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Double-spike solutions for a critical inhomogeneous elliptic problem in domains with small holes

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In this paper we construct solutions which develop two negative spikes as $\varepsilon \to 0^+$ for the problem $-\Delta u = |u|^{4/(N-2)}u + \varepsilon f(x)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain exhibiting a small hole, with $f \geqslant 0$, $f \not\equiv 0$. This result extends a recent work of Clapp *et al.* in the sense that no symmetry assumptions on the domain are required.

1. Introduction

This paper deals with the construction of solutions of the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u + \varepsilon f(x) \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, which has a small hole, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, $f(x)$ is an inhomogeneous perturbation, $f \geqslant 0$, $f \not\equiv 0$ and $\varepsilon > 0$ is a small parameter.

In the case when $1 < p < (N+2)/(N-2)$, it is well known that if $f = 0$, the associated energy functional to problem (1.1) is even and satisfies the Palais–Smale (PS) condition in $H_0^1(\Omega)$, which implies the existence of infinitely many non-trivial solutions by standard Lyusternik–Schnirelman theory. Also known are many results on existence and multiplicity of sign-changing solutions for small and large inhomogeneous perturbations (see [2, 5, 18, 19, 23, 25]), whereas in [16] it was proved that (1.1) does not admit any positive solution if $\varepsilon > 0$ is too large.

In the critical case, $p = (N+2)/(N-2)$, the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is continuous but not compact, so that the (PS) condition does not hold, and serious difficulties in facing the existence question arise. In fact, Pohozaev [17] proved that (1.1) has no solution if $f = 0$ and Ω is strictly star-shaped. In contrast, Brezis and Nirenberg [7] showed that this situation can be reverted by introducing suitable additive perturbations. Rey [20] pointed out that the result in [6] implies that if $f \geqslant 0$, $f \neq 0$ and $f \in H^{-1}(\Omega)$, then at least two positive solutions exist provided that $\varepsilon > 0$ is sufficiently small. Moreover, in [20] it was proven that if $f \geqslant 0$, $f \not\equiv 0$, is sufficiently regular, then at least $\operatorname{cat}(\Omega) + 1$ positive solutions exist for $\varepsilon > 0$ sufficiently small, one of them converging uniformly to 0 while the others

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concentrate at some special points in Ω, depending on f and the regular part of Green's function of the Laplacian on Ω, as $\varepsilon \to 0$. In parallel to Rey's result in [20], but with a different approach, Tarantello [26] proved that (1.1) admits at least two solutions for $f \not\equiv 0$ satisfying $\|\varepsilon f\|_{H^{-1}(\Omega)} < C_N$, where C_N is an explicit constant; such solutions are positive if $f \geqslant 0$. The effect of the symmetries in further multiplicity of solutions has been considered in some works. Ali and Castro [1] proved that the existence result in [7] is optimal for positive solutions in a ball: if Ω is a ball and $f \equiv 1$, problem (1.1) has exactly two positive solutions for all sufficiently small $\varepsilon > 0$. More recently, Clapp *et al*. [9] proved that if Ω is symmetric with respect to 0, $0 \notin \Omega$, and f is even, then at least $\operatorname{cat}(\Omega) + 2$ positive solutions exist provided that $\|\varepsilon f\|_{H^{-1}}$ is sufficiently small. The results in [1, 7, 9, 20, 26] deal with the existence of positive solutions to problem (1.1), provided that $f \geqslant 0$ and $f \neq 0$, where $\varepsilon > 0$ is a small parameter.

Concerning solutions which are not necessarily positive, Clapp *et al*. [10] showed the existence of solutions of (1.1) under certain symmetry assumptions in the domain Ω and the function f. Such solutions develop k negative spikes, for any $k \geqslant k_0(\Omega)$, where $k_0(\Omega)$ is a sufficiently large number depending on Ω.

In this paper we leave aside any symmetry assumptions on the domain Ω and the perturbation f, and we find solutions to problem (1.1) developing a negative double-spike shape. Additionally, we give precise information about the asymptotic profile of the blow-up of these solutions as $\varepsilon \to 0$ and we indicate a clearly delimited region where the spikes are formed.

