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On lookahead equilibria in congestion games

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We investigate the issues of existence and efficiency of lookahead equilibria in congestion games. Lookahead equilibria, whose study has been initiated by Mirrokni *et al.* (2012), correspond to the natural extension of pure Nash equilibria in which the players, when making use of global information in order to predict subsequent reactions of the other ones, have computationally limited capabilities.

1. Introduction

The definition of the process of interaction among self-interested entities is dependent on the context, and in particular on the set of information available to the players. When they have very little knowledge about each others' costs and strategies, one of the most natural and studied dynamics are *sequential best-responses*, where players play sequentially and each player selects a strategy which is a best-response to the current strategy of the others. In such dynamics, the assumption is that each player has no memory about the past and no knowledge about the available strategies and costs of other players and, thus, myopically responds to the current state, without making any prediction about the consequences of the subsequent responses of the remaining players. One of the basic objective of study of game theory is the concept of equilibrium. An equilibrium can be viewed as a steady state of dynamics, where no agent has an incentive to unilaterally deviate from. The steady state of a best-response dynamics is known as *pure Nash equilibrium*. It is well known that the best-response dynamics do not always lead to a pure Nash equilibrium and that the class of congestion games (Rosenthal 1973) is a large class of games guaranteeing convergence under best-responses.

In this work, following the study initiated by Mirrokni *et al.* (2012), we focus on the settings in which each player has full knowledge of the strategies and costs of the other players, so that, based on such a knowledge, she can make predictions about the others' reactions to her move. We also assume that each player is an entity with limited computational abilities, thus she has the ability of making predictions only on the consequences of a fixed constant number of subsequent consecutive moves. In particular, we study the *k-lookahead dynamics* in which the players sequentially perform

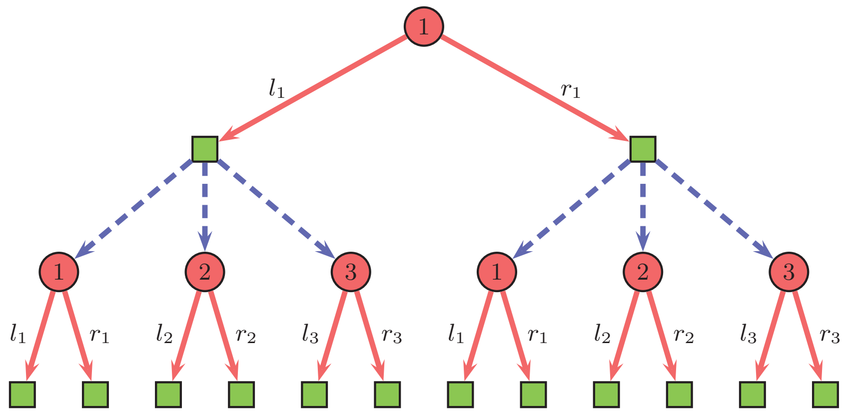

Fig. 1. $N = \{1,2,3\}$, $\Sigma_1 = \{l_1, r_1\}$, $\Sigma_2 = \{l_2, r_2\}$, $\Sigma_3 = \{l_3, r_3\}$.

k-lookahead best-responses. When $k = 1$, the k-lookahead best-response coincides with the best-response. In general, for $k > 1$, the current moving player p evaluates all possible outcomes resulting from $k - 1$ subsequent moves, by taking into account all possible orders in which players move and all of their possible strategies. We say that player p has a long-sightedness of k and she makes a prediction by assuming that any player moving $j < k$ steps after her has a long-sightedness of $k - j$. Thus, player p can compute her best move by backward induction starting from the players having long-sightedness of 1, and proceeding backward up to k. When predicting the strategy chosen by any player q having long sightedness $k - j$, it is necessary to make some assumption on which is the next moving player. We take into account two different models: the worst-case and the average-case ones. In the *worst-case model*, player p assumes that the next move after q is performed by a player providing player q the worst possible cost in the final outcome. In the *average-case model*, player p assumes that the next move after q is taken by a player selected uniformly at random. For each of these models, we finally distinguish between the cases of *consecutive* and *non-consecutive moves*, depending on whether player p assumes that the next move after q may be performed by q itself or not. In Figures 1 and 2, we show the differences between the consecutive and non–consecutive moves in the search tree of player 1 for a strategic game with three players $\{1,2,3\}$ with two strategies each, in the case of 2-lookahead. The square nodes are the player nodes and the round ones are the selection nodes.

We investigate the existence of k-lookahead equilibria and the price of anarchy (that measures the lack of optimality due to the con-cooperativeness of the players) of 2-lookahead equilibria in congestion games with linear latencies (Rosenthal 1973). Congestion games model the settings in which a set of players compete for the usage of a set of common resources. We choose congestion games as representative of a large set of well-studied games for which the existence of pure Nash equilibria is always guaranteed. Moreover, congestion games with linear latencies are able to model a large variety of practical scenarios (traffic routing on communication networks, load balancing, scheduling, competitive resource selection, P2P networks) in which the latency of a resource linearly

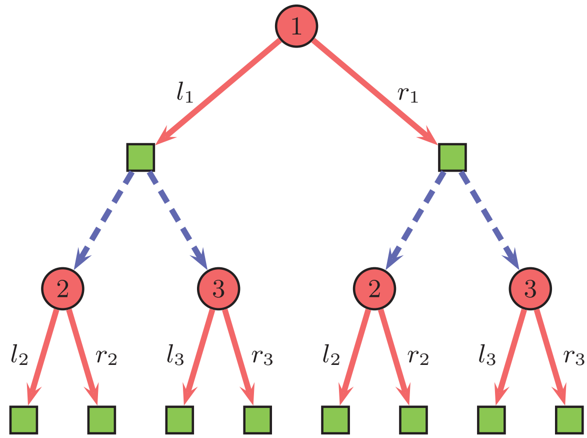

Fig. 2. $N = \{1, 2, 3\}$, $\Sigma_1 = \{l_1, r_1\}$, $\Sigma_2 = \{l_2, r_2\}$, $\Sigma_3 = \{l_3, r_3\}$.

increases as a function of the number of its users. The study of k-lookahead equilibria for small values of k is justified by practical motivations, since notions of equilibria, such as subgame perfect equilibria (Osborne and Rubinstein 1994), defined under the assumption that the players can predict a very high number of opponents moves, make even the computation of a player's single move a $PSPACE$-complete problem. To this aim, considering 2-lookahead equilibria constitutes a first step in the understanding of how a greater long-sightedness (and therefore a higher individual rationality) modifies the final outcomes resulting from the interactions of non-cooperative selfish players possessing limited computational abilities.

