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Turán Number of an Induced Complete Bipartite Graph Plus an Odd Cycle

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Let $k \geqslant 2$ be an integer. We show that if $s = 2$ and $t \geqslant 2$, or $s = t = 3$, then the maximum possible number of edges in a C_{2k+1}-free graph containing no induced copy of $K_{s,t}$ is asymptotically equal to $(t-s+1)^{1/s}(n/2)^{2-1/s}$ except when $k = s = t = 2$.

This strengthens a result of Allen, Keevash, Sudakov and Verstraëte [1], and answers a question of Loh, Tait, Timmons and Zhou [14].

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1. Introduction

Let $\mathcal{F}$ be a family of graphs. A graph is called $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán number* of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices, and is denoted by $\mathrm{ex}(n, \mathcal{F})$.

The classical theorem of Kővári, Sós and Turán [13] concerning the Turán number of complete bipartite graphs states that $\mathrm{ex}(n, K_{s,t}) \leqslant \frac{1}{2}(t-1)^{1/s}n^{2-1/s} + O(n)$, where $s \leqslant t$. Kollár, Rónyai and Szabó [12] provided a lower bound matching the order of magnitude when $t > s!$, and later Alon, Rónyai and Szabó [2] provided a matching lower bound when $t > (s-1)!$.

Let $\mathrm{ex}_{\mathrm{bip}}(n, \mathcal{F})$ denote the maximum number of edges in an n-vertex bipartite $\mathcal{F}$-free graph. Füredi [9] showed that

$$\mathrm{ex}_{\mathrm{bip}}(n, K_{s,t}) \leqslant (1 + o(1))(t-s+1)^{1/s}\left(\frac{n}{2}\right)^{2-1/s}. \tag{1.1}$$

In the cases $s = 2$ and $s = t = 3$ asymptotically sharp values are known: Füredi [10] determined the asymptotics for the Turán number of $K_{2,t}$ by showing that for any fixed $t \geqslant 2$ we have $\mathrm{ex}(n, K_{2,t}) \leqslant \frac{1}{2}(t-1)^{1/2}n^{3/2} + O(n^{4/3})$. Using an example of Brown [5] for the lower bound and a theorem of Füredi [9] for the upper bound, it is known that $\mathrm{ex}(n, K_{3,3}) = \frac{1}{2}n^{5/3} + O(n^{5/3-c})$ for some $c > 0$.

Erdős and Simonovits [7] conjectured that given any family $\mathcal{F}$ of graphs, there exists $k \geqslant 1$ such that $\mathrm{ex}(n, \mathcal{F} \cup \{C_3, C_5, \dots, C_{2k+1}\})$ is equal to $ex_{\mathrm{bip}}(n, \mathcal{F})$ asymptotically. Allen, Keevash, Sudakov and Verstraëte [1] proved this conjecture for complete bipartite graphs $K_{2,t}$ and $K_{3,3}$ in the following stronger form.

Theorem 1.1 (Allen, Keevash, Sudakov and Verstraëte [1]). *Let $k \geqslant 2$ be an integer. If $s = 2$ and $t \geqslant 2$, or $s = t = 3$, then*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{s,t}\}) = (1 + o(1))(t - s + 1)^{1/s}\left(\frac{n}{2}\right)^{2-1/s}.$$

In fact, they proved a more general result for so-called ‘smooth’ families (for more details, see [1]). We also note that the case $s = t = k = 2$ was solved earlier by Erdős and Simonovits [7].

In the rest of the paper we use the following asymptotic notation. Given two functions $f(n)$ and $g(n)$, we write $f(n) \sim g(n)$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Otherwise, we write $f(n) \nsim g(n)$.

1.1. Induced Turán numbers

Loh, Tait, Timmons and Zhou [14] introduced the problem of simultaneously forbidding an induced copy of a graph and a (not necessarily induced) copy of another graph. Let $\mathrm{ex}(n, H, F\text{-ind})$ denote the maximum possible number of edges in an n-vertex graph containing no induced copy of F and no copy of H as a subgraph.

The question of determining $\mathrm{ex}(n, H, F\text{-ind})$ is related to the well-studied areas of Ramsey–Turán theory and the Erdős–Hajnal conjecture. The Ramsey–Turán number $RT(n, H, m)$, is defined as the maximum number of edges that an n-vertex graph with independence number less than m may have without containing H as a subgraph. If we let F be an independent set of order m then $\mathrm{ex}(n, H, F\text{-ind}) = RT(n, H, m)$. So the problem of determining $\mathrm{ex}(n, H, F\text{-ind})$ is a generalization of the Ramsey–Turán problem. We refer the reader to [14] for a detailed discussion of this general problem and its relation to other areas.

