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On functional calculus properties of Ritt operators

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We compare various functional calculus properties of Ritt operators. We show the existence of a Ritt operator $T\colon X \to X$ on some Banach space X with the following property: T has a bounded $\mathcal{H}^\infty$-functional calculus with respect to the unit disc $\mathbb{D}$ (that is, T is polynomially bounded) but T does not have any bounded $\mathcal{H}^\infty$-functional calculus with respect to a Stolz domain of $\mathbb{D}$ with vertex at 1. Also we show that for an R-Ritt operator the unconditional Ritt condition of Kalton and Portal is equivalent to the existence of a bounded $\mathcal{H}^\infty$-functional calculus with respect to such a Stolz domain.

Keywords: Ritt operators; sectorial operators; functional calculus; R-boundedness

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1. Introduction

Ritt operators on Banach spaces have a specific $\mathcal{H}^\infty$-functional calculus that was formally introduced in [13]. This functional calculus is related to various classical notions that play a role in the harmonic analysis of single operators, such as square functions, maximal inequalities, multipliers and dilation properties (see, in particular, the above mentioned paper and [1,2,14]). The aim of the present paper is to compare the $\mathcal{H}^\infty$-functional calculus of Ritt operators with two closely related notions, namely polynomial boundedness and the unconditional Ritt condition from [9].

Let $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disc of the complex field, let X be a (complex) Banach space and recall that a bounded operator $T\colon X \to X$ is called polynomially bounded if there exists a constant $K \geqslant 0$ such that

$$\|P(T)\| \leqslant K \sup\{|P(z)|\colon z \in \mathbb{D}\}$$

for any polynomial P. We say that T is a Ritt operator provided that the spectrum of T is included in $\bar{\mathbb{D}}$ and the set

$$\{(\lambda - 1)R(\lambda, T)\colon |\lambda| > 1\} \tag{1.1}$$

is bounded. (Here $R(\lambda, T) = (\lambda - T)^{-1}$ denotes the resolvent operator.) For any $\gamma \in (0, \frac{1}{2}\pi)$, let B_γ be the open Stolz domain defined as the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$ (see figure 1).

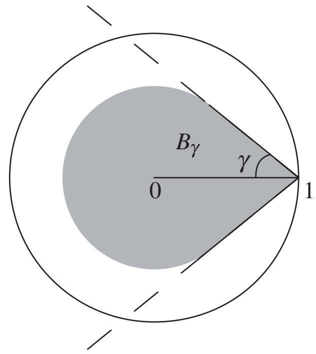

Figure 1. The open Stolz domain.

It is well known that the spectrum of any Ritt operator T is included in the closure $\bar{B}_\gamma$ of one of these Stolz domains. Following [13], we say that T has a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus if there is a constant $K \geqslant 0$ such that

$$\|P(T)\| \leqslant K \sup\{|P(z)|\colon z \in B_\gamma\} \tag{1.2}$$

for any polynomial P. Since $B_\gamma \subset \mathbb{D}$, it is plain that this property implies polynomial boundedness. It was shown in [13] that the converse holds true on Hilbert spaces. Our main result asserts that this does not remain true on all Banach spaces. We shall exhibit a Banach space X and a Ritt operator $T\colon X \to X$ that is polynomially bounded but has no bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus. This will be achieved in § 3 (see theorem 3.2). This example is obtained by first developing and then exploiting a construction of Kalton concerning sectorial operators [8]. Section 2 is devoted to preliminary results and to the main features of Kalton's example.

Following [9] we say that T satisfies the unconditional Ritt condition if there exists a constant $K \geqslant 0$ such that

$$\bigg\| \sum_{k\geqslant 1} a_k(T^k - T^{k-1}) \bigg\| \leqslant K \sup\{|a_k|\colon k \geqslant 1\} \tag{1.3}$$

for any finite sequence $(a_k)_{k\geqslant 1}$ of complex numbers. This property is stronger than the Ritt condition [9, proposition 4.3] and it is easy to check that if T admits a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$, then T satisfies the unconditional Ritt condition (see lemma 4.1). We do not know if the converse holds true. However, we shall show in § 4 that if T is R-Ritt and satisfies the unconditional Ritt condition, then it admits a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$. As a consequence, we generalize [9, theorem 4.7] by showing that on a large class of Banach spaces the unconditional Ritt condition is equivalent to certain square function estimates for R-Ritt operators.

