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A NOTE ON THE AXISYMMETRIC DIFFUSION EQUATION

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Abstract

We consider the explicit solution to the axisymmetric diffusion equation. We recast the solution in the form of a Mellin inversion formula, and outline a method to compute a formula for $u(r, t)$ as a series using the Cauchy residue theorem. As a consequence, we are able to represent the solution to the axisymmetric diffusion equation as a rapidly converging series.

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1. Introduction and the main results

The axisymmetric diffusion equation is given as [3, p. 61]

$$\kappa\nabla^2 u \equiv \kappa\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) = \frac{\partial u}{\partial t}, \tag{1.1}$$

where $t > 0, r \in (0, \infty), u(r, 0) = g(r)$ and κ is the positive diffusivity constant. The boundary conditions $u \to 0$, $\partial u/\partial r \to 0$ as $r \to \infty$ are also assumed. The Hankel transform of a function $f(x)$ is defined as [3, p. 58, equation (1.10.1)]

$$\mathfrak{H}(f(y))(x) := \int_0^\infty yJ_0(xy)f(y)\,dy.$$

We may temporarily drop the integrating variable in denoting integral transforms according to when the context is appropriate throughout. The known explicit solution is obtained by taking Hankel transform of (1.1), which gives

$$\frac{\partial}{\partial t}\mathfrak{H}(u(r,t))(x) + x^2\kappa\mathfrak{H}(u(r,t))(x) = 0 \tag{1.2}$$

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with initial condition $\mathfrak{H}(u(r,0))(x) = \mathfrak{H}(g(r))$. Applying the inverse Hankel transform $\mathfrak{H}^{-1}$ to (1.2) gives the explicit solution [3, p. 62, equation (1.10.25)]

$$u(r,t) = \frac{1}{2\kappa t} e^{-r^2/(4\kappa t)} \int_0^\infty y g(y) I_0\Big(\frac{yr}{2\kappa t}\Big) e^{-y^2/(4\kappa t)}\, dy, \tag{1.3}$$

where the modified Bessel function of the first kind is given by

$$I_v(x) = \sum_{n\geq 0} \frac{1}{n!\,\Gamma(v+n+1)} \Big(\frac{x}{2}\Big)^{2n+v}.$$

Some simple examples include the bell-shaped temperature profile $g(r) = e^{-cr^2}$ or the uniform temperature profile $g(r) = 1$ on $(0,1)$. In both these instances it is a simple task to appeal to the tables for integral transforms.

The purpose of this note is to provide further analysis of (1.3) by means of Mellin inversion [6, p. 80]. In applying methods from Olver [5], we can better understand $u(r,t)$ by providing a method to obtain an infinite series representation involving Laguerre polynomials or a hypergeometric function. One of our motivations for selecting Bessel functions for initial conditions is to apply Watson's lemma [5, p. 336, equations (6.01) and (6.02)]. Indeed, coefficients for the power series representations of Bessel functions are well known and, as a consequence, we may obtain the asymptotic behaviour related to $u(r,t)$. An example of this method will be provided in our last section, which should be compared to our rapidly convergent series obtained in our main theorems. For a general overview of applying Mellin transforms to evaluating integrals involving Bessel functions, see [8, p. 196]. For a recent example of applying Mellin transforms to analyse partial differential equations, see the paper by Boyadjiev and Luchko [2].

Recall that the Mellin transform [6] is given by

$$\mathfrak{M}(g)(s) := \int_0^\infty y^{s-1} g(y)\, dy.$$

Parseval's identity is [6, p. 83, equation (3.1.11)]

$$\int_0^\infty k(y)g(y)\, dy = \frac{1}{2\pi i} \int_{(c)} \mathfrak{M}(k)(s)\mathfrak{M}(g)(1-s)\, ds, \tag{1.4}$$

where (c) is the vertical line where the integrand is analytic.

Recall from [4, p. 709, equation (6.643), #2] (with change of variables $x \to x^2$ and $\mu = s/2$) that

$$\int_0^\infty y^s e^{-\alpha y^2} I_{2v}(2\beta y)\, dy = \frac{\Gamma(s/2+v+1/2)e^{\beta^2/(2\alpha)}}{2\Gamma(2v+1)\beta} \alpha^{-s/2} M_{-s/2,v}\Big(\frac{\beta^2}{\alpha}\Big), \tag{1.5}$$

valid for $\Re(s/2+v+1/2) > 0$.

