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THE SLOT LENGTH OF A FAMILY OF MATRICES

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Abstract

We introduce the notion of the *slot length* of a family of matrices over an arbitrary field $\mathbb{F}$. Using this definition it is shown that, if $n \geq 5$ and A and B are $n \times n$ complex matrices with A unicellular and the pair $\{A, B\}$ irreducible, the slot length s of $\{A, B\}$ satisfies $2 \leq s \leq n - 1$, where both inequalities are sharp, for every n. It is conjectured that the slot length of any irreducible pair of $n \times n$ matrices, where $n \geq 5$, is at most $n - 1$. The slot length of a family of rank-one complex matrices can be equal to n.

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1. Introduction

Let $\mathcal{A}$ be a finite family of distinct symbols. Every word w in $\mathcal{A}$ can be written, using index notation, uniquely in the form $a_1^{r_1} a_2^{r_2} \cdots a_{k-1}^{r_{k-1}} a_k^{r_k}$, where each r_i is a positive integer and adjacent bases are distinct elements of $\mathcal{A}$, that is, $a_i \neq a_{i+1}$ for $1 \leq i \leq k - 1$. We define the *slot length* of the word w to be k. For example, if $a, b, c, x, y, z \in \mathcal{A}$, a^3 has slot length 1; a^2bc^8 has slot length 3, assuming that $a \neq b \neq c$; $x^5y^4z^4a^2bc^5$ has slot length 6, assuming that $x \neq y \neq z \neq a \neq b \neq c$.

Let $\mathbb{F}$ be a field and let $\mathcal{M}$ be a finite family of $n \times n$ matrices over $\mathbb{F}$. Let the identity matrix I be taken as the empty word in $\mathcal{M}$ and let it have slot length 0. For $k \in \mathbb{N}$, let $\mathcal{S}_k(\mathcal{M})$, written as just $\mathcal{S}_k$ if it is clear what $\mathcal{M}$ is, be the linear span of the words in $\mathcal{M}$ of slot length at most k. Clearly $\mathcal{S}_k \subseteq \mathcal{S}_{k+1}$ for every k. If $\mathcal{S}_{k+1} = \mathcal{S}_k$, then $\mathcal{S}_{k+2} = \mathcal{S}_{k+1}$. Thus there exists a positive integer s such that

$$\mathbb{F}I = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \cdots \subset \mathcal{S}_s = \mathcal{S}_p, \quad \text{for every } p \geq s,$$

where '$\subset$' denotes strict inclusion, and where $\mathcal{S}_s$ is the unital algebra generated by $\mathcal{M}$. We define the integer s to be the *slot length* of the family $\mathcal{M}$. Of course, any word in $\mathcal{M}$ of length k in the usual sense has slot length at most k. If the field $\mathbb{F}$ is algebraically closed and $\mathcal{M}$ is irreducible then $\mathcal{S}_s = M_n(\mathbb{F})$, by Burnside's theorem [2].

A great deal of work has been done investigating the notion of 'length of a family of matrices over the field $\mathbb{F}$', much of it directed towards resolving Paz's conjecture [8] that the length of any finite set of $n \times n$ matrices is at most $2n - 2$. This conjecture

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remains unresolved. A recent result [9, Theorem 3] of Shitov shows that the length is at most $2n \log_2 n + 4n - 4$. For other recent results and further references, see [1, 7].

In this paper we will consider the slot lengths of irreducible pairs of $n \times n$ complex matrices where one of the matrices is *unicellular*, that is, its set of invariant subspaces is totally ordered by inclusion (equivalently, it is nonderogatory with singleton spectrum). Here, and in what follows, *irreducible* will mean *having no nontrivial common invariant subspaces*. We considered such pairs in [3] (see also [5]), where it was shown that the length l of such a pair satisfies $l \leq 2n - 2$ and the inequality is sharp for all $n \geq 2$. Here we show that, if $n \geq 5$, the slot length s of such a pair satisfies $s \leq n - 1$, where the inequality is sharp for every $n \geq 5$. We conjecture that this inequality holds, for $n \geq 5$, for any irreducible pair of matrices. Obviously, if $n \geq 2$, the slot length of such a pair is at least 2. We observe that it can actually be 2. An example is given showing that, for any $n \geq 2$, the slot length of a finite, irreducible family of rank-one matrices can be equal to n.

Throughout, we denote the standard basis for $\mathbb{C}^n$ by $\{e_i : 1 \leq i \leq n\}$, and the linear span of a set $\mathcal{E}$ of vectors by $\langle \mathcal{E} \rangle$. If $e, f \in \mathbb{C}^n$ are nonzero vectors, the rank-one matrix $e \otimes f$ is defined by $(e \otimes f)(x) = (x|e)f$, $x \in \mathbb{C}^n$, where '$(\cdot|\cdot)$' denotes the standard inner-product. We then have $T(e \otimes f) = e \otimes Tf$ and $(e \otimes f)T = T^* e \otimes f$, for any matrix T, where T^* denotes the adjoint of T. Note that $e_j \otimes e_i = E_{i,j}$, the usual elementary matrix with a 1 in position (i, j) and 0s elsewhere. The strictly upper triangular Jordan matrix is denoted by J, the $n \times n$ matrix with 1s in positions $\{(i, i + 1) : 1 \leq i \leq n - 1\}$ and 0s elsewhere. It is unicellular and its nonzero invariant subspaces are V_k, where $V_k = \langle e_1, e_2, \ldots, e_k \rangle$, for $1 \leq k \leq n$.

For $-(n - 1) \leq k \leq n - 1$, the subset $\mathcal{D}_k$ of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ defined by $\mathcal{D}_k = \{(i, j) : 1 \leq i, j \leq n, \ \ j - i = k\}$ is called the *kth diagonal*.

DEFINITION 1.1. A matrix $A \in M_n(\mathbb{C})$, $A = (a_{u,v})$, is called an *echelon representative* for the matrix position (i, j) if $a_{i,j} \neq 0$, and $a_{u,v} = 0$ if $v - u < j - i$ and if $v - u = j - i$ with $v < j$.