2. Sectorial operators and Kalton's example

Let X be a Banach space and let $A\colon D(A) \to X$ be a closed operator with dense domain $D(A) \subset X$. We let $\sigma(A)$ denote the spectrum of A, and whenever λ belongs

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to the resolvent set $\mathbb{C} \setminus \sigma(A)$ we let $R(\lambda, A) = (\lambda - A)^{-1}$ denote the corresponding resolvent operator.

For any $\omega \in (0, \pi)$, we let $\Sigma_\omega = \{z \in \mathbb{C}^* : |\arg(z)| < \omega\}$. We also set $\Sigma_0 = (0, \infty)$ for convenience. We recall that, by definition, A is sectorial if there exists an angle ω such that $\sigma(A) \subset \bar{\Sigma}_\omega$ and for any $\nu \in (\omega, \pi)$ the set

$$\{\lambda R(\lambda, A) : \lambda \in \mathbb{C} \setminus \bar{\Sigma}_\nu\} \tag{2.1}$$

is bounded. The smallest $\omega \in [0, \pi)$ with this property is called the sectoriality angle of A.

We shall need a few facts about $\mathcal{H}^\infty$-functional calculus for sectorial operators, which we now recall. For background and complements, we refer the reader to [5, 7, 11, 15].

Let A be a sectorial operator with sectoriality angle $\omega \geqslant 0$. One can naturally define a bounded operator $F(A)$ for any rational function F with nonpositive degree and poles outside $\sigma(A)$. Let $\phi \geqslant \omega$. The operator A is said to admit a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus if there exists a constant K such that, for all functions F as above,

$$\|F(A)\| \leqslant K \sup\{|F(z)| : z \in \Sigma_\phi\}. \tag{2.2}$$

In that case, if μ denotes the infimum of all angles ϕ for which such an estimate holds, then A is said to admit a bounded $\mathcal{H}^\infty$-functional calculus of type μ.

Note that the above definition makes sense even for $\phi = \omega$, which is important for our purpose (see proposition 2.2). If $\phi > \omega$ and A has dense range, it follows from [11, proposition 2.10] that when the estimate (2.2) holds true on rational functions the homomorphism $F \mapsto F(A)$ naturally extends to a bounded operator on $\mathcal{H}^\infty(\Sigma_\phi)$, the Banach algebra of all bounded analytic functions on Σ_ϕ. In particular, for $s \in \mathbb{R}$, the image of the function $z \mapsto z^{\mathrm{i}s}$ under this homomorphism coincides with the classical imaginary power $A^{\mathrm{i}s}$ of A. These imaginary powers hence satisfy the estimate

$$\|A^{\mathrm{i}s}\| \leqslant K \mathrm{e}^{\phi|s|}, \quad s \in \mathbb{R},$$

when (2.2) holds true.

On a Hilbert space, a well-known result of McIntosh [15] asserts that if A is a sectorial operator with sectoriality angle ω that admits bounded imaginary powers or a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for some $\phi > \omega$, then it has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for any $\phi > \omega$. That is, its $\mathcal{H}^\infty$-functional calculus type coincides with its sectoriality angle.

However, on general Banach spaces, this property can fail. Indeed, in [8], Kalton constructs, for any $\theta \in (0, \pi)$, a Banach space X_θ and a sectorial operator A on X_θ with sectoriality angle 0, which admits a bounded $\mathcal{H}^\infty$-functional calculus of type θ.

The construction is as follows. On the classical space $L^2(\mathbb{R})$, consider the norms $\|\cdot\|_\theta$ defined by

$$\|f\|_\theta^2 = \int_{\mathbb{R}} \mathrm{e}^{-2\theta|\xi|} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi. \tag{2.3}$$

Obviously, $\|\cdot\|_0$ is the usual L^2-norm and $\|\cdot\|_\theta$ is a smaller norm. For any $\theta \in (0, \pi)$, we let H_θ denote the completion of $L^2(\mathbb{R})$ for the norm $\|\cdot\|_\theta$; this is a Hilbert space.