Here $M_{\mu,\nu}(x)$ is the Whittaker hypergeometric function [4, p. 1024]

$$M_{\mu,\nu}(x) = x^{\nu+1/2}e^{-x/2}{}_1F_1(\nu - \mu + \tfrac{1}{2}; 2\nu + 1; x) \tag{1.6}$$

and ${}_1F_1(a; b; x)$ is the confluent hypergeometric function.

THEOREM 1.1. *If* $\mathfrak{M}(g)(1-s)$ *is analytic in a subset* S *of the region* $\{s \in \mathbb{C} \mid \mathfrak{R}(s) > -1\}$, *then*

$$u(x,t) = \frac{1}{r}e^{-(r^2 - r^2/2)/(4\kappa t)}\frac{1}{2\pi i}\int_{(c)}\Gamma\Big(\frac{s}{2}+\frac{1}{2}\Big)(4\kappa t)^{s/2}M_{-s/2,0}\Big(\frac{r^2}{4\kappa t}\Big)\mathfrak{M}(g)(1-s)\,ds,$$

$c \in S \cap \{s \in \mathbb{C} \mid \mathfrak{R}(s) > -1\}$.

PROOF. We choose the $k(y)$ to be the integrand in (1.5) with $\nu = 0, \alpha = 1/4\kappa t, \beta = r/4\kappa t$ and apply (1.4). □

Some relevant notes are in order to apply Theorem 1.1. First, Theorem 1.1 requires that $\mathfrak{M}(g)(s)$ is analytic in the region $\{s \in \mathbb{C} \mid \mathfrak{R}(s) < 2\}$. It is known that $M_{\mu,\nu}(x)$ only has simple poles for fixed μ and x at $\nu = -(k+1)/2, k \in \mathbb{N}$. By [4, p. 1026, 9.228],

$$M_{\mu,\nu}(x) \sim \frac{1}{\sqrt{\pi}}\Gamma(2\nu+1)\mu^{-\nu-1/2}x^{1/4}\cos\Big(2\sqrt{\mu x} - \nu\pi - \frac{\pi}{4}\Big)$$

as $|\mu| \to \infty$ and further we have the functional relationship [3, p. 1026, equation (9.231), #2],

$$x^{-1/2-\nu}M_{\mu,\nu}(x) = (-x)^{-1/2-\nu}M_{-\mu,\nu}(-x).$$

We now consider three example initial conditions where we choose Bessel functions and products of Bessel functions of various types. It should be mentioned that our choices of initial conditions were due to their following inherent known properties.

(i) Mellin transforms of Bessel functions are well known and involve ratios of gamma functions, allowing for easy computation in applying Cauchy's residue theorem (see [6] for examples).
(ii) The power series coefficients for Bessel functions are well known and so power series of products of Bessel functions are also readily computable. Thus, we may apply Watson's lemma to obtain asymptotic behaviour of our solutions as well.

EXAMPLE 1.2. In the model with $u(r,0) = J_0(ar)$, the Bessel function of the first kind, we may proceed in the following way. Note that for $-\nu < \mathfrak{R}(s) < 3/2$ [6, p. 407]

$$\mathfrak{M}(J_\nu(ay))(s) = \frac{2^{s-1}\Gamma(\nu/2 + s/2)}{\Gamma(1 + \nu/2 - s/2)}a^{-s}. \tag{1.7}$$

We set $\nu = 0$ and insert (1.7) into the equation in Theorem 1.1 to obtain for $-1/2 < \mathfrak{R}(s) = c < 1$,

$$u(r,t) = \frac{1}{r}e^{-(r^2 - r^2/2)/(4\kappa t)}\frac{1}{2\pi i}\int_{(c)}(4\kappa t)^{s/2}M_{-s/2,0}\Big(\frac{r^2}{4\kappa t}\Big)2^{-s}\Gamma\Big(\frac{1-s}{2}\Big)a^{s-1}\,ds. \tag{1.8}$$