More precisely, our setting in problem (1.1) is as follows: let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B(P, \mu)}, \tag{1.2}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $P \in \mathcal{D}$ and $\mu > 0$ is a small number. Let us consider $f \in C^{0,\gamma}(\bar{\Omega})$, for some $0 < \gamma < 1$, such that $\inf_{x \in \Omega} f(x) > 0$ and, by simplicity, we fix $P = 0$. Then our main result is as follows.

Theorem 1.1. *There exists a constant $\mu_0 = \mu_0(f, \mathcal{D}) > 0$, such that for each $0 < \mu < \mu_0$ fixed, there exists a number $\varepsilon_0 > 0$ and a family of solutions u_ε of (1.1), for $0 < \varepsilon = \varepsilon_n < \varepsilon_0$, with the following property: u_ε has exactly a pair of local minimum points $(\xi_1^\varepsilon, \xi_2^\varepsilon) \in \Omega^2$ with $k_* \mu < |\xi_i^\varepsilon| < k^* \mu$, $i = 1, 2$, for certain constants k_*, k^* independent of μ and such that, for each small $\delta > 0$,*

$$\inf_{\{|x - \xi_i^\varepsilon| > \delta,\ i=1,2\}} u_\varepsilon(x) \to 0 \quad \textit{and} \quad \inf_{\{|x - \xi_i^\varepsilon| < \delta\}} u_\varepsilon(x) \to -\infty, \quad i = 1, 2,$$

as $\varepsilon \to 0$.

Indeed, we will find that u_ε is a non-trivial solution of (1.1) of the form

$$u_\varepsilon(x) = -\alpha_N \sum_{i=1}^{2} \left\{ \frac{\varepsilon^{2/(N-2)} \lambda_{i\varepsilon}}{\varepsilon^{4/(N-2)} \lambda_{i\varepsilon}^2 + |x - \xi_i^\varepsilon|^2} \right\}^{(N-2)/2} + \varepsilon^{-1} \hat{\phi}(x) + \theta_\varepsilon(x),$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, $\hat{\phi}$ is the unique solution of the problem

$$\begin{aligned} -\Delta \hat{\phi}(x) &= \varepsilon^2 f(x) \quad \text{in } \Omega, \\ \hat{\phi} &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}$$

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$\alpha_N = (N(N-2))^{(N-2)/4}$ and the points $\xi_i^\varepsilon \to \xi_i$, up to subsequences, where (ξ_1, ξ_2) is a critical point of the functional

$$\varPhi(x,y) = \frac{1}{2}\left\{\frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}\right\}$$

defined in the region $\{(x,y) \in \varOmega^2 : G(x,y) - H^{1/2}(x,x)H^{1/2}(y,y) > 0,\ x \neq y\}$. Here G and H are, respectively, Green's function of the Laplacian on $\varOmega$ and its regular part, and w is the unique solution of the problem

$$\begin{aligned} -\Delta w &= f \quad \text{in } \varOmega, \\ w &= 0 \quad \text{on } \partial\varOmega. \end{aligned}$$

Additionally, one can identify the limits λ_i of $\lambda_{i\varepsilon}$ as

$$\lambda_i = \left(a_N^{-1}\frac{H(\xi_j,\xi_j)w(\xi_i) + G(\xi_i,\xi_j)w(\xi_j)}{G^2(\xi_i,\xi_j) - H(\xi_i,\xi_i)H(\xi_j,\xi_j)}\right)^{2/(N-2)}, \quad i \neq j,\ i,j = 1,2,$$

where a_N is an explicit constant, and consider the constants k_*, k^* as follows: k_* is the unique solution in $]1, +\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2} + (s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and $K \leqslant k^* = k*(\varOmega, f) < \infty$, where K is the unique solution in $]1, +\infty[$ of the equation

$$\frac{2^{1-N}}{s^N} = \frac{(s^2-1)^{N-1} + (s^2+1)^{N-1}}{(s^4-1)^{N-1}}.$$

In particular, if f is a constant and $\varOmega$ is an annulus, then $k^* = K$.