Loh, Tait, Timmons and Zhou showed the following.

Theorem 1.2 (Loh, Tait, Timmons and Zhou [14]). *For any integers $k \geqslant 2$ and $t \geqslant 2$ there is a constant β_k, depending only on k, such that*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-}ind\}) \leqslant (\alpha(k,t)^{1/2} + 1)^{1/2}\frac{n^{3/2}}{2} + \beta_k n^{1+1/2k},$$

where $\alpha(k,t) = (2k-2)(t-1)((2k-2)(t-1)-1)$.

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They asked whether Theorem 1.2 determines the correct growth rate in k, and showed that it determines the correct growth rate in n and t.

In this paper we answer their question by showing that $\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\})$ does not depend on k asymptotically, thus improving the upper bound in Theorem 1.2 significantly. Our main result determines $\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\})$ asymptotically in all the cases except when $k = t = 2$, and is stated below.

Theorem 1.3. *For any integers $k \geqslant 2$, $t \geqslant 2$ where $(k,t) \neq (2,2)$, we have*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-}ind\}) = (1+o(1))(t-1)^{1/2}\left(\frac{n}{2}\right)^{3/2}.$$

Note that this shows $\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \sim \mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\})$ for all $k \geqslant 2$, $t \geqslant 2$ except in the case $k = t = 2$, which is studied in the next theorem. Therefore, Theorem 1.3 is a strengthening of Theorem 1.1 in the case $s = 2$ (except when $k = t = 2$); thus our proof of Theorem 1.3 provides a new proof of Theorem 1.1 in this case.

For the lower bound in Theorem 1.3, simply consider a bipartite $K_{2,t}$-free graph G with $n/2$ vertices in each colour class and containing $\sqrt{t-1}\cdot(n/2)^{3/2} + O(n^{4/3})$ edges. The existence of such a graph is shown by Füredi in [10]. Clearly, G contains no copy of C_{2k+1} and as it contains no copy of $K_{2,t}$, of course, it contains no induced copy of $K_{2,t}$ either.

In the case $k = t = 2$, we improve the upper bound in Theorem 1.2 as follows.

Theorem 1.4.

$$(1+o(1))\frac{2}{3\sqrt{3}}n^{3/2} \leqslant \mathrm{ex}(n, \{C_5, K_{2,2}\text{-}ind\}) \leqslant (1+o(1))\frac{n^{3/2}}{2}.$$

For the lower bound, just as in [14], we use the following example of Bollobás and Győri from [4], to construct an induced-$K_{2,2}$-free and C_5-free graph G with

$$(1+o(1))\frac{2}{3\sqrt{3}}n^{3/2}$$

edges.

Bollobás–Győri construction. Take a $K_{2,2}$-free bipartite graph G_0 with $n/3$ vertices in each part and $(1+o(1))(n/3)^{3/2}$ edges. In one part, replace each vertex u by a pair of two new vertices u_1 and u_2, where u_1 and u_2 are adjacent to each other, and to all the vertices that were adjacent to u. It is easy to check that this new graph G contains no C_5 and no induced copy of $K_{2,2}$. Moreover, G contains approximately twice as many edges as G_0.

We leave open the question of determining the asymptotics of $\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\})$.

Organization of the proofs of Theorem 1.3 and Theorem 1.4. Our proof of Theorem 1.3 is divided into two cases: $k \geqslant 3$ and $k = 2$ (see Remark 1 for an explanation for why the case $k = 2$ is special). In Section 3, we prove Theorem 1.3 in the case $k \geqslant 3$ and in Section 4 we prove Theorem 1.3 in the case $k = 2$, along with Theorem 1.4.

1.2. $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\textbf{-ind}\})$ **versus** $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\})$

In Section 2 we prove the following proposition which connects $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\text{-ind}\})$ with $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\})$.

Proposition 1.5. *For any $k,s,t \geqslant 2$ we have*

$$\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\text{-}ind\}) \leqslant \mathrm{ex}(n,\{C_3,C_{2k+1},K_{s,t}\}) + 3c_k n^{1+1/k},$$

where c_k is a constant depending only on k.