1.1. *Results*

In Section 3, we discuss our results on the existence of equilibria. We initially focus our attention on the existence of k-lookahead equilibria in strategic games. We are able to show that, in the worst-case model with consecutive moves, for any strategic game, any pure Nash equilibrium is also a k-lookahead equilibrium. This result implicitly shows that the k-lookahead best-responses do not guarantee better performance at equilibrium compared to those achieved by the simple best-responses. In the remainder of Section 3, we focus on the existence of 2-lookahead equilibria in singleton congestion games. We show that in the worst-case model without consecutive moves, any symmetric singleton game always admits 2-lookahead equilibria. For the average-case model, instead, we show that symmetric singleton congestion games do not always admit a 2-lookahead equilibrium regardless of whether consecutive moves are allowed or not.

In Section 4, we present the bounds on the price of anarchy for the 2-lookahead equilibria of linear congestion games, both in the worst-case and in the average-case model. We first show that, in the worst-case model, for any linear congestion game, the price of anarchy is at most 8. For the average-case model, we obtain smaller bounds. In particular, we show that, for any linear congestion game, the price of anarchy is at most 4. This result significantly improves the previous upper bound of $(1+\sqrt{5})^2 \approx 10.47$ given

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in Mirrokni *et al.* (2012). All mentioned bounds hold either with or without consecutive moves. We also show that, when restricting to singleton strategies, the price of anarchy drops to at most 4 in the worst-case model with or without consecutive moves.

1.2. *Related work*

The lookahead search was formally proposed by Shannon (1950), as a practical heuristic for machines to tackle difficult problems and play games. It is not surprising that Shannon applied the method to chess. More recently, the lookahead search has also been presented by Pearl (1984) in his book as the most important heuristic used by game-playing programs. Mirrokni *et al.* (2012) initiated the theoretical examination of the consequences of the decision making determined by the use of lookahead search. The authors formally quantify the deterioration of the outcome when players use lookahead search, by bounding the price of anarchy for several games among which are congestion games.

Our work is also related to many papers on congestion games. Congestion games have been introduced by Rosenthal (1973) and have been proved to be the only class of games admitting an exact potential function by Monderer and Shapley (1996). There is a long series of works investigating the price of anarchy with respect to the pure Nash equilibria (e.g. Aland *et al.* 1987; Bhawalkar *et al.* 2010; Bilò 2012; Christodoulou and Koutsoupias 2005) and studying the best-response and approximate improvement dynamics (e.g. Awerbuch *et al.* 2008; Caragiannis *et al.* 2011; Chien and Sinclair 2007; Fanelli *et al.* 2012; Fanelli and Moscardelli 2011) for congestion games.

2. Model and preliminaries

Definition 2.1 (congestion game, strategies and delay functions). A *congestion game* $\mathcal{G} = (N, E, (\Sigma_i)_{i\in N}, (f_e)_{e\in E}, (c_i)_{i\in N})$ is a non-cooperative strategic game defined by a set E of resources and a set $N = \{1, \ldots, n\}$ of players sharing resources in E.

Any *strategy* $s_i \in \Sigma_i$ of player i is a non-empty subset of resources, i.e. $\emptyset \neq \Sigma_i \subseteq 2^E$. Given a strategy profile $S = (s_1, \ldots, s_n)$ and a resource e, the number of players using e in S, called the congestion on e, is denoted by $n_e(S) = |\{i \in N : e \in s_i\}|$.

A *delay function* $f_e : \mathbb{N} \mapsto \mathbb{R}_+$ associates to resource e a delay depending on the number of players currently using e, so that the cost of player i for the pure strategy s_i is given by the sum of the delays associated with resources in s_i, i.e. $c_i(S) = \sum_{e\in s_i} f_e(n_e(S))$.

We refer to *singleton* congestion games as the games in which all of the players' strategies consist of only a single resource.

In this paper, we will focus on *linear* congestion games, that is having linear delay functions with nonnegative coefficients. More precisely, for every resource $e \in E$, $f_e(x) = \alpha_e x + \beta_e$ with $\alpha_e, \beta_e \geqslant 0$.

Definition 2.2 (social cost). Given the strategy profile $S = (s_1, \ldots, s_n)$, the social cost $C(S)$ of S is defined as the sum of all the players' costs, i.e. $C(S) = \sum_{i\in N} c_i(S) = \sum_{i\in N}\sum_{e\in s_i}(\alpha_e n_e(S) + \beta_e) = \sum_{e\in E}\left(\alpha_e n_e(S)^2 + \beta_e n_e(S)\right)$. An optimal strategy profile $S^* = (s_1^*, \ldots, s_n^*)$ is one with minimum social cost.

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Before introducing the notions of k-lookahead best-response and k-lookahead equilibrium, we briefly define their classical correspondent notions of the best-response and Nash equilibrium.

Each player acts selfishly and aims at choosing the strategy lowering her cost. Given a strategy profile S and a strategy $s_i' \in \Sigma_i$, denote with $S \oplus_i s_i' = (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$ the strategy profile obtained from S if player i changes her strategy from s_i to s_i'.

Definition 2.3 (best response). A *best-response* of player i in S is a strategy $s_i^b \in \Sigma_i$ yielding the minimum possible cost, given the strategic choices of the other players, i.e. $c_i(S \oplus_i s_i^b) \leqslant c_i(S \oplus_i s_i')$ for any other strategy $s_i' \in \Sigma_i$.

Definition 2.4 (Nash equilibrium). A (pure) *Nash equilibrium* is a strategy profile in which every player plays a best-response. Given a strategic game $\mathcal{G}$, we denote as $\mathcal{NE}(\mathcal{G})$ the set of its pure Nash equilibria.

We assume that each player, in order to determine her k-lookahead best-response, exploits k-lookahead search, i.e. she predicts $k-1$ consecutive possible reactions to her move, and selects the best choice according to such a prediction, as shown in the following. More formally, when performing a move starting from a given strategy profile S, player i considers a directed tree game $\mathcal{T} = (V_{\mathcal{T}}^{\text{odd}} \cup V_{\mathcal{T}}^{\text{even}}, A_{\mathcal{T}}^{\text{odd}} \cup A_{\mathcal{T}}^{\text{even}})$ of depth $2k-1$ in which odd levels (with the root belonging to level 1) contain *player nodes* belonging to $V_{\mathcal{T}}^{\text{odd}}$ and even levels contain *selection nodes* belonging to $V_{\mathcal{T}}^{\text{even}}$. Arcs outgoing from nodes in $V_{\mathcal{T}}^{\text{odd}}$ ($V_{\mathcal{T}}^{\text{even}}$, respectively) belong to $A_{\mathcal{T}}^{\text{odd}}$ ($A_{\mathcal{T}}^{\text{even}}$, respectively). Each node $v \in V_{\mathcal{T}}^{\text{odd}}$ is associated to a player $p(v)$ performing an action, with the root being associated to player i, and each arc a outgoing from node v is associated to her strategy $st(a) \in \Sigma_{p(v)}$; there is an outgoing arc for each strategy of player $p(v)$. Each selection node $v \in V_{\mathcal{T}}^{\text{even}}$ is associated to a strategy profile S_v that is obtained in the following way: Initially, S_v is set equal to S. Now, consider the path connecting the root of $\mathcal{T}$ to v; starting from the root, for every arc (u, u') of such a path belonging to $A_{\mathcal{T}}^{\text{odd}}$, S_v is updated to $S_v \oplus_{p(u)} st((u, u'))$. In this paper, we consider two different settings, depending on whether consecutive moves by a same player are allowed or not in the search tree. In the setting *allowing consecutive moves* by the same player, each selection node has n outgoing arcs, one for each player; in the setting in which they are not allowed, each selection node has $n-1$ outgoing arcs.