On the other hand, it will be clear from the proof that the small excised domain does not need to be exactly a ball, and we consider this case just for notational simplicity.

The proof of theorem 1.1 follows a Lyapunov–Schmidt reduction procedure, related with this problem. This method has been used for solving problem (1.1) in the critical case (see [10, 20]) and in the slightly supercritical case with $f = 0$ (see [12, 13], and also [21, 22] for related results).

In the next section we derive some basic estimates for the *reduced energy* associated with this problem. Sections 3 and 4 will be devoted to discussion of the finite-dimensional reduction scheme which we use for the construction of solutions of (1.1). In § 5 we introduce an auxiliary function which will be the key in our min–max scheme, which we develop in § 6 to finally establish theorem 1.1.

2. Basic estimates in the reduced energy

Let $\varOmega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, and let us consider the expanded domain

$$\varOmega_\varepsilon = \varepsilon^{-2/(N-2)}\varOmega, \quad \varepsilon > 0.$$

Using the change of variable

$$v_\varepsilon(x') = -\varepsilon u(\varepsilon^{2/(N-2)}x'), \quad x' \in \varOmega_\varepsilon,$$

we note that u solves (1.1) if and only if v_ε solves

$$\left.\begin{aligned} \Delta v + |v|^{p-1}v &= \varepsilon^{p+1}\tilde{f}(x') \quad \text{in } \Omega_\varepsilon, \\ v &= 0 \quad\quad\quad\quad\quad \text{on } \partial\Omega_\varepsilon, \end{aligned}\right\} \tag{2.1}$$

where $p = (N+2)/(N-2)$ and $\tilde{f}(x') = f(\varepsilon^{2/(N-2)}x')$. It is well known that all positive solutions of equation $\Delta\vartheta + \vartheta^p = 0$ in $\mathbb{R}^N$ are given by the functions

$$\bar{U}_{\lambda,\xi}(x) = \alpha_N\left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{(N-2)/2},$$

with $\lambda > 0$, $\xi \in \mathbb{R}^N$ and $\alpha_N = (N(N-2))^{(N-2)/4}$ [3,7,8,24]. Since Ω_ε is expanding to the whole $\mathbb{R}^N$ as $\varepsilon \to 0$, and $\varepsilon^{p+1}\tilde{f}(x') \to 0$ uniformly as $\varepsilon \to 0$, it is reasonable to assume that, for certain numbers $\lambda_1, \lambda_2 > 0$ and points $\xi_1, \xi_2 \in \Omega$, some solution v_ε of (2.1) becomes

$$v_\varepsilon \sim \bar{U}_{\lambda_1,\xi_1'} + \bar{U}_{\lambda_2,\xi_2'},$$

where $\xi_i' = \varepsilon^{-2/(N-2)}\xi_i \in \Omega_\varepsilon$, and where from now on ξ denotes a point in Ω and ξ' denotes a point in Ω_ε.

From [11], we know that a better approximation to v_ε should be obtained by using the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar{U}_{\lambda,\xi'}$, denoted by $U_{\lambda,\xi'}$, namely the unique solution of the problem

$$\begin{aligned} -\Delta U_{\lambda,\xi'} &= \bar{U}^p_{\lambda,\xi'} \quad \text{in } \Omega_\varepsilon, \\ U_{\lambda,\xi'} &= 0 \quad\quad\; \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

In other words, $U_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} - \bar{v}_{\lambda,\xi'}$, where $\bar{v}_{\lambda,\xi'}$ solves

$$\begin{aligned} -\Delta \bar{v}_{\lambda,\xi'} &= 0 \quad\quad\; \text{in } \Omega_\varepsilon, \\ \bar{v}_{\lambda,\xi'} &= \bar{U}_{\lambda,\xi'} \quad \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