Let $k \geqslant 2$ be an integer. We will compare the asymptotic values of $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\text{-ind}\})$ and $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\})$ in the cases $s=2$, $t \geqslant 2$, and $s=t=3$, and deduce Theorem 1.6, below, from Theorem 1.1, Theorem 1.3 and Proposition 1.5.

First let us consider the case $s=2$, $t \geqslant 2$. As already noted, by comparing Theorem 1.1 and Theorem 1.3, it is easy to see that for all $k \geqslant 2$, $t \geqslant 2$ except when $k=t=2$, we have

$$\mathrm{ex}(n,\{C_{2k+1},K_{2,t}\text{-ind}\}) \sim \mathrm{ex}(n,\{C_{2k+1},K_{2,t}\}). \tag{1.2}$$

Now consider the case $s=t=3$. It follows from Theorem 1.1 that for all $k \geqslant 2$, we have

$$\mathrm{ex}(n,\{C_3,C_{2k+1},K_{3,3}\}) \leqslant (1+o(1))(n/2)^{5/3}.$$

Combining this with Proposition 1.5 (substituting $s=t=3$), we get

$$\mathrm{ex}(n,\{C_{2k+1},K_{3,3}\text{-ind}\}) \leqslant (1+o(1))(n/2)^{5/3} + 3c_k n^{1+1/k}.$$

Now note that $3c_k n^{1+1/k} = o(n^{5/3})$ for all $k \geqslant 2$. Therefore,

$$\mathrm{ex}(n,\{C_{2k+1},K_{3,3}\text{-ind}\}) \leqslant (1+o(1))(n/2)^{5/3} = \mathrm{ex}(n,\{C_{2k+1},K_{3,3}\}).$$

This implies

$$\mathrm{ex}(n,\{C_{2k+1},K_{3,3}\text{-ind}\}) \sim \mathrm{ex}(n,\{C_{2k+1},K_{3,3}\}). \tag{1.3}$$

Therefore, (1.2) and (1.3) imply the following strengthening of Theorem 1.1 in all but one special case.

Theorem 1.6. *Let $k \geqslant 2$ be an integer. If $s=2$ and $t \geqslant 2$, or $s=t=3$, then*

$$\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\text{-}ind\}) = (1+o(1))(t-s+1)^{1/s}\left(\frac{n}{2}\right)^{2-1/s},$$

except when $k=s=t=2$.

Surprisingly, in the special case $k=s=t=2$ the two functions $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\text{-ind}\})$ and $\mathrm{ex}(n,\{C_{2k+1},K_{s,t}\})$ seem to behave quite differently. As noted before, in this case it is known by a theorem of Erdős and Simonovits [7] (or by Theorem 1.1) that

$$\mathrm{ex}(n,\{C_5,K_{2,2}\}) = (1+o(1))(n/2)^{3/2},$$

whereas

$$\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\}) \geqslant (1 + o(1))(2n^{3/2}/3\sqrt{3})$$

by the lower bound in Theorem 1.4. Therefore, in this very special case,

$$\mathrm{ex}(n, \{C_5, K_{2,2}\text{-ind}\}) \not\sim \mathrm{ex}(n, \{C_5, K_{2,2}\}).$$

1.3. Notation and the Blakley–Roy inequality

Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. Given a set $S \subseteq V(G)$, the subgraph of G induced by S is denoted $G[S]$.

Given a graph and a vertex v in it, the first neighbourhood of v, $N_1(v)$ is the set of vertices adjacent to v, and for $i \geqslant 2$, let $N_i(v)$ denote the set of vertices at distance exactly i from v. Notice that for $i \neq j$, $N_i(v) \cap N_j(v) = \emptyset$.

A walk of length k, or a *k-walk*, is a sequence $v_0e_0v_1e_1v_2e_2v_3 \dots v_{k-1}e_{k-1}v_k$ of vertices and edges such that $e_i = v_iv_{i+1}$ for $0 \leqslant i \leqslant k-1$. In this paper, we are only interested in walks of length 3. For convenience, we simply denote a 3-walk by $v_0v_1v_2v_3$ instead of $v_0e_0v_1e_1v_2e_2v_3$. The vertices v_0 and v_3 are called the first and last vertices of the 3-walk respectively, and the edges e_0 and e_2 are called the first and last edges respectively. We say the 3-walk starts with the edge e_0 and ends with the edge e_2. We refer to v_0, v_1, v_2, v_3 as the first, second, third and fourth vertices of the walk respectively. Note that the 3-walks $v_0v_1v_2v_3$ and $v_3v_2v_1v_0$ are generally considered different. Also, note that edges can be repeated in a 3-walk. A 3-*path* is a 3-walk with no repeated vertices or edges.

