This will be done in Sections 4, 5 and 6. But we first return to simple sources, for which we may also apply Rice's methodology to the trie depth study, provided that they fulfil tameness properties. The tameness of simple sources has not been deeply studied, and this explains why the remainder terms in the asymptotic estimates of the mean and variance of trie depth are not precisely given[2] in the literature.

3. Tameness of simple sources

This section has two aims. It studies the simple sources (memoryless sources and Markov chains), and it also introduces and explains the notion of tameness for a general source. We first recall the definition of simple sources (Section 3.1). Then, in Section 3.2, we provide an expression of their Dirichlet generating functions, and describe their analytic properties, first on the half-plane $\Re s > 1$ in Lemma 3.2. In Section 3.3 we consider the situation on the vertical line $\Re s = 1$ and exhibit the periodicity phenomenon in Lemma 3.3, related to arithmetic properties of probabilities. Then, to deal with Rice's methodology, we need precise knowledge of $\Lambda(s)$ on the half-plane $\Re s < 1$, as explained in Section 2.4,

This justifies the general notion of tameness, which describes the behaviour of $\Lambda(s)$ for a general source. In Section 3.4 we introduce three shapes of tameness (S, H, P) that seem *a priori plausible*. We then return to simple sources in Section 3.5, and describe their tameness properties. We observe in Lemma 3.4 that there are only two shapes of tameness (P, H) that are possible for a simple source, and in Section 3.6 we derive a precise expression for the mean and variance of the trie depth in the case when the simple source is P-tame or H-tame.

Finally, in Section 3.7, we focus on sources that are S-tame, and we explain why a natural perturbation of S-tameness may give rise to strong tameness, introduced in Section 2.5.

3.1. Simple sources

The memoryless source (where the random variables Y_i are independent with the same distribution) and the Markov chain (of order 1) (where the emitted symbol can only be correlated with the previous symbol) are the simplest models of sources, where the correlations between symbols may exist but are 'the weakest possible'.

Definition 4 (simple sources).

(a) *Memoryless source.* A source $\mathcal{S}$ is memoryless if the variables Y_k are independent with the same distribution. Such a source is defined by the set p_i of probabilities, where p_i is the probability of emitting the symbol $i \in \Sigma$ at any time k, namely $p_i := \mathbb{P}[Y_k = i]$. In the case when all the probabilities p_i are equal, the source is called *unbiased*.

(b) *Markov chain.* A source on the finite alphabet Σ is a Markov chain of order 1 if, at each time k, and for each pair (i, j) of symbols, the conditional probability of emitting i, knowing that the previously emitted symbol is j, does not depend on k, that is,

$$\text{for all } k \in \mathbb{N}, \quad \mathbb{P}[Y_{k+1} = i | Y_k = j] =: p_{i|j}.$$

[2] and sometimes not even correct ...

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A Markov chain is defined by its transition matrix $\mathbf{P} := (p_{i|j})$ and its initial distribution $V = (v_i)$.

(c) *Good Markov chain.* A Markov chain is *good*[3] if its matrix $\mathbf{P}$ is irreducible and aperiodic. The matrix $\mathbf{P}$ is irreducible if, for all (i, j), there exists an integer n for which the coefficient (i, j) of the matrix $\mathbf{P}^n$ is strictly positive. The matrix $\mathbf{P}$ is aperiodic if

$$d := \gcd(d_i) = 1 \quad \text{with } d_i := \gcd\{n;\ \mathbf{P}^n_{i,i} > 0\}.$$

(d) *Simple source.* A source is simple if it is a memoryless source on a finite alphabet or a good Markov chain.

3.2. Dirichlet series of simple sources

For simple sources, the fundamental probabilities p_w satisfy a multiplicative property. For $w = i_1 i_2 \cdots i_k \in \Sigma^k$, two equalities hold:

$$p_w = p_{i_1} p_{i_2} \cdots p_{i_k} \quad \text{(memoryless)} \quad \text{or} \quad p_w = v_{i_1} p_{i_2|i_1} \cdots p_{i_k|i_{k-1}} \quad \text{(Markov)}.$$

This leads to exact expressions of the Dirichlet series as quasi-inverses.

Lemma 3.1 (expression of the Λ series). *The Λ Dirichlet series of simple sources admit* quasi-inverse *expressions of the following types.*

(a) *For a memoryless source, these are in terms of*

$$\lambda(s) := \sum_{i \in \Sigma} p_i^s \quad \textit{as} \quad \Lambda(s) = \frac{1}{1 - \lambda(s)}, \qquad \Lambda(s, v) = \frac{1}{1 - v\lambda(s)}. \tag{3.1}$$

(b) *For a Markov chain, they are given in terms of the matrix $\mathbf{P}_s$ whose general coefficient is $p^s_{i|j}$, via*

$$\Lambda(s) = 1 + {}^t\mathbf{1} \cdot (I - \mathbf{P}_s)^{-1} \cdot V_s, \quad \Lambda(s, v) = 1 + {}^t\mathbf{1} \cdot (I - v\mathbf{P}_s)^{-1} \cdot V_s. \tag{3.2}$$

Here the vector V_s has components v_i^s, where v_i is the initial distribution of the symbol i.

We now focus on the study of the plain generating function $\Lambda(s)$ and we return to the bivariate generating function $\Lambda(s, v)$ below in Section 3.7.

Lemma 3.2 (properties of the Dirichlet series on $\Re s > 1$ and at $s = 1$).

(a) *The Dirichlet series $\Lambda(s)$ of a simple source is meromorphic on the complex plane and analytic on the half-plane $\Re s > 1$, and has a simple pole at $s = 1$. Moreover, the set $\mathcal{Z}$ of poles is defined by*

$$\mathcal{Z} = \{s;\ \lambda(s) = 1\} \ \text{(memoryless)} \quad \textit{or} \quad \mathcal{Z} = \{s;\ \det(I - \mathbf{P}_s) = 0\} \ \text{(Markov)}. \tag{3.3}$$

[3] We use this terminology because the usual notion of aperiodicity might be confused with non-periodicity, which appears in Section 3.3.

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(b) *Consider* $\lambda(s)$ *defined as in* (3.1) *for a memoryless source, or defined* (*for real* s) *as the dominant eigenvalue of* $\mathbf{P}_s$ *for a good Markov chain. Then two equalities hold,*

$$\mathrm{Res}[\Lambda(s); s=1] = -\frac{1}{\lambda'(1)}, \quad h(\mathcal{S}) = -\lambda'(1), \tag{3.4}$$

and the entropy admits the following expressions:

$$h(\mathcal{S}) = -\sum_{i\in\Sigma} p_i \log p_i \ \ (\text{memoryless}) \quad or \quad h(\mathcal{S}) = -\sum_{(i,j)\in\Sigma^2} \pi^{(j)}\, p_{i|j} \log p_{i|j} \ \ (\text{Markov}), \tag{3.5}$$

where $\pi^{(j)}$ *are the components of the vector* Π *fixed by* $\mathbf{P}$, *whose sum equals* 1.

Proof. (a) For a memoryless source, the function $s \mapsto \lambda(s)$ defined in (3.1) is analytic on the complex plane, and thus the function $s \mapsto \Lambda(s)$ is meromorphic with a set of poles $\mathcal{Z}$ defined in (3.3). Let $\sigma := \Re s$, and assume $\sigma > 1$. Then, the inequality $|\lambda(s)| \leqslant \lambda(\sigma) < \lambda(1) = 1$ entails that the set $\mathcal{Z}$ is contained in the half-plane $\Re s \leqslant 1$.

For a good Markov chain, we use the Perron–Frobenius theorem, which states the following: *A good matrix* $\mathbf{T}$ *with positive coefficients has a unique dominant eigenvalue* λ, *and a unique dominant eigenvector* Π *with positive components* π_i *whose sum equals* 1. We apply this theorem to the matrix $\mathbf{P}_s$ for any real s. Then the matrix $\mathbf{P}_s$ has a unique dominant eigenvalue $\lambda(s)$ and a unique dominant eigenvector Π_s with positive components $\pi_s^{(j)}$ whose sum equals 1. Since the matrix $\mathbf{P}$ is stochastic, the dominant value $\lambda(s)$ satisfies $\lambda(1) = 1$, and the matrix $\mathbf{P} = \mathbf{P}_1$ has a unique (normalized) fixed vector $\Pi := \Pi_1$ with positive components $\pi^{(j)}$, whose sum equals 1.

Moreover, the matrix $\mathbf{P}_s$ decomposes as a sum $\mathbf{P}_s = \lambda(s)\mathbf{Q}_s + \mathbf{N}_s$, where $\mathbf{Q}_s$ is the projection on the dominant eigenspace, and $\mathbf{N}_s$ is the remainder matrix, whose spectral radius $\rho(s)$ satisfies $\rho(s) := \max\{|\lambda|;\ \ \lambda \in \mathrm{Sp}\mathbf{P}_s\} < |\lambda(s)|$. These matrices satisfy $\mathbf{Q}_s \cdot \mathbf{N}_s = \mathbf{N}_s \cdot \mathbf{Q}_s = 0$, so that the previous decomposition extends to any $k \geqslant 1$, namely

$$\mathbf{P}_s^k = \lambda^k(s)\mathbf{Q}_s + \mathbf{N}_s^k, \quad \text{and thus} \quad (I - v\mathbf{P}_s)^{-1} = \frac{v\lambda(s)}{1 - v\lambda(s)}\mathbf{Q}_s + (I - v\mathbf{N}_s)^{-1}. \tag{3.6}$$

This first proves that $\Lambda(s)$ has a simple pole at $s = 1$, and also the asymptotic estimate

$$\Lambda_k(s) = \lambda^{k-1}(s)[{}^t\mathbf{1} \cdot \mathbf{Q}_s \cdot V_s] + {}^t\mathbf{1} \cdot \mathbf{N}_s^k \cdot V_s = \lambda^k(s) w_s [1 + o(\rho^k)] \tag{3.7}$$

for some non-zero constant w_s that satisfies $w_1 = 1$, and some $\rho < 1$.

The function $s \mapsto \mathbf{P}_s$ is analytic on the complex plane, and thus the function $s \mapsto \Lambda(s)$ is meromorphic with a set of poles $\mathcal{Z}$ defined in (3.3). Let $\sigma := \Re s$. Then, the inequality $\|\mathbf{P}_s^k(s)\| \leqslant \|\mathbf{P}_\sigma^k\|$ holds and implies the inequality on the spectral radii $r(s) \leqslant r(\sigma)$. In the case of a good Markov chain, the spectral radius $r(\sigma)$ equals the dominant eigenvalue $\lambda(\sigma)$. We now assume the strict inequality $\sigma > 1$, and wish to prove the strict inequality $\lambda(\sigma) < \lambda(1) = 1$. As the inequality $\lambda(\sigma) \leqslant \lambda(1)$ holds, we assume that the equality $\lambda(\sigma) = \lambda(1)$ holds, and we look for a contradiction. The equalities

$$\sum_j p_{i|j}^{\sigma} \pi_\sigma^{(j)} = \lambda(\sigma)\pi_\sigma^{(i)}, \quad \lambda(1) = 1 = \sum_i p_{i|j} = \sum_j \pi_\sigma^{(j)}$$

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imply the other two equalities,

$$\lambda(\sigma) = \sum_{i,j} p^{\sigma}_{i|j}\pi^{(j)}_{\sigma} = \sum_{j} \pi^{(j)}_{\sigma} \sum_{i} p^{\sigma}_{i|j}, \quad 0 = \lambda(1) - \lambda(\sigma) = \sum_{j} \pi^{(j)}_{\sigma}\left[\sum_{i}(p_{i|j} - p^{\sigma}_{i|j})\right].$$

This implies that for any $i \in \Sigma$ there is a unique $j = \tau(i) \in \Sigma$ for which the probability $p_{i|j} = 1$. When the Markov chain is good, there does not exist such a map $\tau : \Sigma \to \Sigma$.

(b) In both cases, we first prove the equality $h(\mathcal{S}) = -\lambda'(1)$. This is obtained by taking the derivative of the estimate given in (3.7) with respect to k, namely

$$\frac{1}{k}\frac{d}{ds}\Lambda_k(s) \sim_{k\to\infty} \lambda'(s)\lambda^{k-1}(s)w_s \quad \text{and then} \quad \frac{1}{k}\frac{d}{ds}\Lambda_k(s)|_{s=1} \sim_{k\to\infty} \lambda'(1).$$

We now obtain an alternative expression for the derivative $\lambda'(1)$. This is clear in the memoryless case, and, for a good Markov chain, taking the derivative (with respect to s) of the equality $\mathbf{P}_s \cdot \Pi_s = \lambda(s)\,\Pi_s$ leads at $s = 1$ to

$${}^t\mathbf{1} \cdot \mathbf{P}'_1 \cdot \Pi_1 + {}^t\mathbf{1} \cdot \mathbf{P}_1 \cdot \Pi'_1 = \lambda'(1)\,{}^t\mathbf{1} \cdot \Pi_1 + \lambda(1)\,{}^t\mathbf{1} \cdot \Pi'_1.$$

Moreover, since the matrix $\mathbf{P}$ is stochastic, the equality ${}^t\mathbf{1} \cdot \mathbf{P}_1 = {}^t\mathbf{1}$ holds. This implies the expression for the entropy given in (3.5). □

3.3. Properties of $\Lambda(s)$ on the line $\Re s = 1$; periodicity of simple sources

The following result describes the position of the set $\mathcal{Z}$ of poles with respect to the vertical line $\Re s = 1$ and relates it to the rationality of ratios α, which involve logarithms of probabilities, and are defined below.

Definition 5 (ratios α). The ratios α are defined as follows.

(a) In the memoryless case, in terms of probabilities p_i, the ratios are given by

$$\alpha(i,j) := \frac{\log p_i}{\log p_j} \quad \text{for any pair } (i,j) \in \Sigma^2. \tag{3.8}$$

(b) In the case of a good Markov chain, they are given in terms of probabilities of cycles. The probability of a cycle $\mathcal{C} := \{i_1, i_2, \ldots, i_k)$, is $p(\mathcal{C}) := p_{i_2|i_1} \cdots p_{i_k|i_{k-1}} p_{i_1|i_k}$, and

$$\alpha(\mathcal{C},\mathcal{K}) := \frac{\log p(\mathcal{C})}{\log p(\mathcal{K})} \quad \text{for each pair } (\mathcal{C},\mathcal{K}) \text{ of cycles of length at most } r. \tag{3.9}$$

Clearly the ratios of the Markov chain case extend the ratios of the memoryless case.