In other words, A is an echelon representative for (i, j) if its (i, j)th entry is the first nonzero entry of A in the total ordering of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ defined by $(u, v) < (w, x)$ if $v - u < x - w$ or $v - u = x - w$ and $v < x$ (equivalently, if either (u, v) belongs to a 'strictly lower' diagonal than (w, x), or they belong to the same diagonal with (u, v) in an earlier column than (w, x)). This definition was first given in [6] where the linear bijection $\rho : M_n(\mathbb{C}) \to \mathbb{C}^{n^2}$ was defined by

$$\rho(A) = (a_{n,1}, a_{n-1,1}, a_{n,2}, a_{n-2,1}, a_{n-1,2}, a_{n,3}, \ldots, a_{1,n-1}, a_{2,n}, a_{1,n}),$$

where $A = (a_{i,j})$. (Here we are simply writing the matrix A as a row vector starting from the bottom left-hand corner and finishing at the top right-hand corner.) The matrix A is an echelon representative for position (i, j) if the first nonzero entry of its row vector $\rho(A)$ is the same as the position of the nonzero entry of $\rho(E_{i,j})$. Below, we use the fact that, if a set $\mathcal{B}$ of $n \times n$ matrices contains an echelon representative for every one of the matrix positions $\{(i, j) : 1 \leq i, j \leq n\}$, then $\mathcal{B}$ spans $M_n(\mathbb{C})$. This follows from the fact

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that, in such a case, $\rho(\mathcal{B})$ contains a set of n^2 row vectors in echelon so $\rho(\mathcal{B})$ spans $\mathbb{C}^{n^2}$. Then $\langle\mathcal{B}\rangle = \rho^{-1}(\mathbb{C}^{n^2}) = M_n(\mathbb{C})$.

Notice that, if A is an echelon representative for the matrix position (i, j) then J^pAJ^q is an echelon representative for the matrix position $(i + p, j + q)$ if $i + p, j + q \le n$. (The (u, v)th entry of JA is $a_{u+1,v}$ if $1 \le u \le n-1$ and 0 otherwise. The (u, v)th entry of AJ is $a_{u,v-1}$ if $2 \le v \le n$ and 0 otherwise.)

2. Irreducible pairs

EXAMPLE 2.1. Let B be the 3×3 matrix given by $B = e_1 \otimes e_2 + e_2 \otimes e_3$. The pair $\{J, B\}$ is irreducible. We have $B^2 = e_1 \otimes e_3$ and $B^2, JB^2, J^2B^2, B^2J, B^2J^2$ are echelon representatives for the positions $(3, 1), (2, 1), (1, 1), (3, 2), (3, 1)$, respectively. Also J, J^2 are echelon representatives for $(1, 2), (1, 3)$, respectively. Finally, BJ, BJ^2 are echelon representatives for $(2, 2), (2, 3)$, respectively. All of these echelon representatives have slot length at most 2. The slot length of the pair $\{J, B\}$ is 2.

EXAMPLE 2.2. On $\mathbb{C}^n, n \ge 2$, consider the irreducible pair $\{J, J^*\}$, where J^* is the adjoint of J. Now $\{(J^*)^q : 1 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i, 1) : 2 \le i \le n\}$, so $\{(J^*)^qJ^p : 0 \le p \le n-1, 1 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i, j) : 2 \le i \le n, 1 \le j \le n\}$, each of which has slot length 2. Also, $\{J^p : 0 \le p \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(1, j) : 1 \le j \le n\}$. Each of these has slot length 1, with the exception of $J^0 = I$ which has slot length 0. The slot length of $\{J, J^*\}$ is 2.

EXAMPLE 2.3. Let B be the 10×10 matrix

$$B = e_1 \otimes e_3 + e_2 \otimes e_5 + e_4 \otimes e_7 + e_6 \otimes e_9 + e_8 \otimes e_{10}.$$

We have

$$(BJ)^2B = e_1 \otimes e_7 + e_2 \otimes e_9 + e_4 \otimes e_{10}, \quad (BJ)^3B = e_1 \otimes e_9 + e_2 \otimes e_{10},$$
$$(BJ)^4 = e_2 \otimes e_9 + e_3 \otimes e_{10}, \quad (BJ)^4B = e_1 \otimes e_{10}, \quad (BJ)^4J^2B = e_2 \otimes e_{10}.$$

So the following statements hold.

(i) $(BJ)^4B$ is an echelon representative for the position $(10, 1)$. It has slot length 9.
(ii) $\{J^p(BJ)^3BJ^q : 0 \le p \le 8, 0 \le q \le 9\}$ are echelon representatives for the positions $\{(i, j) : 1 \le i \le 9, 1 \le j \le 10\}$. Each of these has slot length at most 9.
(iii) $(BJ)^4J^2B$ is an echelon representative for the position $(10, 2)$. It has slot length 9.
(iv) $(BJ)^4 - (BJ)^2B = e_3 \otimes e_{10} - e_1 \otimes e_7 - e_4 \otimes e_{10}$ is an echelon representative for the position $(10, 3)$ and $\{((BJ)^4 - (BJ)^2B)J^q : 0 \le q \le 7\}$ are echelon representatives for the positions $\{(10, j) : 3 \le j \le 10\}$ and $((BJ)^4 - (BJ)^2B)J^q \in \mathcal{S}_9$.

This shows that the slot length of $\{J, B\}$ is at most 9 and that $\{J, B\}$ is irreducible.

EXAMPLE 2.4. Let $B \in M_4(\mathbb{C})$ be $B = (J^*)^2 = e_1 \otimes e_3 + e_2 \otimes e_4$. Then $\{J, B\}$ is an irreducible pair. Now $BJB = e_1 \otimes e_4$, so $\{J^pBJB : 0 \le p \le 3\}$ is a set of echelon representatives for the matrix positions $\{(i, 1) : 1 \le i \le 4\}$ and $\{BJBJ^q : 0 \le q \le 3\}$ is a set of echelon representatives for $\{(4, j) : 1 \le j \le 4\}$. Now $BJ = e_2 \otimes e_3 + e_3 \otimes e_4$ and

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$\{J^p(BJ)J^q : 1 \le p, q \le 2\}$ is a set of echelon representatives for the matrix positions $\{(i,j) : 1 \le i \le 3, 2 \le j \le 4\}$. So the slot length of the pair is at most 4. We show that it is precisely 4 by showing that $(We_1|e_3) = (We_2|e_4)$ for every word in J, B with slot length at most 3. Since $B^2 = 0$, the only words for which this needs to be verified are $I, B, J^p, BJ^q, J^pB, J^pBJ^q, BJ^pB$, where $1 \le p, q \le 3$, and this is easily done.