We assume that, in the k-lookahead search of player i, a player corresponding to a node of level $2j-1$ in $\mathcal{T}$ (for $j = 1, \ldots, k$), has a long-sightedness equal to $k+1-j$ (player i performs a k-lookahead search, the player moving after her a $(k-1)$-lookahead search and so on).

A k-lookahead best-response can be computed by backward induction on the levels of tree $\mathcal{T}$. First of all, it can be computed under two different models:

— The *worst-case* model, in which each player assumes that the subsequent move is performed by a player providing her the worst possible cost in the final leaf of tree $\mathcal{T}$.

— The *average case* model, in which the player moving at each step is assumed to be selected uniformly at random.

Notice that the worst-case model is a more 'pessimistic' and 'prudent' than the average one, because the strategic choice is performed by taking into account that the next moving player is the one providing her the worst possible cost in the final leaf of tree $\mathcal{T}$.

Moreover, for both models we can consider the two settings in which consecutive moves are or are not allowed.

The basis of the induction is the selection of an arc (marked as red) for each node of the last level of $\mathcal{T}$ (being the odd level $2k-1$): for each node v of this last level, the base case reduces to the selection of a 1-lookahead best-response for player $p(v)$ (i.e. a classical best-response to strategy profile S_v); ties are resolved such that player i's cost in the final strategy profile is maximized.

For each $j \geqslant 1$, given that some outgoing arcs for levels $j+2, \dots, 2k-1$ have been marked as red, we now show how to mark as red an outgoing arc for each node of level j (being an odd level) and, only in the worst-case model, how to mark as red one arc of level $j+1$ (being an even level). In fact, in the average case model, all arcs of level $j+1$ are always marked as red. Given any node v of the odd levels in $\{j+2, \dots, 2k-1\}$, let $Lf(v)$ be the (maximal) set of leaves of $\mathcal{T}$ such that there exists a path of red arcs going from v to a node in $Lf(v)$. Note that for any node v, $|Lf(v)| = 1$ under the worst-case model. Under the worst-case model, a $\left(k - \frac{j-1}{2}\right)$-lookahead best-response for player $p(v)$ (with v being a node of level j) is performed by marking as red an arc (v, v') outgoing from v such that the value $c_{p(v)}(S_u)$, with $u \in Lf(v)$ (notice that, under the worst-case model, $|Lf(v)| = 1$), is minimized taking into account that the worst-case (for player $p(v)$) arc outgoing from v' is also marked as red; ties are resolved such that player i's cost in the final strategy profile is maximized. Under the average model, a $\left(k - \frac{j-1}{2}\right)$-lookahead best-response for player $p(v)$ (with v being a node of level j) is performed by marking as red an arc (v, v') outgoing from v such that the average among values $c_{p(v)}(S_u)$ overall $u \in Lf(v)$ is minimized; again, ties are resolved such that player i's cost in the final strategy profile is maximized.

Suppose that, in a k-lookahead best-response dynamics, player j moves after player i. It is worth noticing that the move performed by j may not be the move anticipated by player i in her own analysis (at the corresponding node of level 3 of $\mathcal{T}$), because in such an analysis of player i, player j was performing a $(k-1)$-lookahead search, while when moving after player i in the 'actual' evolution of the game, she is performing a k-lookahead search.

Definition 2.5 (k-lookahead equilibrium). A *k-lookahead equilibrium*, under the worst or average case model and with or without consecutive moves allowed, is a strategy profile in which every player plays a k-lookahead best-response (under the same setting). Notice that a 1-lookahead best-response corresponds to the classical best-response, and a 1-lookahead equilibrium to a Nash equilibrium.

Definition 2.6 (k-lookahead *price of anarchy*). The k-lookahead *price of anarchy* of a game $\mathcal{G}$, under the worst or average case model and with or without consecutive moves allowed, is the worst-case ratio between the social cost of a k-lookahead Nash equilibrium (under the same setting) and that of an optimal strategy profile, that is, $\mathsf{PoA}(\mathcal{G}) = \max_{S \in \mathcal{LE}_k(\mathcal{G})} \frac{C(S)}{C(S^*)}$, where $\mathcal{LE}_k(\mathcal{G})$ denotes the set of k-lookahead Nash equilibria of $\mathcal{G}$. Roughly speaking, given a social function to be optimized, the price of anarchy measures the degradation of the quality of a game solution with respect to the optimal solution.

3. Existence of lookahead equilibria

We show that, in the worst-case model with consecutive moves, the set of pure Nash equilibria of $\mathcal{G}$ is contained in the set of k-lookahead equilibria of $\mathcal{G}$ for any value of k. This result has a double implication when considering the worst-case model with consecutive moves: from one hand, it shows existence of lookahead equilibria in each game admitting pure Nash equilibria and, from the other hand, it tells us that the price of anarchy can only worsen when moving from the classical definition of myopic rationality to the one based on lookahead search.

Theorem 3.1. For any strategic game $\mathcal{G}$ and for any index $k \geqslant 1$, it holds $\mathcal{NE}(\mathcal{G}) \subseteq \mathcal{LE}_k(\mathcal{G})$ in the worst-case model with consecutive moves.

Proof. First of all, note that, if $\mathcal{G}$ does not possess pure Nash equilibria, then, by definition, $\emptyset = \mathcal{NE}(\mathcal{G}) \subseteq \mathcal{LE}_k(\mathcal{G})$ for any index $k \geqslant 1$ and we are done. Hence, for the remaining on the proof, assume that $\mathcal{NE}(\mathcal{G}) \neq \emptyset$. The proof is by induction on $k \geqslant 1$. Note that, the basic case of $k = 1$ holds by definition since the set of 1-lookahead equilibria coincides with that of pure Nash equilibria. Hence, we only need to show the inductive step.