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It is known that

$$ {}_1F_1(a,1;x) = e^x L_{a-1}(-x), \tag{1.9} $$

where $L_n(x)$ is the Laguerre polynomial [3]. This can be seen by using [4, p. 1001] $L_a(x) = {}_1F_1(-a;1;x)$ together with Kummer's [1, p. 509] ${}_1F_1(a;b;x) = e^x {}_1F_1(1-a;b;-x)$ with $b = 1$. Now (1.6) with (1.8) leads to

$$ \begin{aligned} u(r,t) &= \frac{e^{-r^2/(4\kappa t)}}{\sqrt{4\kappa t}} \frac{1}{2\pi i} \int_{(c)} (4\kappa t)^{s/2} {}_1F_1\Big(\frac{s}{2}+\frac{1}{2};1;\frac{r^2}{4\kappa t}\Big) 2^{-s}\Gamma\Big(\frac{1-s}{2}\Big) a^{s-1}\,ds \\ &= \frac{e^{-r^2/(4\kappa t)}}{\sqrt{4\kappa t}} \frac{1}{2\pi i} \int_{(1-c)} (4\kappa t)^{(1-s)/2} {}_1F_1\Big(1-\frac{s}{2};1;\frac{r^2}{4\kappa t}\Big) 2^{s-1}\Gamma\Big(\frac{s}{2}\Big) a^{-s}\,ds. \end{aligned} $$

Here we made the change of variable $s \to 1-s$. This integrand has simple poles at $s = 0$ and the negative even integers $s = -2n$. We consider a rectangular contour $C_{M,c,T}$, where $M = 2N + 1/2$ with vertices at $(1-c, iT), (-M, iT), (-M, -iT)$ and $(1-c, -iT)$ with $T > 0$. Due to Stirling's formula [6, pp. 31 and 121], the contribution from the horizontal sides tends to zero as $T \to \infty$. Noting that these poles are contained within our $C_{M,c,T}$ and using (1.9) to compute the residues gives

$$ u(r,t) = \sum_{n\geq 0} \frac{L_n(-r^2/4\kappa t)}{n!}(-a^2\kappa t)^n = e^{-a^2\kappa t} J_0(ar). $$

Here we have applied the $\alpha = 0$ case of [7, p. 102, Theorem 5.1 and equation (5.1.16)]

$$ \sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} w^n = e^w (xw)^{-\alpha/2} J_\alpha(2\sqrt{xw}). $$

Next we consider an example of Theorem 1.1 with a function for which it is difficult to evaluate (1.3), and is apparently new.

THEOREM 1.3. *The solution of* (1.1) *with* $u(r,0) = J_0^2(ar)$ *is given by*

$$ u(r,t) = \frac{1}{2} \sum_{n\geq 0} \frac{(2n)!}{(n!)^3} L_n\Big(-\frac{r^2}{4^2\kappa t}\Big)(-a^2\kappa t)^n. $$

PROOF. First, we write down [6, p. 407]

$$ \mathfrak{M}(J_\nu^2(ay))(s) = \frac{2^{s-1}\Gamma(s/2+\nu)\Gamma(1-s)}{\Gamma^2(1-s/2)\Gamma(1+\nu-s/2)} a^{-s}, \tag{1.10} $$

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valid for $-\Re(\nu) < \Re(s) = c' < 1$. We set $\nu = 0$ in (1.10) and insert it into the equation in Theorem 1.1 to find for $0 < c' < 1$,

$$u(r,t) = \frac{1}{r} e^{-r^2/(4\kappa t)} \frac{1}{2\pi i} \int_{(c')} (4\kappa t)^{s/2} M_{-s/2,0}\Big(\frac{r^2}{4\kappa t}\Big) \frac{2^{-s}\Gamma((1-s)/2)\Gamma(s)}{\Gamma^2(1/2+s/2)} a^{s-1}\, ds$$

$$= \frac{e^{-r^2/(4\kappa t)}}{\sqrt{4\kappa t}} \frac{1}{2\pi i} \int_{(1-c')} (4\kappa t)^{(1-s)/2} {}_1F_1\Big(1-s/2; 1; \frac{r^2}{4\kappa t}\Big) \frac{2^{s-1}\Gamma(s/2)\Gamma(1-s)}{\Gamma^2(1-s/2)} a^{-s}\, ds.$$