Hence, if we consider $\bar{U} = \bar{U}_{1,0}$, we obtain

$$\bar{v}_{\lambda,\xi'}(x') = \varepsilon^2\lambda^{(N-2)/2}H(\varepsilon^{2/(N-2)}x', \xi)\int_{\mathbb{R}^N}\bar{U}^p + o(\varepsilon^2) \tag{2.2}$$

and, away from $x' = \xi'$,

$$U_{\lambda,\xi'}(x') = \varepsilon^2\lambda^{(N-2)/2}G(\varepsilon^{2/(N-2)}x', \xi)\int_{\mathbb{R}^N}\bar{U}^p + o(\varepsilon^2) \tag{2.3}$$

uniformly for x' on each compact subset of Ω_ε, where G and H are, respectively, Green's function of the Laplacian with the Dirichlet boundary condition on Ω and its regular part. Now, to simplify notation, we consider the function

$$V(x') = U_1(x') + U_2(x'), \quad x' \in \Omega_\varepsilon,$$

where $U_i = U_{\lambda_i,\xi_i'}$, $i = 1, 2$, and we set $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$. Then, we look for solutions of problem (2.1) of the form

$$v(x') = V(x') + \tilde{\eta}(x'), \quad x' \in \Omega_\varepsilon, \tag{2.4}$$

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which for suitable points ξ and scalars λ will have the remainder term $\tilde{\eta}$ of small order all over Ω_ε. Since solutions of (2.1) correspond to stationary points of its associated energy functional J_ε defined by

$$J_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |v|^{p+1} + \varepsilon^{p+1}\int_{\Omega_\varepsilon} \tilde{f} v, \tag{2.5}$$

we have that if a solution of the form (2.4) exists, then we should have $J_\varepsilon(v) \sim J_\varepsilon(V)$ and the corresponding points $(\boldsymbol{\xi}, \boldsymbol{\lambda})$ in the definition of V also should be 'approximately stationary' for the finite-dimensional functional $(\boldsymbol{\xi}, \boldsymbol{\lambda}) \mapsto J_\varepsilon(V)$. Thus, our first goal is to estimate $J_\varepsilon(V)$. In order to establish the expansion, we consider the function w, which corresponds to the unique solution in $C^{0,\gamma}(\Omega)$ of the problem

$$\left.\begin{aligned} -\Delta w &= f \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.6}$$

and we make the following choice of the points and parameters: we fix $\delta > 0$ and we define the parameters λ_i as

$$\lambda_i = (a_N^{-1}\Lambda_i)^{2/(N-2)}, \quad i = 1, 2,$$

where $a_N = \int_{\mathbb{R}^N} \bar{U}^p$ and $\Lambda_i \in]\delta, \delta^{-1}[$, for $i = 1, 2$. We also define the set

$$\mathcal{M}_\delta = \{(\boldsymbol{\xi}, \boldsymbol{\Lambda}) : |\xi_1 - \xi_2| > \delta,\ \operatorname{dist}(\xi_i, \partial\Omega) > \delta;\ i = 1, 2\}, \tag{2.7}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2) \in]\delta, \delta^{-1}[$.

Lemma 2.1. *Let $\delta > 0$ given. The expansion*

$$J_\varepsilon(V) = 2C_N + \varepsilon^2\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) + o(\varepsilon^2)$$

holds uniformly in the C^1-sense, with respect to $(\boldsymbol{\xi}, \boldsymbol{\Lambda})$ in $\mathcal{M}_\delta$. Here

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1} \tag{2.8}$$

and the function Φ is defined by

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \frac{1}{2}\left\{\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right\} + \sum_{i=1}^{2} \Lambda_i w(\xi_i). \tag{2.9}$$

The proof of the previous lemma is based on (2.2), (2.3) and some estimates established in [4], and follows a similar procedure to that used to prove [13, lemma 3.2] and [10, proposition 1]; it is therefore omitted here.