Blakley and Roy [3] showed that the number of k-walks in any graph of average degree d with n vertices is at least nd^k. In our proofs we will only use this statement with $k = 3$. Note that the Blakley–Roy inequality is in fact a more general statement concerning inner products, which can be seen as a matrix version of Hölder's inequality.

2. Proof of Proposition 1.5

Let G be a C_{2k+1}-free graph containing no induced copy of $K_{s,t}$. Let G_Δ be the subgraph of G consisting of the edges which are contained in the triangles of G. Let $G \setminus G_\Delta$ be the graph obtained after deleting all the edges of G_Δ from G. Of course, $G \setminus G_\Delta$ is triangle-free, and the number of edges in G_Δ is at most three times the number of triangles in G.

Győri and Li [11] showed that in a C_{2k+1}-free graph there are at most $c_kn^{1+1/k}$ triangles where c_k is a constant depending only on k. This implies that $|E(G_\Delta)| \leqslant 3c_kn^{1+1/k}$.

Claim 1. *$G \setminus G_\Delta$ is $K_{s,t}$-free.*

Proof. Assume for a contradiction that there is a (not necessarily induced) copy of $K_{s,t}$ in $G \setminus G_\Delta$, and let A, B be its colour classes. Then since G contains no induced copy of $K_{s,t}$, there must be an edge xy of G which is contained in A or B. In either case, it is easy to see that xy and some two edges of the $K_{s,t}$ form a triangle. However, this means that these two edges of $K_{s,t}$ are contained in G_Δ by definition, contradicting the assumption that this $K_{s,t}$ is contained in $G \setminus G_\Delta$. Therefore, the claim follows. □

Using Claim 1, and the fact that $G \setminus G_\Delta$ is a triangle-free subgraph of G, we have $|E(G \setminus G_\Delta)| \leqslant \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{s,t}\})$. Therefore,

$$|E(G)| = |E(G_\Delta)| + |E(G \setminus G_\Delta)| \leqslant 3c_k n^{1+1/k} + \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{s,t}\}),$$

proving Proposition 1.5.

3. Proof of Theorem 1.3 in the case $k \geqslant 3$

Using Proposition 1.5 for $s = 2$, we have

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \leqslant \mathrm{ex}(n, \{C_3, C_{2k+1}, K_{2,t}\}) + 3c_k n^{1+1/k}, \tag{3.1}$$

where c_k is a constant depending only on k.

Moreover, we have the following proposition, which is proved below.

Proposition 3.1. *For all integers $k \geqslant 2, t \geqslant 2$, we have*

$$\mathrm{ex}(n, \{C_3, C_{2k+1}, K_{2,t}\}) \leqslant (1 + o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2}.$$

Combining Proposition 3.1 and (3.1), we get

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \leqslant (1 + o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2} + 3c_k n^{1+1/k}. \tag{3.2}$$

Since $k \geqslant 3$, clearly $3c_k n^{1+1/k} = o(n^{3/2})$, which completes the proof of Theorem 1.3 in the case $k \geqslant 3$.

Remark 1. Note that when $k = 2$, the number of edges in G_Δ can be as large as $\Theta(n^{3/2})$. For example, observe that this is the case in the Bollobás–Győri construction which is stated after Theorem 1.4 (note that every edge in the construction is contained in a triangle). So using Proposition 1.5, we cannot get an upper bound better than

$$(1 + o(1))\left(3c_2 + \sqrt{t-1}\left(\frac{1}{2}\right)^{3/2}\right)n^{3/2}$$

where $c_2 > 0$ is a constant. Therefore, the current approach does not work in the case $k = 2$; we will give a different proof for this case in Section 4.

Now it only remains to prove Proposition 3.1. Note that Proposition 3.1 follows from Theorem 1.1. However, since our proof of Proposition 3.1 is simpler than the general proof in [1], we present it below. Before we proceed with the details, let us briefly sketch the main ideas of the proof.