DEFINITION 2.5. For any $n \times n$ matrix X and any integer j with $1 \le j \le n$, let $D_j(X)$ be the largest integer k such that the kth element of column j of X is nonzero, taking $D_j(X)$ to be 0 if column j of X is zero. We call $D_j(X)$ the *depth* of column j of X. Also, for any integer i with $1 \le i \le n$, let $L_i(X)$ be the smallest k such that the kth element of row i of X is nonzero, taking $L_i(X)$ to be $n+1$ if row i of X is zero. We call $L_i(X)$ the *length* of row i of X. (We hope this does not confuse the reader.)

Observe that, for any $n \times n$ matrices X, Y and for any integers p, q, r, s such that $1 \le p, q, r, s \le n$,

$$\text{if } L_p(X) = q \text{ and } D_r(Y) < q, \text{ for all } r < s, \text{ and } D_s(Y) = q, \text{ then } L_p(XY) = s. \quad (2.1)$$

Also note that $L_p(XJ) = L_p(X) + 1$ and $D_q(JY) = D_q(Y) - 1$, for any matrices X, Y and any p,q such that $L_p(X) \le n$ and $D_q(Y) \ge 1$.

REMARK 2.6. Notice that if $\mathcal{M} = \{A_i : 1 \le i \le m\}$ is a set of complex matrices and $\{\alpha_i : 1 \le i \le m\}$ is any set of scalars, then $\mathcal{M}^+ = \{A_i + \alpha_i I : 1 \le i \le m\}$ has the same slot length as $\mathcal{M}$. (Any word such as $(A_{i_1} + \alpha_{i_1} I)^{p_1}(A_{i_2} + \alpha_{i_2} I)^{p_2} \cdots (A_{i_k} + \alpha_{i_k} I)^{p_k}$, where $\{p_i : 1 \le i \le m\} \subseteq \mathbb{Z}^+$ and $A_i + \alpha_i \ne A_{i+1} + \alpha_{i+1}$, for $1 \le i \le m-1$, belongs to $\mathcal{S}_k(\mathcal{M})$.)

THEOREM 2.7. *If $\{A, B\}$ is an irreducible pair of $n \times n$ complex matrices, with $n \ge 5$, and A is unicellular, then the slot length s of $\{A, B\}$ satisfies $2 \le s \le n-1$, where both inequalities are sharp for every n.*

PROOF. Since A is unicellular, it is similar to $J + \lambda I$, for some scalar λ. By the remark above, we may suppose that $\lambda = 0$. Since irreducibility and slot length are preserved by similarities (indeed, $\mathcal{S}_k(S^{-1}\mathcal{M}S) = S^{-1}\mathcal{S}_k(\mathcal{M})S$, for every $k \in \mathbb{N}$), we may suppose that $A = J$. We complete the proof by exhibiting, for every matrix position (i,j), with $1 \le i,j \le n$, an echelon representative belonging to $\mathcal{S}_{n-1}$.

As mentioned earlier, the nonzero invariant subspaces of J are the subspaces $V_k = \langle\{e_1, e_2, \ldots, e_k\}\rangle$, $k = 1, 2, \ldots, n-1$. Since B does not leave V_{n-1} invariant, there exists $j < n$ such that column j of B has depth n. Let m_1 be the smallest such j and let d_1 be the depth of column m_1. Then $1 \le m_1 < n$, $d_1 = n$ and column j of B has depth less than n if $j < m_1$. (Column m_1 of B shows that B does not leave any of the subspaces $V_{m_1}, V_{m_1+1}, \ldots, V_{n-1}$ invariant.) Since B does not leave V_{m_1-1} invariant, there exists $j < m_1$ such that column j of B has depth greater than $m_1 - 1$. Let m_2 be the smallest such j and let d_2 be the depth of column m_2. Then $1 \le m_2 < m_1$ and $m_1 \le d_2 < n$. Also, column j of B has depth less than m_1 if $j < m_2$. Continuing in this way, we obtain strictly decreasing sequences $1 = m_t < m_{t-1} < \cdots < m_2 < m_1$ and $d_t < d_{t-1} < \ldots < d_2 < d_1 = n$ with column m_i of B having depth $d_i \ge m_{i-1}$, $i = 2, \ldots, t$.

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We also have $d_i < m_{i-2}$, for $i = 3, 4, \dots, t$, since column j of B has depth less than m_{i-2} if $j < m_{i-1}$, so since $m_i < m_{i-1}$, we have $d_i < m_{i-2}$. Thus

$$1 = m_t < m_{t-1} \le d_t < m_{t-2} \le \cdots < m_{i+1} \le d_{i+2} < m_i \le \cdots \le d_3 < m_1 \le d_2 < d_1 = n.$$

By definition, $D_j(B) < d_i$ if $j < m_i$ for $1 \le i \le t-1$ and $D_{m_i}(B) = d_i$ for $1 \le i \le t$.

Let $u_i = d_i - m_{i-1}$, for $2 \le i \le t$, and define $X_k = BJ^{u_2}BJ^{u_3} \cdots BJ^{u_{k-1}}BJ^{u_k}$, for $2 \le k \le t$, and $Y_k = BJ^{u_3}BJ^{u_4} \cdots BJ^{u_{k-1}}BJ^{u_k}$, for $3 \le k \le t$.

We show that $L_n(X_k) = d_k$, for $2 \le k \le t$. First, observe that $L_n(B) = m_1$, so $L_n(BJ^{u_2}) = m_1 + u_2 = d_2$. Thus the result is true for $k = 2$. Let $2 \le k \le t-1$ and suppose that $L_n(X_k) = d_k$. Then, using observation (2.1) above, $L_n(X_kB) = m_k$ and so $L_n(X_{k+1}) = m_k + u_{k+1} = d_{k+1}$. This completes the induction proof.

