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ARTICLE

An Improved Upper Bound for Bootstrap Percolation in All Dimensions

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Abstract

In r-neighbour bootstrap percolation on the vertex set of a graph G, a set A of initially infected vertices spreads by infecting, at each time step, all uninfected vertices with at least r previously infected neighbours. When the elements of A are chosen independently with some probability p, it is natural to study the critical probability $p_c(G, r)$ at which it becomes likely that all of $V(G)$ will eventually become infected. Improving a result of Balogh, Bollobás and Morris, we give a bound on the second term in the expansion of the critical probability when $G = [n]^d$ and $d \geqslant r \geqslant 2$. We show that for all $d \geqslant r \geqslant 2$ there exists a constant $c_{d,r} > 0$ such that if n is sufficiently large, then

$$p_c([n]^d, r) \leqslant \left(\frac{\lambda(d,r)}{\log_{(r-1)}(n)} - \frac{c_{d,r}}{(\log_{(r-1)}(n))^{3/2}} \right)^{d-r+1},$$

where $\lambda(d, r)$ is an exact constant and $\log_{(k)}(n)$ denotes the k-times iterated natural logarithm of n.

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1. Introduction

Bootstrap percolation on the vertex set of a graph is a cellular automaton in which vertices have two possible states, ‘infected’ and ‘uninfected’. Let $r \in \mathbb{N}$ and let G be a graph. In *r-neighbour bootstrap percolation*, a set $A \subseteq V(G)$ is infected at time 0. At each subsequent time step, all infected vertices remain infected and all uninfected vertices that have at least r infected neighbours become infected. In symbols, letting $A_0 = A$, we have

$$A_{t+1} = A_t \cup \{v : |N(v) \cap A_t| \geqslant r\}$$

for all $t \geqslant 0$. We define the *closure* of A to be $[A] := \bigcup_{t=0}^{\infty} A_t$, the set of vertices that eventually become infected. If $[A] = V(G)$, we say that A *percolates* G, or simply that G *percolates*.

Bootstrap percolation was introduced by Chalupa, Leath and Reich [12] in connection with the Blume–Capel model of ferromagnetism.

†This work was done while the author was at the University of Memphis.

In our context, elements of A are chosen independently with some probability p. Given $p \in [0,1]$, we define $P(G, r, p)$ to be the probability that A percolates G under the r-neighbour rule if the elements of A are chosen in this way. We define $p_\alpha(G, r) = \inf\{p : P(G, r, p) \geqslant \alpha\}$ and let

$$p_c := p_c(G, r) := p_{1/2}(G, r)$$

denote the *critical probability*.

Van Enter [25] and Schonmann [24] showed that for all $d \geqslant 2$, $p_c(\mathbb{Z}^d, r) = 0$ when $r \leqslant d$ and $p_c(\mathbb{Z}^d, r) = 1$ when $r \geqslant d+1$. Since then, much work has focused on $p_c([n]^d, r)$ (where $[n] = \{1, \ldots, n\}$) for $2 \leqslant r \leqslant d$. Aizenman and Lebowitz [2] determined that for all $d \geqslant 2$, $p_c([n]^d, 2) = \Theta((1/\log n)^{d-1})$. (Later, Balogh and Pete [6] independently proved this result for $d = 2$.) Cerf and Cirillo [10] and Cerf and Manzo [11] showed that for all $d \geqslant r \geqslant 2$, $p_c([n]^d, r) = \Theta((1/\log_{(r-1)}(n))^{d-r+1})$, where $\log_{(k)}(n)$ denotes the k-times iterated natural logarithm of n, so that $\log_{(k)}(n) = \log(\log_{(k-1)}(n))$ and $\log_{(1)}(n) = \log n$.

The next breakthrough in the field was due to Holroyd [21], who proved the following sharp threshold result for bootstrap percolation on the two-dimensional grid.

Theorem 1.1. *As $n \to \infty$,*

$$p_c([n]^2, 2) = \frac{\pi^2}{18 \log n} + o\left(\frac{1}{\log n}\right).$$

Later, Gravner and Holroyd [17], Gravner, Holroyd and Morris [18] and Hartarsky and Morris [20] sharpened Holroyd's result. Collectively, they proved the following.

Theorem 1.2. *There exist constants $C \geqslant c > 0$ such that*

$$\frac{\pi^2}{18 \log n} - \frac{C}{(\log n)^{3/2}} \leqslant p_c([n]^2, 2) \leqslant \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}}$$

for all n sufficiently large.

Turning to higher dimensions, Balogh, Bollobás and Morris [5] proved a sharp threshold result for $p_c([n]^3, 3)$ and proved an upper bound on $p_c([n]^d, r)$ for all constant $d \geqslant r \geqslant 2$. Later, Balogh, Bollobás, Duminil-Copin and Morris [4] proved the corresponding lower bound and so established a sharp threshold result for $p_c([n]^d, r)$ for all constant $d \geqslant r \geqslant 2$. The proofs of these results are substantially more difficult than the proof of Theorem 1.1.

Before we state the results of [4, 5], we need to introduce more notation. Given $k \geqslant 1$, define the function $\beta_k : [0,1] \to [0,1]$ by

$$\beta_k(u) = \frac{1}{2}\left(1 - (1-u)^k + \sqrt{1 + (4u-2)(1-u)^k + (1-u)^{2k}}\right) \tag{1.1}$$

and let

$$g_k(z) = -\log(\beta_k(1 - e^{-z})). \tag{1.2}$$

For $d \geqslant r \geqslant 2$, define

$$\lambda(d, r) = \int_0^\infty g_{r-1}(z^{d-r+1})\, dz. \tag{1.3}$$

Holroyd [21] showed that $\lambda(2, 2) = \pi^2/18$. At present, $(2, 2)$ is the only ordered pair (d, r) for which an exact expression for $\lambda(d, r)$ is known. However, it is shown in [5] that $\lambda(d, r) < \infty$ for all $d \geqslant r \geqslant 2$.

Here is the sharp threshold result of Balogh, Bollobás, Duminil-Copin and Morris [4, 5].

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Theorem 1.3. *Let $d \geqslant r \geqslant 2$. With $\lambda(d,r)$ as defined in (1.3),*

$$p_c([n]^d, r) = \left(\frac{\lambda(d,r)+o(1)}{\log_{(r-1)}(n)}\right)^{d-r+1}.$$

A number of variations of the bootstrap process described above have been considered. Holroyd [21, 22] proved, for all $d \geqslant 2$, a sharp threshold result for a modified d-neighbour bootstrap rule on $[n]^d$: in order to become infected, a vertex must have at least one infected neighbour in each dimension. Sharp threshold results have also been proved for other update rules on $\mathbb{Z}^d$ and $[n]^d$ [8, 13, 14, 15, 26]. Similar but weaker results about the threshold behaviour of a very general class of update rules on $\mathbb{Z}^2$ were proved in [3, 7, 9].

Bootstrap percolation has been applied to other fields, especially physics. In particular, there is a strong connection between bootstrap percolation and the Glauber dynamics of the Ising model of ferromagnetism at zero temperature [2, 16, 23]. For other applications in physics, see [1] and the references therein.

In [4], Balogh, Bollobás, Duminil-Copin and Morris suggested that the techniques of [18] could be used to prove an analogue of Theorem 1.2 for $p_c([n]^d, 2)$. We carry out this programme in part. We combine the techniques of [17] and [5] to improve the upper bound on $p_c([n]^d, r)$ given in Theorem 1.3 for all $d \geqslant r \geqslant 2$.

Theorem 1.4. *For all $d \geqslant r \geqslant 2$, there exists a constant $c_{d,r} > 0$ such that*

$$p_c([n]^d, r) \leqslant \left(\frac{\lambda(d,r)}{\log_{(r-1)}(n)} - \frac{c_{d,r}}{(\log_{(r-1)}(n))^{3/2}}\right)^{d-r+1} \tag{1.4}$$

for all n sufficiently large.

We note that when $d = r = 2$, (1.4) reduces to the upper bound in Theorem 1.2, which was proved by Gravner and Holroyd [17].

The rest of the paper is organized as follows. In Section 2, we give an outline of the proof of Theorem 1.4. In Section 3, we introduce additional notation and some preliminary results. In Section 4, we state an important auxiliary result, Theorem 4.1, and also state and prove other auxiliary results. In Section 5, we prove Theorem 4.1 in the case $r = 2$. In Section 6, we complete the proof of Theorem 4.1 and use it to deduce Theorem 1.4. Finally, in Section 7, we conjecture an improved lower bound on $p_c([n]^d, r)$.

2. Outline of the proof of Theorem 1.4

Here we will sketch the proof of Theorem 1.4. Our argument builds on a large body of previous work (in particular, [2], [21], [17], [11] and [5]). We hope that discussing the relevant ideas from these papers at some length will serve to make our proof clearer to the reader.

We begin with a few definitions. In the literature of percolation theory, vertices of a graph are often called *sites*, and we will almost always use this term hereafter. We say that a set $S \subseteq [n]^d$ is *internally spanned* if $[A \cap S] = S$. We say that a set of vertices is *empty* or *unoccupied* if it contains no infected sites and *occupied* otherwise. We say that a sequence of events $E_1, \ldots, E_n$ has a *double gap* if some pair of consecutive events (E_i, E_{i+1}) does not occur. Finally, given $r \geqslant 3$, let $\mathbf{1}^{r-2}$ denote the vector $(1, \ldots, 1) \in \mathbb{R}^{r-2}$ and, for each $i \in [r-2]$, let e_i denote the ith standard basis vector for $\mathbb{R}^{r-2}$.

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2.1 Two dimensions

One might suppose that if $[n]^d$ percolates, then the infected set spreads to all parts of the grid in a fairly uniform manner. In [2], Aizenman and Lebowitz showed that in fact, when the infection probability p is on the order of $(1/\log n)^{d-1}$, whether percolation occurs under the 2-neighbour rule is governed by a more local phenomenon: the existence of a fairly small internally spanned set, called a *critical droplet*. For example, it turns out that for 2-neighbour percolation in $[n]^2$, a good candidate for a critical droplet is a rectangle whose diameter (in the L^∞-norm) is on the order of $\log n$. (A heuristic explanation for this is given in Section 2.2.)

So, in [21], Holroyd proved the upper bound on $p_c([n]^2, 2)$ in Theorem 1.1 by estimating the probability that a square R of side length $B :\approx \log n$ is internally spanned in a certain way.

Let $a \ll B$ and let S denote the copy of $[a]^2$ in the lower left corner of R. If S is fully infected, what conditions imply that the infected set will grow from S to fill R? If the rows $[a] \times \{a+1\}$, $[a] \times \{a+2\}, \ldots, [a] \times \{B\}$ are all occupied, then these rows will iteratively become infected. If the same holds for the columns $\{a+1\} \times [a]$, $\{a+2\} \times [a], \ldots, \{B\} \times [a]$, then all of $[B]^2$ will become infected.

Holroyd observed that we can get away with asking for a bit less. Note that if either of the rows $[a] \times \{a+1\}$ and $[a] \times \{a+2\}$ contains an infected site, then all sites in the row $[a] \times \{a+1\}$ will become fully infected. (Much the same is true for the columns $\{a+1\} \times [a]$ and $\{a+2\} \times [a]$.) Motivated by this observation, we let R_i denote the event that $[i-1] \times \{i\}$ is occupied, let C_i denote the event that $\{i\} \times [i-1]$ is occupied and let D denote the event that the sequences $(R_i)_{i=a+1}^{B+1}$ and $(C_i)_{i=a+1}^{B+1}$ each contain no double gaps. Observe that if D occurs, then the infected set will grow from S to fill R. We think of D as 'diagonal growth' of the infected set, because the infected set iteratively fills the sets $[t]^2$ for $t = a+1, \ldots, B$ (see Figure 1).

As the reader might guess, the probability that D occurs is fairly small. However, it is large enough that if $[n]^2$ is partitioned into squares of side length B, then with high probability D occurs in *some* such square. Furthermore, if such a square is fully infected, then with high probability the infected set will fill the entire grid.