The resulting integrand has simple poles at $s = -2n$ for each integer $n \geq 0$. We consider a rectangular contour $C_{M,c',T}$ where $M = 2N + 1/2$ with vertices at $(1-c', iT), (-M, iT), (-M, -iT)$ and $(1-c', -iT)$ with $T > 0$. Due to Stirling's formula, the contribution from the horizontal sides tends to zero as $T \to \infty$. Noting that these poles are contained within our $C_{M,c',T}$, we use (1.9) to compute the residues. Therefore, computing the residues at these poles gives, by Cauchy's residue theorem and (1.9),

$$u(r,t) = \frac{1}{2} \sum_{n\geq 0} \frac{(2n)!}{(n!)^3} L_n\Big(-\frac{r^2}{4^2\kappa t}\Big)(-a^2\kappa t)^n. \qquad \square$$

It is interesting to note that taking the limit $r \to 0$ of Theorem 1.3 gives

$$\lim_{r\to 0}\Big(\sum_{n\geq 0} \frac{(2n)!}{(n!)^3} L_n\Big(-\frac{r^2}{4^2\kappa t}\Big)(-a^2\kappa t)^n\Big) = e^{-a^2\kappa t} I_0(a^2\kappa t)$$

by means of [4, p. 1024, equation (9.215), #3, $p = 0, z = \mathrm{i}x$]. Next we consider an initial condition involving the modified Bessel function of the second kind $K_\nu(x)$, which has the general relationship [4]

$$K_\nu(x) = \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{2\sin(\pi\nu)}.$$

THEOREM 1.4. *The solution to* (1.1) *with* $u(r,0) = I_\nu(ar)K_\nu(ar)$ *is given by*

$$u(r,t) = e^{-r^2/(4\kappa t)} \frac{(4\kappa t a^2)^\nu}{4\sqrt{\pi}} \Big[\sum_{n\geq 0} {}_1F_1\Big(1+\nu+n; 1; \frac{r^2}{4\kappa t}\Big)$$

$$\times \frac{\Gamma(1+\nu+n)\Gamma(-\nu-n)\Gamma(1/2+n+\nu)}{n!\,\Gamma(2\nu+1+n)}(-a^2 4\kappa t)^n\Big]$$

$$+ \frac{e^{-r^2/(4\kappa t)}}{4\sqrt{\pi}} \sum_{n\geq 0} {}_1F_1\Big(1+n; 1; \frac{r^2}{4\kappa t}\Big) \frac{\Gamma(\nu-n)\Gamma(1/2+n)}{\Gamma(\nu+1+n)}(-a^2 4\kappa t)^n,$$

provided that ν *is not an integer or equal to* 0.

PROOF. From [8, p. 199, equation (7.10.8)] with $0 < \Re(s) = c' < 1$,

$$\mathfrak{M}(I_\nu(ay)K_\nu(ay))(s) = \frac{\Gamma(s/2+\nu)\Gamma(1/2-s/2)\Gamma(s/2)}{4\sqrt{\pi}\Gamma(\nu+1-s/2)} a^{-s}. \tag{1.11}$$

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Setting $v = 0$ in (1.11) and applying Theorem 1.1, we have that $u(r, t)$ is equal to

$$u(r,t) = \frac{1}{r4\sqrt{\pi}} e^{-r^2/(4\kappa t)} \frac{1}{2\pi i} \int_{(c')} \bigg[(4\kappa t)^{s/2} \Gamma\Big(\frac{1}{2} + \frac{s}{2}\Big) M_{-s/2,0}\Big(\frac{r^2}{4\kappa t}\Big) \times \frac{\Gamma((1-s)/2 + v)\Gamma(1/2 - s/2)\Gamma(s/2)}{\Gamma(v + 1/2 + s/2)} a^{s-1}\bigg] ds$$

$$= \frac{e^{-r^2/(4\kappa t)}}{4\sqrt{\pi}\sqrt{4\kappa t}} \frac{1}{2\pi i} \int_{(1-c')} \bigg[(4\kappa t)^{(1-s)/2} {}_1F_1\Big(1 - \frac{s}{2}; 1; \frac{r^2}{4\kappa t}\Big) \times \frac{\Gamma(1 - s/2)\Gamma(s/2 + v)\Gamma(s/2)\Gamma((1-s)/2)}{\Gamma(v + 1 - s/2)} a^{-s}\bigg] ds.$$