Let A be the multiplication operator on $L^2(\mathbb{R})$ defined by

$$Af(x) = \mathrm{e}^{-x} f(x).$$

In the following we shall keep the same notation to denote various extensions of A on some spaces containing $L^2(\mathbb{R})$ as a dense subspace. Note that, for any $\phi > 0$ and any $F \in \mathcal{H}^\infty(\Sigma_\phi)$, $F(A)$ is the multiplication operator associated to $x \mapsto F(\mathrm{e}^{-x})$.

According to [8], A extends to a sectorial operator on H_θ with a bounded $\mathcal{H}^\infty$-functional calculus of type θ. This (non-trivial) fact follows from the following observations. First, for any $f \in L^2(\mathbb{R})$, we have $A^{\mathrm{i}s} f(x) = \mathrm{e}^{-\mathrm{i}sx} f(x)$. Hence,

$$\widehat{A^{\mathrm{i}s} f}(\xi) = \hat{f}(\xi + s) \tag{2.4}$$

for any $s, \xi \in \mathbb{R}$. Second, using the definition of $\|\cdot\|_\theta$, this implies that

$$\|A^{\mathrm{i}s}\|_{H_\theta \to H_\theta} = \mathrm{e}^{\theta|s|}, \quad s \in \mathbb{R}. \tag{2.5}$$

This equality implies, by the above mentioned result of McIntosh, that the operator A on H_θ admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for all $\phi > \theta$.

The next step is to construct a new completion X_θ of $L^2(\mathbb{R})$ on which A has similar $\mathcal{H}^\infty$-functional calculus properties but a 'better' sectoriality angle. We shall point out some important elements of this construction. Consider a new norm on $L^2(\mathbb{R})$ by letting

$$\|f\|_{X_\theta} = \sup_{a \in \mathbb{R}} \|f\chi_{(-\infty,a)}\|_\theta. \tag{2.6}$$

Then let X_θ be the completion of $L^2(\mathbb{R})$ for this norm. Clearly, for any $f \in L^2(\mathbb{R})$, we have

$$\|f\|_\theta \leqslant \|f\|_{X_\theta} \leqslant \|f\|_0.$$

Thus, $L^2(\mathbb{R}) \subset X_\theta \subset H_\theta$ with contractive embeddings. Note that, unlike H_θ, X_θ is not a Hilbert space. Again A extends to a sectorial operator on X_θ. A key fact is that, on this new space, the sectoriality angle of A is equal to 0. This is a consequence of the following computation. For any $f \in L^2(\mathbb{R})$ and any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,

$$(\lambda - \mathrm{e}^{-x})^{-1} f(x) = \int_{\mathbb{R}} \frac{\lambda \mathrm{e}^{-t}}{(\lambda - \mathrm{e}^{-t})^2} f(x)\chi_{(-\infty,t)}(x)\,\mathrm{d}t \tag{2.7}$$

for any $x \in \mathbb{R}$. If we let $\psi = \arg\lambda$, this implies

$$\|\lambda R(\lambda, A) f\|_\theta \leqslant \|f\|_{X_\theta} \int_0^\infty |s - \mathrm{e}^{\mathrm{i}\psi}|^{-2}\,\mathrm{d}s.$$

Applying this with $f\chi_{(-\infty,a)}$ instead of f, we deduce a uniform estimate

$$\|\lambda R(\lambda, A)\|_{X_\theta \to X_\theta} \leqslant K_\psi,$$

which yields the desired sectoriality property.

If $m \in L^\infty(\mathbb{R})$ is such that the multiplication operator $f \mapsto mf$ is bounded on H_θ with norm less than C_m, then the same holds true on X_θ, since

$$\|mf\|_{X_\theta} = \sup_{a \in \mathbb{R}} \|mf\chi_{(-\infty,a)}\|_\theta \leqslant C_m \|f\|_{X_\theta}.$$

Since $F(A)$ is such a multiplication operator for any $F \in \mathcal{H}^\infty(\Sigma_\phi)$, we derive the following.