We also show that if $d_2 = n-1$ then $L_{n-1}(Y_k) = d_k$, $3 \le k \le t$. Now $L_{n-1}(B) = m_2$, so $L_{n-1}(BJ^{u_3}) = m_2 + u_3 = d_3$. Let $3 \le k \le t-1$ and suppose that $L_{n-1}(Y_k) = d_k$. Then $L_{n-1}(Y_kB) = m_k$ (again using (2.1)) and $L_{n-1}(Y_{k+1}) = m_k + u_{k+1} = d_{k+1}$. This completes the induction proof.

Thus $L_n(X_t) = d_t$ and $L_n(X_tB) = m_t = 1$. In other words the entry in position $(n, 1)$ of X_tB is nonzero, and so X_tB is an echelon representative for the position $(n, 1)$. Let w be the cardinality of the set $\{i : 2 \le i \le t \text{ and } d_i = m_{i-1}\}$. The slot length of X_tB is $2t - 1 - 2w$. If $2t - 1 - 2w \le n - 3$ then $\{J^p(X_tB)J^q : 0 \le p, q \le n-1\}$ is a set of echelon representatives for all of the matrix positions, each having slot length at most $n-1$. The slot length of the pair $\{J, B\}$ is at most $n-1$ in this case.

In the remainder of the proof we assume that $2t - 1 - 2w \ge n - 2$. We exhibit, in each case, a set of echelon representatives for all the matrix positions, each belonging to $\mathcal{S}_{n-1}$.

Now consider

$$1 = m_t < m_{t-1} \le d_t < m_{t-2} \le \cdots < m_{i+1} \le d_{i+2} < m_i \le \cdots \le d_3 < m_1 \le d_2 < d_1 = n.$$

In this chain, there are $2t$ points of division of the integer interval $[1, n]$, possibly not all distinct. We have

$$n - 1 = (d_1 - d_2) + (d_2 - m_1) + (m_1 - d_3) + \cdots + (d_t - m_{t-1}) + (m_{t-1} - m_t) = \Sigma + S,$$

where

$$\Sigma = (d_1 - d_2) + (m_{t-1} - m_t) + \sum_{j=1}^{t-2}(m_j - d_{j+2}) \quad \text{and} \quad S = \sum_{i=2}^{t}(d_i - m_{i-1}).$$

All of the bracketed terms in the expression for Σ are positive, while only $t - 1 - w$ of the bracketed terms in the expression for S are positive.

Case 1: $w = 0$. If $w = 0$, since we are assuming that $2t - 1 \ge n - 2$ and since $n - 1 = \Sigma + S \ge t + (t - 1) = 2t - 1$, we have $2t - 1 = n - 1$ or $n - 2$.

Case 1.1: $w = 0$ *and* $2t - 1 = n - 1$. This occurs only on even-dimensional spaces and only when the value of each of the 'jumps'

$$\{m_j - d_{j+2} : 1 \le j \le t-2\} \cup \{d_i - m_{i-1} : 2 \le i \le t\} \cup \{d_1 - d_2, m_{t-1} - m_t\} \qquad (2.2)$$

is equal to 1. (Example 2.3 is of this type.)

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Case 1.2: $w = 0$ *and* $2t - 1 = n - 2$. This case occurs only on odd-dimensional spaces and only when all of the jumps in (2.2) have value 1 with precisely one exception, which has value 2.

Case 2: $w \neq 0$. If $w \neq 0$, since we are assuming that $2t - 1 - 2w \geq n - 2$, we have $2t - 1 - w \geq n - 2 - w$. But $n - 1 = \Sigma + S \geq 2t - 1 - w$. So $n - 1 \geq n - 2 - w$ and so $w = 1$. Then, since we are assuming that $2t - 3 \geq n - 2$ and have shown that $n - 1 \geq 2t - 2$, it follows that $n = 2t - 1$. This case occurs only on odd-dimensional spaces and occurs when precisely one of the jumps $d_{i_0} - m_{i_0 - 1}$, say, is 0 and all of the other jumps

$$\{m_j - d_{j+2} : 1 \leq j \leq t - 2\} \cup \{d_i - m_{i-1} : 2 \leq i \neq i_0 \leq t\} \cup \{d_1 - d_2, m_{t-1} - m_t\}$$

have values equal to 1.

Case 1.1: Echelon representatives. In this case $n = 2t$ and

$$1 = m_t < m_{t-1} < d_t < m_{t-2} < \cdots < m_{i+1} < d_{i+2} < m_i < \cdots < d_3 < m_1 < d_2 < d_1 = n,$$

where each strict inequality indicates a jump of precisely 1. So, for example, $m_{t-1} = 2$, $d_t = 3$ and $d_2 = n - 1, m_1 = n - 2, d_3 = n - 3$.

Position $(n, 1)$. Since each $u_i = d_i - m_{i-1}$ is 1, $X_k = (BJ)^{k-1}$ for $2 \leq k \leq t$. In particular, $X_t = (BJ)^{t-1}$ and $X_t B = (BJ)^{t-1} B$. As noted earlier, the latter is an echelon representative for matrix position $(n, 1)$. It has slot length $2t - 1 = n - 1$.

Positions $\{(i, j) : 1 \leq i \leq n - 1, 1 \leq j \leq n\}$. Now $L_n((BJ)^{t-2}) = L_n(X_{t-1}) = d_{t-1}$, so, using observation (2.1), $L_n((BJ)^{t-2} B) = m_{t-1} = 2$. Consider $L_{n-1}(X_{t-1})$ in this case. We have $Y_t = (BJ)^{t-2}$ for $3 \leq k \leq t$. Since $d_2 = n - 1$, it follows that $L_{n-1}(Y_t) = L_n(X_{t-1}) = L_{n-1}((BJ)^{t-2}) = d_t$ and $L_{n-1}((BJ)^{t-2} B) = 1$. Since $L_n((BJ)^{t-2} B) = 2$ and $L_{n-1}((BJ)^{t-2} B) = 1$, the matrix $(BJ)^{t-2} B$ is an echelon representative for the position $(n - 1, 1)$, with slot length $2t - 3 = n - 3$. Thus, $\{J^p (BJ)^{t-2} BJ^q : 0 \leq p \leq n - 2, 0 \leq q \leq n - 1\}$ is a set of echelon representatives for the matrix positions $\{(i, j) : 1 \leq i \leq n - 1, 1 \leq j \leq n\}$, each having slot length at most $n - 1$ (so belonging to $\mathcal{S}_{n-1}$).