Lemma 3.3 (periodicity of simple sources). *For a memoryless source of probabilities $\mathfrak{P}$, the following conditions are equivalent.*

(a) *The intersection $\mathcal{Z} \cap \{\Re s = 1\}$ contains a point $s \neq 1$.*
(b) *All the ratios $\alpha(i,j)$ defined in* (3.8) *are rational numbers.*
(c) *There exists $\tau > 0$ for which the equality $\mathcal{Z} \cap \{\Re s = 1\} = 1 + 2i\pi\tau\mathbb{Z}$ holds.*
(d) *The function $\lambda(s)$ is periodic of period $2i\pi\tau$.*

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A memoryless source which satisfies one of these conditions is said to be periodic. For a Markov chain with transition matrix $\mathbf{P}$, *the following conditions are equivalent.*

(a) *The intersection* $\mathcal{Z} \cap \{\Re s = 1\}$ *contains a point* $s \neq 1$.
(b) *All the ratios* $\alpha(\mathcal{C}, \mathcal{K})$ *defined in* (3.9) *are rational numbers.*
(c) *There exists* $\tau > 0$ *for which the equality* $\mathcal{Z} \cap \{\Re s = 1\} = 1 + 2i\pi\tau\mathbb{Z}$ *holds.*
(d) *The matrix* $\mathbf{P}_s$ *is periodic of period* $2i\pi\tau$.

A Markov chain which satisfies one of these conditions is said to be periodic.

This result is well known in the memoryless case, and less classical for Markov chains, where Jacquet, Szpankowski and Tang [25] provide such a characterization.

3.4. General definitions for tameness

The two previous sections describe, for simple sources, the position of the set $\mathcal{Z}$ of poles of $\Lambda(s)$ in the half-plane $\{\Re s \geqslant 1\}$. We now focus on the *left half-plane* $\{\Re s < 1\}$, and isolate a region $\mathcal{R} \supset \{\Re s \geqslant 1\}$ where the Λ function is analytic. In fact, we have to reinforce our needs for the region $\mathcal{R}$: to apply Rice's methodology, it is also essential for $\Lambda(s)$ to be of polynomial growth when $s \in \mathcal{R}$ tends to ∞. Such a region will play a central role in the subsequent analyses. We are then led to the following definition, which is proposed for any source. We return to simple sources in the next section.

Definition 6 (tameness region). A *tameness region* for a *general* source $\mathcal{S}$ is a region $\mathcal{R} \supset \{\Re s \geqslant 1\}$ where the Λ series is meromorphic, with a only pole (simple) located at $s = 1$, and is of polynomial growth when $|\Im s| \to \infty$.

We now introduce three shapes for tameness regions, that seem to be *a priori* plausible.

Definition 7 (shape of regions). A region $\mathcal{R} \supset \{\Re s \geqslant 1\}$ has:

(a) an S-*shape* (short for strip shape) if $\mathcal{R}$ is a vertical strip $\Re(s) > 1 - \gamma$ for some $\gamma > 0$,
(b) an H-*shape* (hyperbolic shape) if $\mathcal{R}$ is a hyperbolic region $\mathcal{R}$, defined by

$$\mathcal{R} := \left\{ s = \sigma + it;\quad |t| \geqslant B,\quad \sigma > 1 - \frac{A}{|t|^{\beta}} \right\} \bigcup \left\{ s = \sigma + it;\quad \sigma > 1 - \frac{A}{B^{\beta}}, |t| \leqslant B \right\},$$

for some $A, B, \beta > 0$,
(c) a P-*shape* (periodic shape) if $\mathcal{R}$ is a vertical strip 'with holes', namely

$$\mathcal{R} := \mathcal{R}_0 \setminus \mathcal{R}_1, \quad \mathcal{R}_0 := \{\Re s > 1 - \gamma\}, \quad \mathcal{R}_1 := \{s = 1 + it;\quad t = 2\pi k\tau, k \in \mathbb{Z} \setminus \{0\}\}$$

for some $\gamma, \tau > 0$.

When they exist, γ is the width, β is the hyperbolic exponent, and τ is the period.

Definition 8 (shape of tameness). For $X \in \{P, H, S\}$, a *general* source is X-*tame* if its series $\Lambda(s)$ satisfies the following.

(a) At $s = 1$ it admits a simple pole, with residue equal to $1/h(\mathcal{S})$ (where $h(\mathcal{S})$ is the entropy of the source).

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(b) It admits a tameness region with an X-shape as described in Definition 7.

A vertical strip can be viewed as a region with a zero hyperbolic exponent. We are interested in tameness regions which are the largest possible. Then it is natural to define the hyperbolic exponent of the source $\mathcal{S}$ as the *infimum of all the hyperbolic exponents* of tameness regions of the source $\mathcal{S}$. For instance, if the source admits a vertical strip as tameness region, then the hyperbolic exponent of the source equals 0. There also exist some sources for which the singularities of the Λ function come close to the vertical line $\Re s = 1$ very rapidly, with exponential speed. Such sources have a hyperbolic exponent equal to ∞, and they are not H-tame.

3.5. Tameness of simple sources

We now return to simple sources and examine the possible types of tameness. Even for simple sources, the position of the set of poles $\mathcal{Z}$ with respect to the vertical lines has been investigated only recently. The paper by Fayolle, Flajolet and Hofri [12] seems to have been the first to conduct (in the memoryless case) a detailed discussion of the position of poles. In the memoryless case, Schachinger provides a rigorous and thorough discussion of this geometry of poles [36]. Finally, the paper [14] adapts deep results described in the book by Lapidus and van Frankenhuijsen [28] and precisely relates the shape of the pole-free region to arithmetic properties of probabilities. It proves that 'most' aperiodic memoryless sources are H-tame,

The first result examines the possibilities for a P-shape or an S-shape.

Proposition 3.4.

(a) *A simple source which is periodic is P-tame.*
(b) *A non-periodic simple source is never S-tame.*

Proof. **(a)** In the case of a periodic simple source, the function $s \mapsto \lambda(s)$ is periodic of period $i\tau$. Then there is a vertical strip on the left of the vertical line $\Re s = 1$ where the Λ function is analytic and of polynomial growth. There exists in this case a tameness region of the source which is a 'vertical strip with holes'.

(b) (Sketch.) We now focus on non-periodic simple sources. In this case, the intersection $\mathcal{Z} \cap \{s;\ \Re s = 1\}$ only contains the point $s = 1$, and we now recall why there exist points of $\mathcal{Z}$ which are arbitrarily close to the vertical line $\Re s = 1$. This will entail that an aperiodic simple source is never S-tame. In the aperiodic case, there is, indeed, amongst the coefficients of the matrix α, at least one coefficient $\alpha(i, j)$ which is irrational, and it is then possible to define an approximation function $f : \mathbb{R} \to \mathbb{R}$ which describes the approximability properties of the matrix α by rational numbers. There is a close relation between the approximation function f and the shape of a region which contains no element of $\mathcal{Z}$. The distance between the frontier of this region and the vertical line $\Re s = 1$ can be described with the approximation function f, and it always tends to zero for $|\Im s| \to \infty$. Then the source cannot be S-tame. □

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Informally speaking, the source may be H-tame if the poles of $\mathcal{Z}$ come close to the vertical line $\Re s = 1$, but not too fast, namely at polynomial speed with respect to $|\Im s|$. We now describe *arithmetical conditions* which are sufficient to imply H-tameness. They deal with classical number-theoretic notions, which are now recalled.

Definition 9 (irrationality exponent and Diophantine number).

(a) For an irrational number x, the irrationality exponent is

$$\mu(x) := \sup\left\{ v, \left| x - \frac{p}{q} \right| \leqslant \frac{1}{q^{2+v}} \quad \text{for an infinite number of integer pairs } (p,q) \right\}.$$

(b) An irrational number x is Diophantine if its irrationality exponent is finite.

The irrationality exponent of the irrational x is then a measure of its approximability by rational numbers. Then, a Diophantine irrational number is not too well approximable by rational numbers: it can be viewed (informally) as an irrational number which *strongly differs from a rational number*. The approximability of an irrational number x is closely related to its continued fraction expansion, since truncations of this expansion give rise to the rational numbers that provide the best rational approximations of the irrational x. Instances of Diophantine numbers are irrational numbers whose quotients occurring in the continued fraction expansion of x are bounded.

It is possible to define the irrationality exponent of a finite family of numbers, provided that they are not all rational. The irrationality $\mu(\mathcal{S})$ of a non-periodic simple source $\mathcal{S}$ is then defined as the irrationality exponent of the set $\{\alpha(\mathcal{C},\mathcal{K});\ \mathcal{C},\mathcal{K} \text{ cycles of length} \leqslant r\}$. The source is Diophantine if the irrationality exponent $\mu(\mathcal{S})$ is finite.

The following result, due to Roux and Vallée [34, 33] and based on the general framework described in the book [28], relates the irrationality exponent $\mu(\mathcal{S})$ of the source and its hyperbolic exponent β defined in Section 3.4.

Theorem 2 (Diophantine source and *H*-tameness). *For a simple non-periodic source, the two exponents – the irrationality exponent μ and the hyperbolic exponent β – are related by the equality $\beta = 2\mu + 2$. A Diophantine non-periodic source is H-tame.*

For a memoryless source over an alphabet of size r, the irrationality exponent satisfies *almost everywhere* the inequality $\mu(\mathfrak{P}) + 1 = 1/(r-1)$. Here, 'almost everywhere' means that the probability family $\mathfrak{P}$ is randomly chosen in the subset

$$\{(p_1, p_2, \ldots, p_r) : p_j > 0, p_1 + p_2 + \cdots + p_r = 1\}$$

with respect to the Lebesgue measure. With the previous theorem, this implies that the hyperbolic exponent of a non-periodic memoryless source over an alphabet of size r is 'almost everywhere' equal to $2/(r-1)$. The hyperbolic exponent of a binary source is 'almost everywhere' equal to 2.

3.6. Analysis of trie depth for simple sources

We now make a 'detour' and provide estimates for the mean and the variance of the trie depth for simple sources. We begin with the expression of the mean[4] given in (2.4) and use Rice's method, namely Propositions 2.2 and 2.4 (notably assertion (iii)). We obtain precise remainder terms that depend on the type of tameness of the source.

Theorem 3.5 (classical results revisited). *Consider, for a simple source, $\lambda(s)$ defined as in* (3.1) *for a memoryless source, or defined (for real s) as the dominant eigenvalue of $\mathbf{P}_s$ for a good Markov chain. For the depth of the trie built on a random sequence of n words independently drawn from the source, the following holds.*

(a) *The mean and the variance satisfy*

$$\mathbb{E}[D_n] = -\frac{1}{\lambda'(1)} \log n + c + R_1(n),$$
$$\mathbb{V}[D_n] = \frac{\lambda'^2(1) - \lambda''(1)}{\lambda'^3(1)} \log n + d + R_2(n).$$

The constants c, d also depend on the source. The only case for which the dominant constant of the variance is zero arises for an unbiased memoryless source.

(b) *The type of function $R_i(n)$ depends on the tameness of the source.*

(b1) *If the source is P-tame with width γ and period τ, then $R_i(n) = \Pi_i(n) + O(n^{-\delta})$, where δ satisfies $\delta < \gamma$ and $\Pi_i(n)$ is a periodic function of $\log n$, with period $1/\tau$.*

(b2) *If the source is H-tame with hyperbolic exponent β_0, then*

$$R_i(n) = O\big(\exp[-(\log n)^{\beta}]\big), \quad \textit{with } \beta < 1/(1+\beta_0).$$

With Proposition 3.4 and Theorem B, the previous result applies to *almost all* simple sources, namely all the periodic sources and all the Diophantine sources. However, it does *not* apply to *any* simple source. Indeed, there exist simple sources which are not tame, as their irrationality exponent is infinite. As explained in [14], the function f described in the proof of Proposition 3.4 may *not* be of polynomial order, and it is possible to construct simple sources for which the upper bound for remainder terms $R_i(n)$ tends to 0 arbitrarily slowly.

3.7. Towards strong tameness

We now return to the main purpose of the paper, which deals with distributional studies where the bivariate generating function $\Lambda(s, v)$ plays a central role, and we need tameness properties for $\Lambda(s, v)$. Informally, they may be obtained by perturbation[5] of those of $\Lambda(s)$, provided there is 'enough' space to perturb. This is why S-tameness is certainly easier to perturb than H-tameness, where the distance between the frontier of the hyperbolic region and the vertical line tends to zero when $|\Im s|$ becomes large.

[4] There is a similar expression for the variance involving the function $\widetilde{\Lambda}(s) := (d/dv)\Lambda(s, v)|_{v=1}$ (see, *e.g.*, [20, 19]).
[5] In the sense of perturbation theory (see [26]).

In Section 7 we shall return to possible perturbations of H-tameness, but in the present paper we focus on a possible perturbation of S-tameness which naturally leads to strong tameness, as defined in Definition 3. This involves a vertical strip which is obtained as a perturbation of the vertical strip of $\Lambda(s)$. Moreover, the unique pole of $\Lambda(s,v)$ is also obtained as a perturbation of the unique pole of $\Lambda(s)$, and we postulate that the series $\Lambda(s,v)$ of any 'nice' source behaves like that of a simple source near the point $(s,v) = (1,1)$. Indeed, for simple sources, and with the expression of $\Lambda(s,v)$ described in Lemma 3.1, together with the decomposition (3.6) for good Markov chains, the dominant term of $\Lambda(s,v)$ near $(1,1)$ is closely related to $1/(1-v\lambda(s))$, which defines entropic functions $(\sigma(v), r(v))$ as in Definition 3.

All this explains why the notion of strong tameness is a natural perturbation of S-tameness. The following section exhibits a class of sources which will be proved to be *strongly tame*.

4. Dynamical sources

We first describe in Section 4.1 the general framework of dynamical sources, and then focus on dynamical sources which are complete or Markovian. These sources form an interesting subclass of the general sources and extend the simple sources in a natural way (Section 4.2). In Section 4.3, we present our main tool, the secant transfer operator $\mathbf{H}_s$, which is an extension of the plain (usual) transfer operator of the underlying dynamical system. The importance of this operator becomes clear in Proposition 4.1, which proves that the function $\Lambda(s,v)$ can be expressed as a function of the quasi-inverse $(I - v\mathbf{H}_s)^{-1}$. Then we describe geometric conditions related to the Good Class (Section 4.4) or the UNI Condition (Section 4.5). This defines the Good-UNI Class, which gathers sources that will be proved to be strongly tame, in the following sections.

4.1. Dynamical sources

Definition 10 (dynamical system of the interval). A dynamical system of the interval $\mathcal{I} := [0,1]$ is defined by a mapping $T : \mathcal{I} \to \mathcal{I}$ (called the shift) for which the following holds.

(a) There exists a (finite or denumerable) set Σ, whose elements are called symbols, and a topological partition of $\mathcal{I}$ with disjoint open intervals $(\mathcal{I}_m)_{m\in\Sigma}$, *i.e.*,

$$\overline{\mathcal{I}} = \bigcup_{m\in\Sigma} \overline{\mathcal{I}_m}.$$

(b) The restriction of T to each $\mathcal{I}_m$ is a $\mathcal{C}^2$ bijection from $\mathcal{I}_m$ to $T(\mathcal{I}_m)$.

The system is complete when each restriction is surjective, *i.e.*, $\overline{T(\mathcal{I}_m)} = \mathcal{I}$.

The system is Markovian when each interval $\overline{T(\mathcal{I}_m)}$ is a union of intervals $\overline{\mathcal{I}_j}$.