LEMMA 2.1. *If A admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus on H_θ, then it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus on X_θ as well.*

Finally, and this is the most difficult part of [8], it turns out that the imaginary powers of A have the same norms on X_θ and on H_θ, namely

$$\|A^{\mathrm{i}s}\|_{X_\theta \to X_\theta} = \|A^{\mathrm{i}s}\|_{H_\theta \to H_\theta} = \mathrm{e}^{\theta|s|} \tag{2.8}$$

for any $s \in \mathbb{R}$. Combining this result with lemma 2.1, this implies that, on X_θ, the operator A admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for any $\phi > \theta$ but cannot have a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for some $\phi < \theta$.

We finally consider the case $\phi = \theta$, which is not treated in [8] but is important for our purpose. This requires a new ingredient, namely the next statement, which is implicit in [12].

PROPOSITION 2.2. *Let A be a sectorial operator with dense range on some Hilbert space H. Assume that A admits bounded imaginary powers and that, for some $\theta \in (0, \pi)$, they satisfy an exact estimate $\|A^{\mathrm{i}s}\| \leqslant \mathrm{e}^{\theta|s|}$ for any $s \in \mathbb{R}$. Then A has a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-functional calculus.*

Proof. Let $\mathrm{i}U$ be the generator of the c_0-semigroup $(A^{\mathrm{i}s})_{s\geqslant 0}$. Our assumption ensures that it satisfies both

$$\|\mathrm{e}^{s(\mathrm{i}U-\theta)}\| \leqslant 1 \quad \text{and} \quad \|\mathrm{e}^{s(-\mathrm{i}U-\theta)}\| \leqslant 1$$

for any $s \geqslant 0$. This means that $\mathrm{i}U - \theta$ and $-\mathrm{i}U - \theta$ both generate contractive semigroups on H. Thus, for all $h \in D(U)$, one has

$$\mathrm{Re}\langle(\theta + \mathrm{i}U)h, h\rangle \geqslant 0 \quad \text{and} \quad \mathrm{Re}\langle(\theta - \mathrm{i}U)h, h\rangle \geqslant 0.$$

Hence, the numerical range of U lies in the closed band $\Omega = \{z \in \mathbb{C} \colon |\mathrm{Im}\, z| \leqslant \theta\}$. By [6, theorem 1], this implies the existence of a constant $K > 0$ such that

$$\|G(U)\| \leqslant K \sup\{|G(w)| \colon w \in \Omega\} \tag{2.9}$$

for any rational function G bounded on Ω. The argument in [6] can be extended to more general functions. It is observed in [12] that, in particular, it applies to all functions G of the form $G(w) = F(\mathrm{e}^w)$, where F is a rational function with negative degree and poles off $\bar{\Sigma}_\theta$, and in this case, $G(U) = F(A)$. In this situation, $\sup\{|G(w)| \colon w \in \Omega\}$ coincides with $\sup\{|F(z)| \colon z \in \Sigma_\theta\}$. Hence, we deduce from (2.9) that A admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-functional calculus. □

According to (2.5), the above proposition applies to Kalton's operator A on H_θ. Hence, the latter admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-functional calculus. Applying lemma 2.1, we deduce that the operator A constructed above on X_θ has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for all $\phi \geqslant \theta$ (not just for $\phi > \theta$).

3. Main result

Our main aim is to prove theorem 3.2. We first need to modify Kalton's example discussed in the previous section. Roughly speaking, we need a similar example with

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the additional property that the operator should be bounded in order to obtain a more precise result.

We consider the restriction B of A on $L^2(\mathbb{R}_+)$. More explicitly, $B\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the bounded operator defined by

$$Bf(x) = \mathrm{e}^{-x} f(x), \quad f \in L^2(\mathbb{R}_+).$$

Then we let H_θ^+ be the completion of $L^2(\mathbb{R}_+)$ for the norm $\|\cdot\|_\theta$ defined by (2.3), we let X_θ^+ be the completion of $L^2(\mathbb{R}_+)$ for the norm $\|\cdot\|_{X_\theta}$ defined by (2.6) and we consider extensions of B to those spaces, as in § 2. Of course, X_θ^+ is a closed subspace of X_θ and the operator B on X_θ^+ is the restriction of the operator A on X_θ. Thus, for any $\phi \in (0,\pi)$ and any appropriate $F \in \mathcal{H}^\infty(\Sigma_\phi)$, we have $F(B) = F(A)|_{X_\theta^+ \to X_\theta^+}$, and hence

$$\|F(B)\|_{X_\theta^+ \to X_\theta^+} \leqslant \|F(A)\|_{X_\theta \to X_\theta}. \tag{3.1}$$

Similar comments apply for H_θ and H_θ^+.