Sketch of proof of Proposition 3.1. First we will show that we may assume the minimum degree in our C_{2k+1}-free, induced-$K_{2,t}$-free graph G is at least $d/2$ and the maximum degree in G is at most $4d$. Using the Blakley–Roy inequality, we find a vertex v from which at least d^3 3-walks start. We will then show that at least $d^3 - O(d^2)$ of these 3-walks are of the form $vv_1v_2v_3$ where $v_i \in N_i(v)$. The final edges of such 3-walks form a bipartite graph with parts $N_2(v)$ and $N_3(v)$. It will turn out that this bipartite graph has at least $(d^3 - O(d^2))/(t-1)$ edges but, of course, it can have at most $ex_{\text{bip}}(n, K_{2,t})$ edges. This will give the desired bound on d.

Below we continue with the detailed proof.

Proof of Proposition 3.1. Let d be the average degree of a graph G containing no triangle, C_{2k+1} or $K_{2,t}$. Our aim is to bound d from above. If a vertex has degree smaller than $d/2$, then we can delete the vertex and the edges incident on it without decreasing the average degree. It is easy to see that if the desired upper bound on the average degree holds for the new graph, then it holds for the original graph as well. Therefore, we may assume that G has minimum degree at least $d/2$.

Suppose there is a vertex u in G with degree more than $4d$. That is, $|N_1(u)| > 4d$. Since the minimum degree in G is at least $d/2$, there are at least $(d/2-1)|N_1(u)|$ edges between $N_1(u)$ and $N_2(u)$ as there are no edges contained in $N_1(u)$ (because G is triangle-free). On the other hand, if a vertex $w \in N_2(u)$ is adjacent to t vertices in $N_1(u)$, then u, w and these t vertices form a $K_{2,t}$, a contradiction. So there are at most $|N_2(u)|(t-1)$ edges between $N_1(u)$ and $N_2(u)$. Combining, we get

$$|N_2(u)|(t-1) \geqslant \left(\frac{d}{2}-1\right)|N_1(u)| > \left(\frac{d}{2}-1\right)4d = 2d^2 - 4d.$$

Now, since $n \geqslant |N_1(u)| + |N_2(u)|$, we get

$$n > 4d + (2d^2 - 4d)/(t-1) \geqslant 2d^2/(t-1).$$

Therefore, $d < \sqrt{(t-1)}\sqrt{n/2}$ and the bound in Proposition 3.1 holds because $|E(G)| = dn/2$.

So from now on we can assume that the maximum degree $d_{\max}$ in G is at most $4d$. First let us show that $N_2(v)$ does not induce many edges.

Claim 2. *For any vertex v, the number of edges induced by $N_2(v)$ is at most $(2k-4)16d^2$.*

Proof. For each $q \in N_1(v)$, let S_q be the set of neighbours of q in $N_2(v)$. Of course the sets $\{S_q \mid q \in N_1(v)\}$ cover all the vertices of $N_2(v)$. So we can choose sets $S'_q \subset S_q$ such that $\{S'_q \mid q \in N_1(v)\}$ partition $N_2(v)$. Note that the sets S'_q do not induce any edges as G is triangle-free, so every edge induced by $N_2(v)$ is between some two sets S'_{q_1} and S'_{q_2}, where $q_1, q_2 \in N_1(v)$ with $q_1 \neq q_2$. Colour each set S'_q red or blue with probability 1/2. Each vertex in $N_2(v)$ is coloured by the colour of the set it is contained in. It is easy to see that there exists a colouring of the sets S'_q such that at least half of all the edges induced by $N_2(v)$ are not monochromatic. (Indeed, the probability that an edge of $N_2(v)$ is not monochromatic is 1/2.) If there is a path of length $2k-3$ in the graph B consisting of these non-monochromatic edges, then since $2k-3$ is odd, the end vertices y_1, y_2 of the path are of different colours, so there exist distinct vertices $x_1, x_2 \in N_1(v)$ such that $x_1y_1, x_2y_2 \in E(G)$. However, then $vx_1, vx_2, x_1y_1, x_2y_2$ and this path of length $2k-3$ forms a C_{2k+1}-cycle in G, a contradiction. Therefore, using the Erdős–Gallai theorem [6] and the fact that B contains at least half of the edges induced by $N_2(v)$, we get

$$|G[N_2(v)]| \leqslant \frac{2k-3-1}{2}|N_2(v)| \cdot 2 = (2k-4)|N_2(v)| \leqslant (2k-4)d_{\max}^2 \leqslant (2k-4)16d^2,$$

proving the claim. □

By the Blakley–Roy inequality, there are at least nd^3 3-walks in G, so there exists a vertex v which is the first vertex of at least d^3 3-walks. A 3-walk of the form $vv_1v_2v_3$ where $v_i \in N_i(v)$ for $1 \leqslant i \leqslant 3$ is called *good*. If a 3-walk is not good but has v as its first vertex, then either $v_2 = v$ or $v_2 \in N_2(v), v_3 \in N_1(v)$ or $v_2 \in N_2(v), v_3 \in N_2(v)$. (Note that here we used that $N_1(v)$ does not contain any edges, as G is triangle-free.) Below we show that the number of 3-walks starting from v that are not good is very small.