Now we see that if $v = 0$, then the gamma functions would have a pole of order two at the negative even integers $s = -2n$, which we want to avoid due to the lengthy resulting formula. Hence, we restrict v to be a noninteger and $v \neq 0$, and the poles at $s = -2n - 2v$ and $s = -2n$ are simple. We consider a rectangular contour $C_{M_v,c',T}$, where $M = 2N + 2v + 1/2$ with vertices at $(1 - c', iT), (-M, iT), (-M, -iT)$ and $(1 - c', -iT)$ with $T > 0$. Due to Stirling's formula [6, pp. 31 and 121], the contribution from the horizontal sides tends to zero as $T \to \infty$. Noting that these poles are contained within our $C_{M_v,c',T}$, we use the integrand to compute the residues. For the poles at $s = -2n - 2v$, we have the residue

$$e^{-r^2/(4\kappa t)} \frac{(4\kappa t a^2)^v}{4\sqrt{\pi}} \sum_{n\geq 0} {}_1F_1\Big(1 + v + n; 1; \frac{r^2}{4\kappa t}\Big) \times \frac{\Gamma(1 + v + n)\Gamma(-v - n)\Gamma(1/2 + n + v)}{n!\,\Gamma(2v + 1 + n)} (-a^2 4\kappa t)^n$$

and, for the poles at $s = -2n$, we have the residue

$$\frac{e^{-r^2/(4\kappa t)}}{4\sqrt{\pi}} \sum_{n\geq 0} {}_1F_1\Big(1 + n; 1; \frac{r^2}{4\kappa t}\Big) \frac{\Gamma(v - n)\Gamma(1/2 + n)}{\Gamma(v + 1 + n)} (-a^2 4\kappa t)^n. \qquad \square$$

A nice consequence of our series representations of $u(r, t)$ is that they are rapidly converging and so should be of great interest for numerical calculations. From [4, p. 1003, equation (8.978), #3, $\alpha = 0$], we have the asymptotic expansion for the Laguerre polynomial

$$L_n(x) = \frac{e^{x/2}}{\sqrt{\pi}} (xn)^{-1/4} \cos\Big(2\sqrt{nx} - \frac{\pi}{4}\Big) + O(n^{-3/4}) \tag{1.12}$$

as $n \to \infty$, uniformly in $x > 0$. In conjunction with our series involving Laguerre polynomials, (1.12) may be used to obtain approximations for $u(r, t)$.

2. Some related observations

We mention a method of evaluating (1.3) when $g(y) = h(y)\log(y)$ for a suitable function $h(y)$. It is known [4, p. 919, equation (8.447)] that

$$I_0(x)\log\left(\frac{x}{2}\right) = -K_0(x) + \sum_{n\geq 1}\frac{x^{2n}}{2^{2n}(n!)^2}\psi(n+1), \tag{2.1}$$

where $\psi(x)$ is the digamma function [4]. The formula (2.1) appears to provide an effective way of computing special cases of (1.3). We provide an outline of a method.

THEOREM 2.1. *Let $h(y)$ be a suitable function chosen so the series converges. The solution to (1.1) with initial condition $u(r,0) = h(r)\log(r)$ satisfies*

$$u(r,t) = \frac{1}{2\kappa t}e^{-r^2/(4\kappa t)}\bigg[\log\left(\frac{4\kappa t}{r}\right)\mathfrak{Z}_1(h) - \int_0^\infty yh(y)e^{-y^2/(4\kappa t)}K_0\left(\frac{yr}{2\kappa t}\right)dy$$
$$+\sum_{n\geq 1}\frac{\psi(n+1)}{2^{2n}(n!)^2}\left(\frac{r}{2\kappa t}\right)^{2n}\mathfrak{Z}_{2n+1}(h)\bigg],$$

where

$$\mathfrak{Z}_s(h) := \mathfrak{M}(yh(y)e^{-y^2/(4\kappa t)})(s) = \int_0^\infty h(y)y^s e^{-y^2/(4\kappa t)}\,dy.$$

PROOF. Note that equation (2.1) implies that

$$I_0\left(\frac{yr}{2\kappa t}\right)\log(y) = \log\left(\frac{4\kappa t}{r}\right) - K_0\left(\frac{yr}{2\kappa t}\right) + \sum_{n\geq 1}\frac{\psi(k+1)}{2^{2k}(k!)^2}\left(\frac{yr}{2\kappa t}\right)^{2k}.$$