How might one prove a stronger upper bound on $p_c([n]^2, 2)$? Instead of considering a single event that implies that the square R is internally spanned, one might try to find a set of pairwise disjoint events $E_1, \ldots, E_N$, for some $N = N(p)$, each of which implies that R is internally spanned. If, for each i, we had $\mathbb{P}(E_i) \geqslant (c_1 p)^{p^{-1/2}} \mathbb{P}(D)$, and if $N = (c_2/p)^{p^{-1/2}}$ (where c_1 and c_2 are constants such that $c_1 c_2 > 1$), then a union bound would give

$$\mathbb{P}\left(\bigvee_{i=1}^{N} E_i \right) \geqslant (c_2/p)^{p^{-1/2}} (c_1 p)^{p^{-1/2}} \mathbb{P}(D) = e^{cp^{-1/2}} \mathbb{P}(D), \tag{2.1}$$

where $c := \log(c_1 c_2) > 0$. It turns out that the factor $e^{cp^{-1/2}}$ on the right-hand side of (2.1) is enough to make a difference in the value of $p_c([n]^2, 2)$ and to prove the upper bound in Theorem 1.2.

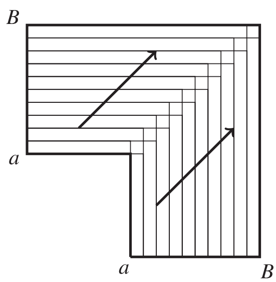

Figure 1. If no two consecutive rows and no two consecutive columns are unoccupied, then the infected set will grow diagonally.

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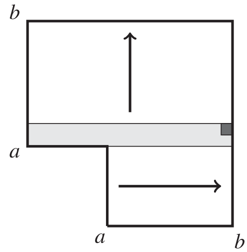

Figure 2. An alternative way of filling a rectangle. The light grey region is unoccupied and the dark grey square represents a single infected site. The arrows depict the growth of the infected set across regions with no double gaps.

Gravner and Holroyd [17] did precisely this. They considered the event that R is internally spanned, but that at some point, a double gap in either (R_i) or (C_i) creates a small 'detour' in the diagonal growth of the infected set. Once again, consider the fully infected square $S=[a]^2$, and suppose that for some $b>a$, the rows $[b-1]\times\{a+1\}$ and $[b-1]\times\{a+2\}$ are both empty. Clearly, this double gap blocks the infected set from growing vertically. However, if the columns to the right of S contain no double gaps until at least column $b+1$, then the infected set can grow horizontally until it fills the rectangle $[b]\times[a]$. If the infected set eventually encounters an infected site above $[b]\times[a]$ (for example $(b,a+2)$), then it can overcome the double gap and fill the rows $[b]\times\{a+1\}$ and $[b]\times\{a+2\}$. Finally, if there are no further double gaps in the rows above $[b]\times[a+2]$, then the infected set can grow vertically until it fills $[b]^2$ (see Figure 2).

It is not hard to show that such a 'detour' is less probable than the event D defined above. However, Gravner and Holroyd showed that if a and b are both on the order of $1/p$ and $b-a=O(p^{-1/2})$, then these detours (or, more precisely, sequences of such detours) are both probable and numerous enough that (2.1) holds.

2.2 Higher dimensions

Now we will describe the proof of the upper bound on $p_c([n]^d,r)$ given in [5] and discuss how we will adapt it to prove Theorem 1.4.

In order to prove the upper bound in Theorem 1.3, Balogh, Bollobás and Morris [5] also used the notion of a critical droplet. They observed that it follows from the results in [11] that a critical droplet for r-neighbour percolation in $[n]^d$ is a d-dimensional box whose diameter is about $\log n$. As a heuristic justification for this, let X_L denote the number of internally spanned cubes of diameter L in $[n]^d$. It is shown in [11] that if L is in a certain range, then the probability that a cube of diameter L is internally spanned is (*very* roughly) e^{-L}. Thus, if $L\approx d\log n$, then $\mathbb{E}X_L\approx n^{d+o(1)}e^{-L}=\Theta(1)$, which suggests that the 'critical diameter' is indeed on the order of $\log n$.

Suppose, then, that a cube $T_0\cong[\log n]^d$ is internally spanned. Under what circumstances is it likely that the infected set can grow from T_0? In particular, when will the infected set grow to fill the $(d-1)$-dimensional 'layer' that is adjacent to T_0 in a given direction? Choose a direction and let S_1 denote the two layers adjacent to T_0 in this direction. Observe that $S_1\cong[\log n]^{d-1}\times[2]$. Crucially, because each site in the layer of S_1 adjacent to T_0, $[\log n]^{d-1}\times\{1\}$, already has an infected neighbour in T_0, each such site requires only $r-1$ additional infected neighbours in S_1 in order to become infected. In contrast, sites in the other layer of S_1, $[\log n]^{d-1}\times\{2\}$, still require r infected neighbours.

Therefore, it makes sense to consider percolation inside S_1, where each site in the first layer has infection threshold $r-1$ and each site in the second layer has threshold r. When is it likely that the first layer of S_1 will percolate? By applying the same heuristic argument as above to S_1, we see that a good candidate for a critical droplet in S_1 is a set of the form $[\log\log n]^{d-1}\times[2]$. (Here, the term 'critical droplet' has a slightly different meaning: it refers to a set whose first layer, if fully infected, will with high probability infect the rest of the first layer of S_1.)

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Suppose, then, that S_1 contains a set $T_1 \cong [\log\log n]^{d-1} \times [2]$ whose first layer is fully infected. What is required for the infected sites in T_1 to fully infect the first layer of S_1? As before, percolation must occur in the first layer of each copy of $[\log\log n]^{d-2} \times [2]^2$ that is adjacent to T_1 (and contained in S_1). Let S_2 be such a copy of $[\log\log n]^{d-2} \times [2]^2$. Observe that sites in the first layer of S_2, $[\log\log n]^{d-2} \times \{(1,1)\}$, need only $r-2$ infected neighbours in S_2, because each such site has one infected neighbour in T_0 and another in T_1. In contrast, sites in the other layers of S_2 still require r infected neighbours in S_2 in order to become infected.[1]

Iterating this argument leads us to consider the probability of percolation in a set of the form $[\log_{(r-2)}(n)]^{d-r+2} \times [2]^{r-2}$, where all sites in $[\log_{(r-2)}(n)]^{d-r+2} \times \{(1,\dots,1)\}$ have threshold 2 and all other sites have threshold r. By induction, if it is likely that all sites in $[\log_{(r-2)}(n)]^{d-r+2} \times \{(1,\dots,1)\}$ become infected, then percolation is likely to occur in $[n]^d$.

Balogh, Bollobás and Morris bounded the probability of percolation in $[\log_{(r-2)}(n)]^{d-r+2} \times [2]^{r-2}$ in much the same way that Holroyd bounded the probability of 2-neighbour percolation in $[n]^2$. Let $B \gg a$ and suppose that a cube $K \cong [a]^{d-r+2} \times \mathbf{1}^{r-2}$ is fully infected. In order to estimate the probability that $[B]^{d-r+2} \times \mathbf{1}^{r-2}$ becomes infected, we would like a fairly simple sufficient condition for all of the sites in a layer adjacent to K (for example $[a]^{d-r+1} \times \{a+1\} \times \mathbf{1}^{r-2}$) to become infected. Let U_i denote the event that $[a]^{d-r+1} \times \{i\} \times \mathbf{1}^{r-2}$ is occupied and, for each $j \in [r-2]$, let $V_i^{(j)}$ denote the event that $[a]^{d-r+1} \times \{i\} \times (\mathbf{1}^{r-2} + e_j)$ is occupied. Observe that because each site in $[a]^{d-r+1} \times \{a+1\} \times \mathbf{1}^{r-2}$ has threshold 2 and already has an infected neighbour in $[a]^{d-r+2} \times \mathbf{1}^{r-2}$, all of the sites in this layer will become infected if one of the events U_{a+1}, U_{a+2}, $V_{a+1}^{(1)}, \dots, V_{a+1}^{(r-2)}$ occurs. The situation in which none of these events occur – or, more generally, in which the event

$$\neg(U_i \vee U_{i+1} \vee V_i^{(1)} \vee \cdots \vee V_i^{(r-2)})$$

occurs for any $i \in [B]$ – is called an *L-gap*.[2] (Note that when $d = r = 2$, an L-gap is simply a double gap.) If the sequence

$$\mathcal{E}_{d-r+2} := (U_i)_{a+1 \leqslant i \leqslant B+1} \cup (V_i^{(j)})_{a+1 \leqslant i \leqslant B, j \in [r-2]}$$

contains no L-gaps, then the infected set will grow in direction $d-r+2$ until it reaches one face of $[B]^{d-r+2} \times \mathbf{1}^{r-2}$. We may similarly define a sequence $\mathcal{E}_t$ for each direction $t \in [d-r+2]$. If none of the $\mathcal{E}_t$ contains an L-gap, then the infected set will fill $[B]^{d-r+2} \times \mathbf{1}^{r-2}$.

To bound the probability of percolation from below, it suffices to show that if $B \approx \log_{(r-1)}(n)$ and we partition $[\log_{(r-2)}(n)]^{d-r+2} \times \mathbf{1}^{r-2}$ into cubes of the form $[B]^{d-r+2} \times \mathbf{1}^{r-2}$, then with high probability, some cube of side length B is spanned. If so, then with high probability, the rest of $[\log_{(r-2)}(n)]^{d-r+2} \times \mathbf{1}^{r-2}$ will become infected.

This yields the upper bound in Theorem 1.3 for $r = 2$. The upper bound for larger r follows from an inductive argument that shows that the full infection of the first layer of

$$[\log_{(r-2)}(n)]^{d-r+2} \times [2]^{r-2}$$

indeed implies r-neighbour percolation in $[n]^d$.

As mentioned above, in order to prove Theorem 1.4, we unite the techniques of [5] and [17]. Again, we consider a fully infected cube $K \cong [a]^{d-r+2} \times \mathbf{1}^{r-2}$. Let D' denote the event that the infected set grows from K without encountering L-gaps until it fills $[B]^{d-r+2} \times \mathbf{1}^{r-2}$. Much as

[1] Sites in $[\log\log n]^{d-2} \times \{(2,1)\}$ actually only need $r-1$ infected neighbours in S_2, because each one has an infected neighbour in T_0, but it turns out that we lose almost nothing by assuming that these sites also have infection threshold r.

[2] When $d = r = 3$, the sets in question are $[a] \times \{(i,1)\}$, $[a] \times \{(i+1,1)\}$ and $[a] \times \{(i,2)\}$. An L-gap is so called because these sets form an L-shape when viewed from the side.

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in the two-dimensional case, we seek a large class of pairwise disjoint events $E'_1, \ldots, E'_{N'}$, each of which implies that $[B]^{d-r+2} \times \mathbf{1}^{r-2}$ becomes fully infected. Suppose that there exist positive constants c_1 and c_2 such that $\mathbb{P}(E'_i) \geqslant (c_1 p)^{p^{-1/2(d-r+1)}} \mathbb{P}(D')$ for all i and such that $N' = N'(p) = (c_2/p)^{p^{-1/2(d-r+1)}}$. Then, similarly to (2.1), we will be able to conclude that

$$\mathbb{P}\left(\bigvee_{i=1}^{N'} E'_i \right) \geqslant e^{cp^{-1/2(d-r+1)}} \mathbb{P}(D') \tag{2.2}$$

for some constant $c > 0$.

Much as in [17], we consider events E'_i that involve small detours in the growth of the infected set. Suppose that an L-gap – for example $\neg(U_{a+1} \vee U_{a+2} \vee V^{(1)}_{a+1} \vee \cdots \vee V^{(r-2)}_{a+1})$ – blocks the infected set from growing from the cube K in direction $d-r+2$. If no L-gaps occur in the other sequences $\mathcal{E}_t$, then the infected set can grow in the other $d-r+1$ directions until it fills a set of the form $[b]^{d-r+1} \times [a] \times \mathbf{1}^{r-2}$, for some $b > a$. If there is an infected site x with $x_1, \ldots, x_{d-r+1} \in [b]$ and $x_{d-r+2} \in \{a+1, a+2\}$, then the infected set can overcome the L-gap and fill $[b]^{d-r+1} \times [a+2] \times \mathbf{1}^{r-2}$. If no further L-gaps occur in direction $d-r+2$, then the infected set will grow in that direction until it fills the cube $[b]^{d-r+2} \times \mathbf{1}^{r-2}$.

We show that when a and b are both on the order of $p^{-1/(d-r+1)}$ and $b - a = O(p^{-1/2(d-r+1)})$, then the number and probability of these detours (or, rather, of sequences of such detours) are both large enough that $\mathbb{P}(\bigvee_{i=1}^{N'} E'_i)$ satisfies (2.2). This yields the claimed improvement in the upper bound on $p_c([n]^d, 2)$.