The number of 3-walks starting from v where $v_2 = v$ is at most $d_{\max}^2 \leqslant 16d^2$. Indeed, there are at most $d_{\max}$ choices for v_1 and v_3 since they are both adjacent to v. Now we estimate the number of 3-walks starting from v where $v_2 \in N_2(v)$, $v_3 \in N_1(v)$. Any given $v_2 \in N_2(v)$ has at most $t-1$ neighbours in $N_1(v)$ for otherwise we can find a copy of $K_{2,t}$ in G, a contradiction. So the number of such 3-walks is at most $d_{\max}^2(t-1) \leqslant 16(t-1)d^2$ because there are at most $d_{\max}, d_{\max}$ and $t-1$ choices for v_1, v_2 and v_3 respectively. Finally, we estimate the number of 3-walks where $v_2 \in N_2(v)$ and $v_3 \in N_2(v)$. So the edge $v_2v_3 \in G[N_2(v)]$. For a given edge $xy \in G[N_2(v)]$, either $v_2 = x, v_3 = y$ or $v_2 = y, v_3 = x$ and for fixed v_2, v_3, there are at most $t-1$ choices for $v_1 \in N_1(v)$. Therefore, the number of such 3-walks is at most $|G[N_2(v)]|2(t-1)$. Now by Claim 2, this is at most $(2k-4)16d^2 \cdot 2(t-1) = 64(k-2)(t-1)d^2$. Therefore, by summing these estimates, we get that the number of 3-walks starting from v that are not good is at most

$$16d^2 + 16(t-1)d^2 + 64(k-2)(t-1)d^2 = 16d^2 + 16(4k-7)(t-1)d^2 \leqslant 32(4k-7)(t-1)d^2.$$

Thus the number of good 3-walks starting at v is at least $d^3 - 32(4k-7)(t-1)d^2$. Let us consider the graph H consisting of the last edges of these good 3-walks. Observe that H is a $K_{2,t}$-free bipartite graph with colour classes $N_2(v)$ and $N_3(v)$. It is easy to see that an edge of H belongs to at most $t-1$ good 3-walks, otherwise we can find a $K_{2,t}$. Therefore there are at least $(d^3 - 32(4k-7)(t-1)d^2)/(t-1)$ edges in H. It follows from (1.1) (or by a simple double counting of number of paths of length two) that a bipartite $K_{2,t}$-free graph on n vertices (or less) contains at most $(1+o(1))\sqrt{t-1}(n/2)^{3/2}$ edges, so H contains at most this many edges. Combining the two estimates, we get

$$(d^3 - 32(4k-7)(t-1)d^2)/(t-1) \leqslant (1+o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2}.$$

Rearranging, we get

$$(d - c_{k,t})^3 \leqslant (1+o(1))(t-1)^{3/2}(n/2)^{3/2}$$

where

$$c_{k,t} = \frac{32}{3}(4k-7)(t-1).$$

Therefore

$$d \leqslant (1+o(1))\sqrt{t-1}(n/2)^{1/2},$$

which implies that

$$|E(G)| = \frac{nd}{2} \leqslant (1+o(1))\sqrt{t-1}\left(\frac{n}{2}\right)^{3/2},$$

completing the proof of Proposition 3.1, and the proof of Theorem 1.3 in the case $k \geqslant 3$. □

Remark 2. Note that a very useful idea in the proof of Theorem 1.3 was to remove all the edges contained in triangles first (see Proposition 1.5). This helped us avoid technicalities that would otherwise arise in the proof. For example, after destroying all of the triangles, it is straightforward that for any vertex v in the resulting graph, $N_1(v)$ does not induce any edges; moreover, it becomes very easy to argue that $N_2(v)$ does not induce many edges, which is an important step of the proof.