A dynamical system, together with a distribution G on the unit interval $\mathcal{I}$, defines a probabilistic source, called a dynamical source, which is now described. The map T is used as a shift mapping, and the mapping τ, whose restriction to each $\mathcal{I}_m$ is equal to

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m, is used for coding. The words are emitted as follows. With each real x (except for a denumerable set), we associate the word $W(x) \in \Sigma^{\mathbb{N}}$:

$$W(x) = (m_1(x), m_2(x), \ldots, m_n(x), \ldots) \quad \text{with } m_j(x) = \tau(T^{j-1}(x)).$$

Given a prefix $w \in \Sigma^\star$, the set $\mathcal{I}_w$ denotes the set of all reals x for which the word $W(x)$ begins with the prefix w. The set $\mathcal{I}_w$ turns out to be an interval,[6] of the form $]a_w, b_w[$, which is called the fundamental interval associated with w, and the measure of this interval (with respect to distribution G) equals (by definition) the fundamental probability p_w:

$$p_w = G(b_w) - G(a_w).$$

In the case of a complete system, we let $h_{[m]}$ denote the local inverse of T restricted to $\mathcal{I}_m$, extended by continuity to the whole interval $\mathcal{I}$, and we let $\mathcal{H}$ denote the set $\mathcal{H} := \{h_{[m]}, m \in \Sigma\}$ of all the local inverses. Each local inverse of the kth iterate T^k is associated with a prefix w of length k, of the form $w = m_1 \cdots m_k \in \Sigma^k$, and is written as

$$h_{[w]} := h_{[m_1]} \circ h_{[m_2]} \cdots \circ h_{[m_k]}.$$

Then the set of all the inverse branches of T^k is

$$\mathcal{H}^k = \{h = h_{[m_1]} \circ h_{[m_2]} \cdots \circ h_{[m_k]} ; m_i \in \Sigma\} = \{h_{[w]} ; \quad w \in \Sigma^k\}.$$

Each fundamental interval $\mathcal{I}_w$ is then simply equal to $\mathcal{I}_w = h_{[w]}(\overset{\circ}{\mathcal{I}})$, and the fundamental probability satisfies

$$p_w = |G(h_{[w]}(1)) - G(h_{[w]}(0))|. \tag{4.1}$$

For $h \in \mathcal{H}^k$, the number k is called the *depth* of h, and is denoted by $|h|$. We let $\mathcal{H}^\star := \bigcup_{k \geqslant 0} \mathcal{H}^k$ denote the set of all the inverse branches of any depth.

4.2. Simple sources seen as dynamical sources

Simple sources are related to the case when the branches of the system are affine, and the initial distributions are uniform. More precisely:

(a) a complete dynamical source, with affine branches and a uniform initial distribution, defines a memoryless source,

(b) a Markovian dynamical source, with affine branches and a family of uniform initial distributions on each $\mathcal{I}_m$, defines a Markov chain.

As soon as the derivatives h' of the branches are not constant, there exist correlations between successive symbols, and the dynamical source is no longer simple. A primary example is the dynamical source relative to the Gauss map, which underlies Euclid's algorithm and is defined on the unit interval via the shift T:

$$T(0) = 0, \quad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad (x \neq 0). \tag{4.2}$$

[6] Up to a denumerable set.

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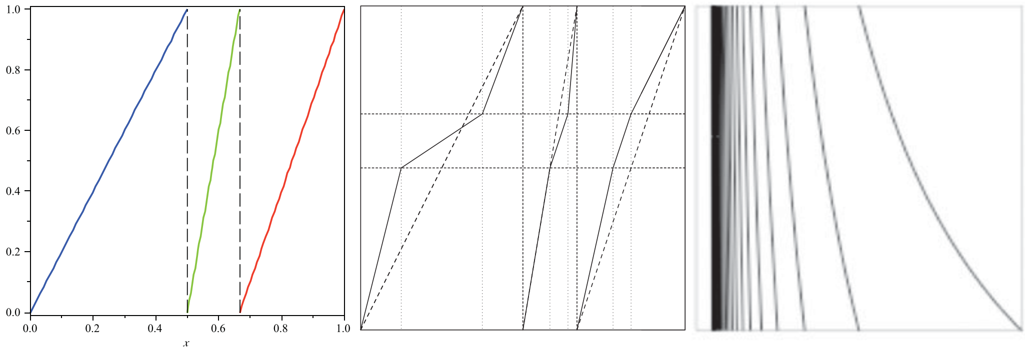

(a) (b) (c)

Figure 3. (Colour online) Three different dynamical sources: two with affine branches (one complete, and one Markovian), and the third one related to Euclid's algorithm.

4.3. Transfer operators

One of the main tools in dynamical systems theory is the transfer operator introduced by Ruelle [35], denoted by H_s. It generalizes the density transformer H that describes the evolution of the density. Here, as in [39], we describe a generalized version of the transfer operator – the secant operator – which gives rise to an expression of the Dirichlet series $\Lambda(s)$ defined in (1.2) as a quasi-inverse (see Proposition 4.1), in a way that generalizes expressions obtained in (3.1) or in (3.2). We now limit ourselves to a complete dynamical system. There are easy extensions to a Markovian system, with heavier computations.

If $f = f_0$ denotes the initial density on $\mathcal{I}$, and f_1 the density on $\mathcal{I}$ after one iteration of T, then f_1 can be written as $f_1 = H[f_0]$, where the operator H (called the density transformer) is defined by

$$H[f](x) := \sum_{h\in\mathcal{H}} |h'(x)|\, f \circ h(x).$$

The transfer operator H_s extends the density transformer; it depends on a complex parameter s, coincides with H when $s = 1$, and is defined by

$$H_s[f](x) = \sum_{h\in\mathcal{H}} |h'(x)|^s \cdot f \circ h(x). \tag{4.3}$$

With multiplicative properties of derivatives, the kth iterate of the transfer operator involves the set $\mathcal{H}^k$ in the form

$$H_s^k[f](x) = \sum_{h\in\mathcal{H}^k} |h'(x)|^s \cdot f \circ h(x).$$

Here we are interested in the fundamental probabilities, whose expression is provided in (4.1) in the case of a complete dynamical system. We now introduce the main tool for generating these probabilities, namely the secant transfer operator. This operator involves the secant function of inverse branches (instead of their derivatives), and acts on functions

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F of two variables; for $s \in \mathbb{C}$, it is defined by

$$\mathbf{H}_s[F](x, y) := \sum_{h \in \mathcal{H}} \left| \frac{h(x) - h(y)}{x - y} \right|^s \cdot F(h(x), h(y)). \tag{4.4}$$

The secant operator is then an extension of the plain transfer operator. On the diagonal $x = y$, the equality

$$\mathbf{H}_s[F](x, x) = H_s[\operatorname{diag} F](x) \tag{4.5}$$

holds and involves the 'diagonal' function $\operatorname{diag} F$ defined by $\operatorname{diag} F(x) := F(x, x)$. As for usual transfer operators, multiplicative properties of secants then give the relation

$$\mathbf{H}_s^k[F](x, y) = \sum_{h \in \mathcal{H}^k} \left| \frac{h(x) - h(y)}{x - y} \right|^s F(h(x), h(y)).$$

For $w \in \Sigma^k$, the probability p_w^s is written as a function of the inverse branch $h_{[w]}$, in the form

$$p_w^s = |G(h_{[w]}(1)) - G(h_{[w]}(0))|^s = \left| \frac{h_{[w]}(1) - h_{[w]}(0)}{1 - 0} \right|^s \cdot \left| \frac{G(h_{[w]}(1)) - G(h_{[w]}(0))}{h_{[w]}(1) - h_{[w]}(0)} \right|^s.$$

Then, if L is the secant of the distribution G, defined by

$$L(x, y) := \frac{G(x) - G(y)}{x - y}, \tag{4.6}$$

then the series $\Lambda_k(s)$ and $\Lambda(s, v)$ are expressed as follows:[7]

$$\Lambda_k(s) := \sum_{w \in \Sigma^k} p_w^s = \mathbf{H}_s^k[L^s](1, 0), \quad \Lambda(s) = (1 - v\mathbf{H}_s)^{-1}[L^s](1, 0).$$

Finally, we have proved the following result, which provides an extension of formulae already obtained in (3.1) and (3.2) for the case of simple sources.

Proposition 4.1. *The Dirichlet series of a dynamical source, relative to a shift T and a distribution G, admit alternative expressions which involve the quasi-inverse of the secant operator, defined in* (4.4)*, applied to the function L^s, where L is the secant of the distribution G, described in* (4.6)*. We have*

$$\Lambda_k(s) = \mathbf{H}_s^k[L^s](0, 1), \quad \Lambda(s, v) = (I - v\mathbf{H}_s)^{-1}[L^s](0, 1).$$

4.4. The Good Class

We now consider particular complete dynamical systems belonging to the so-called Good Class, for which the transfer operator has spectral properties that are similar to those of a good Markov chain (see Proposition 5.1). This will entail, with Proposition 4.1, nice properties for the function $\Lambda(s, v)$.

[7] The formula extends to the Markovian case, replacing the operators with a matrix of operators.

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Definition 11 (Good Class). A dynamical system of the interval $(\mathcal{I}, T)$ belongs to the Good Class if it is complete, with a set $\mathcal{H}$ of inverse branches which satisfies the following.

(G1) The set $\mathcal{H}$ is uniformly contracting, *i.e.*, the constant ρ defined by

$$\rho = \limsup_{n\to\infty} \left(\sup_{h\in\mathcal{H}^n} \beta_h \right)^{1/n} \quad \text{with} \quad \beta_h := \max_{x\in\mathcal{I}} |h'(x)| \tag{4.7}$$

satisfies $\rho < 1$ and is called the contraction constant.

(G2) There is a constant $A > 0$ such that every inverse branch $h \in \mathcal{H}$ satisfies $|h''| \leqslant A|h'|$.

(G3) There exists $\sigma_0 < 1$ for which the series $\sum_{h\in\mathcal{H}} \beta_h^s$ converges on $\Re s > \sigma_0$.

There exists a stronger version ($\underline{\text{G}}$1) of condition (G1), which also seems more natural:

$$\exists \rho < 1, \quad \forall h \in \mathcal{H}, \quad \forall x \in \mathcal{I}, \quad |h'(x)| \leqslant \rho.$$

However, condition ($\underline{\text{G}}$1) is not satisfied for the Euclidean dynamical system, for instance, since there exist $x \in \mathcal{I}$ and $h \in \mathcal{H}$ for which $|h'(x)| = 1$, while condition (G1) holds for this system. Condition (G1) implies the following property: for any $\widehat{\rho}$ with $\rho < \widehat{\rho} < 1$, there exists an integer $N \geqslant 1$ for which

$$|h'(x)| \leqslant \widehat{\rho}^n, \quad \text{for any } n \geqslant N, \quad h \in \mathcal{H}^n, \quad x \in \mathcal{I}. \tag{4.8}$$

The bounded distortion property (G2) and the property (G3) are always fulfilled for a finite alphabet Σ. Properties (G1) and (G2) together imply the existence of a constant $K > 0$, for which the following inequalities are true for all $x, y \in \mathcal{I}$ and all $h \in \mathcal{H}^\star$:

$$|h''(x)| \leqslant K|h'(x)|, \quad |h'(x)| \leqslant K|h'(y)|, \quad \left| \frac{h(x) - h(y)}{x - y} \right| \leqslant K|h'(x)|. \tag{4.9}$$

4.5. The UNI Condition

We now consider a subclass of the Good Class which gathers sources which *strongly differ from sources with affine branches.*

We first define a probability $\mathbb{P}_n$ on each set $\mathcal{H}^n$ in a natural way. We let $\mathbb{P}_n\{h\} := |h(\mathcal{I})|$, where $|\mathcal{J}|$ denotes the length of the interval $\mathcal{J}$. Furthermore, $\Delta(h, k)$ denotes the 'distance' between two inverse branches h and k of same depth, defined by

$$\Delta(h, k) = \inf_{x\in\mathcal{I}} |\Psi'_{h,k}(x)| \quad \text{with } \Psi_{h,k}(x) = \log \left| \frac{h'(x)}{k'(x)} \right|. \tag{4.10}$$

The distance $\Delta(h, k)$ is a measure of the difference between the 'form' of the two branches h, k. The *UNI Condition*, stated as follows, expresses that the probability of two inverse branches having almost the same form is very small.

Property (UNI Condition). A dynamical system $(\mathcal{I}, T)$ satisfies the UNI Condition if its set $\mathcal{H}$ of inverse branches satisfies the following.

(U1) For any $\widehat{\rho} \in]\rho, 1[$, there exists $C > 0$, such that, for any integer n, and for any $h \in \mathcal{H}^n$, we have

$$\mathbb{P}_n[k;\ \Delta(h, k) \leqslant \widehat{\rho}^n] \leqslant C\,\widehat{\rho}^n.$$

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(U2) Each $h \in \mathcal{H}$ is of class $\mathcal{C}^3$ and for each integer n, there exists B_n for which $|h'''| \leqslant B_n |h'|$ for any $h \in \mathcal{H}^n$.

A source with affine branches never satisfies the UNI Condition: in this case, the ‘distance’ Δ is always zero, and the probabilities of assertion (U1) are all equal to 1. More generally, a dynamical source of the Good Class which satisfies the UNI Condition cannot be conjugate to a source with affine branches, as is (easily) proved in Proposition 1 of [2]. Then, the UNI Condition excludes all the simple sources which cannot be strongly tame (see Proposition 3.4). The interest of the UNI Condition is due to the fact that it is sufficient to imply strong tameness, as shown in the rest of the paper, in particular in Theorem 6.2.

Moreover, there are natural instances of sources that belong to the Good-UNI Class, for instance the Euclidean dynamical system defined in (4.2), together with two other dynamical systems, of Euclidean type, described in [2].

4.6. Strong tameness of a dynamical source of the Good-UNI Class

We will see that, on convenient functional spaces, the two operators (the plain operator H_s and the secant operator $\mathbf{H}_s$) fulfil two kinds of properties which together imply that the dynamical source is strongly tame.

(a) When the dynamical system belongs to the Good Class, these operators admit dominant spectral properties for s near the real axis, together with a spectral gap. This implies that, for v near 1, the function $s \mapsto \Lambda(s, v)$ is meromorphic for s with small imaginary part, and admits a simple pole at $s = 1 + \sigma(v)$ (see Propositions 5.5 and 6.1).

(b) When the dynamical system satisfies the UNI Condition, the function $(v, s) \mapsto \Lambda(s, v)$ is analytic and of polynomial growth, for v near 1 and s with large imaginary part (see Theorem 6.2).

The next theorem summarizes these main facts about dynamical sources of the Good-UNI Class and precisely describes the distribution of the depth of a trie built on the Good-UNI Class.