The rest of the proof of Theorem 1.4 consists of an inductive argument that is very similar to the inductive argument of [5] mentioned above, albeit with additional technical complications.

3. Notation and preliminaries

In this section we will introduce further notation and definitions, state a useful correlation inequality and make preliminary observations.

For the most part, our notation and terminology follow that of [5]. In order to reduce clutter, we will omit floor signs throughout the paper. All logarithms are taken with base e.

We say that a set S is *occupied* if it contains at least one infected site, and *empty* or *unoccupied* otherwise. If all of the sites in S are infected, we say that S is *full*.

We will denote the vector $(1, \ldots, 1) \in \mathbb{R}^\ell$ by $\mathbf{1}^\ell$. For each $j \in [\ell]$, we let e_j denote the jth standard basis vector for $\mathbb{R}^\ell$.

Given a set S, we write $A \sim \mathrm{Bin}\,(S, p)$ to denote that the elements of A are chosen from S independently with probability p.

Harris's lemma [19] will play an important role in the proof. We define a partial order $\leqslant$ on $\{0, 1\}^n$ by writing $x \leqslant y$ if, for all $i \in [n]$, $x_i \leqslant y_i$. We say that an event $E \subseteq \{0, 1\}^n$ is *increasing* if, for $x, y \in \{0, 1\}^n$, $x \in E$ and $x \leqslant y$ imply that $y \in E$. Given $p \in [0, 1]$, let $\mathbb{P}_p$ denote the product measure on $\{0, 1\}^n$ with $\mathbb{P}_p(i = 1) = p$ for all $i \in [n]$. (We will almost always suppress the dependence on p and simply write $\mathbb{P}(\,\cdot\,)$.)

Theorem 3.1. (**Harris's lemma**). *If E and F are increasing events in $\{0, 1\}^n$ and $p \in [0, 1]$, then*

$$\mathbb{P}_p(E \cap F) \geqslant \mathbb{P}_p(E)\mathbb{P}_p(F).$$

We conclude this section by discussing properties of the functions β_k and g_k defined in (1.1) and (1.2), respectively. Given $p \in (0, 1)$, we let

$$q = -\log\,(1 - p). \tag{3.1}$$

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Note that for p sufficiently small, we have

$$p \leqslant q \leqslant p + p^2 \leqslant 2p.$$

Equation (3.1) allows us to write

$$\beta_k(1-(1-p)^n) = e^{-g_k(nq)}. \tag{3.2}$$

We also observe that (1.1) implies that

$$\beta_k(u)^2 = (1-(1-u)^k)\beta_k(u) + u(1-u)^k. \tag{3.3}$$

Straightforward calculations show that for all k, β_k is positive, continuous, increasing and differentiable on $(0, 1)$, and g_k is positive, continuous, decreasing and differentiable on $(0, \infty)$.

We will need a further result about the behaviour of g_k.

Proposition 3.2. *For all $k \geqslant 1$ and all $z \geqslant 1$,*

$$|g_k'(z)| \leqslant \frac{1}{2}. \tag{3.4}$$

The proof of Proposition 3.2 is given in the Appendix.

4. Percolation in an auxiliary bootstrap structure

In this section we will state the key result, Theorem 4.1, that we will use to prove Theorem 1.4. We will also define the important notion of an L-gap and prove a lower bound on the probability that no L-gaps occur in a sequence of independent events.

In Section 2.2, we related the probability of r-neighbour percolation in $[n]^d$ to the probability of percolation in a set of the form $[\log_{(r-2)}(n)]^{d-r+2} \times [2]^{r-2}$ in which not all vertices have the same infection threshold. A *bootstrap structure* is an ordered pair $(G, (r(v))_{v\in V(G)})$, where G is a graph and $r : V(G) \to \mathbb{N}$. Given a vertex v, the value $r(v)$ is called the *threshold* of v. This means that if we consider bootstrap percolation in $(G, (r(v))_{v\in V(G)})$ and let $A_0 = A$ as before, then

$$A_{t+1} = A_t \cup \{v : |N(v) \cap A_t| \geqslant r(v)\}$$

for each $t \geqslant 0$.

Let $B([n]^d, r)$ denote the usual r-neighbour bootstrap structure on $[n]^d$. In order to prove Theorem 1.4, we will also consider an auxiliary bootstrap structure that was defined in [5]. Let $C^*([n]^d \times [2]^\ell, r)$ be the subgraph of $\mathbb{Z}^{d+\ell}$ induced by $[n]^d \times [2]^\ell$ in which all vertices of the form $(a_1, \ldots, a_d) \times \mathbf{1}^\ell$ have threshold r and all other vertices have threshold $r + \ell$. Note that if $\ell = 0$, then $C^*([n]^d \times [2]^\ell, r)$ is identical to $B([n]^d, r)$.

Recall that A denotes the set of initially infected vertices and that $[A]$ denotes the closure of A, the set of vertices that ultimately become infected. We say that A *semi-percolates* in $C^*([n]^d \times [2]^\ell, r)$ if $[A] \supseteq [n]^d \times \mathbf{1}^\ell$ and that a set S is *internally semi-spanned* if $[S \cap A] \supseteq S \cap ([n]^d \times \mathbf{1}^\ell)$. Letting $A \sim \text{Bin}\,([n]^d \times [2]^\ell, p)$, we set

$$P(n, d, \ell, r, p) := \mathbb{P}(A \text{ semi-percolates in } C^*([n]^d \times [2]^\ell, r)). \tag{4.1}$$

(The quantity $P(n, d, \ell, r, p)$ was originally defined in [5]. The definition given here is slightly simpler.)

Theorem 4.1 is a lower bound on $P(n, d, \ell, r, p)$. Before we state the theorem, we need to define several constants.

For all $d \geqslant 2$ and $\ell \geqslant 0$, let

$$\zeta(d, \ell) = e^{-(\ell+2)2^{2d-1}}(1-e^{-1})^{2d} \tag{4.2}$$

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and let

$$\gamma(d,\ell)=\zeta(d,\ell)e^{-d(d-1)2^{2d-4}}. \tag{4.3}$$

Observe that for all $d\geqslant 2$ and $\ell\geqslant 0$,

$$\gamma(d,\ell)\leqslant\gamma(2,0)=e^{-18}(1-e^{-1})^4<10^{-8}. \tag{4.4}$$

Finally, given $d\geqslant r\geqslant 2$ and $\ell\geqslant 0$, let

$$c_{d,\ell,r}=\begin{cases}\gamma(d,\ell) & r=2,\\ \gamma(d-r+2,\ell+r-2)\big(1-\sum_{s=0}^{r-3}2^{-r+s+1}\big) & r\geqslant 3.\end{cases} \tag{4.5}$$

We observe for use in Section 6 that (4.5) implies that if $r\geqslant 3$, then

$$c_{d,\ell,r}=c_{d-1,\ell+1,r-1}-2^{-r+1}\gamma(d-r+2,\ell+r-2). \tag{4.6}$$

We are at last ready to state our key result about semi-percolation in $C^*([n]^d\times[2]^\ell,r)$.

Theorem 4.1. *Let $d\geqslant r\geqslant 2$, let $\ell\geqslant 0$ and let $c_{d,\ell,r}$ be as in (4.5). If*

$$p\geqslant\left(\frac{\lambda(d+\ell,\ell+r)}{\log_{(r-1)}(n)}-\frac{c_{d,\ell,r}}{(\log_{(r-1)}(n))^{3/2}}\right)^{d-r+1} \tag{4.7}$$

for all n sufficiently large, then

$$P(n,d,\ell,r,p)\to 1$$

as $n\to\infty$.

Let us say a few words about how we will use this result to prove Theorem 1.4. In Section 6, we will show that if p is such that (4.7) holds for d, ℓ, r and n, then (4.7) also holds for $d-1$, $\ell+1$, $r-1$ and (roughly) $\log n$. It will follow by induction that for all $j\leqslant r-2$, (4.7) holds for $d-j$, $\ell+j$, $r-j$ and $\log_{(j)}(n)$. Turning this around, we see that if the bound on p in (4.7) is sufficient for semi-percolation in

$$C^*([\log_{(r-2)}(n)]^{d-r+2}\times[2]^{r-2},2),$$

then it is also sufficient for semi-percolation in

$$C^*([\log_{(r-2-i)}(n)]^{d-r+2+i}\times[2]^{r-2-i},i+2)\quad\text{for all } i\leqslant r-2.$$

In particular, when $i=r-2$, this is the same as saying that the bound on p implies that, with high probability, percolation occurs in $B([n]^d,r)$.

Observe also that in order to prove Theorem 1.4, it is enough to apply Theorem 4.1 in the case when $\ell=0$ (see (1.4)).

Now let us turn to the notion of an L-gap. For $m\geqslant -1$ and $\ell\geqslant 0$, let

$$\mathcal{E}=(U_i)_{i\in[m+1]}\cup(V_i^{(j)})_{i\in[m],j\in[\ell]}$$

be a sequence of events. (When $m=-1$, $\mathcal{E}=\emptyset$.) An *L-gap* in $\mathcal{E}$ is an event of the form

$$\neg(U_i\vee U_{i+1}\vee V_i^{(1)}\vee\cdots\vee V_i^{(\ell)})$$

for some $i\in[m]$. (As mentioned in Section 2.2, L is not a variable, but rather refers to the shape of an L-gap when $d=2$ and $\ell=1$.) In this paper, the events in the sequence $\mathcal{E}$ will all be of the form 'a certain set of sites is occupied'. Thus, an L-gap in $\mathcal{E}$ will mean that a certain collection of sets are all unoccupied. In particular, as was the case in Section 2.2, an L-gap will block the set of infected sites from growing in a specific direction.

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We will need a lower bound on the probability that no L-gaps occur in a sequence of independent events. We can express this bound in terms of the function β_k defined in (1.1). (Similar results were proved in [5, Lemma 6] and [17, Proposition 10].)

Lemma 4.2. *Let $m \geqslant -1$ and $\ell \geqslant 0$ be integers and let $u_1, \ldots, u_{m+1} \in (0, 1)$. Let*

$$\mathcal{E}_{m+1} := (U_i)_{i\in[m+1]} \cup (V_i^{(j)})_{i\in[m],j\in[\ell]}$$

be a sequence of independent events such that for each i, the events $U_i, V_i^{(1)}, \ldots, V_i^{(\ell)}$ each occur with probability u_i. Let $\mathbf{u} = (u_i)_{i=1}^{m+1}$ and let $L_\ell(m, \mathbf{u})$ denote the probability that no L-gap occurs in $\mathcal{E}_{m+1}$. If the sequence $(u_i)_{i=1}^{m+1}$ is increasing in i, then

$$L_\ell(m, \mathbf{u}) \geqslant \prod_{i=1}^{m+1} \beta_{\ell+1}(u_i).$$

In order to prove Lemma 4.2, we need another result about β_k.

Lemma 4.3. *If $0 < u \leqslant v \leqslant 1$, then*

$$(1-(1-u)^k)\beta_k(v) + (1-u)^k v \geqslant \beta_k(u)\beta_k(v).$$

Proof. For $0 < u \leqslant v \leqslant 1$, define

$$h(u, v) = (1-(1-u)^k)\beta_k(v) + (1-u)^k v - \beta_k(u)\beta_k(v).$$

and observe that we are done if we can show that $h(u, v) \geqslant 0$ for $0 < u \leqslant v \leqslant 1$.

Observe that by (3.3),

$$\beta_k(u)h(u, v) = (1-u)^k(v\beta_k(u) - u\beta_k(v)).$$

Equivalently,

$$\frac{\beta_k(u)}{uv}h(u, v) = (1-u)^k\left(\frac{\beta_k(u)}{u} - \frac{\beta_k(v)}{v}\right),$$

so it is enough to show that $\beta_k(u)/u$ is decreasing on $(0, 1)$. Let $B = (1-(1-u)^k)/u$ and let $C = (1-u)^k/u$. By (3.3), $\beta_k(u)/u$ is the positive root of $X^2 - BX - C = 0$. Observe that both B and C are decreasing in u. It follows that

$$\frac{\beta_k(u)}{u} = \frac{B + \sqrt{B^2 + 4C}}{2}$$

is also decreasing, as claimed. □

Proof of Lemma 4.2. We prove the result by induction on m. Observe that $\mathcal{E}_0 = \emptyset$ and that an L-gap is undefined for $\mathcal{E}_1$. So, for all $u \in (0, 1)$, we may take $L_\ell(-1, \mathbf{u}) = L_\ell(0, \mathbf{u}) = 1$. Then, because $\beta_{\ell+1}$:$[0, 1] \to [0, 1]$, the result holds for $m \in \{-1, 0\}$.