4. Proof of Theorem 1.3 in the case $k = 2$ and Proof of Theorem 1.4

Before we proceed with the proof, let us quickly explain the differences between the proof in this section (for the case $k = 2$) and in the previous section (for the case $k \geqslant 3$). Firstly, as explained in Remark 1, we cannot apply Proposition 1.5 in this case, so we cannot assume our C_5-free, induced-$K_{2,t}$-free graph G is triangle-free as in the previous section. Moreover, if we pick a vertex v from which at least d^3 3-walks start (where d is the average degree of G), as in the previous section, then we cannot claim that the bipartite graph H with parts $N_2(v)$ and $N_3(v)$ is $K_{2,t}$-free. Therefore, our idea is to use the following approach instead.

Sketch of the proof. As usual, we will show that we can assume the minimum degree in G is $d/2$ and the maximum degree is at most $O(d)$.

We pick an edge xy in G which is the first edge of at least $2d^2 - O(d)$ 3-paths (applying the Blakley–Roy inequality). Then we show that most of these 3-paths (*i.e.* at least $2d^2 - O(d)$ of them) are such that the second and fourth vertex in them are not adjacent in G. On the other hand, it will turn out that for any vertex w different from x and y, at most $\max(3,t) - 1$ such 3-paths start with the edge xy, and have w as their last vertex. This means that there are at least $(2d^2 - O(d))/(\max(3,t) - 1)$ vertices in G but the number of vertices in G is at most n. This gives the desired upper bound on d.

Now we continue with the detailed proof.

Proof. Let G be a C_{2k+1}-free graph containing no induced copy of $K_{2,t}$. Let d be the average degree of G. It suffices to show that

$$d \leqslant (1+o(1))\sqrt{(\max(3,t)-1)}\sqrt{n/2}.$$

As usual we can assume that each vertex of G has degree at least $d/2$, for otherwise we can delete it and the edges incident on it to obtain a new graph with average degree at least d.

Claim 3. *Any two non-adjacent vertices u, v in G have at most* $\max(3,t)-1$ *common neighbours.*

Proof. Suppose for a contradiction that u and v have $\max(3,t)$ common neighbours and let S denote the set of these common neighbours. Then since u, v and vertices in S cannot form an induced copy of $K_{2,t}$, there must be an edge xy among the vertices in S. However, then $uxyvw$ is a 5-cycle in G for some $w \in S \setminus \{x,y\}$, a contradiction. □

Now we will show that we can assume the maximum degree $d_{\max}$ of G is at most $6d$. Suppose that there is a vertex v in G with degree more than $6d$. Of course $|N_1(v)| > 6d$. Since G is C_5-free, there is no path on four vertices in the subgraph of G induced by $N_1(v)$. Therefore, the number of edges induced by $N_1(v)$ is at most $(4-2)/2 \cdot |N_1(v)| = |N_1(v)|$ by the Erdős–Gallai theorem [6]. Since the minimum degree is at least $d/2$, the sum of degrees of the vertices in $N_1(v)$ is at least $\frac{d}{2}|N_1(v)|$. In this sum, the edges between v and $N_1(v)$ are counted once, the edges induced by $N_1(v)$ are counted twice, so the number of edges between $N_1(v)$ and $N_2(v)$ is at least

$$\frac{d}{2}|N_1(v)| - |N_1(v)| - 2|N_1(v)| = \left(\frac{d}{2}-3\right)|N_1(v)|.$$

On the other hand, a vertex u in $N_2(v)$ is adjacent to at most $\max(3,t)-1$ vertices in $N_1(v)$ by applying Claim 3 to u and v. So there are at most $|N_2(v)|(\max(3,t)-1)$ edges between $N_1(v)$ and $N_2(v)$. So combining, we get

$$|N_2(v)|(\max(3,t)-1) \geqslant \left(\frac{d}{2}-3\right)|N_1(v)| > \left(\frac{d}{2}-3\right)6d = 3d^2 - 18d.$$

Since $n \geqslant |N_1(v)| + |N_2(v)|$, we have that

$$n > 6d + (3d^2-18d)/(\max(3,t)-1) \geqslant (3d^2-6d)/(\max(3,t)-1) \geqslant 2d^2/(\max(3,t)-1)$$

whenever $d \geqslant 6$. Therefore

$$d < (1+o(1))\sqrt{(\max(3,t)-1)}\sqrt{n/2},$$

so

$$|E(G)| = \frac{nd}{2} < (1+o(1))\sqrt{(\max(3,t)-1)}\left(\frac{n}{2}\right)^{3/2},$$

proving Theorem 1.3 in the case $k=2$ and $t \geqslant 3$ and Theorem 1.4.