Proposition 3.1. *On the Banach space X_θ^+, the operator B is sectorial, its sectoriality angle is equal to 0, its spectrum $\sigma(B)$ lies in $[0,1]$, it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for all $\phi \geqslant \theta$ and*

$$\|B^{\mathrm{i}s}\|_{X_\theta^+ \to X_\theta^+} = \mathrm{e}^{\theta|s|}, \quad s \in \mathbb{R}. \tag{3.2}$$

Proof. It is clear from (3.1) and the results established for A in § 2 that, on X_θ^+, B is sectorial with a sectoriality angle equal to 0, and it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus for all $\phi \geqslant \theta$.

To show the spectral inclusion $\sigma(B) \subset [0,1]$, consider $\lambda \in \mathbb{C} \setminus [0,1]$. As in (2.7), we have

$$(\lambda - \mathrm{e}^{-x})^{-1} f(x) = \int_0^\infty \frac{\mathrm{e}^{-t}}{(\lambda - \mathrm{e}^{-t})^2} f(x)\chi_{(-\infty,t)}(x)\,\mathrm{d}t$$

for any $f \in L^2(\mathbb{R}_+)$ and any $x \geqslant 0$. Note that, contrary to (2.7), integration is now taken over $(0,\infty)$. We can therefore deduce that

$$\|(\lambda - B)^{-1} f\|_{X_\theta} \leqslant \|f\|_{X_\theta} \int_0^\infty \frac{\mathrm{e}^{-t}}{|\lambda - \mathrm{e}^{-t}|^2}\,\mathrm{d}t$$

for any $f \in L^2(\mathbb{R}_+)$, which ensures that $\lambda - B$ is invertible on X_θ^+.

It remains to prove (3.2). We shall establish it by appealing to (2.8) and by showing that, for any $s \in \mathbb{R}$,

$$\|B^{\mathrm{i}s}\|_{X_\theta^+ \to X_\theta^+} = \|A^{\mathrm{i}s}\|_{X_\theta \to X_\theta}.$$

Let us start with a simple observation. Let τ_a denote the translation operator defined by $\tau_a f(x) = f(x-a)$. Then, for any $f \in L^2(\mathbb{R})$ and for any $a \in \mathbb{R}$, we have

$$\widehat{\tau_a f}(\xi) = \mathrm{e}^{-\mathrm{i}a\xi} \hat{f}(\xi) \quad \text{for any } \xi \in \mathbb{R}.$$

Looking at the definition (2.3), we deduce that

$$\|\tau_a f\|_\theta = \|f\|_\theta. \tag{3.3}$$

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For any $t \in \mathbb{R}$, we have $\chi_{(-\infty,t)}\tau_a f = \tau_a(\chi_{(-\infty,t-a)}f)$. Hence, the above implies

$$\|\tau_a f\|_{X_\theta} = \|f\|_{X_\theta}. \tag{3.4}$$

Now take a function f in $L^2(\mathbb{R})$ with bounded support included in some compact interval $[-M, M]$. Given any $t \in \mathbb{R}$, we have

$$\begin{aligned}\|\chi_{(-\infty,t)}A^{\mathrm{i}s}f\|_\theta &= \|\tau_M(\chi_{(-\infty,t)}A^{\mathrm{i}s}f)\|_\theta \\ &= \|\chi_{(-\infty,t+M)}\tau_M(A^{\mathrm{i}s}f)\|_\theta \\ &\leqslant \|\tau_M(A^{\mathrm{i}s}f)\|_{X_\theta}\end{aligned}$$

by (3.3). Furthermore, $A^{\mathrm{i}s}f(x) = \mathrm{e}^{-\mathrm{i}sx}f(x)$; hence,

$$[\tau_M(A^{\mathrm{i}s}f)](x) = \mathrm{e}^{\mathrm{i}sM}A^{\mathrm{i}s}(\tau_M f)(x) \quad \text{for any real } x.$$