Let $m \geqslant 0$ and suppose that the result holds for values smaller than $m+1$. Observe that $\mathcal{E}_{m+1}$ has no L-gaps if (i) at least one of the events $U_1, V_1^{(1)}, \ldots, V_1^{(\ell)}$ occurs and $(U_i)_{2\leqslant i\leqslant m+1} \cup (V_i^{(j)})_{2\leqslant i\leqslant m, j\in[\ell]}$ has no L-gaps, or (ii) none of the events $U_1, V_1^{(1)}, \ldots, V_1^{(\ell)}$ occur, but the event U_2 occurs and $(U_i)_{3\leqslant i\leqslant m+1} \cup (V_i^{(j)})_{3\leqslant i\leqslant m, j\in[\ell]}$ has no L-gaps. Hence, by induction,

$$L_\ell(m, \mathbf{u}) \geqslant (1-(1-u_1)^{\ell+1}) \prod_{i=2}^{m+1} \beta_{\ell+1}(u_i) + (1-u_1)^{\ell+1} u_2 \prod_{i=3}^{m+1} \beta_{\ell+1}(u_i). \tag{4.8}$$

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Because $u_1 \leqslant u_2$, Lemma 4.3 implies that

$$(1-(1-u_1)^{\ell+1})\beta_{\ell+1}(u_2)+(1-u_1)^{\ell+1}u_2 \geqslant \beta_{\ell+1}(u_1)\beta_{\ell+1}(u_2).$$

Combining this with the right-hand side of (4.8) yields the claimed inequality. □

5. Proof of Theorem 4.1 for $r=2$

Our aim in this section is to prove a result that implies Theorem 4.1 in the case $r=2$.

Lemma 5.1. *Let $d \geqslant 2$, let $\ell \geqslant 0$ and let $\gamma(d,\ell)$ be as in (4.3). If, for some constant c with*

$$0 < c < \frac{3}{2}\gamma(d,\ell), \tag{5.1}$$

we have

$$p \geqslant \left(\frac{\lambda(d+\ell,\ell+2)}{\log n} - \frac{c}{(\log n)^{3/2}}\right)^{d-1} \tag{5.2}$$

for all n sufficiently large, then

$$P(n,d,\ell,2,p) \to 1$$

as $n \to \infty$.

Remark 5.2. By (4.5), $c_{d,\ell,2}$ certainly satisfies (5.1) for all $d \geqslant 2$ and $\ell \geqslant 0$.

Here is a sketch of the proof of Lemma 5.1. Below, we will define an event $\mathcal{D}_a^b$ that implies that if $[a]^d \times \mathbf{1}^\ell$ is internally spanned, then the infected set grows to fill $[b-1]^d \times \mathbf{1}^\ell$ 'diagonally', *i.e.* by iteratively filling sets of the form $[i]^d \times \mathbf{1}^\ell$. The main step in the proof of the upper bound on $p_c([n]^d,2)$ given in [5] amounts to a lower bound on $\mathbb{P}(\mathcal{D}_a^b)$. In order to prove a stronger bound on $p_c([n]^d,2)$, we will define an event $\mathcal{T}_a^{\mathbf{b}}$ (the vector superscript is explained below) that implies that the infected set grows from $[a]^d \times \mathbf{1}^\ell$ to $[b]^d \times \mathbf{1}^\ell$ not diagonally but via a 'detour'.

We will show that $\mathcal{T}_a^{\mathbf{b}}$ is not too much less probable than $\mathcal{D}_a^b$ (Lemma 5.6). As the infected set grows, it may make a detour and then resume diagonal growth several times. So, we think of the growth of the infected set as diagonal growth interrupted by a sequence of detours. We will show that different 'growth sequences' of this sort are disjoint events (Lemma 5.7) and that the number of growth sequences is fairly large (Lemma 5.8). Furthermore, we will use these results to show that if p satisfies (5.2), then with high probability some cube of the form $[B]^d \times [2]^\ell$, where $B=B(p)$ is sufficiently large, is internally semi-spanned. Finally, we will show that with high probability, the fact that such a cube is internally semi-spanned leads to semi-percolation in $C^*([n]^d \times [2]^\ell, 2)$, which will complete the proof of Lemma 5.1.

Recall the definition of an L-gap from Section 4 and recall that for $j \in [\ell]$, e_j denotes the jth standard basis vector of $\mathbb{R}^\ell$. For all $i, s \in \mathbb{N}$ and $t \in [d]$, let

$$U_i(t,s) = \{[s]^{t-1} \times \{i\} \times [s]^{d-t} \times \mathbf{1}^\ell \text{ is occupied}\}, \tag{5.3}$$

and for all $j \in [\ell]$, let

$$V_i^{(j)}(t,s) = \{[s]^{t-1} \times \{i\} \times [s]^{d-t} \times (\mathbf{1}^\ell + e_j) \text{ is occupied}\}. \tag{5.4}$$

Let $\mathcal{D}_a^b$ be the event that for each $t \in [d]$, the sequence

$$(U_i(t,i-1))_{a+1 \leqslant i \leqslant b} \cup (V_i^{(j)}(t,i-1))_{a+1 \leqslant i \leqslant b-1,\, j \in [\ell]} \tag{5.5}$$

has no L-gaps.

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The next result shows that, as mentioned above, the event $\mathcal{D}_a^b$ means that the infected set grows 'diagonally' from $[a]^d \times \mathbf{1}^\ell$ to $[b-1]^d \times \mathbf{1}^\ell$. Recall that we say that a set S is internally semi-spanned if $[S \cap A] \supseteq S \cap ([n]^d \times \mathbf{1}^\ell)$.

Lemma 5.3. *If $[a]^d \times [2]^\ell$ is internally semi-spanned and $\mathcal{D}_a^b$ occurs, then $[b-1]^d \times [2]^\ell$ is internally semi-spanned.*

Proof. We will show by induction on i that if $[a]^d \times \mathbf{1}^\ell$ is internally spanned and $\mathcal{D}_a^b$ occurs, then for each i with $a+1 \leqslant i \leqslant b-1$, the set $[i]^d \times \mathbf{1}^\ell$ is internally spanned. Let $i \geqslant a+1$ and suppose that $[i-1]^d \times \mathbf{1}^\ell$ is internally spanned. By hypothesis, for each $t \in [d]$, the sequence in (5.5) does not have an L-gap at i, which means that for each t, all of the sites in $[i-1]^{t-1} \times \{i\} \times [i-1]^{d-t} \times \mathbf{1}^\ell$ become infected. (Note that each such site already has one infected neighbour in $[i-1]^d \times \mathbf{1}^\ell$.) Therefore, all of $[i]^d \times \mathbf{1}^\ell$ becomes infected. The claim follows by induction. □

Let

$$G_a^b = \exp\left[-\sum_{i=a}^{b-1} g_{\ell+1}(i^{d-1}q)\right], \tag{5.6}$$

where q is as defined in (3.1). Observe that if $a < b < c$, then

$$G_a^c = G_a^b G_b^c. \tag{5.7}$$

Lemma 5.4. *For all $d \geqslant 2$ and all $b > a \geqslant 2$,*

$$\mathbb{P}(\mathcal{D}_a^b) \geqslant (G_a^b)^d.$$

Proof. Observe from (5.3) and (5.4) that, for each i and t, the events $U_i(t, i-1)$, $U_{i+1}(t, i)$, $V_i^{(1)}(t, i-1), \ldots, V_i^{(\ell)}(t, i-1)$ concern pairwise disjoint sets of sites and are therefore independent. Furthermore, the probability that these events occur is increasing in i. Hence, the sequence (5.5) satisfies the hypotheses of Lemma 4.2. The lemma and (3.2) imply that

$$\begin{aligned}\mathbb{P}(\mathcal{D}_a^b) &\geqslant \left(\prod_{i=a+1}^{b} \beta_{\ell+1}\big(1-(1-p)^{(i-1)^{d-1}}\big)\right)^d \\ &= \exp\left[-d \sum_{i=a+1}^{b} g_{\ell+1}((i-1)^{d-1}q)\right] \\ &= (G_a^b)^d,\end{aligned}$$

as claimed. □

Now we will define the 'detour' mentioned above. In [17], Gravner and Holroyd defined an event $\mathcal{T}_a^b$ that describes another way for the infected set to grow from $[a]^2$ to $[b]^2$. (A simplified version of $\mathcal{T}_a^b$ is shown in Figure 2.) If the rows $[b-1] \times \{a+2\}$ and $[b-1] \times \{a+3\}$ are empty, then the infected set is prevented from growing vertically. However, if there are no double gaps in the columns to the right of $[a]^2$, then the infected set grows horizontally until it fills the rectangle $[b] \times [a+1]$. If the site $(b, a+3)$ is infected, then the infected set overcomes the double gap and resumes vertical growth, ultimately filling $[b]^2$.

We will define a similar event $\mathcal{T}_a^{\mathbf{b}}$, where $\mathbf{b} := \{b_1, \ldots, b_{d-1}\}$. In this event, an L-gap prevents the fully infected cube $[a+1]^d \times \mathbf{1}^\ell$ from growing in direction d (parts (ii), (iii) and (v) of the

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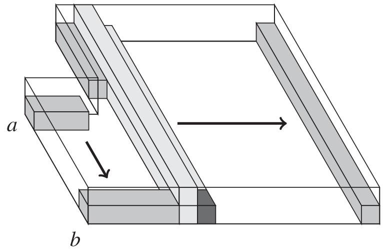

Figure 3. The event $\mathcal{T}_a^{\mathbf{b}}$ for $d = 2$ and $\ell = 1$. The grey regions are occupied (parts (i), (vii) and (ix) of the definition) and the light grey regions are unoccupied (parts (ii), (iii) and (v)). The dark grey cube is an infected site (part (iv)). The arrows depict the growth of the infected set across regions with no L-gaps (parts (vi) and (viii)).

definition below). However, the infected set continues to grow in the other $d-1$ directions (parts (vi) and (vii)) until it meets the infected site $\{b_1, \ldots, b_{d-1}, a+3\} \times \mathbf{1}^\ell$ (part (iv)). This site allows the infected set to overcome the L-gap. Finally, letting $b = \max\{b_i : i \in [d-1]\}$, the infected set continues to grow in direction d until it fills a cube of side length b (parts (viii) and (ix)).

Recall the definitions of the events $U_i(t, s)$ and $V_i^{(j)}(t, s)$ from (5.3) and (5.4), respectively. Let $a, b_1, \ldots, b_{d-1}$ be such that $b := \max\{b_i : i \in [d-1]\}$ satisfies $b \geqslant a+4$ and let $\mathbf{b} = \{b_1, \ldots, b_{d-1}\}$. We define $\mathcal{T}_a^{\mathbf{b}}$ to be the event that all of the following hold (see Figure 3, which depicts the case $d = 2$, $\ell = 1$).

(i) For all $t \in [d]$, the cuboid $[a-1]^{t-1} \times \{a+1\} \times [a-1]^{d-t} \times \mathbf{1}^\ell$ is occupied.

(ii) The cuboid $[b]^{d-1} \times \{a+2\} \times \mathbf{1}^\ell$ is empty.

(iii) For all $j \in [\ell]$, the cuboid $[b]^{d-1} \times \{a+2\} \times (\mathbf{1}^\ell + e_j)$ is empty.

(iv) The site $\{b_1, \ldots, b_{d-1}, a+3\} \times \mathbf{1}^\ell$ is infected.

(v) The cuboid $[b]^{d-1} \times \{a+3\} \times \mathbf{1}^\ell$ contains no other infected sites.

(vi) For all $t \neq d$, the sequence

$$(U_i(t, a+1))_{a+2 \leqslant i \leqslant b-1} \cup (V_i^{(j)}(t, a+1))_{a+2 \leqslant i \leqslant b-2,\, j \in [\ell]}$$

has no L-gaps.

(vii) For all $t \neq d$, the cuboid $[a+1]^{t-1} \times \{b\} \times [a+1]^{d-t} \times \mathbf{1}^\ell$ is occupied.

(viii) The sequence

$$(U_i(d, b))_{a+4 \leqslant i \leqslant b-1} \cup (V_i^{(j)}(d, b))_{a+4 \leqslant i \leqslant b-2,\, j \in [\ell]}$$

has no L-gaps.