So from now on, we may assume $d_{\max} \leqslant 6d$. A 3-path is called *good* if the second and fourth vertex in it are not adjacent.

Claim 4. *There is an edge xy in G such that the number of good 3-paths starting with xy is at least $2d^2 - 84d$.*

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Proof. By the Blakley–Roy inequality, there are at least nd^3 3-walks in G, so there is an edge xy that is the first edge of at least $nd^3/|E(G)| = nd^3/(nd/2) = 2d^2$ 3-walks. First let us show that most of these $2d^2$ 3-walks are 3-paths. Suppose $xyzw$ is a 3-walk that is not a 3-path. Then either $z = x$, or $w = y$, or $w = x$. In each of these cases, there are at most $d_{\max}$ 3-walks, so in total there are at most $3d_{\max} \leqslant 18d$ such 3-walks. Similarly there are at most $18d$ 3-walks of the form $yxzw$ that are not 3-paths. Therefore, there are at least $2d^2 - 36d$ 3-paths starting with the edge xy.

If a 3-path $xyzw$ or $yxzw$ is not good, then the edge $zw \in G[N_1(y)]$ or $zw \in G[N_1(x)]$ respectively. Moreover, given an edge $zw \in G[N_1(y)]$ or $zw \in G[N_1(x)]$ there are at most four paths starting with the edge xy and containing zw as its last edge, so the total number of 3-paths starting with the edge xy that are not good is at most $4(|G[N_1(x)]| + |G[N_1(y)]|)$. Since the neighbourhood of a vertex does not contain a path on four vertices,

$$|G[N_1(x)]| + |G[N_1(y)]| \leqslant \frac{(4-2)}{2} \cdot |N_1(x)| + \frac{(4-2)}{2} \cdot |N_1(y)| \leqslant 2d_{\max} \leqslant 12d$$

by the Erdős–Gallai theorem [6]. So the number of good 3-paths starting with the edge xy is at least $2d^2 - 36d - 48d = 2d^2 - 84d$. □

Claim 5. *For any vertex w different from x and y, at most* $\max(3,t) - 1$ *good* 3-*paths start with the edge xy and have w as their last vertex.*

Proof. Suppose for a contradiction that $\max(3,t)$ good 3-paths start with the edge xy and have w as their last vertex.

Let us suppose that among these good 3-paths, there is one of the form xyz_1w and another of the form yxz_2w. Now if $z_1 \neq z_2$, then xyz_1wz_2 forms a C_5 in G, a contradiction. Therefore $z_1 = z_2$. Now since $\max(3,t) \geqslant 3$, there must be another good 3-path starting with the edge xy and having w as its last vertex: it is either of the form $xyz'w$ or of the form $yxz'w$ for some $z' \neq z_1$. In the first case, $xyz'wz_1$ is a C_5 and in the second case $yxz'wz_1$ is a C_5 in G, a contradiction.

Therefore, all the $\max(3,t)$ good 3-paths starting with the edge xy and having w as their last vertex are of the form $xyv_1w, xyv_2w, \ldots, xyv_{\max(3,t)}w$ or $yxv_1w, yxv_2w, \ldots, yxv_{\max(3,t)}w$, where $v_1, v_2, \ldots, v_{\max(3,t)}$ are distinct vertices. However, in the first case, y and w are non-adjacent (as these are good 3-paths) and in the second case x and w are non-adjacent. Moreover, in both of these cases they have $v_1, v_2, \ldots, v_{\max(3,t)}$ as their common neighbours, contradicting Claim 3. □

It follows from Claim 4 and Claim 5 that there are at least $(2d^2 - 84d)/(\max(3,t) - 1)$ vertices in G. Therefore $(2d^2 - 84d)/(\max(3,t) - 1) \leqslant n$. Simplifying, we get

$$(d-21)^2 \leqslant \frac{n}{2}(\max(3,t)-1) + 441.$$

So,

$$d \leqslant (1+o(1))\sqrt{(\max(3,t)-1)}\sqrt{n/2},$$

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implying that

$$|E(G)| = \frac{nd}{2} \leqslant (1+o(1))\sqrt{(\max(3,t)-1)}(n/2)^{3/2},$$

completing the proof of Theorem 1.3 in the cases $k = 2$ and $t \geqslant 3$ and Theorem 1.4 (after substituting $k = t = 2$). □

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