Thus,

$$\|\tau_M(A^{\mathrm{i}s}f)\|_{X_\theta} = \|A^{\mathrm{i}s}(\tau_M f)\|_{X_\theta}.$$

Since $\tau_M f$ has support in $\mathbb{R}_+$, the above equality leads to

$$\|\tau_M(A^{\mathrm{i}s}f)\|_{X_\theta} \leqslant \|B^{\mathrm{i}s}\|_{X_\theta^+\to X_\theta^+}\|\tau_M f\|_{X_\theta}.$$

According to (3.4) and the preceding inequalities, we deduce that

$$\|\chi_{(-\infty,t)}A^{\mathrm{i}s}f\|_\theta \leqslant \|B^{\mathrm{i}s}\|_{X_\theta^+\to X_\theta^+}\|f\|_{X_\theta}.$$

Taking the supremum over $t \in \mathbb{R}$, one obtains $\|A^{\mathrm{i}s}f\|_{X_\theta} \leqslant \|B^{\mathrm{i}s}\|_{X_\theta^+\to X_\theta^+}\|f\|_{X_\theta}$. Hence,

$$\|A^{\mathrm{i}s}\|_{X_\theta\to X_\theta} \leqslant \|B^{\mathrm{i}s}\|_{X_\theta^+\to X_\theta^+}.$$

The reverse inequality is clear (see (3.1)). □

We now turn to Ritt operators. Recall the definition of a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus from § 1 (see also [13]).

Theorem 3.2. *There exists a Ritt operator T on a Banach space X that is polynomially bounded but does not admit a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for any $\gamma < \frac{1}{2}\pi$.*

Proof. We take for X the Banach space $X_{\pi/2}^+$ considered above and we let $B\colon X \to X$ be the operator considered in proposition 3.1. Then we let

$$T = (I - B)(I + B)^{-1}.$$

We note that $z \mapsto (1-z)/(1+z)$ maps $\Sigma_{\pi/2}$ onto $\mathbb{D}$ and $[0, 1]$ into itself. Thus,

$$\sigma(T) \subset [0, 1].$$

To show that T is a Ritt operator, we consider $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. One can write $\lambda = (1-z)/(1+z)$ with $z \notin \bar{\Sigma}_{\pi/2}$. It is easy to check that

$$(\lambda - 1)(\lambda - T)^{-1} = z(z - B)^{-1}(I + B).$$

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Since the sectoriality angle of B is 0, the set $\{z(z-B)^{-1}\colon z \notin \bar{\Sigma}_{\pi/2}\}$ is bounded. Since B is bounded, we deduce that the set defined in (1.1) is bounded.

The fact that B has a bounded $\mathcal{H}^\infty(\Sigma_{\pi/2})$-functional calculus on X implies that T is polynomially bounded. Indeed, if P is a polynomial, then $P(T) = F(B)$ for the rational function F defined by

$$F(z) = P\biggl(\frac{1-z}{1+z}\biggr).$$

Hence, for some constant K, we have

$$\|P(T)\| = \|f(B)\| \leqslant K \sup\{|F(z)|\colon z \in \Sigma_{\pi/2}\},$$

and, moreover,

$$\sup\{|F(z)|\colon z \in \Sigma_{\pi/2}\} = \sup\{|P(w)|\colon w \in \mathbb{D}\}.$$

Now assume that T has a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$. Consider the auxiliary operator

$$C = I - T = 2B(I+B)^{-1}.$$

By [13, proposition 4.1], C is a sectorial operator that admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ for some $\theta \in (0, \frac{1}{2}\pi)$. Thus, there exists a constant $K > 0$ such that

$$\|C^{\mathrm{i}s}\| \leqslant K\mathrm{e}^{\theta|s|}, \quad s \in \mathbb{R}.$$