For any index $k \geqslant 2$ assume, for the sake of induction, that $\mathcal{NE}(\mathcal{G}) \subseteq \mathcal{LE}_j(\mathcal{G})$ for each index j such that $1 \leqslant j \leqslant k-1$. Consider a pure Nash equilibrium $S \in \mathcal{NE}(\mathcal{G})$ and a player i. If i does not change her strategy, then, since S is a $(k-1)$-lookahead Nash equilibrium for $\mathcal{G}$, no player possesses a $(k-1)$-lookahead improving deviation in S and so, the resulting state of i's search tree is S, where i pays $c_i(S)$. If i changes her strategy to s_i', let $S' = S \oplus_i s_i'$ be the resulting state. It holds $c_i(S') \geqslant c_i(S)$ since S is a pure Nash equilibrium for $\mathcal{G}$. Note that, if the adversary always selects i for the successive $k-1$ moves, the game can never reach a state in which i pays less than $c_i(S)$ (if such a deviation existed, it would contradict the fact that S is a pure Nash equilibrium for $\mathcal{G}$). It follows that, after player i's deviation, the adversary can always select a sequence of player so as to generate a final state S'' such that $c_i(S'') \geqslant c_i(S)$. Hence, i does not possess any k-lookahead improving deviation from S and the claim is proved. □

For the worst-case model without consecutive moves, we show existence of 2-lookahead Nash equilibria in symmetric singleton congestion games, that is, singleton games in which all players share the same set of strategies.

Theorem 3.2. Any symmetric singleton congestion game always admits 2-lookahead Nash equilibria in the worst-case model without consecutive moves.

Proof. Fix a symmetric singleton game $\mathcal{G}$ and consider the following two cases.

Case 1. $\mathcal{G}$ admits a pure Nash equilibrium S such that there exists two resources with a congestion of at least 2. Consider a player i, using a resource e, whose cost is $c_i(S)$. Since S is a pure Nash equilibrium for $\mathcal{G}$, if i migrates to another resource e', she gets a cost of at least $c_i(S)$. If the adversary selects a player j currently using a resource different than e', the current cost of player i cannot decrease. Since player j always exists under our hypothesis, S has to be a 2-lookahead Nash equilibrium.

Case 2. $\mathcal{G}$ admits a pure Nash equilibrium S such that there exists three resources with a congestion of at least 1. With a similar argument as in the previous case, it is possible to show that S has to be a 2-lookahead Nash equilibrium.

If both Cases 1 and 2 do not occur, then, there exists a pure Nash equilibrium S of $\mathcal{G}$ in which there are two resources e and e' with $n_e(S) = 1$ and $n_{e'}(S) = n-1$, $f_e(2) > f_{e'}(n-1)$ and $f_{e''}(1) > f_{e'}(n-1)$ for each $e'' \in E \setminus \{e, e'\}$. Note also that, since S is a pure Nash equilibrium, $f_e(1) \leqslant f_{e''}(1)$ for each $e'' \in E \setminus \{e, e'\}$.

Consider the strategy profile S' such that $n_{e'}(S') = n$. We claim that either S or S' is a 2-lookahead Nash equilibrium.

If S' is a pure Nash equilibrium for $\mathcal{G}$, then it is also a 2-lookahead Nash equilibrium. Hence, we can assume that $f_e(1) < f_{e'}(n)$. Consider any player: If she does not change her strategy, then, no matter which is the other player selected by the adversary, she ends up paying $f_{e'}(n-1)$. If she changes her strategy, then, no matter which is the other player selected by the adversary, she ends up paying at least $f_e(1)$. Thus, S' is a 2-lookahead Nash equilibrium when $f_{e'}(n-1) \leqslant f_e(1)$.

On the other hand, since S is a pure Nash equilibrium, player i using resource e in S, ends up paying $f_e(1)$ when not changing her strategy, while any player j using resource e' in S ends up paying $f_{e'}(n-1)$ when not changing her strategy. If player i changes her strategy, no matter which is the other player selected by the adversary, she ends up paying at least $\min\{f_e(1), f_{e'}(n-1)\}$. If any player j changes her strategy, she ends up paying at least $\min\{f_e(2), f_{e''}(1)\}$. Thus, S is a 2-lookahead Nash equilibrium when $f_{e'}(n-1) \geqslant f_e(1)$ and this concludes the proof. □

For the average-case model, we show that there exists a very simple game $\mathcal{G}$ with 4 symmetric players and 3 singleton strategies admitting no 2-lookahead Nash equilibria independently of whether consecutive moves are allowed or not.

Theorem 3.3. In both variants of the average-case model, no 2-lookahead Nash equilibria are guaranteed to exist even in symmetric singleton games.

Proof. Let $\mathcal{G}$ be the symmetric singleton game in which there are four players and three resources, namely e_1, e_2 and e_3 such that $f_{e_1}(x) = 6x$, $f_{e_2}(x) = 7x$ and $f_{e_3}(x) = 10x + \epsilon$, where $\epsilon > 0$ is an arbitrarily small quantity. We show, by inspection, that $\mathcal{G}$ does not admit any 2-lookahead Nash equilibrium in both variants of the average-case model.

Assume, by contradiction, that a 2-lookahead Nash equilibrium S exists. We divide the proof into the following cases.

Case 1. S is such that there exists a resource e with congestion 4, i.e. all players share the same resource. In the variant with consecutive moves, the expected cost of any player i when playing e is at least $\frac{1}{4}6 + \frac{3}{4}18 = 15$. If player i switches to another resource $e' \neq e$, her expected cost is at most $10+\epsilon$. In the variant without consecutive moves, the expected cost of any player i when playing e is at least 18. If player i switches to another resource

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$e' \neq e$, her expected cost is at most $10+\epsilon$. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 2. S is such that there exists a resource e with congestion 3. In the variant with consecutive moves, the expected cost of any player i when playing e is at least $\frac{1}{4}6+\frac{1}{4}18+\frac{1}{2}12=12$. If player i switches to another resource $e' \neq e$, her expected cost is at most $\frac{1}{2}(10+\epsilon)+\frac{1}{2}12<12$. In the variant without consecutive moves, the expected cost of any player i when playing e is at least $\frac{1}{3}18+\frac{2}{3}12=14$. If player i switches to another resource $e' \neq e$, her expected cost is at most $\frac{1}{3}(10+\epsilon)+\frac{2}{3}12<14$. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 3. S is such that there exists two resources e and e' with congestion 2. Assume, without loss of generality, that e is more expensive than e'.

Case 3.1. $e=e_2$.

In the variant with consecutive moves, the expected cost of any player i when playing e is $\frac{1}{4}7+\frac{1}{4}(10+\epsilon)+\frac{1}{2}14>11$. If player i switches to e_3, her expected cost is $10+\epsilon$. In the variant without consecutive moves, the expected cost of any player i when playing e is $\frac{1}{3}7+\frac{2}{3}14>11$. If player i switches to e_3, her expected cost is $10+\epsilon$. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 3.2. $e \neq e_2 \Rightarrow e=e_3$.