Theorem 4.2. *Consider a dynamical source, defined by a dynamical system of the Good-UNI Class and a distribution G of class $\mathcal{C}^2$ whose secant equals L. Then we have the following.*

(i) *This source is strongly tame. Moreover, the entropic functions $\sigma(v), r(v)$ from Definition 3 are expressed with the dominant spectral objects*[8] *of the secant operator $\mathbf{H}_s$, defined in* (4.4). *The function $\sigma(v)$ is defined from the dominant eigenvalue $\lambda(s)$ via the implicit equation*

$$v\lambda(1 + \sigma(v)) = 1 \quad \textit{with } \sigma(1) = 0. \tag{4.11}$$

[8] In the following section they are proved to exist in the functional space $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$.

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The residue $r(v)$ of $s \mapsto \Lambda(s,v)$ at $s = 1+\sigma(v)$ involves spectral objects of $\mathbf{H}_s$ at $s = 1+\sigma(v)$, namely, the dominant eigenvalue $\lambda(s)$ and the dominant projector $\mathbf{Q}_s$:

$$r(v) := \mathrm{Res}[s \mapsto \Lambda(s,v); 1+\sigma(v)] = \frac{-1}{v\lambda'(1+\sigma(v))}\mathbf{Q}_{1+\sigma(v)}[L^{1+\sigma(v)}](0,1). \tag{4.12}$$

(ii) *The depth of a random trie built on this source asymptotically follows a Gaussian law with speed of convergence $O((\log n)^{-1/2})$. Moreover, the mean and the variance satisfy*

$$\mathbb{E}[D_n] = -\frac{1}{\lambda'(1)} \log n + c(\mathcal{S}) + O(n^{-\delta}),$$
$$\mathbb{V}[D_n] = \frac{\lambda'^2(1) - \lambda''(1)}{\lambda'^3(1)} \log n + d(\mathcal{S}) + O(n^{-\delta}).$$

for any δ strictly less than the tameness width of the source.

Assertion (ii) provides asymptotic expansions for the mean and the variance of trie depth for UNI sources, which can be compared with similar results obtained for simple sources in Theorem 3.5. We note that the main terms are of the same type and only involve the dominant eigenvalue $\lambda(s)$. However, the remainder term is different and reflects the strong tameness of a source of the Good-UNI Class.

The remainder of the paper is devoted to proving this theorem. Section 5 studies the spectral properties of transfer operators for parameters s with small or moderate imaginary part, whereas Section 6 deals with values of s with large imaginary part. This study aims to compare the two transfer operators, the usual one, H_s, and the secant one, $\mathbf{H}_s$. There already exist studies of this type for secant operators acting on spaces of analytic functions (see [39] or [38], for instance, with corrections in [6]). However, we need to study secant operators when they act on spaces of functions of class $\mathcal{C}^1$, since we wish to use and extend estimates *à la* Dolgopyat obtained on such functional spaces.

5. Spectral properties of transfer operators of the Good Class: case of parameters s with small imaginary part

We first define the convenient functional spaces (Section 5.1), together with the notion of quasi-compactness (Section 5.2). We then recall in Section 5.3 the main spectral properties of the plain transfer operator H_s. Then Theorem 5.1, in Section 5.4, states the main spectral properties of the secant transfer operator. The following four subsections are devoted to the proof of this theorem. Finally, in Sections 5.9 and 6.1 we draw the conclusions of this section, namely Propositions 5.5 and 6.1.

5.1. Functional spaces

We first define the functional spaces used for the plain transfer operator H_s and the secant transfer operator $\mathbf{H}_s$.

Consider the real σ_0 defined in property (G3) of Definition 11 and the half-plane

$$\Sigma_0 := \{s,\ \Re s > \sigma_0\}. \tag{5.1}$$

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For $s \in \Sigma_0$, the operator H_s acts on the space $\mathcal{C}^1(\mathcal{I})$ endowed with the norm

$$\|f\|_{1,1} = \|f\|_0 + \|f\|_1, \quad \text{with} \quad \|f\|_0 = \sup_{\mathcal{I}} |f(x)|, \quad \|f\|_1 = \sup_{\mathcal{I}} |f'(x)|,$$

and H_s also acts on $(\mathcal{C}^0(\mathcal{I}), \|.\|_0)$. For $s \in \Sigma_0$, the secant operator acts on the space $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$ endowed with the norm

$$\|F\|_{1,1} = \|F\|_0 + \|F\|_1,$$
$$\text{with} \quad \|F\|_0 = \sup_{\mathcal{I}\times\mathcal{I}} |F(x,y)|, \quad \|F\|_1 = \sup_{\mathcal{I}\times\mathcal{I}} [|F_x(x,y)| + |F_y(x,y)|].$$

We note the inequalities

$$\|\operatorname{diag} F\|_0 \leqslant \|F\|_0, \quad \|\operatorname{diag} F\|_1 \leqslant \|F\|_1 \quad \text{so that} \quad \|\operatorname{diag} F\|_{1,1} \leqslant \|F\|_{1,1}.$$

5.2. Quasi-compact operators

Our operators of interest will be quasi-compact. We first recall this notion.

For an operator $\mathbf{L}$ which acts on a Banach space, we let $\operatorname{Sp}\mathbf{L}$ denote the spectrum of $\mathbf{L}$, $R(\mathbf{L})$ its spectral radius, and $R_{(e)}(\mathbf{L})$ its *essential spectral radius*, *i.e.*, the smallest $r \geqslant 0$ such that any $\lambda \in \operatorname{Sp}(\mathbf{L})$ with modulus $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity. An operator $\mathbf{L}$ is *quasi-compact* if the equality $R_e(\mathbf{L}) < R(\mathbf{L})$ holds.

The following theorem due to Hennion provides sufficient conditions under which an operator is quasi-compact. These conditions generalize previous conditions due to Ionescu-Tulcea and Marinescu, and Lasota and Yorke [18]. It deals with a space $\mathcal{B}$ endowed with two norms.

Theorem 3. *Let $\mathcal{B}$ be a space endowed with a strong norm $\|\cdot\|$ and a weak norm $|\cdot|$. Assume that the space $(\mathcal{B}, \|\cdot\|)$ is Banach and the unit ball of $\mathcal{B}$ is precompact in $(\mathcal{B}, |\cdot|)$. Consider a bounded operator L on $(\mathcal{B}, \|\cdot\|)$ and assume that there are two sequences r_n and t_n of positive numbers such that, for all $n \geqslant 1$, the following bound, called the Lasota–Yorke bound, holds:*

$$\|\mathbf{L}^n[f]\| \leqslant r_n\|f\| + t_n|f|.$$

Then the essential spectral radius of the operator $\mathbf{L}$ on $(\mathcal{B}, \|\cdot\|)$ satisfies

$$R_e(\mathbf{L}) \leqslant r := \liminf_{n\to\infty} (r_n)^{1/n}.$$

If, moreover, the spectral radius $R(\mathbf{L})$ in $(\mathcal{B}, \|\cdot\|)$ satisfies $R(\mathbf{L}) > r$, then the operator $\mathbf{L}$ is quasi-compact on $(\mathcal{B}, \|\cdot\|)$.

5.3. Spectral properties for the plain transfer operator

The spectral properties of the plain transfer operator H_s, when the parameter s has small imaginary part, are summarized below in Theorem D. Proofs of these results can be found in [1, 4, 39].

Theorem 4 (classical spectral properties of the plain transfer operator). *Consider a dynamical system $(\mathcal{I}, T)$ of the Good Class with contraction constant ρ, and let H_s (for $s \in \Sigma_0$) denote its plain transfer operator.*

(i) Quasi-compactness. *If $s \in \Sigma_0$ defined in (5.1), then H_s acts on $\mathcal{C}^1(\mathcal{I})$. The spectral radius of H_s and its essential spectral radius satisfy, with $\sigma := \Re s$,*

$$R(H_s) \leqslant R(H_\sigma), \quad R_e(H_s) \leqslant \rho \cdot R(H_\sigma).$$

In particular, H_s is quasi-compact for real s.

(ii) Unique dominant eigenvalue. *For real $\sigma \in \Sigma_0$, H_σ has a unique eigenvalue $\lambda(\sigma)$ of maximal modulus, which is simple and strictly positive, called the* dominant eigenvalue. *There exists an associated eigenfunction f_σ which is strictly positive, and the associated eigenvector μ_σ of the adjoint operator H_σ^* is a Radon measure. With the normalization conditions, $\mu_\sigma[1] = 1$, $\mu_\sigma[f_\sigma] = 1$, the measure μ_σ and the dominant eigenfunction f_σ are defined in a unique way. In particular, μ_1 is Lebesgue measure, with $\lambda(1) = 1$.*

(iii) Spectral gap. *For a real parameter $\sigma \in \Sigma_0$, there is a spectral gap, i.e., the subdominant spectral radius $r(\sigma)$, defined by*

$$r(\sigma) := \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}(\mathbf{H}_\sigma), \lambda \neq \lambda(\sigma)\},$$

satisfies $r(\sigma) < \lambda(\sigma)$.

(iv) Analyticity on compact sets. *The operator H_s depends analytically on s for $s \in \Sigma_0$. Thus, $\lambda(\sigma)^{\pm 1}$, $f_\sigma^{\pm 1}$, f'_σ, depend analytically on $\sigma \in \Sigma_0$.*

(v) Decomposition of the quasi-inverse. *For s close enough to the real axis and $s \in \Sigma_0$, the operator H_s has a dominant eigenvalue $\lambda(s)$ which is simple and separated from the rest of the spectrum by a spectral gap. The quasi-inverse of the operator H_s splits as*

$$(I - vH_s)^{-1}[f] = \frac{v\lambda(s)}{1 - v\lambda(s)} Q_s[f] + (I - vN_s)^{-1}[f],$$

where Q_s is the projector onto the dominant eigensubspace and the spectral radius of N_s is strictly smaller than $|\lambda(s)|$. The projector Q_s satisfies $Q_s[f](x) := f_s(x) \cdot \mu_s[f]$, where f_s is the dominant eigenvalue and μ_s is the corresponding eigenvector of the adjoint operator. In particular, at $s = 1$, the equality $\mu_1[f] = \int_{\mathcal{I}} f(x)dx$ holds.

(vi) Dominant eigenvalue as a function of σ. *The map $\sigma \mapsto \lambda(\sigma)$ is decreasing, its derivative $-\lambda'(1)$ equals the entropy $h(\mathcal{S})$, and it is weakly log-convex, i.e., $\lambda''(1) - \lambda'(1)^2 \geqslant 0$.*

5.4. Spectral properties for the secant transfer operator

The sets Σ_1, Σ_2 defined below will play a central role below.

Definition 12. Let ρ be the contraction constant. The sets Σ_1 and Σ_2 are defined by

$$\Sigma_1 := \Sigma_0 \cap \{s := \sigma + it :\quad R(\mathbf{H}_s) > \rho \cdot R(H_\sigma)\}, \tag{5.2}$$

$$\Sigma_2 := \Sigma_1 \cap \{s;\ H_s \text{ has a unique simple dominant eigenvalue}\}. \tag{5.3}$$

Our first main result extends the properties of the plain transfer operator stated in Theorem D to the secant transfer operator.

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Theorem 5.1 (spectral properties for the secant transfer operator). *Consider a dynamical system $(\mathcal{I}, T)$ of the Good Class with contraction constant ρ, and let H_s (for $s \in \Sigma_0$) denote the usual transfer operator and $\mathbf{H}_s$ the secant transfer operator. Let $R(\mathbf{H}_s)$ be its spectral radius and $R_e(\mathbf{H}_s)$ its essential spectral radius.*

(ia) Quasi-compactness. *If $s \in \Sigma_0$, then $\mathbf{H}_s$ acts on $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$. Its spectral radius and its essential spectral radius satisfy*

$$R(H_s) \leqslant R(\mathbf{H}_s) \leqslant R(\mathbf{H}_\sigma) = R(H_\sigma) \quad \textit{and} \quad R_e(\mathbf{H}_s) \leqslant \rho \cdot R(H_\sigma).$$

In particular, the line $\Sigma_0 \cap \mathbb{R}$ is included in Σ_1, and $\mathbf{H}_s$ is quasi-compact for $s \in \Sigma_1$.

(ib) Comparison of spectra. *Any eigenvalue λ of $\mathbf{H}_s$ with $|\lambda| > \rho\, R(H_\sigma)$ is an eigenvalue of H_s. The following inclusion holds:*

$$\operatorname{Sp} \mathbf{H}_{\sigma+it} \cap \{z;\ |z| > \rho\, R(H_\sigma)\} \subset \operatorname{Sp} H_{\sigma+it} \cap \{z;\ |z| > \rho\, R(H_\sigma)\}.$$

Moreover, for $s \in \Sigma_1$, the two spectral radii coincide, i.e., $R(\mathbf{H}_s) = R(H_s)$.

(ii) Unique dominant eigenvalue. *For $s \in \Sigma_2$, the operator $\mathbf{H}_s$ has a unique dominant eigenvalue, equal to the dominant eigenvalue $\lambda(s)$ of the plain transfer operator H_s. Moreover, the diagonal of a dominant eigenfunction F_s is a dominant eigenfunction f_s of H_s. For real $\sigma \in \Sigma_2$, there exists a strictly positive dominant eigenfunction F_σ.*

(iii) Spectral gap. *For $s \in \Sigma_2$, there is a spectral gap, i.e., the* subdominant spectral radius *$r(\mathbf{H}_s)$, defined by*

$$r(\mathbf{H}_s) := \sup\{|\lambda|; \lambda \in \operatorname{Sp}(\mathbf{H}_s), \lambda \neq \lambda(s)\},$$

satisfies $r(\mathbf{H}_s) < R(\mathbf{H}_s)$. Moreover, the inequality $r(\mathbf{H}_s) \leqslant \max\,[r(H_s),\ \rho\, R(H_\sigma)]$ holds.

(iv) Analyticity in compact sets. *The operator $\mathbf{H}_s$ depends analytically on s for $s \in \Sigma_0$. Thus, $\lambda(s)^{\pm 1}$, $F_s^{\pm 1}$, and DF_s depend analytically on s, and are uniformly bounded when s belongs to any compact subset of Σ_2.*

(v) Decomposition of the quasi-inverse. *For s close enough to the real axis and $s \in \Sigma_2$, the operator $\mathbf{H}_s$ has a dominant eigenvalue $\lambda(s)$ which is simple and separated from the rest of the spectrum by a spectral gap. The quasi-inverse of the operator $\mathbf{H}_s$ splits as*

$$(I - v\mathbf{H}_s)^{-1}[F] = \frac{v\lambda(s)}{1 - v\lambda(s)}\, \mathbf{Q}_s[F] + (I - v\mathbf{N}_s)^{-1}[F], \tag{5.4}$$

where $\mathbf{Q}_s$ is the projector onto the dominant eigensubspace, and the spectral radius of $\mathbf{N}_s$ is strictly smaller than $|\lambda(s)|$. The projector $\mathbf{Q}_s$ satisfies $\mathbf{Q}_s[F](x, y) := F_s(x, y) \cdot \mu_s[\operatorname{diag} F]$, where F_s is the dominant eigenvalue and μ_s is the corresponding eigenvector of the adjoint of the plain operator H_s. In particular, for $s = 1$, we have

$$\mathbf{Q}_1[F](0, 1) = \int_{\mathcal{I}} F(x, x)dx.$$

Analytic properties of the secant operator have already been studied in [39], but in other functional spaces. The proofs of assertions (iv) and (v) follow the same lines as in [39]. The following four subsections are devoted to proving assertions (i)–(iii) of Theorem 5.1.