Position $(n, 2)$. Continuing, $L_n(X_t J^2) = L_n((BJ)^{t-1} J^2) = d_t + 2 = d_{t-1}$, so by (2.1), $L_n((BJ)^{t-1} J^2 B) = m_{t-1} = 2$. Since $L_{n-1}((BJ)^{t-2}) = d_t$, we have $L_{n-1}((BJ)^{t-1}) = m_t + 1 = m_{t-1}$, so $L_{n-1}((BJ)^{t-1} J^2) = m_{t-1} + 2 = m_{t-2} > d_t$ and $L_{n-1}((BJ)^{t-1} J^2 B) > 1$. Thus $(BJ)^{t-1} J^2 B$ is an echelon representative for the matrix position $(n, 2)$. It has slot length $2t - 1 = n - 1$.

Positions $\{(n, j) : 3 \leq j \leq n\}$. Finding suitable echelon representatives for the remaining matrix positions is more difficult. (We need $n \geq 5$ to find such representatives.) Consider the matrices $(BJ)^{t-1}$ and $(BJ)^{t-3} B$. We have $L_n((BJ)^{t-1}) = L_n(X_t) = d_t = 3$ and (from above) $L_{n-1}((BJ)^{t-1}) = m_{t-1} = 2$. Since $L_p(XJ) > 1$, for every integer $1 \leq p \leq n$ and every matrix $X \in M_n(\mathbb{C})$, we also have $L_{n-2}((BJ)^{t-1}) > 1$. In comparison $L_n((BJ)^{t-3}) = L_n(X_{t-2}) = d_{t-2}$ so $L_n((BJ)^{t-3} B) = m_{t-2} = 4$. Since $L_{n-1}((BJ)^{t-3}) = L_{n-1}(Y_{t-1}) = d_{t-1}$, we have $L_{n-1}((BJ)^{t-3} B) = m_{t-1} = 2$. Also, $L_{n-2}(B) > m_3$, so $L_{n-2}(BJ) > m_3 + 1 = d_4$. By induction, $L_{n-2}((BJ)^k) > d_{k+3}$, for $1 \leq k \leq t - 3$. In particular, $L_{n-2}((BJ)^{t-3}) > d_t$, so $L_{n-2}((BJ)^{t-3} B) > m_t = 1$. Comparing

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$$L_n((BJ)^{t-1}) = 3, \quad L_{n-1}((BJ)^{t-1}) = 2, \quad L_{n-2}((BJ)^{t-1}) > 1,$$
$$L_n((BJ)^{t-3}B) = 4, \quad L_{n-1}((BJ)^{t-3}B) = 2, \quad L_{n-2}((BJ)^{t-3}B) > 1,$$

we see that there exists a scalar λ such that $(BJ)^{t-1} - \lambda(BJ)^{t-3}B$ is an echelon representative for the position $(n, 3)$. (The scalar λ is equal to x/y where x and y are the $(n-1, 2)$th elements of $(BJ)^{t-1}$ and $(BJ)^{t-3}B$, respectively.) It follows that $\{((BJ)^{t-1} - \lambda(BJ)^{t-3}B)J^q : 0 \le q \le n-3\}$ is a set of echelon representatives for the matrix positions $\{(n, j) : 3 \le j \le n\}$, each belonging to $\mathcal{S}_{n-1}$.

Case 1.2: Echelon representatives. In this case $n = 2t + 1$ and all of the jumps in (2.2) have value 1 with precisely one exception, which has value 2.

(i) Suppose that $m_{t-1} = m_t + 2$.

Positions $\{(n, j) : 1 \le j \le n\}$. Since $L_n(X_tB) = L_n((BJ)^{t-1}B) = 1$ and the slot length of $(BJ)^{t-1}B$ is $2t - 1 = n - 2$, we see that $\{(BJ)^{t-1}BJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(n, j) : 1 \le j \le n\}$, each having slot length at most $n - 1$.

Positions $\{(i, j) : 1 \le i \le n-1, 1 \le j \le n\}$. Once again we have $L_{n-1}((BJ)^{t-2}) = d_t$, so $L_{n-1}((BJ)^{t-2}B) = 1$. Since $L_n((BJ)^{t-2}) = d_{t-1}$, we have $L_n((BJ)^{t-2}B) = m_{t-1} = 3$. It follows that $(BJ)^{t-2}B$ is an echelon representative for the position $(n-1, 1)$. It has slot length $2t - 1 = n - 4$. Hence $\{J^p(BJ)^{t-2}BJ^q : 0 \le p \le n-2, 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i, j) : 1 \le i \le n-1, 1 \le j \le n\}$, each having slot length at most $n - 1$.

(ii) Suppose that $d_1 = d_2 + 2$.

Positions $\{(i, 1) : 1 \le i \le n\}$. Here, $L_n((BJ)^{t-1}B) = 1$ so $\{J^p(BJ)^{t-1}B : 0 \le p \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i, 1) : 1 \le i \le n\}$, each having slot length at most $n - 1$.

Positions $\{(i, j) : 1 \le i \le n, 2 \le j \le n\}$. Now $L_n((BJ)^{t-2}) = d_{t-1}$ and it follows that $L_n((BJ)^{t-2}B) = m_{t-1} = 2$. Since $L_{n-1}(B) > m_2$, we have $L_{n-1}(BJ) > d_3$. By induction $L_{n-1}((BJ)^k) > d_{k+2}$ for $1 \le k \le t-2$. In particular, $L_{n-1}((BJ)^{t-2}) > d_t$ so $L_{n-1}((BJ)^{t-2}B) > m_t = 1$. Hence $(BJ)^{t-2}B$ is an echelon representative for the matrix position $(n, 2)$. It has slot length $n - 4$. Hence

$$\{J^p(BJ)^{t-2}BJ^q : 0 \le p \le n-1, 0 \le q \le n-2\}$$

is a set of echelon representatives for the matrix positions $\{(i, j) : 1 \le i \le n, 2 \le j \le n\}$, each having slot length at most $n - 2$.