(ix) The cuboid $[b]^{d-1} \times \{b\} \times \mathbf{1}^\ell$ is occupied.

Lemma 5.5.

(i) *Events (i)–(ix) in the definition of $\mathcal{T}_a^{\mathbf{b}}$ are independent.*

(ii) *If $[a-1]^d \times [2]^\ell$ is internally semi-spanned and $\mathcal{T}_a^{\mathbf{b}}$ occurs, then $[b]^d \times [2]^\ell$ is also internally semi-spanned.*

Proof. (i) This follows from the fact that events (i)–(ix) in the definition of $\mathcal{T}_a^{\mathbf{b}}$ concern pairwise disjoint sets of sites. Indeed, all of the sites in the sets described in parts (i), (vi) and (vii) have dth coordinate at most $a+1$. Moreover, all of the sites in the sets described by the events $U_i(t, a+1)$ and $V_i^{(j)}(t, a+1)$ have ith coordinate t. Similarly, all of the sites in the sets described in parts (ii), (iii), (iv) and (v) have dth coordinate in $\{a+2, a+3\}$, and it is easy to see that these four sets are

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pairwise disjoint. Finally, all of the sites in parts (viii) and (ix) have dth coordinate at least $a+4$, and it is again easy to see that the sets mentioned in these parts are pairwise disjoint.

(ii) If $[a-1]^d \times [2]^\ell$ is internally semi-spanned, then part (i) implies that each set of the form $[a-1]^{t-1} \times \{a\} \times [a-1]^{d-t} \times \mathbf{1}^\ell$ becomes fully infected. This in turn guarantees that all of $[a]^d \times \mathbf{1}^\ell$, and then all of $[a+1]^d \times \mathbf{1}^\ell$, becomes infected. Parts (vi) and (vii) then guarantee that $[b]^{d-1} \times [a+1] \times [2]^\ell$ is internally semi-spanned. Finally, parts (iv), (viii) and (ix) imply that $[b]^d \times [2]^\ell$ is internally semi-spanned. □

Now we will show that if $b = \max\{b_i : i \in [d-1]\}$, then $\mathcal{T}_a^{\mathbf{b}}$ is not too much less probable than $\mathcal{D}_a^b$. It will be convenient to compare $\mathbb{P}(\mathcal{T}_a^{\mathbf{b}})$ not to $\mathbb{P}(\mathcal{D}_a^b)$ but to G_a^b.

Lemma 5.6. *Let $d \geqslant 2$, let $\ell \geqslant 0$ and let $\zeta = \zeta(d,\ell)$ be the constant defined in (4.2). If $p > 0$ is sufficiently small, if $a, b_1, \ldots, b_{d-1}$ are integers in the interval $[p^{-1/(d-1)} + 1, 4p^{-1/(d-1)}]$ such that $b := \max\{b_i : i \in [d-1]\}$ satisfies $b \geqslant a+4$, and if $\mathbf{b} = \{b_1, \ldots, b_{d-1}\}$, then*

$$\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) \geqslant \zeta p \exp\left[-pd(b-a)(b^{d-1} - a^{d-1})\right](G_a^b)^d.$$

The key to the proof of Lemma 5.6 is that if p is sufficiently small and s is on the order of $p^{-1/(d-1)}$, then $(1-p)^{s^{d-1}}$ is bounded away from both 0 and 1.

Proof of Lemma 5.6. By Lemmas 4.2 and 5.5(i) and the definition of $\mathcal{T}_a^{\mathbf{b}}$,

$$\begin{aligned}\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) \geqslant\ & (1-(1-p)^{(a-1)^{d-1}})^d (1-p)^{b^{d-1}} (1-p)^{\ell b^{d-1}} p (1-p)^{b^{d-1}-1} \\ & \times \beta_{\ell+1}(1-(1-p)^{(a+1)^{d-1}})^{(d-1)(b-a-2)} (1-(1-p)^{(a+1)^{d-1}})^{d-1} \\ & \times \beta_{\ell+1}(1-(1-p)^{b^{d-1}})^{b-a-4}(1-(1-p)^{b^{d-1}}).\end{aligned}$$

Because $b > a$, we have

$$\begin{aligned}\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) \geqslant\ & p(1-p)^{(\ell+2)b^{d-1}-1}(1-(1-p)^{(a-1)^{d-1}})^{2d} \\ & \times \beta_{\ell+1}(1-(1-p)^{(a+1)^{d-1}})^{(d-1)(b-a)} \beta_{\ell+1}(1-(1-p)^{b^{d-1}})^{b-a}. \end{aligned} \tag{5.8}$$

If x is sufficiently small, then $e^{-x} \geqslant 1 - x \geqslant e^{-2x}$. So, because $b \leqslant 4p^{-1/(d-1)}$ and $a - 1 \geqslant p^{-1/(d-1)}$, if p is sufficiently small, then

$$(1-p)^{(\ell+2)b^{d-1}-1}(1-(1-p)^{(a-1)^{d-1}})^{2d} \geqslant e^{-(\ell+2)2^{2d-1}}(1-e^{-1})^{2d} = \zeta.$$

When we plug this into (5.8) and use (3.2), we see that

$$\begin{aligned}\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) &\geqslant \zeta p \beta_{\ell+1}(1-(1-p)^{(a+1)^{d-1}})^{(d-1)(b-a)} \beta_{\ell+1}(1-(1-p)^{b^{d-1}})^{b-a} \\ &= \zeta p \exp\left[-(d-1)(b-a)g_{\ell+1}((a+1)^{d-1}q) - (b-a)g_{\ell+1}(b^{d-1}q)\right].\end{aligned}$$

Finally, since $g_{\ell+1}$ is decreasing, we have

$$\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) \geqslant \zeta p \exp\left[-d(b-a)g_{\ell+1}(a^{d-1}q)\right]. \tag{5.9}$$

Observe from (5.6) that

$$G_a^b = \exp\left[-\sum_{i=a}^{b-1} g_{\ell+1}(i^{d-1}q)\right] \leqslant \exp\left[-(b-a)g_{\ell+1}(b^{d-1}q)\right].$$

Thus, we may rewrite (5.9) as

$$\mathbb{P}(\mathcal{T}_a^{\mathbf{b}}) \geqslant \zeta p \exp\left[-d(b-a)(g_{\ell+1}(a^{d-1}q) - g_{\ell+1}(b^{d-1}q))\right](G_a^b)^d. \tag{5.10}$$

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Now

$$g_{\ell+1}(a^{d-1}q) - g_{\ell+1}(b^{d-1}q) \leqslant (b^{d-1}q - a^{d-1}q) \max_{x\in[a^{d-1}q, b^{d-1}q]} |g'_{\ell+1}(x)|.$$

Recall that $p \leqslant q$, which, by our assumptions on a and b, means that $1 \leqslant a^{d-1}q < b^{d-1}q$. So, Proposition 3.2 and the fact that $q \leqslant 2p$ for p sufficiently small imply that

$$g_{\ell+1}(a^{d-1}q) - g_{\ell+1}(b^{d-1}q) \leqslant (b^{d-1} - a^{d-1})q \cdot \frac{1}{2} \leqslant p(b^{d-1} - a^{d-1}).$$

Combining this with (5.10) gives the desired result. □

If semi-percolation occurs in $C^*([n]^d \times [2]^\ell, 2)$, then, as the infected set grows, it may encounter and overcome several L-gaps. We order the L-gaps by the associated values of a and, for each i, define $\mathbf{b}_i$ to be the vector associated with the ith L-gap.

Now we will define the event that the infected set grows from $[2]^d \times \mathbf{1}^\ell$ to $[B]^d \times \mathbf{1}^\ell$ (where $B = B(p)$ is a large value to be chosen later) with periods of diagonal growth interrupted by a specified sequence of events of the form $\mathcal{T}_a^{\mathbf{b}}$.

For each $t \in [d]$, let $x^{(t)} = \{1\}^{t-1} \times \{2\} \times \{1\}^{d-t} \times \mathbf{1}^\ell$ and let $y^{(t)} = \{B\}^{t-1} \times \{1\} \times \{B\}^{d-t} \times \mathbf{1}^\ell$. Let $m \in \mathbb{N}$ and let $2 \leqslant a_1 < b_1 \leqslant \cdots \leqslant a_m < b_m$ be such that for all $i \in [m]$, we have $b_i - a_i \geqslant 4$, and such that $B > b_m$. For each $i \in [m]$, let $\mathbf{b}_i = \{b_{i,1}, \ldots, b_{i,d-1}\}$ be such that $b_i = \max\{b_{i,t} : t \in [d-1]\}$. Define

$$\begin{aligned}\mathcal{G}(a_1, \mathbf{b}_1, \ldots, a_m, \mathbf{b}_m) = (\{1\}^d \times \mathbf{1}^\ell \text{ is infected}) &\cap \left(\bigcap_{t=1}^{d} (x^{(t)} \text{ is infected})\right) \\ &\cap \mathcal{D}_2^{a_1} \cap \mathcal{T}_{a_1}^{\mathbf{b}_1} \cap \cdots \cap \mathcal{D}_{b_{m-1}}^{a_m} \cap \mathcal{T}_{a_m}^{\mathbf{b}_m} \cap \mathcal{D}_{b_m}^{B-1} \\ &\cap \left(\bigcap_{t=1}^{d} (y^{(t)} \text{ is infected})\right).\end{aligned}$$

Lemma 5.7.

(i) *The events in the definition of* $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ *are independent.*

(ii) *If* $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ *occurs then* $C^*([B]^d \times [2]^\ell, 2)$ *is internally semi-spanned.*

(iii) *Events of the form* $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ *are pairwise disjoint, that is, they correspond to pairwise disjoint subsets of* $\{0, 1\}^{B^d 2^\ell}$.

Proof. (i) It follows from the definition of $\mathcal{D}_a^b$ and from Lemma 5.5(i) that the events in the definition of $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ involve pairwise disjoint sets of sites. Thus, they are independent.

(ii) First, if all of the sites $\{1\}^d \times \mathbf{1}^\ell, x^{(1)}, \ldots, x^{(d)}$ are infected, then $[2]^{d+\ell}$ is internally semi-spanned. Next, observe that by Lemmas 5.3 and 5.5(ii), if the events $\mathcal{D}_2^{a_1}$, $\mathcal{T}_{a_1}^{\mathbf{b}_1}, \ldots, \mathcal{D}_{b_{m-1}}^{a_m}$, $\mathcal{T}_{a_m}^{\mathbf{b}_m}$ and $\mathcal{D}_{b_m}^{B-1}$ all occur, then $[B-2]^d \times [2]^\ell$ is internally semi-spanned. Finally, if all of the sites $y^{(1)}, \ldots, y^{(d)}$ are infected, then $[B]^d \times [2]^\ell$ is internally semi-spanned.

(iii) Consider two sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ and $(a'_i, \mathbf{b}'_i)_{i=1}^m$ and the associated events $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ and $\mathcal{G}((a'_i, \mathbf{b}'_i)_{i=1}^m)$. Recall the definitions of the events $U_i(t,s)$ and $V_i^{(j)}(t,s)$ from (5.3) and (5.4), respectively. Given $i \geqslant 1$, it follows from the definition of $\mathcal{D}_a^b$ and parts (ii), (iii) and (v) of the definition of $\mathcal{T}_a^{\mathbf{b}}$ that $a_i + 2$ is the least $s \geqslant b_{i-1}$ such that the events $U_s(d, s-1)$, $U_{s+1}(d,s)$, $V_s^{(1)}(d, s-1), \ldots, V_s^{(\ell)}(d, s-1)$ all do not occur. (Here, we set $b_0 = 2$.) This means that if $a_i \neq a'_i$,

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then $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ and $\mathcal{G}((a'_i, \mathbf{b}'_i)_{i=1}^m)$ are disjoint. Similarly, parts (iv) and (v) of the definition of $\mathcal{T}_a^{\mathbf{b}}$ imply that if $\mathbf{b}_i \neq \mathbf{b}'_i$, then $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ and $\mathcal{G}((a'_i, \mathbf{b}'_i)_{i=1}^m)$ are disjoint. Thus, the two events are disjoint unless they are identical, as claimed. □

Parts (ii) and (iii) of Lemma 5.7 indicate that if we can bound from below the probability that an event of the form $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ occurs, then a union bound will give us a lower bound on the probability of semi-percolation. To this end, we wish to enumerate those sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ that satisfy certain conditions. We will be interested in sequences such that

$$p^{-1/(d-1)} + 1 \leqslant a_1 < b_1 \leqslant \cdots \leqslant a_m < b_m \leqslant 4p^{-1/(d-1)} \tag{5.11}$$

and

$$4 \leqslant b_i - a_i \leqslant p^{-1/(2d-2)} \quad \text{for all } i \in [m]. \tag{5.12}$$

Let us explain these conditions. First, the lower bound on the probability of $\mathcal{T}_a^{\mathbf{b}}$ in Lemma 5.6 requires that a and b both be on the order of $p^{-1/(d-1)}$, which corresponds to (5.11).