5.5. Quasi-compactness and Lasota–Yorke bounds

Here, the sup-norm $\|\cdot\|_0$ is the weak norm and the $\|\cdot\|_{1,1}$-norm is the strong norm. The following lemma proves that the secant operator satisfies a Lasota–Yorke bound, which will be used to prove quasi-compactness via Hennion's theorem.

Lemma 5.2 (Lasota–Yorke bounds). *Let ρ be the contraction ratio defined in* (4.7). *There exists $C > 0$ such that, for any $\widehat{\rho}$ with $\rho < \widehat{\rho} < 1$, there exists an integer N such that, for all $n \geqslant N$, for all $s = \sigma + it \in \Sigma_0$, and all $F \in \mathcal{C}^1(\mathcal{I} \times \mathcal{I})$, we have*

$$\|\mathbf{H}_s^n[F]\|_{1,1} \leqslant \|\mathbf{H}_\sigma^n\|_0\big(C\,|s|\,\|F\|_0 + \widehat{\rho}^n\|F\|_1\big). \tag{5.5}$$

Proof. With the inequality

$$\|\mathbf{H}_s^n[F]\|_0 \leqslant \|\mathbf{H}_\sigma^n[1]\|_0 \cdot \|F\|_0 \leqslant \|\mathbf{H}_\sigma^n\|_0 \cdot \|F\|_0, \tag{5.6}$$

it is sufficient to deal with $\|\mathbf{H}_s^n[F]\|_1$. The function $\mathbf{H}_s^n[F]$ can be written as a sum over $h \in \mathcal{H}^n$ of terms of the form

$$\left|\frac{h(x)-h(y)}{x-y}\right|^s F(h(x),h(y)),$$

and we begin by considering the partial derivative of each term with respect to x, which is written as $p_h + q_h$, with

$$|p_h| \leqslant |s|\left|\frac{h(x)-h(y)}{x-y}\right|^{\sigma-1} \cdot \left|\frac{h'(x)(x-y)-(h(x)-h(y))}{(x-y)^2}\right| \cdot |F(h(x),h(y))|$$

and

$$|q_h| \leqslant \left|\frac{h(x)-h(y)}{x-y}\right|^{\sigma} \cdot |F'_x(h(x),h(y))| \cdot |h'(x)|.$$

The distortion assumption (4.9) is used to bound p_h: the inequality

$$\left|\frac{x-y}{h(x)-h(y)}\,\frac{h'(x)(x-y)-(h(x)-h(y))}{(x-y)^2}\right| \leqslant \sup_{(x,y)\in\mathcal{I}\times\mathcal{I}} \frac{|h''(x)|}{|h'(y)|} \leqslant L$$

implies the bound

$$|p_h| \leqslant L|s|\left|\frac{h(x)-h(y)}{x-y}\right|^{\sigma} |F(h(x),h(y))|.$$

Finally, property (4.8) provides an estimate for q_h, via the inequality (valid for $n \geqslant N$)

$$|F'_x(h(x),h(y))| \cdot |h'| \leqslant \widehat{\rho}^n \cdot |F'_x(h(x),h(y))|.$$

We then obtain

$$|\mathbf{H}_s^n[F]'_x| \leqslant L\,|s|\,\|\mathbf{H}_\sigma^n\|_0\,\|F\|_0 + \widehat{\rho}^n\,\|\mathbf{H}_\sigma^n\|_0\,\|F'_x\|_0.$$

As the partial derivative with respect to y can be bounded in the same vein, one obtains the bound

$$\|\mathbf{H}_s^n[F]\|_1 \leqslant 2L|s|\,\|\mathbf{H}_\sigma^n\|_0\,\|F\|_0 + \widehat{\rho}^n\,\|\mathbf{H}_\sigma^n\|_0\,\|F\|_1,$$

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and, with (5.6), the final result. □

Remarks. For an operator $\mathbf{L}$ which acts on a Banach space $(\mathcal{B}, \|\cdot\|)$, the Spectral Radius Theorem $R(\mathbf{L})$ asserts the equality $R(\mathbf{L}) = \lim_{n\to\infty} \|\mathbf{L}^n\|^{1/n}$. In particular, this implies

$$R(\mathbf{H}_s) = \lim_{n\to\infty} \|\mathbf{H}_s^n\|_{1,1}^{1/n} \quad \text{and} \quad R(H_s) = \lim_{n\to\infty} \|H_s^n\|_{1,1}^{1/n}. \tag{5.7}$$

For $s := \sigma + it$, the Lasota–Yorke bounds give the inequality

$$R(\mathbf{H}_s) \leqslant \lim_{n\to\infty} \|\mathbf{H}_\sigma^n\|_0^{1/n}. \tag{5.8}$$

The inequality $\|F\|_{1,1} \geqslant \|F\|_0$ implies that inequality (5.8) is an equality for real s.

5.6. Proof of assertion (ia) of Theorem 5.1

The following lemma compares the spectral radii of secant and plain operators.

Lemma 5.3. *For $s = \sigma + it \in \Sigma_0$, the following inequalities hold:*

$$R(H_s) \leqslant R(\mathbf{H}_s) \leqslant R(\mathbf{H}_\sigma) = R(H_\sigma). \tag{5.9}$$

Proof. The diagonal relation (4.5) and the inequality $\|F\|_{1,1} \geqslant \|\operatorname{diag} F\|_{1,1}$ together give

$$\|\mathbf{H}_s^n\|_{1,1} := \sup_{\|F\|_{1,1}\leqslant 1} \|\mathbf{H}_s^n[F]\|_{1,1} \geqslant \sup_{\|F\|_{1,1}\leqslant 1} \|\operatorname{diag} \mathbf{H}_s^n[F]\|_{1,1} = \sup_{\|F\|_{1,1}\leqslant 1} \|H_s^n[\operatorname{diag} F]\|_{1,1}. \tag{5.10}$$

Now observe that the diagonal of any function F of $\mathcal{C}^1(\mathcal{I}\times\mathcal{I})$ is also the diagonal of the function $\widehat{F}$ of $\mathcal{C}^1(\mathcal{I}\times\mathcal{I})$, defined by

$$\widehat{F}(x,y) = F(x,x) = \operatorname{diag} F(x), \quad \text{for any } (x,y) \in \mathcal{I}\times\mathcal{I}, \tag{5.11}$$

which furthermore satisfies the relation $\|\widehat{F}\|_{1,1} = \|\operatorname{diag} F\|_{1,1}$. This implies the equalities

$$\sup_{\|F\|_{1,1}\leqslant 1} \|H_s^n[\operatorname{diag} F]\|_{1,1} = \sup_{f\in\mathcal{C}^1(\mathcal{I}),\, \|f\|_{1,1}\leqslant 1} \|H_s^n[f]\|_{1,1} = \|H_s^n\|_{1,1},$$

and thus the inequality $\|\mathbf{H}_s^n\|_{1,1} \geqslant \|H_s^n\|_{1,1}$. With (5.7), the first inequality is proved. The second inequality follows easily from (5.8) and the inequality $\|F\|_{1,1} \geqslant \|F\|_0$.

Now, for a real σ, the bounded distortion property (4.9) implies the inequalities

$$\left|\frac{h(x)-h(y)}{x-y}\right|^\sigma \leqslant L|h'(x)|^\sigma \quad \text{for all } (x,y) \in \mathcal{I}\times\mathcal{I} \text{ and } h \in \mathcal{H}^\star,$$

which imply, for any $F \in \mathcal{C}^1(\mathcal{I}\times\mathcal{I})$ and $n \geqslant 1$, that

$$\|\mathbf{H}_\sigma^n[F]\|_0 \leqslant L\,\|H_\sigma^n[1]\|_0\,\|F\|_0 \leqslant L\,\|H_\sigma^n[1]\|_{1,1}\,\|F\|_0.$$

With (5.7) and (5.8), it follows that

$$R(\mathbf{H}_\sigma) \leqslant \lim_{n\to\infty} \|\mathbf{H}_\sigma^n\|_0^{1/n} \leqslant \lim_{n\to\infty} \|H_\sigma^n\|_{1,1}^{1/n} \leqslant R(H_\sigma),$$

which completes the proof of Lemma 5.3. □

Now, Lemma 5.2, Hennion's theorem and Lemma 5.3 entail the inequality

$$R_e(\mathbf{H}_s) \leqslant \widehat{\rho} \cdot R(\mathbf{H}_\sigma) = \widehat{\rho} \cdot R(H_\sigma) \quad \text{for any } \widehat{\rho} > \rho, \quad \text{and thus}$$
$$R_e(\mathbf{H}_s) \leqslant \rho \cdot R(\mathbf{H}_\sigma) = \rho \cdot R(H_\sigma).$$

In particular, the operator $\mathbf{H}_s$ is quasicompact for real s. This completes the proof of assertion (ia) of Theorem 5.1. □

5.7. Proof of assertion (ib) of Theorem 5.1

Eigenvalues of the plain operator H_s and those of the secant operator are closely related. Suppose that F is an eigenfunction of $\mathbf{H}_s$ relative to the eigenvalue λ. Then the diagonal relation (4.5) proves the equalities

$$\lambda \operatorname{diag} F = \operatorname{diag}(\lambda F) = \operatorname{diag}(\mathbf{H}_s[F]) = H_s[\operatorname{diag} F]. \tag{5.12}$$

Then, the function $\operatorname{diag} F$ is an eigenfunction of H_s relative to λ provided that F is not identically zero on the diagonal $\mathcal{D} := \{(x,x),\ x \in \mathcal{I}\}$. The next result shows that this is not possible when the inequality $|\lambda| > \rho\, R(\mathbf{H}_\sigma)$ holds.

Lemma 5.4. *Let $\rho < 1$ be the contraction ratio, and consider any pair (s,α) where $s := \sigma + it$ belongs to Σ_0, and α satisfies $|\alpha| > \rho \cdot R(H_\sigma)$. Consider a function F for which $\mathbf{H}_s[F] = \alpha F$ and $\operatorname{diag} F \equiv 0$. Then $F \equiv 0$ on $\mathcal{I} \times \mathcal{I}$.*

Proof. Consider $\widehat{\rho}$ with $\rho < \widehat{\rho} < 1$. Then, the inequality which relates the function F to the function $\widehat{F}$ defined in (5.11), together with property (4.8), gives the bound, for $n \geqslant N$, $h \in \mathcal{H}^n$, $(x,y) \in \mathcal{I} \times \mathcal{I}$,

$$|F(h(x),h(y)) - \widehat{F}(h(x),h(y))| \leqslant \|DF\|_0\, \|h'\|_0 \leqslant \|DF\|_0\, \widehat{\rho}^n, \tag{5.13}$$

which implies, for $n \geqslant N$, that

$$|\mathbf{H}_s^n[F](x,y) - \mathbf{H}_s^n[\widehat{F}](x,y)| \ \leqslant \|\mathbf{H}_\sigma^n\|_{1,1}\, \|DF\|_0\, \widehat{\rho}^n.$$

Now, consider an eigenfunction F relative to the eigenvalue α whose diagonal function is zero. Then, the function $\widehat{F}$ is zero and, for any $n \geqslant N$,

$$|\mathbf{H}_s^n[F](x,y)| \ \leqslant \|\mathbf{H}_\sigma^n\|_{1,1}\, \|DF\|_0\, \widehat{\rho}^n.$$

Finally, for any $n \geqslant N$,

$$|\alpha^n|\|F\|_0 = \|\alpha^n F\|_0 = \|\mathbf{H}_s^n[F]\|_0 \leqslant \|DF\|_0\, \widehat{\rho}^n\, \|\mathbf{H}_\sigma^n\|_{1,1}. \tag{5.14}$$

Assume now that F is not zero. Then, $\|F\|_0$ is not zero, and, with (5.14), the same is true for $\|DF\|_0$. Inequalities (5.14), with the Spectral Radius Theorem, imply the inequality $|\alpha| \leqslant \widehat{\rho} \cdot R(\mathbf{H}_\sigma)$ for any $\widehat{\rho} > \rho$, and then $|\alpha| \leqslant \rho \cdot R(\mathbf{H}_\sigma)$. With the equality $R(\mathbf{H}_\sigma) = R(H_\sigma)$, this provides a contradiction to the hypothesis. Then F is zero. □

Completion of the proof of assertion (ib). Assume that λ is an eigenvalue of $\mathbf{H}_s$ with $|\lambda| > \rho\, R(H_\sigma)$ and let F be an eigenfunction relative to $\mathbf{H}_s$. Lemma 5.4 ensures that the

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diagonal function of F is non-zero. Now, (5.12) proves that $\operatorname{diag} F$ is an eigenfunction relative to λ of H_s.

For $s \in \Sigma_1$, the secant operator $\mathbf{H}_s$ is quasicompact, with assertion (ia). Hence, there exists an eigenvalue of $\mathbf{H}_s$ whose modulus equals $R(\mathbf{H}_s)$. As this eigenvalue satisfies the hypothesis of Lemma 5.4, this is also an eigenvalue for the plain operator H_s, and the inequality $R(\mathbf{H}_s) \leqslant R(H_s)$ holds. Furthermore, with assertion (ia), the inequality $R(\mathbf{H}_s) \geqslant R(H_s)$ holds. This finally proves the equality between the two spectral radii.

5.8. Proof of assertions (ii) and (iii) of Theorem 5.1

(ii) Let us begin with assertion (ii). For $s \in \Sigma_2$, there exists an eigenvalue λ of $\mathbf{H}_s$ whose modulus equals $R(\mathbf{H}_s)$. With assertion (ib), λ is an eigenvalue of H_s, and coincides with the dominant eigenvalue $\lambda(s)$ of H_s. Again, assertion (ib) entails that $\lambda(s)$ is the unique eigenvalue with maximal modulus. If not, the operator H_s would have an eigenvalue of maximal modulus different from $\lambda(s)$.

We now prove that $\lambda(s)$ is simple. Suppose that F_1 and F_2 are two eigenfunctions of $\mathbf{H}_s$ related to $\lambda(s)$. By (5.12), the diagonal functions $\operatorname{diag} F_1$ and $\operatorname{diag} F_2$ are eigenfunctions of H_s relative to $\lambda(s)$. Since this eigenvalue is simple for H_s, the diagonal functions $\operatorname{diag} F_1$ and $\operatorname{diag} F_2$ are linearly dependent, *i.e.*, there are non-zero numbers α_1 and α_2 such that

$$0 = \alpha_1 \operatorname{diag} F_1 + \alpha_2 \operatorname{diag} F_2 = \operatorname{diag}(\alpha_1 F_1 + \alpha_2 F_2) \quad \text{for all } x \in \mathcal{I}.$$

The function $F = \alpha_1 F_1 + \alpha_2 F_2$ is an eigenfunction of $\mathbf{H}_s$ relative to $\lambda(s)$, whose diagonal $\operatorname{diag} F$ is identically zero. With Lemma 5.4, the function F is identically zero on $\mathcal{I} \times \mathcal{I}$. This proves that F_1 and F_2 are linearly dependent, and $\lambda(s)$ is also simple for $\mathbf{H}_s$.