(iii) Suppose that $m_{j_0} - d_{j_0+2} = 2$, where $1 \le j_0 \le t - 2$.

Positions $\{(n, j) : 1 \le j \le n\}$. Now $L_n(X_tB) = L_n((BJ)^{t-1}B) = 1$. Since the slot length of $(BJ)^{t-1}B$ is $2t - 1 = n - 2$, the set $\{(BJ)^{t-1}BJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(n, j) : 1 \le j \le n\}$, each having slot length at most $n - 1$.

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Positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$. Here, $L_{n-1}(Y_t) = L_{n-1}((BJ)^{t-2}) = d_t$, since $d_2 = n-1$, so $L_{n-1}((BJ)^{t-2}B) = 1$. Also $L_n((BJ)^{t-2}) = L_n(X_{t-1}) = d_{t-1}$, so $L_n((BJ)^{t-2}B) = m_{t-1} = 2$. Thus $(BJ)^{t-2}B$ is an echelon representative for the matrix position $(n-1,1)$. It has slot length $2t-3 = n-4$. Thus

$$\{J^p(BJ)^{t-2}BJ^q : 0 \le p \le n-2, 0 \le q \le n-1\}$$

is a set of echelon representatives for the positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$, each having slot length at most $n-2$.

(iv) Suppose that $d_{i_1} - m_{i_1-1} = 2$, where $2 \le i_1 \le t$.

Positions $\{(n,j) : 1 \le j \le n\}$. Once again, $L_n(X_tB) = 1$ and $\{X_tBJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(n,j) : 1 \le j \le n\}$, each having slot length at most $n-1$.

Positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$. Here, $L_{n-1}(Y_t) = d_t$ since $d_2 = n-1$. Thus $L_{n-1}(Y_tB) = 1$. Since $L_n(B) = m_1$, we have $L_n(BJ^{u_3}) = L_n(Y_3) = m_1 + u_3 = m_1 + 1 = d_2$, if $i_1 \ne 2,3$. Then $L_n(Y_4) = m_2 + u_4 = m_2 + 1 = d_3$, if $i_1 \ne 2,3,4$. By induction, $L_n(Y_k) = d_{k-1}$, if $2 \le k \le i_1 - 1$. Then $L_n(Y_{i_1}) = L_n(Y_{i_1-1}BJ^{u_{i_1}}) = m_{i_1-2} + u_{i_1} = m_{i_1-2} + 2 = m_{i_1-3} > d_{i_1-1}$. Then $L_n(Y_{i_1+1}) = L_n(Y_{i_1}BJ^{u_{i_1+1}}) > m_{i_1-1} + u_{i_1+1} = m_{i_1-1} + 1 > d_{i_1+1}$. By induction, $L_n(Y_k) > d_k$, for $i_1 + 1 \le k \le t$. In particular, $L_n(Y_t) > d_t$, and so $L_n(Y_tB) > m_t = 1$. This shows that Y_tB is an echelon representative for the position $(n-1,1)$. It has slot length $2t-3 = n-4$. So $\{J^pY_tBJ^q : 0 \le p \le n-2, 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$, each having slot length at most $n-2$.

Case 2: Echelon representatives. In this case $n = 2t-1$ and $d_{i_0} = m_{i_0-1}$, with all other jumps equal to 1. In particular, $d_2 = n-1$, so we have $L_{n-1}(Y_k) = d_k$, for $3 \le k \le t$.

(i) Suppose that $i_0 = 2$.

Positions $\{(n,j) : 1 \le j \le n\}$. In this case, $L_n(X_t) = L_n(B(BJ)^{t-2}) = d_t$ and $L_n(B(BJ)^{t-2}B) = 1$. The slot length of $B(BJ)^{t-2}B$ is $2t-3 = n-2$ and so $\{B(BJ)^{t-2}BJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(n,j) : 1 \le j \le n\}$, each having slot length at most $n-1$.

Positions $\{(n-1,j) : 1 \le j \le n\}$. Since $L_{n-1}(Y_k) = d_k$, for $3 \le k \le t$ and $u_k = 1$, for $3 \le k \le t$, this becomes $L_{n-1}((BJ)^{k-2}) = d_k$ and taking $k = t$ gives $L_{n-1}((BJ)^{t-2}) = d_t$. So $L_{n-1}((BJ)^{t-2}B) = 1$. Now $L_n(B) = m_1$, so $L_n(BJ) = m_1 + 1 = d_1$. Thus $L_n((BJ)^k) = d_1$, for all k. In particular, $L_n((BJ)^{t-2}) = d_1$ and so $L_n((BJ)^{t-2}B) = m_1 > 1$. Thus $(BJ)^{t-2}B$ is an echelon representative for the position $(n-1,1)$. It has slot length $2t-3 = n-2$, so $\{(BJ)^{t-2}BJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives of the matrix positions $\{(n-1,j) : 1 \le j \le n\}$, each having slot length at most $n-1$.

Positions $\{(i,j) : 1 \le i \le n-2, 1 \le j \le n\}$. Since $L_n((BJ)^{t-3}) = d_1$ we have $L_n((BJ)^{t-3}B) = m_1 = n-1 > 2$. Since $L_{n-1}((BJ)^{t-3}) = L_{n-1}(Y_{t-1}) = d_{t-1}$, we have $L_{n-1}((BJ)^{t-3}B) = m_{t-1} = 2$. Also, $L_{n-2}(B) = m_3$ since $d_3 = n-2$, so $L_{n-2}(BJ) = d_4$ and,

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by induction $L_{n-2}((BJ)^k) = d_{k+3}$, for $1 \le k \le t-3$. In particular, $L_{n-2}((BJ)^{t-3}) = d_t$. Hence $L_{n-2}((BJ)^{t-3}B) = 1$. This shows that $(BJ)^{t-3}B$ is an echelon representative for the position $(n-2,1)$. It has slot length $2t-5 = n-4$. Thus $\{J^p(BJ)^{t-3}BJ^q : 0 \le p \le n-3, 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i,j) : 1 \le i \le n-2, 1 \le j \le n\}$, each having slot length at most $n-2$.