Second, we wish to show that there are many sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ such that for all i, we have $b_i - a_i \leqslant K = K(p)$. What, then, should K and m be? Observe that (5.11) implies that $Km = O(p^{-1/(d-1)})$. Moreover, we will show that, given K and m, the number of sequences of the desired form is roughly $(K/mp)^m$, which is maximized when K and m have the same order of magnitude. Thus, we will take both K and m to be on the order of $p^{-1/(2d-2)}$; the former requirement is the second inequality in (5.12). Finally, $\mathcal{T}_a^{\mathbf{b}}$ is defined only if $b \geqslant a + 4$.

Lemma 5.8. *Let $d \geqslant 2$, let $\ell \geqslant 0$ and let $p > 0$. Let $\gamma = \gamma(d, \ell)$ be as in (4.3), let*

$$m = \gamma p^{-1/(2d-2)},$$

and let $\mathcal{S}$ denote the set of sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ that satisfy (5.11) and (5.12). If p is sufficiently small, then

$$|\mathcal{S}| \geqslant \left(\frac{8}{\gamma p}\right)^m.$$

Proof. We construct sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ satisfying (5.11) and (5.12) as follows: we start by choosing $a_1, \ldots, a_m$ such that $a_1 \geqslant p^{-1/(d-1)} + 1$, such that $a_{i+1} \geqslant a_i + p^{-1/(2d-2)}$ for all $i \in [m-1]$, and such that $a_m \leqslant 4p^{-1/(d-1)} - p^{-1/(2d-2)}$. Then, for each i, we choose $\mathbf{b}_i$ as follows. First, we choose an element of $\{a_i + 4, \ldots, a_i + p^{-1/(2d-2)}\}$ and call it b_i. Then, to complete the vector $\mathbf{b}_i$, we choose $d-2$ elements of $[b_i]$ with replacement. Observe that a sequence chosen in this way indeed satisfies (5.11) and (5.12) and so is an element of $\mathcal{S}$.

Let $\mathcal{S}'$ denote the set of sequences chosen above and observe that by Stirling's approximation, if p is sufficiently small, then

$$\begin{aligned} |\mathcal{S}'| &\geqslant \binom{3p^{-1/(d-1)} - 1 - mp^{-1/(2d-2)}}{m} (p^{-1/(2d-2)} - 3)^m \prod_{i=1}^m b_i^{d-2} \\ &\geqslant \left(\frac{e(3-2\gamma)p^{-1/(d-1)}}{m}\right)^m ((1-\gamma)p^{-1/(2d-2)})^m \prod_{i=1}^m b_i^{d-2}. \end{aligned}$$

For each $i \in [m]$, we have $b_i \geqslant p^{-1/(d-1)}$. It follows from (4.4) that

$$|\mathcal{S}| \geqslant |\mathcal{S}'| \geqslant \left(\frac{e(3-5\gamma)p^{-1/(2d-2)}}{m}\right)^m (p^{-1/(d-1)})^m (p^{-(d-2)/(d-1)})^m \geqslant \left(\frac{8}{\gamma p}\right)^m.$$

This completes the proof. □

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Remark 5.9. We make two further remarks regarding Lemma 5.8. First, one might also count sequences of fewer than m L-gaps, but it turns out that this would not significantly affect the total. This is because if $M \gg m$, then $\sum_{j=1}^{m} (M/j)^j \leqslant m(M/m)^m$; for our purposes, the extra factor of m represents a negligible increase.

Second, recall that in part (iv) of the definition of $\mathcal{T}_a^{\mathbf{b}}$ we required that the site $(b_1, \ldots, b_{d-1}, a+3) \times \mathbf{1}^\ell$ be infected. One might be tempted to define $\mathcal{T}_a^{\mathbf{b}}$ so that the site $(b, \ldots, b, a+3) \times \mathbf{1}^\ell$ is infected. However, with this alternative definition, there does not exist a constant $c > 0$ such that the number of sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ satisfying (5.11) and (5.12) is at least $(c/p)^m$ – and, as the proof of Lemma 5.10 will show, this bound is exactly what we need.

Recall the definition of $P(n, d, \ell, r, p)$ from (4.1). Now we will combine the results of this section to prove a lower bound on $P(B, d, \ell, 2, p)$ for $B > 4p^{-1/(d-1)}$. Once we have done so, we will be ready to prove Lemma 5.1.

Lemma 5.10. *Let $d \geqslant 2$, let $\ell \geqslant 0$ and let $\gamma = \gamma(d, \ell)$ be as in (4.3). If $p > 0$ is sufficiently small and $B > 4p^{-1/(d-1)}$, then*

$$P(B, d, \ell, 2, p) \geqslant \exp\left[\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda(d+\ell, \ell+2)}{p^{1/(d-1)}}\right]. \tag{5.13}$$

Proof. We begin by bounding from below the probability that a single event of the form $\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)$ occurs. Let m be as in the statement of Lemma 5.8 and let $(a_i, \mathbf{b}_i)_{i=1}^m$ be a sequence that satisfies (5.11) and (5.12). (It follows from (5.11) and our assumption on B that $B > b_m$.) By Lemma 5.7(i), we have

$$\mathbb{P}(\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)) = p^{2d+1}\mathbb{P}(\mathcal{D}_2^{a_1})\mathbb{P}(\mathcal{T}_{a_1}^{\mathbf{b}_1}) \cdots \mathbb{P}(\mathcal{D}_{b_{m-1}}^{a_m})\mathbb{P}(\mathcal{T}_{a_m}^{\mathbf{b}_m})\mathbb{P}(\mathcal{D}_{b_m}^{B-1}).$$

Recall that in Lemmas 5.4 and 5.6, we bounded $\mathbb{P}(\mathcal{D}_a^b)$ and $\mathbb{P}(\mathcal{T}_a^{\mathbf{b}})$, respectively, in terms of G_a^b. It follows from these results and (5.7) that

$$\mathbb{P}(\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)) \geqslant p^{2d+1}(G_2^{B-1})^d \prod_{i=1}^{m} (\zeta p e^{-pd(b_i-a_i)(b_i^{d-1}-a_i^{d-1})}). \tag{5.14}$$

By the Mean Value Theorem, for each i, there exists $\alpha_i \in [a_i, b_i]$ such that

$$b_i^{d-1} - a_i^{d-1} = (b_i - a_i)(d-1)\alpha_i^{d-2}.$$

It then follows from (5.11) and (5.12) that

$$(b_i - a_i)(b_i^{d-1} - a_i^{d-1}) \leqslant (d-1)(b_i - a_i)^2 b_i^{d-2} \leqslant (d-1)2^{2d-4}p^{-1}.$$

Plugging this into (5.14) and recalling the definition of γ from (4.3) shows that

$$\begin{aligned}\mathbb{P}(\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)) &\geqslant p^{2d+1}(G_2^{B-1})^d \prod_{i=1}^{m} (\zeta p e^{-d(d-1)2^{2d-4}}) \\ &= p^{2d+1}(G_2^{B-1})^d(\gamma p)^m. \end{aligned} \tag{5.15}$$

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Now let $\lambda = \lambda(d+\ell, \ell+2)$ be as in (1.3). Observe that (5.6), the fact that $g_{\ell+1}$ is decreasing and the fact that $p \leqslant q$ imply that

$$\begin{aligned} G_2^{B-1} &= \exp\left[-\sum_{i=2}^{B-2} g_{\ell+1}(i^{d-1}q)\right] \\ &\geqslant \exp\left[-\frac{1}{p^{1/(d-1)}}\int_0^\infty g_{\ell+1}(z^{d-1})\,dz\right] \\ &= \exp\left[-\frac{\lambda}{p^{1/(d-1)}}\right]. \end{aligned}$$

Plugging this into (5.15), we see that

$$\mathbb{P}(\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)) \geqslant p^{2d+1}(\gamma p)^m \exp\left[-\frac{d\lambda}{p^{1/(d-1)}}\right]. \tag{5.16}$$

Now we are ready to prove our lower bound on $P(B, d, \ell, 2, p)$. Let $\mathcal{S}$ denote the set of sequences $(a_i, \mathbf{b}_i)_{i=1}^m$ that satisfy (5.11) and (5.12) and recall from Lemma 5.8 that

$$|\mathcal{S}| \geqslant \left(\frac{8}{\gamma p}\right)^m.$$

It then follows from Lemma 5.7(ii), (iii) and from (5.16) that

$$P(B, d, \ell, 2, p) \geqslant \sum_{(a_i,\mathbf{b}_i)_{i=1}^m \in \mathcal{S}} \mathbb{P}(\mathcal{G}((a_i, \mathbf{b}_i)_{i=1}^m)) \geqslant p^{2d+1} 8^m \exp\left[-\frac{d\lambda}{p^{1/(d-1)}}\right].$$

Recall from Lemma 5.8 that $m = \gamma p^{-1/(2d-2)}$. Since $p^{1/(2d-2)} \log(1/p) \to 0$ as $p \to 0$, it follows that

$$\begin{aligned} P(B, d, \ell, 2, p) &\geqslant p^{2d+1} \exp\left[\frac{\gamma \log 8}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}}\right] \\ &\geqslant \exp\left[\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}}\right], \end{aligned}$$

as claimed. □

Now we will show that the right-hand side of (5.13) is large enough that it is very likely that some fairly large cube in $[n]^d \times [2]^\ell$ is internally semi-spanned. In particular, the $2\gamma p^{-1/(2d-2)}$ term in the exponent on the right-hand side of (5.13) will allow us to prove Lemma 5.1.

Proof of Lemma 5.1. Recall that we want to show that if c satisfies (5.1) and p satisfies (5.2), then $P(n, d, \ell, 2, p) \to 1$ as $n \to \infty$. A standard coupling argument shows that $P(n, d, \ell, 2, p)$ is increasing in p, so it is enough to prove the lemma under the assumption that

$$p \leqslant \left(\frac{\lambda(d+\ell, \ell+2)^2}{d^2 \log n}\right)^{d-1}. \tag{5.17}$$

Let $B = p^{-3/(d-1)}$ and partition $[n]^d \times [2]^\ell$ into cubes of the form $[B]^d \times [2]^\ell$. We want to show that with high probability at least one of these cubes is internally semi-spanned. To do this, we use the following claim, whose proof we postpone to the Appendix.

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Claim 5.11. *Let $d \geqslant 2$, let $\ell \geqslant 0$ and let $\gamma = \gamma(d,\ell)$ be as in (4.3). If c satisfies (5.1) and p satisfies (5.2) and (5.17), then there exists a constant $\alpha > 0$ such that*

$$\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda(d+\ell,\ell+2)}{p^{1/(d-1)}} \geqslant \alpha(\log n)^{1/2} - d\log n$$

for all n sufficiently large.