Hence,

$$\int_0^\infty yh(y)I_0\left(\frac{yr}{2\kappa t}\right)\log(y)e^{-y^2/(4\kappa t)}\,dy = \log\left(\frac{4\kappa t}{r}\right)\int_0^\infty yh(y)e^{-y^2/(4\kappa t)}\,dy$$
$$-\int_0^\infty yh(y)e^{-y^2/(4\kappa t)}K_0\left(\frac{yr}{2\kappa t}\right)dy$$
$$+\sum_{n\geq 1}\frac{\psi(k+1)}{2^{2k}(k!)^2}\left(\frac{r}{2\kappa t}\right)^{2k}\mathfrak{Z}_{2k}(h(y)),$$

provided $yh(y)\log(y)$ satisfies certain growth conditions. In particular, by [4, p. 920], $K_0(t) = O(e^{-t}/\sqrt{t})$ when $t \to \infty$ in $|\arg(t)| < 3\pi/2$ and so we require the very mild necessary condition that for a positive constant c_1 and any $t > 0$,

$$|yh(y)| < c_1 e^{y^2/(4\kappa t)}$$

by the first integral on the right-hand side. □

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3. Asymptotic analysis of $u(r,t)$

First, we write down Watson's lemma (see [5, p. 336, equations (6.01) and (6.02)] or [6, p. 5, Lemma 1.2]). Suppose that $f(y)$ has the power series expansion

$$f(y) \sim \sum_{n\geq 0} a_n y^{(n+\lambda-\mu)/\mu} \tag{3.1}$$

as $y \to 0^+$. Then, if $\lambda > 0, \mu > 0$,

$$\int_0^\infty e^{-xy} f(y)\, dy \sim \sum_{n\geq 0} \Gamma\Big(\frac{n+\lambda}{\mu}\Big)\frac{a_n}{x^{(n+\lambda)/\mu}} \tag{3.2}$$

as $x \to \infty$, assuming that the integral converges. In this section, we illustrate an attractive aspect of working with Bessel functions for $u(r,0)$. Namely, exploiting the power series representation to find an asymptotic formula related to the integral representation of $u(r,t)$. First, in our initial example following Theorem 1.1, with $f(r) = J_0(ar)$, we can use the power series formula

$$J_0(ay)I_0\Big(\frac{yr}{2\kappa t}\Big) = \sum_{n\geq 0}\Big(\frac{1}{2^{2n}}\sum_{0\leq k\leq n}\frac{(-a^2)^k}{(k!)^2((n-k)!)^2}\Big(\frac{r}{2\kappa t}\Big)^{2(n-k)}\Big)y^{2n}, \tag{3.3}$$

which is easily obtained by equating coefficients after taking the product of power series. Now we replace y by $\sqrt{y}$ in (3.3), put $\mu = \lambda = 1$ in (3.1)–(3.2) and make the change of variable $y \to y^2$ in (3.2) to obtain the following theorem.

THEOREM 3.1. *We have*

$$2\int_0^\infty ye^{-xy^2} J_0(ay)I_0\Big(\frac{yr}{2\kappa t}\Big)\, dy \sim \sum_{n\geq 0}\frac{n!\, b_n}{x^{(n+1)}}$$

as $x \to \infty$, *where*

$$b_n = \frac{1}{2^{2n}}\sum_{0\leq k\leq n}\frac{(-a^2)^k}{(k!)^2((n-k)!)^2}\Big(\frac{r}{2\kappa t}\Big)^{2(n-k)}.$$

In the example provided in Theorem 1.3, we will need the known power series [4, p. 918, equation (8.442), #1, $v = \mu = 0, z = $ ay],

$$J_0^2(ay) = \sum_{n\geq 0}\frac{(-1)^n(n+1)_n}{(n!)^3}\Big(\frac{ay}{2}\Big)^{2n}. \tag{3.4}$$

Proceeding precisely in the same way as we did for Theorem 3.1 but with (3.4), we obtain the following theorem.

THEOREM 3.2. *We have*

$$2\int_0^\infty ye^{-xy^2} J_0^2(ay)I_0\Big(\frac{yr}{2\kappa t}\Big)\, dy \sim \sum_{n\geq 0}\frac{n!\, c_n}{x^{(n+1)}}$$

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as $x \to \infty$, *where*

$$c_n = \frac{1}{2^{2n}} \sum_{0 \le k \le n} \frac{(-a^2)^k (k+1)_k}{(k!\,)^3((n-k)!\,)^2} \left(\frac{r}{2\kappa t}\right)^{2(n-k)}.$$

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