(ii) Suppose that $i_0 \ne 2$.

Positions $\{(n,j) : 1 \le j \le n\}$. Again $L_n(X_tB) = 1$, with $X_tB = (BJ)^{i_0-2}B(BJ)^{t-i_0}B$ having slot length $n-2$. So $\{X_tJ^q : 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(n,j) : 1 \le j \le n\}$, each having slot length at most $n-1$.

Positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$. Now $L_{n-1}(Y_t) = L_{n-1}((BJ)^{i_0-3}B(BJ)^{t-i_0}) = d_t$, so $L_{n-1}(Y_tB) = m_t = 1$. Also, $L_n(X_{i_0-2}) = L_n((BJ)^{i_0-3}) = d_{i_0-2}$ so $L_n((BJ)^{i_0-3}B) = m_{i_0-2} > d_{i_0}$ and $L_n((BJ)^{i_0-3}B(BJ)) > d_{i_0+1}$. By induction, $L_n(Y_k) > d_{i_0+k}$, for $1 \le k \le t-i_0$. In particular, $L_n(Y_t) > d_t$ and $L_n(Y_tB) > 1$. This shows that $Y_tB = (BJ)^{i_0-3}B(BJ)^{t-i_0}B$ is an echelon representative for the position $(n-1,1)$, so $\{J^pY_tBJ^q : 0 \le p \le n-2, 0 \le q \le n-1\}$ is a set of echelon representatives for the matrix positions $\{(i,j) : 1 \le i \le n-1, 1 \le j \le n\}$. Since the slot length of $(BJ)^{i_0-3}B(BJ)^{t-i_0}B$ is $2t-5 = n-4$, each has slot length at most $n-2$.

Finally, we establish the sharpness of the inequality $2 \le s \le n-1$ as in the statement of the theorem. Example 2.2 above shows that the lower bound is sharp. That the upper bound is sharp is shown by Example 2.8 below. □

EXAMPLE 2.8. (a) Let $n = 2t$ and $t \ge 3$. Let the strictly decreasing sequences $\{m_i : 1 \le i \le t\}$ and $\{d_i : 1 \le i \le t\}$ be as in Case 1.1: $w = 0$, all jumps equal to 1. Let $B = \sum_{i=1}^{t} e_{m_i} \otimes e_{d_i}$. Then $\{J, B\}$ is an irreducible pair and, by the theorem, its slot length is at most $n-1$. In fact its slot length is equal to $n-1$, as we now show.

We show that every word W in J, B of slot length $n-2$ or less satisfies $(We_1|e_n) = 0$. It follows that $(Te_1|e_n) = 0$, for every $T \in \mathcal{S}_{n-2}$, so $\mathcal{S}_{n-2} \ne M_n(\mathbb{C})$. Observe that $Be_i, Je_i \in \{0\} \cup \{e_j : 1 \le j \le n\}$, for $1 \le i \le n$. In fact, $Be_{m_i} = e_{d_i}$ for $1 \le i \le t$ and $\text{Range}(B) = \langle\{e_{d_i} : 1 \le i \le t\}\rangle$. Clearly $B^2 = 0$. Note also that

$$\{e_j : j > m_k\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le k+1\}, \quad \text{for } 1 \le k \le t.$$

Suppose that $(We_1|e_n) \ne 0$, where W is a word in J, B of slot length $n-2$ or less. Since $Je_1 = J^*e_n = 0$, the word W cannot begin or end with a term J^p with $p > 0$. Since $W \ne I$, W has odd positive slot length, say $2r+1$, where $r \le t-2$ (since $2r+1 \le n-2 = 2t-2$). Let $W = BJ^{p_1}BJ^{p_2}\cdots BJ^{p_r}B$ and $W_k = J^{p_k}BJ^{p_{k+1}}B\cdots J^{p_r}B$, for $1 \le k \le r$.

We claim that $W_ke_1 \in \{e_{m_j} : 1 \le j \le k\}$, for $1 \le k \le r$. Now $We_1 = BW_1e_1 = e_n = e_{d_1}$, so $W_1e_1 = e_{m_1}$, so the result is true for $k = 1$. Let $1 \le k \le r-1$ and assume that the result is true for k. Then $W_ke_1 = J^{p_k}BW_{k+1}e_1 \in \{e_{m_j} : 1 \le j \le k\}$, so that $BW_{k+1}e_1 \in \{e_j : j > m_k\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le k+1\}$. Thus $W_{k+1}e_1 \in \{e_{m_j} : 1 \le j \le k+1\}$. This completes the proof by induction.

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In particular, $W_r e_1 = J^{p_r} B e_1 \in \{e_{m_j} : 1 \le j \le r\}$, so

$$Be_1 \in \{e_j : j > m_r\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le r+1\}.$$

But $Be_1 = Be_{m_t} = e_{d_t}$ and, since $r+1 < t$, we get a contradiction. This contradiction shows that the slot length of the pair $\{J, B\}$ is precisely $n-1$.

(b) Let $n = 2t+1, t \ge 2$. Let the strictly decreasing sequences $\{m_i : 1 \le i \le t\}$ and $\{d_i : 1 \le i \le t\}$ be as in Case 1.2 (ii): $w = 0$, all jumps equal to 1 except that $d_1 = d_2 + 2$. Let $B = \sum_{i=1}^{t} e_{m_i} \otimes e_{d_i}$. Then $\{J, B\}$ is an irreducible pair, and by the theorem, its slot length is at most $n-1$. In fact its slot length is equal to $n-1$, as we now show.