Let I_B denote the event that at least one cube of the form $[B]^d \times [2]^\ell$ is internally semi-spanned and let $\lambda = \lambda(d+\ell,\ell+2)$. By Lemma 5.10, Claim 5.11 and the fact that $e^{(\log n)^{1/3}} \gg B$, we have

$$\begin{aligned}
\mathbb{P}(I_B) &\geqslant 1 - \left(1 - \exp\left[\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}}\right]\right)^{(n/B)^d} \\
&\geqslant 1 - \exp\left[-\left(\frac{n}{B}\right)^d \exp\left(\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}}\right)\right] \\
&\geqslant 1 - \exp\left[-\left(\frac{n}{B}\right)^d \exp\left(\alpha(\log n)^{1/2} - d\log n\right)\right] \\
&\geqslant 1 - \exp\left[-\exp\left(\alpha(\log n)^{1/2} - (\log n)^{1/3}\right)\right] \\
&= 1 - o(1).
\end{aligned}$$

It is easy to see that if a cube of the form $[B]^d \times [2]^\ell$ is internally semi-spanned and every cuboid of the form $[B]^{t-1} \times \{1\} \times [B]^{d-t} \times \mathbf{1}^\ell$ is occupied, then the initially infected set A semi-percolates in $C^*([n]^d \times [2]^\ell, 2)$. By (5.17), for all $d \geqslant 2$, we have $p \ll (\log n)^{-1/2}$. So, by the definition of B, the probability that some cuboid of the form $[B]^{t-1} \times \{1\} \times [B]^{d-t} \times \mathbf{1}^\ell$ is unoccupied is at most

$$dn^d(1-p)^{B^{d-1}} \leqslant dn^d e^{-p^{-2}} = o(1).$$

Finally, because the events 'a cube of the form $[B]^d \times [2]^\ell$ is internally semi-spanned' and 'a cuboid of the form $[B]^{t-1} \times \{1\} \times [B]^{d-t} \times \mathbf{1}^\ell$ is occupied' are increasing, Harris's lemma implies that $P(n,d,\ell,2,p) \to 1$ as $n \to \infty$. □

6. Proofs of Theorems 1.4 and 4.1

In this section we complete the proof of Theorem 4.1 and use it to deduce Theorem 1.4. Note that Lemma 5.1 proves Theorem 4.1 for $r=2$, all $d \geqslant 2$ and all $\ell \geqslant 0$. The rest of the proof of Theorem 4.1 is an inductive argument that is very similar to the one used in [5]. However, we face certain technical complications that are not present in [5]. So, in spite of the many similarities, we will give almost all of the details of the proof.

We wish to show that for all $d \geqslant r$ and all $\ell \geqslant 0$, if p satisfies (4.7), then $P(n,d,\ell,r,p) \to 1$ as $n \to \infty$. We will assume that Theorem 4.1 holds for all $r' < r$, all $d \geqslant r'$ and all $\ell \geqslant 0$. In order to carry out the induction, we need two lemmas. The first is due to Holroyd [22, Lemma 2]. We will need it for the case $r = 3$.

Lemma 6.1. *For any $d \geqslant 3$, $\ell \geqslant 0$ and $\varepsilon > 0$, if n is sufficiently large and $p^{-2d} \leqslant n^\varepsilon$, then*

$$P(n,d,\ell,3,p) \geqslant \exp(-n^{1+\varepsilon}).$$ □

The second lemma is due to Balogh, Bollobás and Morris [5, Lemma 12].

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Lemma 6.2. *For each $d \geqslant r \geqslant 2$ and each $\ell \geqslant 0$, there exists a constant $\eta = \eta(d, \ell, r) > 0$ such that the following holds. Let $\varepsilon, p > 0$, let $n, m \in \mathbb{N}$ and let $A \sim \text{Bin}([n]^d \times [2]^\ell, p)$. If*

$$P(m, d-i, \ell+i, r-i, p) \geqslant 1-\eta \quad \text{for all } i \in [r-2] \tag{6.1}$$

and if $M \leqslant n$ is such that M/m is sufficiently large (depending on d, ℓ, r and ε), then

$$\mathbb{P}([n]^d \times \mathbf{1}^\ell \subseteq [A \cup ([M]^d \times \mathbf{1}^\ell)]) \geqslant 1-\varepsilon; \tag{6.2}$$

in particular,

$$P(n, d, \ell, r, p) \geqslant (1-\varepsilon) P(M, d, \ell, r, p). \qquad \square$$

Remark 6.3. Lemma 6.2 simply provides a lower bound on the the probability that the infected set grows from a smaller cuboid to a larger one. However, the role of m in the statement of the lemma deserves some explanation.

Let $t \geqslant m$ and suppose that $[t]^d \times \mathbf{1}^\ell$ is internally spanned. It follows from an observation in [2] that if (6.1) holds, then the probability that $[t+1]^d \times \mathbf{1}^\ell$ is not internally spanned is exponentially small in t/m. (For a proof of this statement, see *e.g.* [5, Lemma 11].) So, Harris's lemma and the assumption on M/m imply that there exists $C > 0$ such that

$$\mathbb{P}([n]^d \times \mathbf{1}^\ell \subseteq [A \cup ([M]^d \times \mathbf{1}^\ell)]) \geqslant \prod_{t=M}^{n-1} (1 - Ce^{-t/m}) \geqslant 1-\varepsilon,$$

which is exactly (6.2).

To prove Theorem 4.1, we will need to define quantities m, M and N such that $1 \ll m \ll M \ll N \ll n$. We will bound from below the probability of filling a cuboid of side length M. Then, using our induction hypothesis and Lemma 6.2, we will bound the probability that this cuboid grows to fill a cuboid of side length $N := (\log n)^3$. Once we have bounded $P(N, d, \ell, r, p)$, it will be easy to show that, with high probability, there exists a copy of $[N]^d \times \mathbf{1}^\ell$ in $[n]^d \times \mathbf{1}^\ell$ that is internally spanned and that, with high probability, this copy of $[N]^d \times \mathbf{1}^\ell$ grows to fill all of $[n]^d \times \mathbf{1}^\ell$.

In order to apply Lemma 6.2 in the proof of Theorem 4.1, we must take some care in choosing the values of m and M. We want to define m such that, for all $i \in [r-2]$, $P(m, d-i, \ell+i, r-i, p)$ is sufficiently close to 1. If m is such that

$$p \geqslant \left(\frac{\lambda(d+\ell, \ell+r)}{\log_{(r-i-1)}(m)} - \frac{c_{d-i,\ell+i,r-i}}{(\log_{(r-i-1)}(m))^{3/2}} \right)^{d-r+1}, \tag{6.3}$$

then the desired lower bound on $P(m, d-i, \ell+i, r-i, p)$ follows from the induction hypothesis. Comparing (6.3) to the bound on p in (4.7) suggests that it is reasonable to define m such that $\log_{(r-2)}(m)$ is close to $\log_{(r-1)}(n)$. Recall also from Lemma 6.2 that we want to define M such that $M/m \to \infty$ as $n \to \infty$. Furthermore, it will turn out that we want $\log_{(r-2)}(M)$ to be slightly less than $\log_{(r-1)}(n)$. However, how close $\log_{(r-2)}(m)$ and $\log_{(r-2)}(M)$ must be to $\log_{(r-1)}(n)$ depends on n, which complicates the argument slightly.

First, let

$$N = (\log n)^3.$$

Next, given d, ℓ and r, let $\lambda = \lambda(d+\ell, \ell+r)$ be as in (1.3). We define M, m and a third quantity δ, such that M and m are the largest positive values such that

$$\delta = \frac{2^{-r}\gamma(d-r+2, \ell+r-2)}{\lambda} (\log_{(r-2)}(M))^{-1/2}, \tag{6.4}$$

$$\log_{(r-2)}(M) = (1-\delta)\log_{(r-1)}(n), \tag{6.5}$$

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and

$$\log_{(r-2)}(m) = (1-2\delta)\log_{(r-1)}(n). \tag{6.6}$$

It is not necessarily obvious from (6.4)–(6.6) that δ, M and m are well-defined. To see that these quantities are indeed well-defined for n sufficiently large, let $c = 2^{-r}\gamma(d-r+2, \ell+r-2)/\lambda$ and observe that by (6.4), we may rewrite (6.5) as

$$\log_{(r-2)}(M) = (1 - c(\log_{(r-2)}(M))^{-1/2})\log_{(r-1)}(n). \tag{6.7}$$

Now let $y = \log_{(r-1)}(n)$ and let

$$f(x) = x - (1 - cx^{-1/2})y.$$

Elementary calculations show that for n sufficiently large, f has at least one and at most two positive real roots, at least one of which is larger than 1. Let x_0 be the larger (or only) positive real root of f. Then we may define M by $\log_{(r-2)}(M) = x_0$, which is exactly (6.7). Thus, δ, M and m are well-defined.

On another note, (6.4) and (6.5) imply that

$$\delta = \Theta((\log_{(r-1)}(n))^{-1/2}). \tag{6.8}$$

So, $\delta \to 0$ as $n \to \infty$. This convergence to 0, which we have not been able to avoid, is the source of most of the technical complications in the proof of Theorem 4.1.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. As stated above, Lemma 5.1 gives the claimed result for $r = 2$. So, suppose that $r \geqslant 3$ and that for all $r' < r$, the result holds for all $d \geqslant r'$ and for all $\ell \geqslant 0$.

We begin by proving a lower bound on $P(M, d, \ell, r, p)$.

Claim 6.4. *We have $P(M, d, \ell, r, p) \geqslant 1/n$ as $n \to \infty$.*

Proof. To prove the claim for $r = 3$, we first show that

$$p^{-2d} \leqslant (\log\log n)^{4d^2} \leqslant M^{\delta}. \tag{6.9}$$

The first inequality follows from (4.7). To see the second inequality, observe that (6.5) and (6.8) imply that

$$\delta \log M \gg (\log\log n)^{1/2} \gg \log\log\log n.$$

By (6.9), we may apply Lemma 6.1 to $P(M, d, \ell, 3, p)$. The lemma and (6.5) imply that for n sufficiently large,

$$P(M, d, \ell, 3, p) \geqslant \exp(-M^{1+\delta}) = \exp(-(\log n)^{1-\delta^2}) \geqslant 1/n.$$

To prove the claim for $r \geqslant 4$, it suffices to bound $P(M, d, \ell, r, p)$ from below by the probability that $[M]^d \times \mathbf{1}^\ell$ is full. To do this, we first show that

$$\log M \ll \log\log n. \tag{6.10}$$

Indeed, exponentiating both sides of (6.5) and applying (6.8) gives

$$\log_{(r-3)}(M) = \exp((1-\delta)\log_{(r-1)}(n)) \ll \exp(\log_{(r-1)}(n)) = \log_{(r-2)}(n).$$

This proves (6.10) for $r = 4$. For $r \geqslant 5$, (6.10) follows by repeatedly exponentiating both sides of the inequality above.

We then observe that, by (6.10),

$$P(M, d, \ell, r, p) \geqslant p^{M^d} \gg \exp(-\log(1/p)(\log n)^{1-\delta}),$$

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which means that we are done if we can show that

$$\exp\left(-\log(1/p)(\log n)^{1-\delta}\right) \gg \frac{1}{n}. \tag{6.11}$$

If we take logarithms twice in (6.11), we see that it is enough to show that $\log\log(1/p) \ll \delta\log\log n$. To do this, we observe that (6.8) and the fact that $1/p \ll \log_{(r-1)}(n)$ imply that for all $r \geqslant 4$,

$$\delta \log\log n \gg \frac{\log\log n}{(\log_{(r-1)}(n))^{1/2}} \gg \log_{(r+1)}(n) \geqslant \log\log(1/p).$$

This proves (6.11), which proves the claim. □

Now we wish to use Lemma 6.2 to show that $P(N, d, \ell, r, p) \geqslant 1/2n$ for all n sufficiently large.

First, we claim that $M/m \to \infty$ as $n \to \infty$. For $r \geqslant 4$, this is easy to see. For $r = 3$, we observe that by (6.5), (6.6) and (6.8),

$$\frac{M}{m} = (\log n)^{\delta} = \exp(\delta \log\log n) \geqslant \exp(C(\log\log n)^{1/2})$$

for some constant $C > 0$. So, M/m indeed tends to infinity as $n \to \infty$.

Next, we show that our induction hypothesis implies that (6.1) holds – that is, A is likely to semi-percolate in the lower-threshold sets adjacent to $[m]^d \times [2]^\ell$. Once we have shown this, we will be ready to apply Lemma 6.2.