With the diagonal relation (4.5), $\operatorname{diag} F_s$ is an eigenfunction of H_s which coincides with f_s (with a convenient normalization).

We now prove that, for real σ, there exists a dominant eigenfunction which is strictly positive on $\mathcal{I} \times \mathcal{I}$. The operator H_σ has a dominant eigenfunction f_σ which is strictly positive on $\mathcal{I}$, and we consider the eigenfunction F_σ of $\mathbf{H}_\sigma$ whose diagonal function $\operatorname{diag} F_\sigma$ coincides with f_σ. As F_σ is continuous, there is a neighbourhood $\mathcal{E}$ of the diagonal $\mathcal{D}$ where F_σ is positive. Consider an inverse branch $h \in \mathcal{H}^n$ and a point $(x, y) \in \mathcal{I} \times \mathcal{I}$. The distance of the point $(h(x), h(y))$ to the diagonal satisfies

$$d((h(x), h(y)), \mathcal{D}) \leqslant |h(x) - h(y)| \leqslant \widehat{\rho}^n, \quad \text{for } n \geqslant N,$$

and then all the points $(h(x), h(y))$ belong to $\mathcal{E}$ as soon as the depth $|h|$ is large enough. Then, with the definitions of $\mathcal{E}$ and F_σ, the relation

$$F_\sigma(x, y) = \frac{1}{\lambda(\sigma)^n} \mathbf{H}_\sigma^n[F_\sigma](x, y) > 0$$

holds for any $(x, y) \in \mathcal{I} \times \mathcal{I}$, and implies that F_σ is strictly positive on $\mathcal{I} \times \mathcal{I}$.

(iii) The existence of a spectral gap is just a consequence of the definition of Σ_2 and assertion (ia). Now, suppose that the inequality $r(\mathbf{H}_s) > \rho R(H_\sigma)$ holds. Then the inequality $r(\mathbf{H}_s) > R_e(\mathbf{H}_s)$ holds, and there exists an eigenvalue of $\mathbf{H}_s$ whose modulus equals $r(\mathbf{H}_s)$. By assertion (ib), this is an eigenvalue of H_s. Hence, in this case, the inequality $r(H_s) \geqslant r(\mathbf{H}_s)$ between the subdominant spectral radii holds.

The proof of Theorem 5.1 is now complete. □

5.9. A first conclusion: properties of the quasi-inverse near the real axis

We now show how Theorem 5.1 entails the following two propositions, which are the first two steps (the easiest ones) for proving Theorem 4.2.

Proposition 5.5. *Consider a dynamical system of the Good Class and denote by* $\mathbf{H}_s$ *the secant transfer operator. Then there exist a rectangle* $\mathcal{R}_1 := \{s : |\sigma - 1| \leqslant \gamma_1,\ |t| \leqslant t_1\}$*, with* $t_1, \gamma_1 > 0$*, and a neighbourhood* $\mathcal{V}$ *of* $v = 1$*, for which the following holds.*

(i) *For any* $v \in \mathcal{V}$*, the Dirichlet generating function* $\Lambda(s, v)$ *has a unique pole in* $\mathcal{R}_1$*, located at* $s := 1 + \sigma(v)$*, with residue* $r(v)$*. The function* $\sigma(v)$ *is an analytic function defined by the implicit equation described in* (4.11) *and the residue* $r(v)$ *is described in* (4.12)*. At* $s = 1$*, we have* $\sigma'(1) = r(1) = -1/\lambda'(1)$*.*

(ii) *The series* $\Lambda(s, v)$ *is bounded on the vertical segment* $\sigma = 1 - \gamma_1, |t| \leqslant t_1$*, uniformly when* $v \in \mathcal{V}$*.*

Proof. **(i)** Assertions (iv) and (v) of Theorem 5.1 together with analytic perturbation theory [26] imply the existence of a neighbourhood of the line $\Sigma_0 \cap \mathbb{R}$ where the quasi-inverse splits as in (5.4). As soon as the subdominant spectral radius $r(\mathbf{H}_s)$ is strictly less than $|1/v|$, the second term in (5.4) is analytic, and the singularities come from the first term in (5.4). Indeed, with the equality $\lambda(1) = 1$, Theorem 5.1 shows the existence of complex neighbourhoods $\mathcal{U}$ of $s = 1$ and $\mathcal{V}$ of $v = 1$ where the subdominant spectral radius $r(H_s)$ is strictly less than $|1/v|$ for $s \in \mathcal{U}$. Now, choose $\gamma_1 > 0$ and $t_1 > 0$ such that the rectangle $\mathcal{R}_1 := [1 - \gamma_1, 1 + \gamma_1] \times [-t_1, t_1]$ satisfies $\mathcal{R}_1 \subset \mathcal{U}$. Finally, the spectral decomposition (5.4) holds on this rectangle. Moreover, with the analyticity of $s \mapsto \lambda(s)$, together with the inequality $\lambda'(1) \neq 0$, the Implicit Function Theorem applies.

(ii) Restricting $\mathcal{V}$, if necessary, we can assume that $\Re\sigma(v) > -\gamma_1 + \epsilon$ for a small $\epsilon > 0$. In this case, the map $(v, s) \mapsto (I - v\mathbf{H}_s)^{-1}$ is continuous and thus uniformly bounded when s belongs to the segment $\sigma = 1 - \gamma_1, |t| \leqslant t_1$ and v belongs to $\mathcal{V}$. □

6. Spectral properties of transfer operators of the Good-UNI Class: case of parameters s with large or moderate imaginary part

We now consider a dynamical system of the Good-UNI Class. We first prove that the quasi-inverse is well-behaved in 'intermediate' regions. Then, the following of the section is devoted to extending Dolgopyat-type estimates to the secant operator for parameters s with large imaginary part.

6.1. Properties of the quasi-inverse in any intermediate region

We first deal with the intermediate region and establish the following result, which constitutes the second step for Theorem 4.2.

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Proposition 6.1. *Consider a dynamical system of the Good-UNI Class and let* $\mathbf{H}_s$ *denote the secant transfer operator.*

(i) *For any* $t \neq 0$, *the distance* $d(1, \operatorname{Sp} \mathbf{H}_{1+it})$ *is strictly positive.*

(ii) *Consider two positive reals* t_1 *and* t_2 *with* $t_1 \leqslant t_2$. *Then, there exists a rectangle* $\mathcal{R}_2 := \{s;\ |\sigma - 1| \leqslant \gamma_2,\ t_1 \leqslant |t| \leqslant t_2\}$ *with* $\gamma_2 > 0$ *for which*

$$d(1, \operatorname{Sp} \mathbf{H}_s) \geqslant \beta > 0 \quad \text{for } s \in \mathcal{R}_2.$$

(iii) *There exists a neighbourhood* $\mathcal{V}_2$ *of* $v = 1$ *for which the quasi-inverse* $(v, s) \mapsto (I - v\mathbf{H}_s)^{-1}$ *is well defined on* $\mathcal{V}_2 \times \mathcal{R}_2$ *and uniformly bounded.*

Proof. **(i)** There are two possibilities, according to whether $s = 1 + it$ belongs to Σ_1. For $s \notin \Sigma_1$, the inequality $R(\mathbf{H}_s) \leqslant \rho\, R(H_1) = \rho$ holds, and implies the inequality

$$d(1, \operatorname{Sp} \mathbf{H}_{1+it}) \geqslant 1 - \rho > 0.$$

Consider now $s \in \Sigma_1$. With assertion (ia) of Theorem 5.1, the operator $\mathbf{H}_s$ is quasi-compact. Then, if the distance $d(1, \operatorname{Sp} \mathbf{H}_s)$ is zero, $\mathbf{H}_s$ has an eigenvalue equal to 1. The inequality $\rho < 1$, together with assertion (ib) of Theorem 5.1, implies that 1 is also an eigenvalue of the plain operator H_s. But Proposition 1 in [2] ensures that this is not possible for a system of the Good-UNI Class. Finally, the secant operator $\mathbf{H}_s$ does not possess 1 as an eigenvalue and thus $d(1, \operatorname{Sp}\mathbf{H}_{1+it})$ is strictly positive.

(ii) Continuity of the superior part of the spectrum implies the existence of $\gamma_2 > 0$ and $\beta > 0$, for which the inequality $d(1, \operatorname{Sp}\mathbf{H}_s) \geqslant \beta_1 > 0$ holds if $|\sigma - 1| < \gamma$ and $t_1 \leqslant |t| \leqslant t_2$.

(iii) Part (iii) is clear. □

6.2. When s is far from the real axis

Results of Dolgopyat [10], generalized by Baladi and Vallée [2], provide estimates for the quasi-inverse of the plain transfer operator when s is far from the real axis. This section aims to prove that secant operators also satisfy Dolgopyat-type estimates. This will constitute the third (and last) step for proving Theorem 4.2. In the statement, we use the following family of equivalent norms on $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$:

$$\|F\|_{1,t} := \|F\|_0 + \frac{1}{|t|}\|F\|_1 := \sup|F| + \frac{1}{|t|}\sup||F_x| + |F_y|| \quad t \neq 0. \tag{6.1}$$

Theorem 6.2 (Dolgopyat-type estimates for secant operators). *Consider a dynamical system of the Good-UNI Class and its secant transfer operator* $\mathbf{H}_s$ *acting on* $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$. *Then, there are* $\delta < 1$, *a (complex) neighbourhood* $\mathcal{V}$ *of* $v = 1$, *an unbounded rectangle of the form* $\mathcal{R}_3 := \{s; |\sigma - 1| \leqslant \gamma_3, |t| \geqslant t_2\}$ *with* $\gamma_3 > 0$, *and a real* $D_1 > 0$ *such that, for all* $v \in \mathcal{V}$, *and for all* $s = \sigma + it \in \mathcal{R}_3$, *we have*

$$\|(I - v\mathbf{H}_s)^{-1}\|_{1,t} \leqslant D_1 \cdot |t|^{\delta}. \tag{6.2}$$

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On $\mathcal{R}_3$, the function $s \mapsto \Lambda(s,v)$ satisfies, for some positive constant D,

$$|\Lambda(s,v)| \leqslant \|(I - v\mathbf{H}_s)^{-1}[L^s]\|_{1,t} \leqslant D_1 \cdot |t|^\delta \cdot \|L^s\|_{1,t} \leqslant D|t|^\delta.$$

6.3. Return to the proof of Theorem 4.2

Before proving Theorem 6.2, we explain how Theorem 6.2, together with Propositions 5.5 and 6.1, entail Theorem 4.2.

Assertion (i). Here we describe the properties of the dynamical source.

Properties of functions $r(v)$ and $\sigma(v)$. The definition of $\sigma(v)$ and the expression for the residue $r(v)$ given in Theorem 4.2 are provided in Proposition 5.5. Taking derivatives with respect to v leads to the expressions

$$\sigma'(1) = -\frac{1}{\lambda'(1)}, \quad \sigma''(1) + \sigma'(1) = \frac{\lambda'(1)^2 - \lambda''(1)}{\lambda'(1)^3}.$$

Properties of the derivatives of the dominant eigenvalue at $s = 1$ have been widely studied. In particular, it is well known that $-\lambda'(1)$ equals the entropy $h(\mathcal{S})$ (see, *e.g.*, [39]). A proof of strict log-convexity (namely $\lambda''(1) - \lambda'(1)^2 > 0$) can be found in [4].

Analyticity on half-planes to the right of $\Re s = 1$. The following facts are well known. For $\sigma > \sigma_0$, the map $\sigma \mapsto R(H_\sigma)$ is decreasing, and $R(H_1) = 1$. For s with $\Re s \geqslant 1 + \gamma$ (with $\gamma > 0$), the previous facts together with assertion (ia) of Theorem 5.1 prove the inequality

$$R(\mathbf{H}_s) \leqslant R(H_{1+\gamma}) < 1.$$

Then, in a small neighbourhood $\mathcal{V}$ of $v = 1$, the map $\Lambda(s,v)$ is analytic and uniformly bounded on $\mathcal{V} \times \{s, \Re s \geqslant 1 + \gamma\}$.

Analytic properties and polynomial growth on a vertical strip to the left of $\Re s = 1$. We first choose rectangles $\mathcal{R}_1$ from Proposition 5.5 and $\mathcal{R}_3$ from Theorem 6.2 defined by the pairs (γ_1, t_1) and (γ_3, t_2), and consider the corresponding neighbourhoods $\mathcal{V}_1$ and $\mathcal{V}_3$ of $v = 1$. Then, Proposition 6.1, defines a real γ_2 and a neighbourhood $\mathcal{V}_2$. Finally, the vertical strip of Theorem 4.2 is given by $|\Re s - 1| \leqslant \gamma$ with $\gamma = \min(\gamma_1, \gamma_2, \gamma_3)$, whereas the final convenient neighbourhood is $\mathcal{V} := \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$.

Assertion (ii). This is a immediate consequence of assertion (i) and Theorem 2.5.

6.4. Description of the main steps of the proof of Theorem 6.2

It will be convenient to associate with the secant transfer operator $\mathbf{H}_s$ a *normalized operator* $\mathbb{H}_s$ defined by

$$\mathbb{H}_s[F] = \frac{1}{\lambda(\sigma)F_\sigma} \mathbf{H}_s[F_\sigma \cdot F], \quad s = \sigma + it. \tag{6.3}$$

By construction, for $\sigma \in \Sigma_0$, the operator $\mathbb{H}_\sigma$ acting on $\mathcal{C}^1(\mathcal{I} \times \mathcal{I})$ has a spectral radius equal to 1, and fixes the constant function 1. Also, the spectrum $\mathrm{Sp}\,\mathbf{H}_{\sigma+it}$ satisfies $\mathrm{Sp}\,\mathbf{H}_{\sigma+it} = \lambda(\sigma)\mathrm{Sp}\,\mathbb{H}_{\sigma+it}$. Then, the inequality $\|\mathbb{H}_s[F]\|_0 \leqslant \|F\|_0\,\mathbb{H}_\sigma[1] = \|F\|_0$ implies the

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useful bound

$$\|\mathbb{H}_s\|_0 \leqslant 1. \tag{6.4}$$

The proof of Theorem 6.2 follows the same lines as in [2]. We deal with the $(1, t)$-norm defined in (6.1). We begin in Section 5.4 (see Lemma 6.3) with estimates on the L^2-norm of the secant operator, directly obtained from estimates on the usual operator. We transfer these estimates into bounds for the convenient norm $(1, t)$ in Section 5.6, after stating useful lemmas in Section 5.5: the first one (Lemma 6.4) compares the operators $\mathbb{H}_1^k$ and $\mathbb{H}_\sigma^k$, while the second one (Lemma 6.5) provides Lasota–Yorke bounds for the operator $\mathbb{H}_s$, which explain the introduction of the $(1, t)$-norm.