3. The finite-dimensional reduction

We first introduce some notation to be used in what follows. For functions u, v defined in Ω_ε we set

$$\langle u, v\rangle = \int_{\Omega_\varepsilon} uv.$$

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Let us fix a small number $\delta > 0$ and consider points $(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ in

$$\mathcal{M}_\delta^\varepsilon = \{(\boldsymbol{\xi}', \mathbf{\Lambda}) \in \Omega_\varepsilon^2 \times]\delta, \delta^{-1}[^2 \colon |\xi_1' - \xi_2'| > \delta_\varepsilon,\ \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta_\varepsilon;\ i = 1, 2\}, \tag{3.1}$$

where $\delta_\varepsilon = \delta\varepsilon^{-2/(N-2)}$, $\boldsymbol{\xi}' = (\xi_1', \xi_2')$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2)$. Since all solutions ϑ of the problem $\Delta\vartheta + p\bar{U}_{\Lambda,0}^{p-1}\vartheta = 0$ in $\mathbb{R}^N$ which satisfy $|\vartheta(x)| < C|x|^{2-N}$ belong to

$$\operatorname{span}\left\{\frac{\partial \bar{U}_{\Lambda,0}}{\partial x_j}, \frac{\partial \bar{U}_{\Lambda,0}}{\partial \Lambda}\right\}_{j=1,\dots,N+1}$$

(see [8]), it is convenient to consider, for $i = 1, 2$, the functions

$$\bar{Z}_{ij}(x') = \frac{\partial \bar{U}_i}{\partial \xi_{ij}'}(x'), \quad j = 1, \dots, N, \qquad \bar{Z}_{i(N+1)}(x') = \frac{\partial \bar{U}_i}{\partial \Lambda_i}(x'),$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely, the unique solutions of

$$\begin{aligned} \Delta Z_{ij} &= \Delta \bar{Z}_{ij} \quad \text{in } \Omega_\varepsilon, \\ Z_{ij} &= 0 \qquad\ \ \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

In order to simplify notation, we will define

$$V = U_1 + U_2 \quad \text{and} \quad \bar{V} = \bar{U}_1 + \bar{U}_2.$$

We start by studying a linear problem which is the basis for the reduction of (2.1): given $h \in L^\infty(\bar{\Omega}_\varepsilon)$, find a function η and constants c_{ij} such that

$$\left.\begin{aligned} \Delta\eta + p|V|^{p-1}\eta &= h + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \eta &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \langle \eta, U_i^{p-1} Z_{ij} \rangle &= 0 && \text{for all } i, j. \end{aligned}\right\} \tag{3.2}$$

We want to prove that this problem is uniquely solvable with uniform bounds in certain appropriate norms. In other words, we want study the linear operator L_ε associated with (3.2), namely

$$L_\varepsilon(\eta) = \Delta\eta + p|V|^{p-1}\eta, \tag{3.3}$$

under the previous orthogonality conditions. In order to achieve this goal, we introduce the following L^∞-norms with weight. Let $\omega_i = (1 + |x' - \xi_i'|^2)^{-(N-2)/2}$, $i = 1, 2$; for a function θ defined in Ω_ε, we consider the norms

$$\|\theta\|_* = \|(\omega_1 + \omega_2)^{-\sigma}\theta(x')\|_\infty + \|(\omega_1 + \omega_2)^{-\sigma-1}\nabla\theta(x')\|_\infty,$$

where $\sigma = \frac{1}{2}$ if $3 \leqslant N \leqslant 6$, $\sigma = 2/(N-2)$ if $N \geqslant 7$ and

$$\|\theta\|_{**} = \|(\omega_1 + \omega_2)^{-\varsigma}\theta(x')\|_\infty,$$

where $\varsigma = \frac{1}{2}p$ if $3 \leqslant N \leqslant 6$ and $\varsigma = 4/(N-2)$ if $N \geqslant 7$. These norms are similar to those defined in [10] for $N \geqslant 7$ but, for $3 \leqslant N \leqslant 6$, we have modified them, something apparently necessary in this case, since $p \geqslant 2$. Now, we study the invertibility of the linear operator L_ε defined in (3.3). Hence, it is also important to understand the differentiability of L_ε in the variables $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

PROPOSITION 3.1. *Assume that $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. There then exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ and for all $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, the problem (3.2) admits a unique solution $\eta \equiv M_\varepsilon(h)$. Moreover, the map $(\boldsymbol{\xi}', \boldsymbol{\Lambda}, h) \mapsto \eta \equiv M_\varepsilon(h)$ is of class C^1 and satisfies*

$$\|\eta\|_* \leqslant C\|h\|_{**} \quad \textit{and} \quad \|\nabla_{(\boldsymbol{\xi}', \boldsymbol{\Lambda})}\eta\|_* \leqslant C\|h\|_{**}.$$