Claim 6.5. *For all $i \in [r-2]$, $P(m, d-i, \ell+i, r-i, p) \to 1$ as $n \to \infty$.*

Proof. Let $\lambda = \lambda(d+\ell, \ell+r)$ and let $c_{d,\ell,r}$ be as in (4.5). By induction, it is enough to show that for all $i \in [r-2]$, we have

$$\begin{aligned} p &\geqslant \left(\frac{\lambda(1-2\delta)}{\log_{(r-2)}(m)} - \frac{c_{d,\ell,r}(1-2\delta)^{3/2}}{(\log_{(r-2)}(m))^{3/2}}\right)^{d-r+1} \\ &\geqslant \left(\frac{\lambda}{\log_{(r-i-1)}(m)} - \frac{c_{d-i,\ell+i,r-i}}{(\log_{(r-i-1)}(m))^{3/2}}\right)^{d-r+1}. \end{aligned} \tag{6.12}$$

The first inequality in (6.12) follows from (4.7) and (6.6). For $i \geqslant 2$, the second inequality in (6.12) is easy to see. For $i = 1$, we need to show that

$$\frac{\lambda(1-2\delta)}{\log_{(r-2)}(m)} - \frac{c_{d,\ell,r}(1-2\delta)^{3/2}}{(\log_{(r-2)}(m))^{3/2}} \geqslant \frac{\lambda}{\log_{(r-2)}(m)} - \frac{c_{d-1,\ell+1,r-1}}{(\log_{(r-2)}(m))^{3/2}}.$$

Because $(1-2\delta)^{3/2} < 1$, it is enough to show that

$$2\delta\lambda(\log_{(r-2)}(m))^{1/2} + c_{d,\ell,r} \leqslant c_{d-1,\ell+1,r-1}.$$

Indeed, (6.4), the fact that $m \leqslant M$ and (4.6) imply that

$$2\delta\lambda(\log_{(r-2)}(m))^{1/2} + c_{d,\ell,r} \leqslant 2^{-r+1}\gamma(d-r+2, \ell+r-2) + c_{d,\ell,r} = c_{d-1,\ell+1,r-1},$$

which proves (6.12).

It then follows from the induction hypothesis that for each $i \in [r-2]$,

$$P(m, d-i, \ell+i, r-i, p) \to 1 \quad \text{as } n \to \infty,$$

as claimed. □

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By Claim 6.5, we may apply Lemma 6.2 to $P(N, d, \ell, r, p)$. The lemma and Claim 6.4 imply that

$$P(N, d, \ell, r, p) \geqslant (1-\varepsilon)P(M, d, \ell, r, p) \geqslant \frac{1}{2n}$$

for all n sufficiently large. Since $(n/N)^d \gg 2n$, with high probability, there exists a cuboid $K \times \mathbf{1}^\ell \subseteq [A]$ with $|K| \geqslant N^d$. So, by applying Lemma 6.2 again (this time with N in place of M) and Harris's lemma, we have

$$P(n, d, \ell, r, p) \geqslant (1-o(1))\mathbb{P}([n]^d \times \mathbf{1}^\ell \subseteq [A \cup (K \times \mathbf{1}^\ell)]) = 1-o(1).$$

This completes the proof of Theorem 4.1. □

The proof of Theorem 1.4 is immediate.

Proof of Theorem 1.4. Let $d \geqslant r \geqslant 2$ and let n be sufficiently large. Let $c_{d,r} = c_{d,0,r}$ and note that (4.5) implies that $c_{d,r} > 0$. Applying Theorem 4.1 with $\ell = 0$ shows that

$$p_c([n]^d, r) \leqslant \left(\frac{\lambda(d,r)}{\log_{(r-1)}(n)} - \frac{c_{d,r}}{(\log_{(r-1)}(n))^{3/2}}\right)^{d-r+1},$$

as claimed. □

7. Open questions

It remains to improve the lower bound on the critical probability $p_c([n]^d, r)$ for values of $d \geqslant r \geqslant 2$ other than $d = r = 2$. Given the difficulty of the proof of the lower bound in Theorem 1.3, this is likely to be much harder than the proof of Theorem 1.4, especially for $r \geqslant 3$. Note, however, that the upper bound on $p_c([n]^2, 2)$ in [17] gave the correct order of magnitude of the second term. Because the proof of Theorem 1.4 can be seen as a fairly natural generalization of the arguments in [17] to higher dimensions, we conjecture that the theorem gives the correct order of magnitude of the second term in $p_c([n]^d, r)$ for all $d \geqslant r \geqslant 2$.

Conjecture 7.1. Let $d \geqslant r \geqslant 2$. As $n \to \infty$,

$$p_c([n]^d, r) = \left(\frac{\lambda(d,r)}{\log_{(r-1)}(n)} - \Theta\left(\frac{1}{(\log_{(r-1)}(n))^{3/2}}\right)\right)^{d-r+1}.$$

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Appendix

Here we give the proofs of results from the paper that, while straightforward, rely on somewhat lengthy calculations.

Proof of Proposition 3.2. Recall that we wish to bound $|g_k'(z)|$ from above for all $k \geqslant 1$ and all $z \geqslant 1$.

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By (1.2),

$$|g_k'(z)| = \left| \frac{e^{-z}\beta_k'(1-e^{-z})}{\beta_k(1-e^{-z})} \right|. \tag{A.1}$$

First, we bound $\beta_k'(1-e^{-z})$ from above. (Recall that β_k is increasing on $(0, 1)$.) Differentiating both sides of (3.3) gives

$$2\beta_k(u)\beta_k'(u) = k(1-u)^{k-1}\beta_k(u) + (1-(1-u)^k)\beta_k'(u) + (1-u)^k - ku(1-u)^{k-1}.$$

So,

$$\beta_k'(u) = \frac{k(1-u)^{k-1}\beta_k(u) + (1-u)^k - ku(1-u)^{k-1}}{2\beta_k(u) - 1 + (1-u)^k}.$$

It follows from (1.1) and the fact that $\beta_k(u) < 1$ for all $u \in (0, 1)$ that

$$\begin{aligned}\beta_k'(u) &= \frac{k(1-u)^{k-1}\beta_k(u) + (1-u)^k - ku(1-u)^{k-1}}{\sqrt{1+(4u-2)(1-u)^k + (1-u)^{2k}}} \\ &\leqslant \frac{k(1-u)^{k-1} + (1-u)^k - ku(1-u)^{k-1}}{\sqrt{1+(4u-2)(1-u)^k + (1-u)^{2k}}} \\ &= \frac{(k+1)(1-u)^k}{\sqrt{1+(4u-2)(1-u)^k + (1-u)^{2k}}}.\end{aligned} \tag{A.2}$$

Observe that the denominator of the right-hand side of (A.2) is at least 1 for all $u \geqslant 1/2$. If $z \geqslant 1$ then $1 - e^{-z} \geqslant 1/2$, so for all such z we have

$$\beta_k'(1-e^{-z}) \leqslant (k+1)e^{-zk}. \tag{A.3}$$

Next, observe that for $u \geqslant 0$, the quantity under the square root on the right-hand side of (1.1) is at least $(1-(1-u)^k)^2$, which means that

$$\beta_k(u) \geqslant 1-(1-u)^k \tag{A.4}$$

for all $u \in (0, 1)$.

When we combine (A.3) and (A.4) with (A.1), we find that

$$|g_k'(z)| \leqslant \frac{(k+1)e^{-z(k+1)}}{1-e^{-zk}} = \frac{k+1}{e^{z(k+1)} - e^z} \leqslant \frac{2}{e^2-2} < \frac{1}{2},$$

which is what we wanted. □

Proof of Claim 5.11. The claim is a lower bound on $\log(P(B, d, \ell, 2, p))$. Recall that c is the constant from Lemma 5.1 and that

$$\left(\frac{\lambda(d+\ell, \ell+2)}{\log n} - \frac{c}{(\log n)^{3/2}}\right)^{d-1} \leqslant p \leqslant \left(\frac{\lambda(d+\ell, \ell+2)^2}{d^2 \log n}\right)^{d-1}, \tag{A.5}$$

where the upper bound is the assumption (5.17).

Let $\lambda = \lambda(d+\ell, \ell+2)$. By (A.5),

$$\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}} \geqslant 2\gamma\left(\frac{\lambda^2}{d^2\log n}\right)^{-1/2} - d\log n\left(1 - \frac{c}{\lambda(\log n)^{1/2}}\right)^{-1}.$$

Let $\varepsilon > 0$ be sufficiently small. If x is sufficiently small, then $(1-x)^{-1} \leqslant 1 + (1+\varepsilon)x$. Hence, for n sufficiently large, we have

$$\frac{2\gamma}{p^{1/(2d-2)}} - \frac{d\lambda}{p^{1/(d-1)}} \geqslant d\log n \cdot \frac{2\gamma}{\lambda(\log n)^{1/2}} - d\log n\left(1 + \frac{(1+\varepsilon)c}{\lambda(\log n)^{1/2}}\right). \tag{A.6}$$

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By (5.1),

$$(1+\varepsilon)c < \frac{4c}{3} < 2\gamma.$$

It follows that there exists $\alpha > 0$ such that the right-hand side of (A.6) is at least $\alpha(\log n)^{1/2} - d \log n$, which is what we wanted. □

References

[1] Adler, J. and Lev, U. (2003) Bootstrap percolation: Visualizations and applications. *Braz. J. Phys.* **33** 641–644.

[2] Aizenman, M. and Lebowitz, J. L. (1988) Metastability effects in bootstrap percolation. *J. Phys.* A **21** 3801–3813.

[3] Balister, P., Bollobás, B., Przykucki, M. and Smith, P. (2016) Subcritical $\mathcal{U}$-bootstrap percolation models have non-trivial phase transitions. *Trans. Amer. Math. Soc.* **368** 7385–7411.

[4] Balogh, J., Bollobás, B., Duminil-Copin, H. and Morris, R. (2012) The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701.

[5] Balogh, J., Bollobás, B. and Morris, R. (2009) Bootstrap percolation in three dimensions. *Ann. Probab.* **37** 1329–1380.

[6] Balogh, J. and Pete, G. (1998) Random disease on the square grid. *Random Struct. Alg.* **134** 409–422.

[7] Bollobás, B., Duminil-Copin, H., Morris, R. and Smith, P. (2019) Universality of two-dimensional critical cellular automata. *Proc. London Math. Soc.*, to appear.

[8] Bollobás, B., Duminil-Copin, H., Morris, R. and Smith, P. (2017) The sharp threshold for the Duarte model. *Ann. Probab.* **45** 4222–4272.

[9] Bollobás, B., Smith, P. J. and Uzzell, A. J. (2015) Monotone cellular automata in a random environment. *Combin. Probab. Comput.* **24** 687–722.

[10] Cerf, R. and Cirillo, E. N. M. (1999) Finite size scaling in three-dimensional bootstrap percolation. *Ann. Probab.* **27** 1837–1850.

[11] Cerf, R. and Manzo, F. (2002) The threshold regime of finite volume bootstrap percolation. *Stoch. Process. Appl.* **101** 69–82.

[12] Chalupa, J., Leath, P. L. and Reich, G. R. (1979) Bootstrap percolation on a Bethe lattice. *J. Phys.* C **12** L31–L35.

[13] Duminil-Copin, H. and Holroyd, A. E. (2012) Finite volume Bootstrap Percolation with balanced threshold rules on $\mathbb{Z}^2$ http://www.ihes.fr/~duminil/

[14] Duminil-Copin, H. and van Enter, A. C. D. (2013) Sharp metastability threshold for an anisotropic bootstrap percolation model. *Ann. Probab.* **41** 1218–1242.

[15] Duminil-Copin, H., van Enter, A. C. D. and Hulshof, T. (2018) Higher order corrections for anisotropic bootstrap percolation. *Probab. Theory Related Fields* **172** 191–243.

[16] Fontes, L. R., Schonmann, R. H. and Sidoravicius, V. (2002) Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228** 495–518.

[17] Gravner, J. and Holroyd, A. E. (2008) Slow convergence in bootstrap percolation. *Ann. Appl. Probab.* **18** 909–928.

[18] Gravner, J., Holroyd, A. E. and Morris, R. (2012) A sharper threshold for bootstrap percolation in two dimensions. *Probab. Theory Related Fields* **153** 1–23.

[19] Harris, T. E. (1960) A lower bound for the critical probability in a certain percolation process. *Math. Proc. Cambridge Philos. Soc.* **26** 13–20.

[20] Hartarsky, I. and Morris R. (2018) The second term for two-neighbour bootstrap percolation in two dimensions. *Trans. Amer. Math. Soc.*

[21] Holroyd, A. E. (2003) Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224.

[22] Holroyd, A. E. (2006) The metastability threshold for modified bootstrap percolation in d dimensions. *Electron. J. Probab.* **11** 418–433.

[23] Morris, R. (2011) Zero-temperature Glauber dynamics on $\mathbb{Z}^d$. *Probab. Theory Related Fields* **149** 417–434.

[24] Schonmann, R. H. (1992) On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20** 174–193.

[25] van Enter, A. C. D. (1987) Proof of Straley's argument for bootstrap percolation. *J. Statist. Phys.* **48** 943–945.

[26] van Enter, A. C. D. and Fey, A. (2012) Metastability threshold for anisotropic bootstrap percolation in three dimensions. *J. Statist. Phys.* **147** 97–112.

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