We show that every word W in J, B of slot length $n-2$ or less satisfies $(We_1 | e_{n-1}) = 0$. Again observe that $Be_i, Je_i \in \{0\} \cup \{e_j : 1 \le j \le n\}$, for $1 \le i \le n$. In fact $Be_{m_i} = e_{d_i}$, for $1 \le i \le t$ and $\text{Range}(B) = \langle\{e_{d_i} : 1 \le i \le t\}\rangle$. Clearly $B^2 = 0$. Again note that

$$\{e_j : j > m_k\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le k+1\}, \quad \text{for } 1 \le k \le t.$$

Suppose that $(We_1 | e_{n-1}) \ne 0$, where W is a word in J, B of slot length $n-2$ or less. Clearly $W \ne I$. Since $Je_1 = 0$, W cannot end with a term J^p with $p > 0$. Since $B^* e_{n-1} = 0$, W cannot begin with B either. Let $W = J^{p_1} B J^{p_2} B \cdots J^{p_r} B$ and let $W_k = J^{p_k} B J^{p_{k+1}} B \cdots J^{p_r} B$, for $1 \le k \le r$.

We claim that $W_k e_1 \in \{e_{m_j} : 1 \le j \le k-1\}$, for $2 \le k \le r$. Now $We_1 = J^{p_1} B W_2 e_1 = e_n = e_{d_1}$, so $BW_2 e_1 = e_n = e_{d_1}$ and $W_2 e_1 = e_{m_1}$, so the result is true for $k = 1$. Let $2 \le k \le r-1$ and assume that the result is true for k. Then $W_k e_1 = J^{p_k} B W_{k+1} e_1 \in \{e_{m_j} : 1 \le j \le k-1\}$, so

$$BW_{k+1} e_1 \in \{e_j : j > m_{k-1}\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le k\}.$$

Thus $W_{k+1} e_1 \in \{e_{m_j} : 1 \le j \le k\}$. This completes the proof by induction.

In particular, $W_r e_1 = J^{p_r} B e_1 \in \{e_{m_j} : 1 \le j \le r-1\}$, so

$$Be_1 \in \{e_j : j > m_{r-1}\} \cap \text{ Range } (B) = \{e_{d_j} : 1 \le j \le r\}.$$

But $Be_1 = Be_{m_t} = e_{d_t}$ and, since $r < t$, we get a contradiction. This contradiction shows that the slot length of the pair $\{J, B\}$ is precisely $n-1$.

CONJECTURE 2.9. If $n \ge 5$, every irreducible pair of complex matrices has slot length at most $n-1$.

REMARK 2.10. (1) If $\mathcal{R}$ is an irreducible family of rank-one matrices, then slot length$(\mathcal{R})$ = length$(\mathcal{R})$, since for every rank-one matrix R we have $R^2 = \lambda R$ for some scalar λ. Irreducible families of rank-one matrices are considered in [4]. In particular, it is shown there that, if $n \ge 2$ and $\mathcal{R} \subseteq M_n(\mathbb{C})$ then $\mathcal{R}$ has slot length at most n.

The slot length of such a family can equal n. For example, the set of elementary matrices $\mathcal{E} = \{E_{i,j} : j - i = -1 \text{ or } n-1, 1 \le i, j \le n\}$ has slot length n. Here every element of $\mathcal{E}$ has square 0. The linear span of the words of slot length $n-1$ is $\mathcal{S}_{n-1} = \langle\{E_{i,j} : j \ne i, 1 \le i, j \le n\}\rangle + \mathbb{C}I$. Since $E_{i,i+1} E_{i+1,i} = E_{i,i}$, for $1 \le i \le n-1$, and $E_{n,n-1} E_{n-1,n} = E_{n,n}$, where $E_{i,i+1}, E_{n-1,n} \in \mathcal{S}_{n-1}$ and $E_{i+1,i}, E_{n,n-1} \in \mathcal{E}$, it follows that $\mathcal{S}_n = M_n(\mathbb{C})$, so the slot length of $\mathcal{E}$ is n.

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(2) Examples of irreducible pairs of matrices with slot length 3 follow from [6, Theorem 1]. By this theorem, if $\{A, B\}$ is an irreducible pair and B has rank 1, then $\{A^pBA^q; 0 \le p, q \le n-1\}$ is a basis for $M_n(\mathbb{C})$. Clearly, the slot length of the pair is at most 3, and since $\mathcal{S}_2$ is spanned by $\{I, B, A^p, BA^q, A^rB : 0 \le p, q, r \le n-1\}$, which has $3n-1$ elements and $3n-1 < n^2$, the slot length equals 3, if $n \ge 3$.

(3) In the example in (2) above, A need not be unicellular. For example, if A is a diagonal matrix with distinct nonzero diagonal entries it is well known that every nonzero invariant subspace of A has the form $\langle\{e_i : i \in \mathcal{F}\}\rangle$, for some nonempty subset $\mathcal{F}$ of $\{1, 2, \dots, n\}$. Thus if $B = e \otimes e$, where $e = (1, 1, \dots, 1)$, the pair $\{A, B\}$ is irreducible. It is easy to see that this pair has slot length 3, assuming that $n \ge 3$. This is so because, for $1 \le i \le n$, $E_{i,i} = p_i(A)$ for some polynomial $p_i(z)$. So $p_i(A)Bp_j(A)$ is the elementary matrix $E_{i,j}$, for all i, j. Every polynomial in A belongs to $\mathcal{S}_1(\{A, B\})$.

(4) Finally, notice that if $A \in M_n(\mathbb{C})$ has a minimum nonzero invariant subspace M_0 and a maximum proper invariant subspace N_0, any rank-one matrix B satisfying $(Bu|v) \neq 0$ for some $u \in M_0, v \in N_0^\perp$ has no nontrivial invariant subspaces in common with A. Indeed, if the subspace $M \neq (0)$ or $\mathbb{C}^n$ is invariant under A, then $M_0 \subseteq M$ and $M \subseteq N_0$. Thus $u \in M$ and $v \in M^\perp$. The fact that $(Bu|v) \neq 0$ now shows that B does not leave M invariant. The slot length of the pair $\{A, B\}$ is 3, if $n \ge 3$. (For example, $A = J$, $M_0 = \langle e_1 \rangle$, $N_0 = \langle\{e_1, e_2, \dots, e_{n-1}\}\rangle$, $B = e_1 \otimes e_n$ and $u = e_1$, $v = e_n$.)

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