In the variant with consecutive moves, the expected cost of any player i when playing e_3 is at least $\frac{1}{4}6+\frac{1}{4}(10+\epsilon)+\frac{1}{2}(20+\epsilon)>14$. If player i switches to the empty resource, her expected cost is at most 14. In the variant without consecutive moves, the expected cost of any player i when playing e_3 is $\frac{1}{3}(10+\epsilon)+\frac{2}{3}(20+\epsilon)>16$. If player i switches to the empty resource, her expected cost is at most 14. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 4. S is such that there exists only one resource e with congestion 2 (thus, both other resources have congestion 1).

Case 4.1. $e=e_1$.

In the variant with consecutive moves, the expected cost of any player i when playing e_1 is 12. If player i switches to e_2, her expected cost is $\frac{1}{4}12+\frac{1}{4}7+\frac{1}{2}14<12$. In the variant without consecutive moves, the expected cost of any player i when playing e_1 is 12. If player i switches to e_2, her expected cost is $\frac{1}{3}7+\frac{2}{3}14<12$. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 4.2. $e=e_2$.

In the variant with consecutive moves, the expected cost of player i playing e_3 is $10+\epsilon$. If player i switches to e_1, her expected cost is $\frac{1}{4}(10+\epsilon)+\frac{1}{4}6+\frac{1}{2}12=10+\frac{\epsilon}{4}$. In the variant without consecutive moves, the expected cost of player i playing e_3 is $10+\epsilon$. If player i switches to e_1, her expected cost is $\frac{1}{3}6+\frac{2}{3}12=10$. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium.

Case 4.3. $e=e_3$.

In the variant with consecutive moves, the expected cost of any player i when playing e_3 is $\frac{1}{4}12+\frac{1}{4}(10+\epsilon)+\frac{1}{2}20+\epsilon>15$. If player i switches to e_1, her expected cost is 12. In the variant without consecutive moves, the expected cost of any player i when playing e_3 is $\frac{1}{3}(10+\epsilon)+\frac{2}{3}(20+\epsilon)>16$. If player i switches to e_1, her expected cost is 12. In both cases, we get a contradiction to the fact that S is a 2-lookahead Nash equilibrium. □

4. Bounds on the price of anarchy

In this section, we give upper bounds on the price of anarchy of 2-lookahead Nash equilibria of linear congestion games both in the worst-case model and in the average-case model either with or without consecutive moves. To this aim, we use the primal-dual method introduced in Bilò (2012). Denoted with $K=(k_1,\ldots,k_n)$ and $O=(o_1,\ldots,o_n)$ the worst 2-lookahead Nash equilibrium and the social optimum, respectively, this method aims at formulating the problem of maximizing the ratio $\frac{C(K)}{C(O)}$ via linear programming. The two strategy profiles K and O play the role of fixed constants, while, for each $e \in E$, the values α_e and β_e defining the delay functions are variables that must be suitably chosen so as to satisfy two constraints: the first, assures that K is a 2-lookahead Nash equilibrium, while the second normalizes to 1 the value of the social optimum $C(O)$. The objective function aims at maximizing the social value $C(K)$ which, being the social optimum normalized to 1, is equivalent to maximizing the ratio $\frac{C(K)}{C(O)}$. Let us denote with $LP(K,O)$ such a linear program. By the weak duality theorem, each feasible solution to the dual program of $LP(K,O)$ provides an upper bound on the optimal solution of $LP(K,O)$. Hence, by providing a feasible dual solution, we obtain an upper bound on the ratio $\frac{C(K)}{C(O)}$. Anyway, if the provided dual solution is independent on the particular choice of K and O, we obtain an upper bound on the ratio $\frac{C(K)}{C(O)}$ for any possible pair of profiles K and O, which means that we obtain an upper bound on the price of anarchy of 2-lookahead Nash equilibria.

For the sake of brevity, throughout this section, for each $e \in E$, we set $K_e := n_e(K)$ and $O_e := n_e(O)$. Moreover, note that a simplificative argument widely exploited in the literature of linear congestion games states that we do not lose in generality by assuming $\beta_e = 0$ for each $e \in E$ (as long as we are not interested in singleton strategies). Finally, we denote by $c_i'(S,t)$ the cost that player i foresees in her search tree when selecting, in state S, strategy t.

4.1. *Worst-case model*

For the worst-case model without consecutive moves, for any player $i \in N$, strategy profile K and strategy $t \in \Sigma_i$, it holds

$$c_i'(K,k_i) \geqslant \sum_{e\in k_i}(\alpha_e K_e) - \sum_{e\in k_i : K_e\geqslant 2}\alpha_e \tag{1}$$

and

$$c_i'(K,t) \leqslant \sum_{e\in t}(\alpha_e(K_e+2)). \tag{2}$$

In fact, with 2-lookahead best-responses, when selecting strategy k_i, player i has to suffer, for every used resource e for which $K_e \geqslant 2$, a congestion at least equal to $K_e - 1$,

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where the decrease of one unit is due to the possibility that the player performing the next move could leave resource e; moreover, when selecting any strategy t, player i can suffer for every used resource e, a congestion at most equal to $K_e + 2$, where the increase of 2 units is due to the fact that player i is selecting e and also the player moving after her could select e.

For the case with consecutive moves, the same inequalities apply as well, since the fact that the adversary can also select again player i can only increase the cost $c_i'(K, k_i)$, whereas the value $\sum_{e \in t}(\alpha_e(K_e + 2))$ is already the maximum possible one that can be suffered by a migrating player in any model of 2-lookahead rationality.

Hence, for each player $i \in N$, since K is a 2-lookahead Nash equilibrium, by combining inequalities 1 and 2, it holds

$$\sum_{e \in k_i}(\alpha_e K_e) - \sum_{e \in k_i : K_e \geqslant 2} \alpha_e \leqslant \sum_{e \in o_i}(\alpha_e(K_e + 2)). \tag{3}$$

Such an inequality was already exploited in Mirrokni *et al.* (2012) in order to study the price of anarchy in the average-case model without consecutive moves. Anyway, as we will see later, in this case a more significant inequality can be derived. When embedded into the primal-dual technique, inequality (3) gives life to the following primal formulation $LP(K, O)$.

$$\begin{aligned}
&\textit{maximize} \sum_{e \in E}\left(\alpha_e K_e^2\right) \\
&\textit{subject to} \\
&\sum_{e \in k_i}(\alpha_e K_e) - \sum_{e \in k_i : K_e \geqslant 2} \alpha_e - \sum_{e \in o_i}(\alpha_e(K_e + 2)) \leqslant 0, \quad && \forall i \in N \\
&\sum_{e \in E}\left(\alpha_e O_e^2\right) = 1, \\
&\alpha_e \geqslant 0, && \forall e \in E
\end{aligned}$$