In the following, the notation $A(x) \ll B(x)$ means that A is less than B up to absolute multiplicative constants, or there exists some absolute constant k such that, for any x of interest, the inequality $A(x) \leqslant k \cdot B(x)$ holds. It is synonymous of $A(x) = O(B(x))$ with an absolute O-term.

6.5. UNI Condition and L^2-estimates

The next result summarizes Lemmas 4 and 5 of Baladi and Vallée's paper, which provide L^2-estimates for plain transfer operators. Using the diagonal relation (4.5), we rewrite this result and transfer to a result on L^2- estimates for the (normalized) secant transfer operator.

Lemma 6.3. *Consider a dynamical system of the Good-UNI Class, with contraction ratio $\rho < 1$. Letting $\lceil x \rceil$ denote the smallest integer greater than x, let us associate with $s = \sigma + it$ the integer $n_0(t)$ defined by*

$$n_0 := \left\lceil \frac{1}{|\log \rho|} \log |t| \right\rceil. \tag{6.5}$$

Let $\mathbb{H}_s$ denote the normalized version of the secant transfer operator. Then, for any interval $[1 - \gamma, 1 + \gamma]$, for any s with $\sigma = \Re s \in [1 - \gamma, 1 + \gamma]$, $|t| \geqslant 1/\rho^2$ and a with $(2/5) < a < 1/2$, we have, for any function $F \in \mathcal{C}^1(\mathcal{I} \times \mathcal{I})$,

$$\int_{\mathcal{I}} |\operatorname{diag} \mathbb{H}_s^{n_0}[F](x)|^2 \, dx \ll \rho^{(1-2a)n_0} \, \| \operatorname{diag} F \|_{1,t}^2. \tag{6.6}$$

We recall the main ideas of the proof. The expression of $|\operatorname{diag} \mathbb{H}_s^n[F](x)|^2$ involves a sum over all the pairs $(h, k) \in \mathcal{H}^n$. There are two parts to this sum. The first part of the sum is relative to pairs (h, k) which are sufficiently close with respect to the distance Δ defined in (4.10), and the UNI Condition (U1) entails that this sum is small enough. The second part is relative to pairs (h, k) for which the distance Δ admits a lower bound. Then the Van Der Corput Lemma, together with condition (U2), provides an upper bound for this second part.

6.6. Useful lemmas

We state three lemmas which follow the same lines as in [2]. The first lemma relates the behaviour of the iterate $\mathbb{H}_\sigma^k$ to the iterate $\mathbb{H}_1^k$, for any $\sigma \in \Sigma_0$, and any integer k.

Lemma 6.4. *For real σ such that σ and $2\sigma - 1$ belong to Σ_0, define A_σ as*

$$A_\sigma := \frac{\lambda(2\sigma-1)^{1/2}}{\lambda(\sigma)}.$$

Then, for any compact subset $\mathcal{L}$ of Σ_0, and for any $\sigma \in \mathcal{L}$, for any $F \in \mathcal{C}^1(\mathcal{I} \times \mathcal{I})$, for any integer $k \geqslant 1$, the inequality

$$\|\mathbb{H}_\sigma^k[F]\|_0^2 \ll A_\sigma^{2k} \ \|\mathbb{H}_1^k[|F|^2]\|_0 \tag{6.7}$$

holds and involves absolute constants that only depend on $\mathcal{L}$. The map $\sigma \mapsto A_\sigma$ is continuous and satisfies $A_1 = 1$.

Proof. Now consider $F \in \mathcal{C}^1(\mathcal{I} \times \mathcal{I})$. The relation

$$|\mathbb{H}_\sigma^k[F](x,y)| \ll \frac{1}{\lambda(\sigma)^k} \sum_{h \in \mathcal{H}^k} \left| \frac{h(x)-h(y)}{x-y} \right|^\sigma \cdot |F|(h(x),h(y))$$

is valid if σ belongs to $\mathcal{L}$, and, by the Cauchy–Schwarz inequality, with

$$\left| \frac{h(x)-h(y)}{x-y} \right|^{\sigma-1/2} \quad \text{and} \quad \left| \frac{h(x)-h(y)}{x-y} \right|^{1/2} \cdot |F|(h(x),h(y)),$$

we obtain

$$\left(\sum_{h \in \mathcal{H}^k} \left| \frac{h(x)-h(y)}{x-y} \right|^\sigma \cdot |F|(h(x),h(y)) \right)^2$$
$$\leqslant \left(\sum_{h \in \mathcal{H}^k} \left| \frac{h(x)-h(y)}{x-y} \right|^{2\sigma-1} \right) \cdot \left(\sum_{h \in \mathcal{H}^k} \left| \frac{h(x)-h(y)}{x-y} \right| \cdot |F|^2(h(x),h(y)) \right).$$

The second factor is exactly $\mathbf{H}_1^k[|F|^2](x,y)$, which is less than $\mathbb{H}_1^k[|F|^2](x,y)$ (up to absolute multiplicative constants). Thanks to dominant spectral properties, the first factor is easily related to $\lambda(2\sigma-1)^k$. □

The normalized secant transfer operator admits Lasota–Yorke bounds, easily derived from Lasota–Yorke bounds for the secant operator.

Lemma 6.5. *For every compact subset $\mathcal{L}$ of Σ_0, there exists $C > 0$ such that, for any $\widehat{\rho}$ with $\rho < \widehat{\rho} < 1$, there exists an integer N for which, for any $n \geqslant N$, for all s with $\Re s \in \mathcal{L}$, and all $F \in \mathcal{C}^1(\mathcal{I} \times \mathcal{I})$,*

$$\|\mathbb{H}_s^n F\|_1 \leqslant C\left(|s| \, \|F\|_0 + \widehat{\rho}^n \, \|F\|_1\right), \quad \textit{for all } n \geqslant N. \tag{6.8}$$

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Proof. The two derivatives (normalized operator and non-normalized operator) are related as follows:

$$D(\mathbb{H}_s^n[F]) = \frac{1}{\lambda(\sigma)^n}\left(\frac{-1}{F_\sigma^2}\mathbf{H}_s^n[F\cdot F_\sigma]\,D[F_\sigma] + \frac{1}{F_\sigma}\,D[\mathbf{H}_s^n[F\cdot F_\sigma]]\right).$$

Recall that F_σ and its derivatives are uniformly bounded from above and below when σ belongs to a compact set $\mathcal{L}$. Furthermore, the inequality $R(\mathbf{H}_s) \leqslant \lambda(\sigma)$ holds between the spectral radius of $\mathbf{H}_s$ and the dominant eigenvalue $\lambda(\sigma)$ for $\Re s = \sigma$. Hence, Lasota–Yorke bounds for non-normalized operators entail, for $\widehat{\rho} > \rho$ and all $n \geqslant N$,

$$\|\mathbb{H}_s^n[F]\|_1 \ll \lambda(\sigma)^{-n}\big(\|\mathbf{H}_s^n\|_0\|F\|_0 + \|\mathbf{H}_s^n[F\cdot F_\sigma]\|_1\big) \leqslant C\big(|s|\|F\|_0 + \widehat{\rho}^n\|F\|_1\big),$$

where the constant C depends only on $\mathcal{L}$. □

First use of the $(1,t)$-norm

In the bound (6.8) of Lemma 6.5, there appear two terms: one contains a factor $|s|$, the other a decreasing exponential in n. In order to suppress the effect of the factor $|s|$, in the same spirit as in Dolgopyat's works, we use the family of equivalent norms $\|.\|_{1,t}$ already defined in (6.1). With these norms and Lemma 6.5, together with (6.4), we obtain the first (easy) result.

Lemma 6.6. *For any $t_1 > 0$, for every compact neighbourhood $\mathcal{K}$ of $\sigma = 1$, there exists $M_0 > 0$ such that, for all $n \geqslant 1$, and all s for which $\Re s \in \mathcal{K}$, $|\Im s| \geqslant t_1$, we have*

$$\|\mathbb{H}_s^n\|_{1,\Im s} \leqslant M_0.$$

6.7. Completion of the proof of Theorem 6.2

We now operate transfers between various norms.

From the L^2-norm to the sup-norm. Since the normalized density transformer $\mathbb{H}_1$ is quasi-compact with respect to the $(1,1)$-norm, and fixes the constant function 1, the spectral decomposition (5.4) gives

$$\|\mathbb{H}_1^k[|G|^2]\|_0 = \left(\int_{\mathcal{I}} |\operatorname{diag} G(x)|^2\,dx\right) + O(r_1^k)\|G^2\|_{1,1}, \tag{6.9}$$

where r_1 is the subdominant spectral radius of $\mathbb{H}_1$.

Consider an iterate $\mathbb{H}_s^n$ with $n \geqslant n_0$ (n_0 defined in Lemma 6.3). Then

$$\|\mathbb{H}_s^n[F]\|_0^2 \ll \|\mathbb{H}_\sigma^{n-n_0}[G]\|_0^2 \quad \text{with } G = |\mathbb{H}_s^{n_0}[F]|.$$

Now, using (6.7) from Lemma 6.4 and (6.9) with $k := n - n_0$, together with the bound (6.6) for the L^2-norm and finally Lemma 6.5 to evaluate $\|G^2\|_{1,1}$, we obtain, for any t with $|t| \geqslant t_1$,

$$\|\mathbb{H}_s^n[F]\|_0^2 \ll A_\sigma^{2(n-n_0)}\big[\rho^{(1-2a)n_0} + r_1^{n-n_0}\,|t|\big]\,\|F\|_{1,t}^2.$$

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We now choose $n = n_1$ as a function of t so that the two terms $\rho^{(1-2a)n_0}$ and $r_1^{n-n_0}|t|$ are almost equal:

$$n_1 = (1+\eta)n_0 \quad \text{with } \eta := 2(1-a)\frac{\log \rho}{\log r_1} > 0. \tag{6.10}$$

Now choose d such that $0 < \eta(5a-2) < d < 1-2a < 1/5$ (which is possible if a is of the form $a = 2/5+\epsilon$, with a small $\epsilon > 0$). Recalling (6.6), where a first neighbourhood was defined, and considering a (real) neighbourhood $\mathcal{R}$ of $s=1$ for which

$$A_\sigma^\eta < \rho^{-\eta(5a/2-1)} < \rho^{-d/2} \quad \text{and} \quad \lambda(\sigma)^{1+\eta} < \rho^{-\frac{1}{4}(1-2a-d)}, \tag{6.11}$$

we finally obtain, for $n_1(t)$ and η defined in (6.10),

$$\|\mathbb{H}_s^{n_1}[F]\|_0 \ll \rho^{n_1 b}\,\|F\|_{1,t}, \quad \text{with } b := \frac{1-2a-d}{2(1+\eta)}. \tag{6.12}$$

From the sup-norm to the $\|.\|_{1,t}$-norm. Using (6.12), applying Lemma 6.5 twice with a given $\widehat{\rho}$, and choosing t sufficiently large for the integer $n_1(t)$ of (6.10) to be larger than the integer N of (4.8), we obtain the inequality

$$\begin{aligned}
\|\mathbb{H}_s^{2n_1}[F]\|_1 &\ll |s|\,\|\mathbb{H}_s^{n_1}[F]\|_0 + \widehat{\rho}^{\,n_1}\,\|\mathbb{H}_s^{n_1}[F]\|_1 \\
&\ll |s|\,\rho^{n_1 b}\|F\|_{1,t} + \widehat{\rho}^{\,n_1}|t|\left(\frac{|s|}{|t|}\|F\|_0 + \widehat{\rho}^{\,n_1}\frac{\|F\|_1}{|t|}\right) \\
&\ll |t|\widehat{\rho}^{\,n_1 b}\|F\|_{1,t},
\end{aligned} \tag{6.13}$$

which finally entails for $n_2 = 2n_1$ (and $n_1(t)$ as above)

$$\|\mathbb{H}_s^{n_2}\|_{1,t} \leqslant C\,\widehat{\rho}^{\,n_2 b/2}. \tag{6.14}$$

Now choose t sufficiently large, namely $|t| \geqslant t_2 := C^{4/(1-2a-d)}$, to ensure the inequality $C < \widehat{\rho}^{\,-n_2 b/4}$ for any $n_2(t)$ with $|t| \geqslant t_2$. Finally we have

$$\|\mathbb{H}_s^{n_2}\|_{1,t} \leqslant \widehat{\rho}^{\,n_2 b/4} \quad (\Re s \in \mathcal{R}, |t| \geqslant t_2). \tag{6.15}$$

6.8. The last step in Theorem 6.2

For fixed t with $|t| > t_2$, any integer n can be written $n = kn_2 + \ell$ with $\ell < n_2(t)$. Then (6.14) and Lemma 6.6 imply

$$\|\mathbb{H}_s^{n}\|_{1,t} \leqslant M_0\,\|\mathbb{H}_s^{n_2}\|_{1,t}^k \leqslant M_0\,\widehat{\rho}^{\,bkn_2/4} \leqslant M_0\,\widehat{\rho}^{\,bn/4}\,\widehat{\rho}^{\,-bn_2/4}.$$

Since $bn_2/4 = bn_1/2 = (1-2a-d)n_0/4$, with n_0 defined in (6.5), we finally obtain

$$\|\mathbb{H}_s^{n}\|_{1,t} \leqslant M_0\,|t|^\delta\,\gamma^n,$$
$$\text{with} \quad \delta := \frac{1-2a-d}{4}, \quad b := \frac{2\delta}{1+\eta}, \quad \gamma := \widehat{\rho}^{\,b/4}.$$

Therefore, returning to the operator $\mathbf{H}_s$, we have shown that

$$\|\mathbf{H}_s^n\|_{1,t} \leqslant D_3 \cdot \gamma^n \cdot |t|^\delta \cdot \lambda(\sigma)^n, \quad \forall n, \quad \forall t, \quad \text{with } |t| \geqslant t_2. \tag{6.16}$$

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Finally, with

$$a \in]2/5, 1/2[, \quad \eta := 2(1-a)\frac{\log \rho}{\log r_1}, \quad \eta(5a-2) < d < 1-2a, \quad \delta := \frac{1-2a-d}{4},$$

we take a refinement of the $\mathcal{R}$ defined in (6.11) and $\mathcal{K}$ defined in Lemma 6.6, with a small neighbourhood $\mathcal{V}$ of $v = 1$, we define the rectangle $\mathcal{R}_3$ as

$$\mathcal{R}_3 := \{s = \sigma + it;\ |t| \geqslant t_2,\ A_\sigma < \rho^{-(2-5a)/2},\ |v|\lambda(\sigma) < \rho^{-(1-2a-d)/16(1+\eta)}\}.$$

Then, for $s \in \mathcal{R}_3$ and $v \in \mathcal{V}$, we have

$$\gamma|v|\lambda(\sigma) \leqslant \widehat{\rho}^{(1-2a-d)/16(1+\eta)} = \widehat{\gamma} < 1.$$

This finally proves Theorem 6.2 with $D_1 := D_3/(1-\widehat{\gamma})$.

We have shown in Section 6.3 how Theorem 6.2, together with Propositions 5.5 and 6.1, entails Theorem 4.2, which proves that tries built on dynamical sources of the Good-UNI Class have a depth which follows an asymptotic Gaussian law, with a speed of convergence of order $(\log n)^{-1/2}$.