Furthermore, $\sigma(I+B) \subset [1,2]$. Thus, $I+B$ is bounded and invertible, and hence it admits a bounded $\mathcal{H}^\infty$-functional calculus of any type. Thus, for any $\theta' > 0$ there exists $K' > 0$ such that

$$\|(I+B)^{\mathrm{i}s}\| \leqslant K'\mathrm{e}^{\theta'|s|}.$$

Since B and C commute, we have

$$B^{\mathrm{i}s} = 2^{-\mathrm{i}s} C^{\mathrm{i}s}(I+B)^{\mathrm{i}s}.$$

Hence,

$$\|B^{\mathrm{i}s}\| \leqslant KK'\mathrm{e}^{(\theta+\theta')|s|}$$

for any $s \in \mathbb{R}$. Applying this with θ' small enough so that $\theta + \theta' < \frac{1}{2}\pi$, this contradicts (3.2) on $X^+_{\pi/2}$. □

Remark 3.3. A Ritt operator T on Banach space X is called R-Ritt if the bounded set in (1.1) is actually R-bounded. This notion was introduced in [3], in relation to the study of discrete maximal regularity (see also [4, 9, 13, 16], in which background and references on R-boundedness can also be found).

The existence of Ritt operators that are not R-Ritt goes back to Portal [16]. According to [13, proposition 7.6], a polynomially bounded R-Ritt operator has a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$. Thus, the operator T constructed in theorem 3.2 is a Ritt operator that is not R-Ritt. This example is of a different nature to those in [16].

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4. Unconditional Ritt operators

We now investigate the links between the unconditional Ritt condition and the $\mathcal{H}^\infty$-functional calculus. It is observed in [9] that the unconditional Ritt condition (1.3) is equivalent to the existence of a constant $K > 0$ such that

$$\sum_{k\geqslant 1} |\langle (T^k - T^{k-1})x, y\rangle| \leqslant K\|x\|\|y\|, \quad x \in X,\ y \in X^*. \tag{4.1}$$

Moreover, it is stronger than the Ritt property. Condition (4.1) should be regarded as a discrete analogue of the weak quadratic estimate (or L^1-condition) introduced in [5, § 4]. We shall now show that the unconditional Ritt condition is weaker than the existence of a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$.

LEMMA 4.1. *If T admits a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$, then T satisfies the unconditional Ritt condition.*

Proof. Assume that T admits a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$. Consider a finite sequence $(a_k)_{k\geqslant 1}$. Since

$$\sum_{k\geqslant 1} a_k(T^k - T^{k-1}) = P(T)$$

for the polynomial P defined by

$$P(z) = \sum_{k\geqslant 1} a_k(z^k - z^{k-1}),$$

(1.2) implies that

$$\bigg\| \sum_{k\geqslant 1} a_k(T^k - T^{k-1}) \bigg\| \leqslant K \sup\{|P(z)|\colon z \in B_\gamma\}.$$

Now we have

$$\begin{aligned} |P(z)| &\leqslant \sup_{k\geqslant 1} |a_k| \sum_{k\geqslant 1} |z^k - z^{k-1}| \\ &= \sup_{k\geqslant 1} |a_k| \bigg(\frac{|z-1|}{1-|z|}\bigg). \end{aligned}$$

Since $z \mapsto |z-1|/(1-|z|)$ is bounded on B_γ, this implies the unconditional Ritt condition (1.3). □

We now show a partial converse. See remark 3.3 for the notion of the R-Ritt operator. We shall use the companion notion of the R-sectorial operator. We recall that a sectorial operator A on Banach space is called R-sectorial if there exists an angle ω such that $\sigma(A) \subset \bar{\Sigma}_\omega$ and for any $\nu \in (\omega, \pi)$ the set (2.1) is R-bounded. In accordance with terminology in § 2, the smallest $\omega \in [0, \pi)$ with this property will be called the R-sectoriality angle of A. We refer the reader to [3, 4, 10, 13] and the references therein for information on R-sectoriality.