The dual program $DLP(K, O)$ is

$$\begin{aligned}
&\textit{minimize}\ \gamma \\
&\textit{subject to} \\
&\sum_{i : e \in k_i}(x_i(K_e - 1)) - \sum_{i : e \in o_i}(x_i(K_e + 2)) + \gamma O_e^2 \geqslant K_e^2, \quad && \forall e \in E : K_e \geqslant 2 \\
&\sum_{i : e \in k_i}(x_i K_e) - \sum_{i : e \in o_i}(x_i(K_e + 2)) + \gamma O_e^2 \geqslant K_e^2, && \forall e \in E : K_e < 2 \\
&x_i \geqslant 0, && \forall i \in N
\end{aligned}$$

Theorem 4.1. For any linear congestion game $\mathcal{G}$, it holds $\mathsf{PoA}(\mathcal{G}) \leqslant 8$ in the worst-case model.

Proof. We show the claim by proving that the dual solution such that $x_i = 2$ for each $i \in N$ and $\gamma = 8$ is feasible.

The first dual constraint becomes $f_1(K_e, O_e) \geqslant 0$ with $f_1(K_e, O_e) := K_e^2 - 2K_e(O_e+1) + 4O_e(2O_e - 1)$. It holds $f_1(K_e, 0) = K_e^2 - 2K_e$ which implies $f_1(K_e, 0) \geqslant 0$ for any $K_e \geqslant 2$. For $O_e \geqslant 1$, note that the discriminant of the equation $f_1(K_e, O_e) = 0$, when solved for K_e, is $1 + 6O_e - 7O_e^2$ which is always non-positive when $O_e \geqslant 1$. This implies that $f_1(K_e, O_e) \geqslant 0$ for each pair of real numbers (K_e, O_e) with $O_e \geqslant 1$. Hence, it follows that the first dual constraint is always verified for any pair of non-negative integers (K_e, O_e) with $K_e \geqslant 2$.

The second dual constraint becomes $f_2(K_e, O_e) \geqslant 0$ with $f_2(K_e, O_e) := K_e^2 - 2K_eO_e + 4O_e(2O_e - 1)$. Note that the discriminant of the equation $f_2(K_e, O_e) = 0$, when solved for K_e, is $4O_e - 7O_e^2$ which is always non-positive when $O_e \geqslant 0$. This implies that the second dual constraint is always verified for any pair of non-negative reals (K_e, O_e). □

4.2. *Average-case model*

For the average-case model without consecutive moves, for any player $i \in N$, strategy profile K and strategy $t \in \Sigma_i$, it holds

$$c_i'(K, k_i) \geqslant \sum_{e \in k_i} (\alpha_e K_e) - \sum_{e \in k_i : K_e \geqslant 2} \frac{\alpha_e(K_e - 1)}{n-1} \tag{4}$$

and

$$c_i'(K, t) \leqslant \sum_{e \in t} \left(\alpha_e \left(K_e + 2 - \frac{K_e}{n-1} \right) \right). \tag{5}$$

In fact, with 2-lookahead best-responses, when selecting strategy k_i, player i has to suffer, for every used resource e for which $K_e \geqslant 2$, a congestion at least equal to $K_e - 1$, where the decrease of one unit is due to the event, having probability at most $\frac{K_e - 1}{n-1}$, that the player performing the next move leave resource e (because such a player has to belong to the set of players selecting e in K); moreover, when selecting any strategy t, in order to evaluate the congestion player i can suffer on every used resource e, we have to distinguish between two different cases: (i) If player i is using resource e also in K, i.e. $e \in k_i$, she can suffer on e a congestion at most equal to $K_e + 1$, where the increase of one unit is due to the event, having probability at most $\frac{n - K_e}{n-1} = 1 - \frac{K_e - 1}{n-1}$, that also the player moving after i selects e (because such a player has not to belong to the set of players selecting e in K). (ii) If player i is not using resource e in K, i.e. $e \notin k_i$, she can suffer on e a congestion at most equal to $K_e + 2$, where the increase of one unit is due to the fact that player i is selecting e and the increase of another unit is due to the event, having probability at most $\frac{n-1-K_e}{n-1} = 1 - \frac{K_e}{n-1}$, that also the player moving after i selects e. Therefore, since $K_e + 2 - \frac{K_e}{n-1} \geqslant K_e + 1 - \frac{K_e - 1}{n-1}$, inequality 5 holds. Hence, for each player $i \in N$, since K is a 2-lookahead Nash equilibrium, by combining inequalities 4 and 5 it holds

$$\sum_{e \in k_i} (\alpha_e K_e) - \sum_{e \in k_i : K_e \geqslant 2} \frac{\alpha_e(K_e - 1)}{n-1} \leqslant \sum_{e \in o_i} \left(\alpha_e \left(K_e + 2 - \frac{K_e}{n-1} \right) \right). \tag{6}$$

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When embedded into the primal-dual technique, inequality (6) gives life to the following primal formulation $LP(K,O)$.

$$\begin{aligned}
&maximize \sum_{e\in E}\left(\alpha_e K_e^2\right)\\
&subject\ to\\
&\sum_{e\in k_i}(\alpha_e K_e) - \sum_{e\in k_i:K_e\geqslant 2}\frac{\alpha_e(K_e-1)}{n-1}\\
&\quad -\sum_{e\in o_i}\left(\alpha_e\left(K_e+2-\frac{K_e}{n-1}\right)\right)\leqslant 0, \qquad \forall i\in N\\
&\sum_{e\in E}\left(\alpha_e O_e^2\right)=1,\\
&\alpha_e\geqslant 0, \qquad \forall e\in E
\end{aligned}$$

The dual program $DLP(K,O)$ is

$$\begin{aligned}
&minimize\ \gamma\\
&subject\ to\\
&\sum_{i:e\in k_i}\left(x_i\left(K_e-\frac{K_e-1}{n-1}\right)\right)\\
&\quad -\sum_{i:e\in o_i}\left(x_i\left(K_e+2-\frac{K_e}{n-1}\right)\right)+\gamma O_e^2\geqslant K_e^2, \qquad \forall e\in E: K_e\geqslant 2\\
&\sum_{i:e\in k_i}(x_i K_e)-\sum_{i:e\in o_i}\left(x_i\left(K_e+2-\frac{K_e}{n-1}\right)\right)+\gamma O_e^2\geqslant K_e^2, \qquad \forall e\in E: K_e<2\\
&x_i\geqslant 0, \qquad \forall i\in N
\end{aligned}$$

The following result significantly improves the previous upper bound of $(1+\sqrt{5})^2\approx 10.47$ (Mirrokni *et al.* 2012).