7. Conclusion and extensions

Simple sources and Good-UNI sources. The probabilistic properties of a random trie, built on n words independently drawn from a source, *a priori* depend on the probabilistic properties of the underlying source. In the general context of dynamical sources, there is a close relationship between the form of the branches and the analytic properties of the Dirichlet generating functions of the source, in particular their *tameness* properties. From this point of view, there are two extreme cases in the Good Class: the simple sources and the sources which satisfy the UNI Condition. The simple sources are defined by dynamical systems all of whose branches have the same form, as they are all affine. On the contrary, for a Good-UNI source, the probability that different branches have 'almost the same form' is exponentially small. The Good-UNI Class gathers sources which 'strongly differ' from the simple sources. This implies different tameness properties: the simple sources are never strongly tame, whereas the sources of the Good-UNI Class are always strongly tame. Then, the properties of random tries built on these two subclasses of sources may be *a priori* different. The present paper shows that this is not actually the case: the dominant terms in the asymptotic expansions of the expectation and the variance are the same, and tameness only has an influence on remainder terms.

Good-DIOP sources. It is also interesting to study the probabilistic behaviour of tries when they are built on other sources, for instance other dynamical sources of the Good Class. Dolgopyat [11] introduced another class of dynamical systems, the *Good-DIOP Class*. This class gathers dynamical sources which extend the *Diophantine* simple sources. It is defined by arithmetical conditions on branches, of Diophantine type, and it contains both simple sources and dynamical sources which are not conjugated to simple sources. Dolgopyat showed in [11] that the quasi-inverse of the (plain) transfer operator of such a dynamical system admits a pole-free region of hyperbolic shape, where it is of polynomial growth. The works by Roux [33] and Roux and Vallée [34] use properties of the Good

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Class that are established here, extend Dolgopyat's results to the secant operator, and prove that such a source is hyperbolic tame: its Dirichlet generating function admits a pole-free region, of hyperbolic shape, where it is of polynomial growth. It is then possible to study the probabilistic behaviour of tries when they are built on Good-DIOP sources. Using Rice's method, it is proved that the trie depth for general sources of the Good-DIOP Class behaves as for particular simple Diophantine sources.

Distributional results for the trie depth. We also prove here that the trie depth of a Good-UNI source asymptotically follows a Gaussian law, with an optimal speed of convergence of order $(\log n)^{-1/2}$. These results are based on tameness properties of the bivariate generating function $\Lambda(s, v)$, which are obtained via a perturbation of the series $\Lambda(s)$ in a complex neighbourhood of $v = 1$. In the case of H-tameness, as the distance between the frontier of the hyperbolic region and the vertical line tends to zero (for $|\Im s| \to \infty$), it is not possible to use such a strong perturbation (in a whole complex neighbourhood of $v = 1$), but there exist weakest notions of perturbation which are sufficient to obtain Gaussian laws with an optimal speed of convergence. Instead of the Quasi-Powers Theorem, we use the Goncharov theorem, followed by the Berry–Esseen inequality (see the recent paper [20] and the thesis [19]).

Similar studies for the digital search tree. All the previous results about tries can be extended to another type of digital trees, the digital search tree (DST for short). The DST is more difficult to deal with, but the recent paper [20] and the thesis [19] show how to conduct a similar study for typical depth, in a parallel way, for both tries and DSTs, which leads to very similar results for the two types of digital tree.

Importance of source tameness. This paper is among the first[9] to introduce this notion and to show its importance, specifically in the analysis of trie depth. This notion appears to be central to many other studies that deal with sources, either directly or indirectly, in the analysis of data structures on words (for instance DST as in [20] or [19]) or algorithms on words (for instance sorting algorithms as in [9] or searching algorithms as in [7]).

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[9] In fact the first version of the present paper was written in 2008, and has already been cited in many articles.

Appendix

This Appendix is devoted to proving Proposition 2.4, which arises in the proofs of Proposition 2.3 and Theorem 3.5. The main arguments[10] are due to Flajolet and Sedgewick and are summarized in [16].

Proposition A.1 (Proposition 2.4 restated).

(i) *For any fixed s with $s \notin \mathbb{Z}_{\geqslant 0}$, we have*

$$L_n(s) := \frac{n!(-1)^n}{s(s-1)\cdots(s-n)} = -n^s\Gamma(-s)\left[1 + O\left(\frac{1}{n}\right)\right].$$

The O-term is uniform for s in a bounded set.

(ii) *Consider a vertical line $\Re(s) = \alpha$ with $\alpha \notin \mathbb{Z}_{\leqslant 0}$ and assume that $\varpi(s)$ is continuous on $\Re(s) = \alpha$ and of at most polynomial growth there, i.e., $\omega(s) = O(s^r)$ as $|s| \to \infty$ on $\Re(s) = \alpha$. Then, the integral admits the following estimate, as $n \to \infty$:*

$$\int_{\Re s=\alpha} \varpi(s)\frac{n!}{s(s-1)\cdots(s-n)}ds = O(n^\alpha).$$

(iii) *Consider a curve ρ of hyperbolic type, namely of the form*

$$\rho := \left\{s = \sigma + it, |t| \geqslant B, \sigma = \sigma_0 - \frac{A}{|t|^{\beta_0}}\right\} \cup \left\{s = \sigma + it, \sigma = \sigma_0 - \frac{A}{B^{\beta_0}}, |t| \leqslant B\}\right\},$$

for some strictly positive constants (A, B, β_0) and some real σ_0. Assume further, and assume that $\varpi(s)$ is continuous on ρ and of at most polynomial growth there, i.e., $\varpi(s) = O(|s|^r)$ as $|s| \to \infty$. Then the integral of $\varpi(s)L_n(s)$ on the curve ρ admits the following estimate, as $n \to \infty$:

$$\int_\rho \varpi(s)L_n(s)ds = n^{\sigma_0} \cdot \; O(\exp[-(\log n)^\beta]), \quad \text{with } \beta < \frac{1}{1+\beta_0}.$$

The proof is based on two main lemmas. The first lemma is useful for proving assertion (i) and estimates the integrals of assertions (ii) and (iii) near the real axis, whereas the second lemma is a main step for estimating the integrals of assertions (ii) and (iii) near the imaginary infinity. Then, the proof has three main steps: the proofs of the two lemmas, and then their use in the proof of Proposition A.1.

A.1. Estimates near the real axis

Lemma A.1. *For s outside a fixed sector containing the negative real axis in its interior, and under the condition $|s| \leqslant \sqrt{n}$, we have, as $n \to \infty$,*

$$L_n(s) = \frac{n!(-1)^n}{s(s-1)\cdots(s-n)} = -n^s\Gamma(-s)\left(1 + O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{s^2}{n}\right)\right). \tag{A.1}$$

Also for any fixed s with $s \notin \mathbb{N}$, we have

$$L_n(s) = -n^s\Gamma(-s)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{A.2}$$

[10] Many thanks are due to Philippe Flajolet for discussions on this proof.

Proof. We have

$$\frac{n!(-1)^n}{s(s-1)\cdots(s-n)} = -\frac{n!}{-s(-s+1)\cdots(-s+n)} = -\frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n-s+1)}.$$

Stirling's formula holds in the complex plane, provided a sector around the negative real axis is avoided. Under this condition, we have

$$\Gamma(w+1) = w^w e^{-w}\sqrt{2\pi w}\left(1+O\left(\frac{1}{n}\right)\right), \quad |w| \to +\infty. \tag{A.3}$$

With the Stirling formula,

$$\begin{aligned}\frac{\Gamma(n+1)}{\Gamma(n-s+1)} &= \frac{n^n e^{-n}\sqrt{2\pi n}}{(n-s)^{s-n}e^{s-n}\sqrt{2\pi(n-s)}}\left(1+O\left(\frac{1}{n}\right)\right)\\ &= \exp[n\log n-(n-s)\log(n-s)-s]\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\\ &= \exp[s\log n-(n-s)\log(1-s/n)-s]\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right).\end{aligned}$$

In the region under consideration, we have $s/n = O(1/\sqrt{n})$, which is a small quantity, so that $\log(1+s/n) = s/n + O(s^2/n^2)$. Consequently,

$$\begin{aligned}\frac{\Gamma(n+1)}{\Gamma(n-s+1)} &= n^s\exp\left[O\left(\frac{s^2}{n}\right)\right]\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\\ &= n^s\left(1+O\left(\frac{1}{\sqrt{n}}\right)+O\left(\frac{s^2}{n}\right)\right),\end{aligned}$$

and we obtain (A.1). The proof of (A.2) is similar, indeed simpler, via the relation $s/n = O(1/n)$. □

A.2. Far from the real axis

Lemma A.2. *Fix any number $m > 0$. Then, there exists a computable constant $K_m > 0$ such that, for n large enough, $s = b + it$, b fixed and $t \geqslant \sqrt{n}$, we have*

$$|L_n(s)| \leqslant \frac{K_m}{t^m}e^{-B\sqrt{n}}, \quad \textit{with } B = \log(\sqrt{2}).$$

Proof. The proof is given for $b = 0$, but extends to any fixed value of b. Choose an integer $m > 0$ and set $A = \lfloor\sqrt{n}\rfloor$. We write

$$|L_n(s)| = \left|\frac{n!}{s(s-1)(s-2)\cdots(s-n)}\right| = \frac{1}{|s|}\prod_{a=1}^{m}\left|\frac{a}{a-s}\right|\prod_{a=m+1}^{m+A}\left|\frac{a}{a-s}\right|\prod_{a=m+A+1}^{n}\left|\frac{a}{a-s}\right|$$

The first product has a trivial bound:

$$\prod_{a=1}^{m}\left|\frac{a}{a-s}\right| < \frac{m!}{t^m}. \tag{A.4}$$

For the second product, the complex s is close to the imaginary axis when $n \to \infty$. The triangle $(a, 0, s)$ is approximately a right-angled triangle. The angle β at a satisfies, for n

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large,

$$\tan(\beta) \sim \frac{|s|}{|a|} \geqslant 1, \quad \text{and thus} \quad \left|\frac{a}{a-s}\right| = \cos(\beta) < \cos\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^A.$$

resulting in

$$\prod_{a=m+1}^{m+A} \left|\frac{a}{s-a}\right| < \left(\frac{1}{\sqrt{2}}\right)^A. \tag{A.5}$$

For the third product, we use the triangle inequality, which gives $|a/(a-s)| < 1$ and

$$\prod_{a=m+A+1}^{n} \left|\frac{a}{a-s}\right| < 1. \tag{A.6}$$

Collecting (A.4), (A.5), (A.6), we have

$$|L_n(s)| < \frac{m!}{t^m}\left(\frac{1}{\sqrt{2}}\right)^A = \frac{m!}{t^m}e^{-B\sqrt{n}}.$$

Then, $K_m = m!$ and $B = \log(\sqrt{2})$. □

A.3. Proof of Proposition A.1

It remains to prove assertions (ii) and (iii). We only need to consider the integrals in the upper half-plane. We use $T = \sqrt{n}$ as a cut-off point and decompose each positive part $\tilde{\rho}$ of the curve – the vertical line or the hyperbolic curve ρ – into two parts.

Case of a vertical line. We use the decomposition

$$\int_{\tilde{\rho}} \varpi(s)\, L_n(s)ds = \int_{s=\alpha}^{\alpha+iT} \varpi(s)\, L_n(s)ds + \int_{s=\alpha+iT}^{\alpha+i\infty} \varpi(s)\, L_n(s)ds.$$

Near the real axis, namely for $s \in [\alpha, \alpha + iT]$, we apply Lemma A.1:

$$\int_{s=\alpha}^{\alpha+iT} \varpi(s)\, L_n(s)ds = -\int_{s=\alpha}^{\alpha+iT} n^s\Gamma(-s)\varpi(s)(1 + O(n^{-1}))ds. \tag{A.7}$$

As the fast decay of $\Gamma(s)$ compensates more for the polynomial growth of $\varpi(s)$ and $|n^s| = n^\alpha$, the integral is $O(n^\alpha)$.

Far from the real axis, namely for $s \in [\alpha + iT, \alpha + \infty t]$, we apply Lemma A.2,

$$\int_{s=\alpha+iT}^{\alpha+i\infty} |L_n(s)|ds < K_m e^{-L\sqrt{n}} \int_{t=T}^{\infty} \frac{t^r}{t^m}dt = O(e^{-L\sqrt{n}}), \tag{A.8}$$

for n large enough, provided m has been chosen such that $m > r + 2$. The combination of equations (A.7) and (A.8) yields the claimed estimate in the case of a vertical line.

Case of a hyperbolic curve. Now consider the case of a hyperbolic curve, and consider the two parts of the curve $\tilde{\rho}$: the curve ρ^- (near the real axis) and the curve ρ^+ (near imaginary infinity):

$$\int_{\tilde{\rho}} \varpi(s)\, L_n(s)ds = \int_{\rho^+} \varpi(s)\, L_n(s)ds + \int_{\rho^-} \varpi(s)\, L_n(s)ds. \tag{A.9}$$

In the case of the curve ρ^+, which resembles a vertical line, we apply Lemma A.1,

$$\left| \int_{\rho^+} \varpi(s)\, L_n(s) ds \right| < K_m \int_T^\infty O(t^r) \cdot O(t^{-m}) \cdot e^{-L\sqrt{n}} dt = O(e^{-L\sqrt{n}}), \tag{A.10}$$

for n large enough, provided that m has been chosen such that $m > r + 2$.

Now, near the real axis, Lemma A.2 gives

$$\int_{\rho^-} \varpi(s)\, L_n(s) ds = \left(\int_{\rho^-} n^s \Gamma(-s) \varpi(s) ds \right) \left(1 + O(n^{-1})\right). \tag{A.11}$$

Letting $s := \sigma + it$, and $L := \log n$, we use the estimates

$$|n^s| = n^\sigma = n^{\sigma_0} \exp[-ALt^{-\beta_0}], \quad |\varpi(s)\Gamma(-s)| \leqslant \exp[-Kt]$$

(for some $K > 0$). The first one is due to the definition of the curve whereas the second one uses the fast decay of $\Gamma(-s)$, which more than compensates for the polynomial growth of $\varpi(s)$. With $L := \log n$, the modulus of the integrand is at most

$$|L_n(s)| \leqslant n^{\sigma_0} \exp[-Kt - ALt^{-\beta_0}].$$

When n (and then L) is fixed, the minimum of the function $t \mapsto Kt + ALt^{-\beta_0}$ is reached for $t^{\beta_0+1} = \beta_0 L/K$. Then the maximum of $|L_n(s)|$ is of order $n^{\sigma_0} \exp[-(\log n)^\beta]$ with $\beta < 1/(1+\beta_0)$. Using the same principles as in the Laplace method, we obtain the estimate

$$\int_{\rho^-} L_n(s) ds = n^{\sigma_0} O(\exp[-(\log n)^\beta]) \quad \text{with } \beta < 1/(1+\beta_0).$$

This yields the claimed estimate in the case of a hyperbolic curve. The proof of Proposition A.1 is now complete.

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