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THEOREM 4.2. *Let T be an R-Ritt operator that satisfies the unconditional Ritt condition. It then admits a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$.*

Proof. We consider the operator

$$C = I - T.$$

According to [3, theorem 1.1] and its proof, the assumption that T is R-Ritt implies that C is R-sectorial, with an R-sectoriality angle less than $\frac{1}{2}\pi$. On the other hand, the unconditional Ritt condition (1.3) for T implies the so-called L_1-condition for C:

$$\int_0^\infty |\langle C\mathrm{e}^{-tC}x, y\rangle| \frac{\mathrm{d}t}{t} \leqslant K\|x\|\|y\|, \quad x \in X,\ y \in X^*.$$

Indeed, for any $t > 0$,

$$C\mathrm{e}^{-tC} = (I-T)\mathrm{e}^{-t}\mathrm{e}^{tT} = \sum_{n\geqslant 0}(I-T)\mathrm{e}^{-t}\frac{t^nT^n}{n!}.$$

Thus, for any $x \in X$ and $y \in X^*$, we have

$$\langle C\mathrm{e}^{-tC}x, y\rangle = \sum_{n\geqslant 0}\mathrm{e}^{-t}\frac{t^n}{n!}\langle (I-T)T^n x, y\rangle.$$

This implies, using (4.1), that

$$\begin{aligned}\int_0^\infty |\langle C\mathrm{e}^{-tC}x, y\rangle| \frac{\mathrm{d}t}{t} &\leqslant \sum_{n\geqslant 0}\frac{1}{n!}\int_0^\infty |\langle (I-T)T^n x, y\rangle| \mathrm{e}^{-t}t^{n-1}\,\mathrm{d}t \\ &= \sum_{n\geqslant 0}|\langle (I-T)T^n x, y\rangle| \\ &\leqslant K\|x\|\|y\|.\end{aligned}$$

Now, by the results of [5, § 4], the L_1-condition implies that C admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-functional calculus for all $\theta > \frac{1}{2}\pi$. Since C is R-sectorial with an R-sectoriality angle less than $\frac{1}{2}\pi$, it follows from [10, proposition 5.1] that C actually admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$-functional calculus for some $\theta < \frac{1}{2}\pi$. By [13, proposition 4.1], this is equivalent to the fact that T has a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$. □

It is shown in [9, theorem 4.7] that when X is a Hilbert space the unconditional Ritt condition is equivalent to certain square function estimates. We can now extend that result to L^p-spaces. In the next statement, we let $p' = p/(p-1)$ denote the conjugate number of p.

COROLLARY 4.3. *Let Ω be a measure space, let $1 < p < \infty$ and let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a power-bounded operator. The following assertions are equivalent.*

(i) *T is R-Ritt and satisfies the unconditional Ritt condition.*

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(ii) *There exists a constant $C > 0$ such that*

$$\Big\| \Big(\sum_{k=1}^{\infty} k |T^k(x) - T^{k-1}(x)|^2 \Big)^{1/2} \Big\|_p \leqslant C\|x\| \tag{4.2}$$

for any $x \in L^p(\Omega)$ and

$$\Big\| \Big(\sum_{k=1}^{\infty} k |T^{*k}(y) - T^{*(k-1)}(y)|^2 \Big)^{1/2} \Big\|_{p'} \leqslant C\|y\| \tag{4.3}$$

for any $y \in L^{p'}(\Omega)$.

Proof. If the square function estimates in (ii) hold true, then T is an R-Ritt operator by [13, theorem 5.3]. Furthermore, T has a bounded $\mathcal{H}^\infty(B_\gamma)$-functional calculus for some $\gamma < \frac{1}{2}\pi$ by [13, theorem 1.1]. Hence, lemma 4.1 ensures that T satisfies the unconditional Ritt condition. The converse assertion that (i) implies (ii) is obtained by combining theorem 4.2 and [13, theorem 1.1]. □

It is clear from [13] that corollary 4.3 also holds on reflexive Banach lattices with finite cotype. Further generalizations hold true on more Banach spaces, using the abstract square functions introduced and discussed in [13], to which we refer the reader for more information. Combining the results from [13] with theorem 4.2, one obtains that, when X has finite cotype and $T : X \to X$ is an R-Ritt operator, T satisfies the unconditional Ritt condition if and only if T and T^* admit square function estimates.

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