Theorem 4.2. For any linear congestion game $\mathcal{G}$, it holds $\mathsf{PoA}(\mathcal{G})\leqslant 4$ in the average-case model without consecutive moves.

Proof. For $n=2$, we show that the dual solution such that $x_i=2$ for each $i\in N$ and $\gamma=4$ is feasible. The first dual constraint, since $n=2$ implies $K_e=2$, becomes $O_e(O_e-1)\geqslant 0$ which is always satisfied for any integer value O_e. The second constraint becomes $K_e^2+4O_e(O_e-1)\geqslant 0$ which is always satisfied for any integer value O_e when $K_e\in\{0,1\}$.

For $n\geqslant 3$, we show that the dual solution such that $x_i=\frac{3(n-1)}{2n-3}$ for each $i\in N$ and $\gamma=4$ is feasible.

The first dual constraint becomes $\frac{f_1(K_e,O_e)}{2n-3}\geqslant 0$ with $f_1(K_e,O_e):=K_e^2(n-3)-3K_e(nO_e-2O_e-1)+2O_e(4nO_e-3n-6O_e+3)$. Since $2n-3>0$ for any $n\geqslant 3$, we need to show that $f_1(K_e,O_e)\geqslant 0$ for any pair of non-negative integers (K_e,O_e) with $K_e\geqslant 2$ when $n\geqslant 3$. It holds $f_1(K_e,0)=K_e^2(n-3)+3K_e$ which implies $f_1(K_e,0)\geqslant 0$ for any $K_e\geqslant 2$

when $n \geqslant 3$, moreover, $f_1(K_e, 1) = (K_e^2 - 3K_e + 2)(n-3)$ which implies $f_1(K_e, 1) \geqslant 0$ for any integer K_e when $n \geqslant 3$. The discriminant of the equation $f_1(K_e, O_e) = 0$, when solved for K_e, is $n^2O_e(24 - 23O_e) + 6nO_e(18O_e - 19) - 9(12O_e^2 - 12O_e - 1)$. Note that $-9(12O_e^2 - 12O_e - 1) \leqslant 0$ when $O_e \geqslant 2$ and $n^2O_e(24 - 23O_e) + 6nO_e(18O_e - 19) \leqslant 0$ when $O_e \geqslant 2$ and $n \geqslant 5$. For $n = 4$, the discriminant becomes $-44O_e^2 + 36O_e + 9$ which is always non-positive when $O_e \geqslant 2$. Finally, for $n = 3$, $f_1(K_e, O_e)$ becomes $(K_e - 4O_e)(1 - O_e)$ which is always non-negative since $O_e \geqslant 2$ and $K_e \leqslant n = 3$. Hence, it follows that the first dual constraint is always verified for any pair of non-negative integers (K_e, O_e) with $K_e \geqslant 2$ when $n \geqslant 3$.

The second dual constraint becomes $\frac{f_2(K_e,O_e)}{2n-3} \geqslant 0$ with $f_2(K_e, O_e) := K_e^2n - 3K_eO_e(n-2) + 2O_e(4nO_e - 3n - 6O_e + 3)$. It holds $f_2(0, O_e) = O_e(4nO_e - 3n - 6O_e + 3)$ which is always non-negative for any integer value O_e when $n \geqslant 3$ and $f_2(1, O_e) = n(8O_e^2 - 9O_e + 1) - 12O_e(O_e - 1)$ which is always non-negative for any integer value O_e when $n \geqslant 3$. Hence, it follows that the second dual constraint is always verified for any pair of non-negative integers (K_e, O_e) with $K_e < 2$ when $n \geqslant 3$. □

For the average-case model with consecutive moves, for any player $i \in N$, strategy profile K, strategy $t \in \Sigma_i$ and best-response t^* for player i in K, it holds

$$c_i'(K, k_i) \geqslant \frac{1}{n} c_i(K \oplus_i t^*) + \frac{n-1}{n} \left(\sum_{e \in k_i} (\alpha_e K_e) - \sum_{e \in k_i : K_e \geqslant 2} \frac{\alpha_e (K_e - 1)}{n-1} \right),$$

and

$$c_i'(K, t) \leqslant \frac{1}{n} c_i(K \oplus_i t^*) + \frac{n-1}{n} \sum_{e \in t} \left(\alpha_e \left(K_e + 2 - \frac{K_e}{n-1} \right) \right).$$

In fact, if consecutive moves are allowed, with probability $\frac{1}{n}$ a 2-lookahead best-response of player i coincides with a classical best-response, and with probability $\frac{n-1}{n}$ the same arguments exploited for the case without repetitions apply.

Hence, the same inequality characterizing the case without repetition occurs also in this case and we can claim the following theorem.

Theorem 4.3. For any linear congestion game $\mathcal{G}$, it holds $\mathsf{PoA}(\mathcal{G}) \leqslant 4$ in the average-case model with consecutive moves.

4.3. *Singleton strategies*

In this subsection, we show that better results can be achieved for the worst-case model when restricting to singleton linear congestion games.

For the worst-case model without consecutive moves, for any player $i \in N$, strategy profile K and strategy $t \in \Sigma_i$, it holds, unless all the players share the same resource in K,

$$c_i'(K, k_i) \geqslant \sum_{e \in k_i} (\alpha_e K_e + \beta_e), \tag{7}$$

and

$$c_i'(K, t) \leqslant \sum_{e \in t} (\alpha_e (K_e + 2) + \beta_e). \tag{8}$$

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In fact, with 2-lookahead best-responses, when selecting strategy k_i consisting of resource e, in the worst-case model the adversary can always select a player not selecting e for the next move (unless all the players share the same resource in K); moreover, when selecting any strategy t consisting of resource e', player i can suffer a congestion at most equal to $K_{e'} + 2$, where the increase of 2 units is due to the fact that player i is selecting e' and also the player moving after her could select e'.

For the case with consecutive moves, the same inequalities apply as well, since the fact that the adversary can also select again player i can only increase the cost $c_i'(K, k_i)$, whereas the value $\sum_{e\in t}(\alpha_e(K_e+2)+\beta_e)$ is already the maximum possible one that can be suffered by a migrating player in any model of 2-lookahead rationality.

Hence, for each player $i \in N$, since K is a 2-lookahead Nash equilibrium, by combining inequalities (7) and (8), it holds

$$\sum_{e\in k_i}(\alpha_e K_e + \beta_e) \leqslant \sum_{e\in o_i}(\alpha_e(K_e+2)+\beta_e), \tag{9}$$

unless all the players share the same resource in K.

When embedded into the primal-dual technique, inequality (9) gives life to the following primal formulation $LP(K, O)$.

$$\begin{aligned}
&\textit{maximize} \sum_{e\in E}\left(\alpha_e K_e^2 + \beta_e K_e\right) \\
&\textit{subject to} \\
&\sum_{e\in k_i}(\alpha_e K_e + \beta_e) - \sum_{e\in o_i}(\alpha_e(K_e+2)+\beta_e) \leqslant 0, && \forall i \in N \\
&\sum_{e\in E}\left(\alpha_e O_e^2 + \beta_e O_e\right) = 1, \\
&\alpha_e \geqslant 0, && \forall e \in E
\end{aligned}$$

The dual program $DLP(K, O)$ is

$$\begin{aligned}
&\textit{minimize}\ \gamma \\
&\textit{subject to} \\
&\sum_{i:e\in k_i}(x_i K_e) - \sum_{i:e\in o_i}(x_i(K_e+2)) + \gamma O_e^2 \geqslant K_e^2, && \forall e \in E \\
&\sum_{i:e\in k_i} x_i - \sum_{i:e\in o_i} x_i + \gamma O_e \geqslant K_e, && \forall e \in E \\
&x_i \geqslant 0, && \forall i \in N
\end{aligned}$$

Theorem 4.4. For any singleton linear congestion game $\mathcal{G}$, it holds $\mathsf{PoA}(\mathcal{G}) \leqslant 4$ in the worst-case model.

Proof. Set $x_i = 4/3$ for each $i \in N$ and $\gamma = 4$.

The first dual constraint becomes $f_1(K_e, O_e) \geqslant 0$ with $f_1(K_e, O_e) := K_e^2 - 4K_eO_e + 4O_e(3O_e - 2)$. The discriminant of the equation $f_1(K_e, O_e) = 0$, when solved for K_e, is

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$8O_e - 8O_e^2$ which is always non-positive when $O_e \geqslant 0$. Hence, it follows that the first dual constraint is always verified for any pair of non-negative reals (K_e, O_e).

The second dual constraint becomes $K_e^2 + 8O_e \geqslant 0$, which is always verified for any pair of non-negative reals (K_e, O_e).

In order to complete the proof, we have to show that, for each 2-lookahead Nash equilibrium K in which all players share the same resource e, it holds $C(K) \leqslant 4C(O)$. By the definition of K, for each player $i \in N$, it holds $\alpha_e(n-1) + \beta_e \leqslant 2\alpha_{o_i} + \beta_{o_i}$. By summing for each $i \in N$, we obtain

$$\alpha_e n(n-1) + \beta_e n \leqslant \sum_{i \in N} \left(2\alpha_{o_i} + \beta_{o_i}\right) \leqslant 2C(O).$$

By the fact that $n^2 \leqslant 2n(n-1)$ for each $n \geqslant 2$, it follows that $C(K) = \alpha_e n^2 + \beta_e n \leqslant 4C(O)$. □

For the average-case model, no improved bounds, with respect to the ones holding for the case of general strategies, seem possible using our analysis technique. However, the fact that, for singleton strategies, the upper bound on the price of anarchy in the worst-case model matches the one holding for the average-case model might appear counterintuitive. To this aim, in the following example, we show that this is not the case, since there are games with singleton strategies in which the performance of 2-lookahead Nash equilibria in the worst-case model are better than the one achieved in the average-case model.

Example 4.1. Let $\mathcal{G}$ be the symmetric singleton game in which there are three players and two resources, namely e_1 and e_2, such that $f_{e_1}(x) = \frac{4}{3}x$ and $f_{e_2}(x) = x$.

Let S be the strategy profile in which two players choose e_1 and one player chooses e_2 and consider the average-case model. In the variant with consecutive moves, the expected cost of any player i choosing e_1 is $\frac{1}{3}\left(2 + \frac{4}{3} + \frac{8}{3}\right) = 2$. If player i switches to resource e_2, her expected cost is 2. Moreover, the expected cost of the player choosing e_2 is $\frac{1}{3}(1 + 2 + 2) = \frac{5}{3}$. If she switches to resource e_1, her expected cost is $\frac{1}{3}\left(2 + \frac{8}{3} + \frac{8}{3}\right) = \frac{22}{9}$. In the variant without consecutive moves, the expected cost of any player i playing e_1 is $\frac{1}{2}\left(\frac{4}{3} + \frac{8}{3}\right) = 2$. If player i switches to resource e_2, her expected cost is 2. Moreover, the expected cost of the player choosing e_2 is $\frac{1}{2}(2 + 2) = 2$. If she switches to resource e_1, her expected cost is $\frac{1}{3}\left(\frac{8}{3} + \frac{8}{3}\right) = \frac{8}{3}$. Thus, in both variants of the average-case model, S is a 2-lookahead Nash equilibrium for $\mathcal{G}$.

Consider now the worst-case model.

First of all, we show that S is not a 2-lookahead Nash equilibrium for $\mathcal{G}$ in both variants of the model. In fact, in both variants, the cost of any player i choosing e_1 is $\frac{8}{3}$. If she switches to resource e_2, her cost is 2. Thus, in both variants, S is not a 2-lookahead Nash equilibrium for $\mathcal{G}$.

Now, let S' be the strategy profile in which one player chooses e_1 and two players choose e_2. In both variants, the cost of the player choosing e_1 is $\frac{4}{3}$. If she switches to resource e_2, her cost is 2. Moreover, in both variants, the cost of any player i choosing e_2 is 2. If she switches to resource e_1, her cost is $\frac{8}{3}$. Thus, in both variants, S' is a 2-lookahead Nash equilibrium for $\mathcal{G}$.

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Finally, it is not difficult to see that any profile in which all three players choose the same resource cannot be a 2-lookahead Nash equilibrium for $\mathcal{G}$, again in both variants.

Hence, since S' is the only 2-lookahead Nash equilibrium for $\mathcal{G}$ in the worst-case model, S is a 2-lookahead Nash equilibrium for $\mathcal{G}$ in the average-case model, and $C(S) > C(S')$, we can conclude that the price of anarchy of $\mathcal{G}$ in the average-case model is higher than the one of the worst-case model regardless of whether consecutive moves are allowed or not.

5. Conclusion

In this paper, we have investigated the existence of lookahead equilibria in congestion games and we have provided upper bounds to the performance of 2-lookahead equilibria in linear congestion games under several settings.

The main open problem is that of studying the performance of k-lookahead equilibria with $k > 2$. In particular, it would be interesting to understand how the performance of k-lookahead equilibria, for increasing values of k, is related to the one of subgame perfect equilibria, that is known to be, in many settings, worst than that of Nash equilibria (Bilò *et al.* 2015).

Moreover, finding significant lower bounds to the price of anarchy appears to be a challenging question even for 2-lookahead equilibria: providing such bounds calls for future investigation.

Finally, investigating the performance of equilibria induced by farsighted players in the context of other delay functions or other games constitutes another interesting research direction.

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