The proof of this proposition follows from a slight variation of the arguments in the proof of [13, propositions 4.1 and 4.2] with the necessary modifications in [14], so we omit it here. In what follows, C represents a generic positive constant that is independent of ε and of the particular points $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

Now, we are ready to begin the finite-dimensional reduction. We want to solve the following nonlinear problem: find a function $\tilde{\eta}$ such that, for certain constants c_{ij}, $i = 1, 2$, $j = 1, \dots, N + 1$, one has

$$\left.\begin{aligned} \Delta(V + \tilde{\eta}) + |V + \tilde{\eta}|^{p-1}(V + \tilde{\eta}) - \varepsilon^{p+1}\tilde{f} &= \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \tilde{\eta} &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \langle \tilde{\eta}, U_i^{p-1} Z_{ij}\rangle &= -\langle \phi, U_i^{p-1} Z_{ij}\rangle && \text{for all } i, j, \end{aligned}\right\} \tag{3.4}$$

where ϕ solves the problem

$$\left.\begin{aligned} -\Delta\phi &= \varepsilon^{p+1}\tilde{f} && \text{in } \Omega_\varepsilon, \\ \phi &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned}\right\} \tag{3.5}$$

Note that $V + \tilde{\eta}$ is a solution of (2.1) if the scalars c_{ij} in (3.4) are all zero. Also, we note that the partial differential equation in (3.4) is equivalent in Ω_ε to

$$\Delta\eta + p|V|^{p-1}\eta = -N_\varepsilon(\eta) - R_\varepsilon + \sum_{i,j} c_{ij} U_i^{p-1},$$

where $\eta = \tilde{\eta} - \phi$,

$$N_\varepsilon(\eta) = |V + \eta - \phi|^{p-1}(V + \eta - \phi)_+ - |V|^{p-1}V - p|V|^{p-1}(\eta - \phi) \tag{3.6}$$

and

$$R_\varepsilon = |V|^{p-1}V - \bar{U}_1^p - \bar{U}_2^p - p|V|^{p-1}\phi. \tag{3.7}$$

A first step to solve (3.4) consists of dealing with the following nonlinear problem: find a function φ that, for certain constants c_{ij}, $i = 1, 2$, $j = 1, \dots, N + 1$, solves

$$\left.\begin{aligned} \Delta(V + \tilde{\eta}) + |V + \tilde{\eta}|^{p-1}(V + \tilde{\eta})_+ - \varepsilon^{p+1}\tilde{f} &= \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \varphi &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1} Z_{ij}\rangle &= 0 && \text{for all } i, j, \end{aligned}\right\} \tag{3.8}$$

where $\tilde{\eta} = \psi + \varphi - \phi$, with ϕ satisfying (3.5), and the function ψ is chosen as

$$\psi = -M_\varepsilon(R_\varepsilon), \tag{3.9}$$

where M_ε is defined as in proposition 3.1 and R_ε is given by (3.7). Actually, it is easy to check that, for points $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$, one has

$$\|\psi\|_* \leqslant C\varepsilon^2.$$

Now, in (3.8) we rewrite the equation of interest as

$$\Delta\varphi + p|V|^{p-1}\varphi = -N_\varepsilon(\eta) - (\Delta\psi + p|V|^{p-1}\psi + R_\varepsilon) + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij},$$

where $\eta = \psi + \varphi$.

Lemma 3.2. *Assume that $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$. There then exists $C > 0$ such that, for all $\varepsilon > 0$ small enough and $\|\varphi\|_* \leqslant \frac{1}{4}$, one has*

$$\|N_\varepsilon(\psi+\varphi)\|_{**} \leqslant \begin{cases} C(\|\varphi\|_*^2 + \varepsilon\|\varphi\|_* + \varepsilon^{p+1}) & \text{if } 3 \leqslant N \leqslant 6, \\ C(\varepsilon^{2(p-2)}\|\varphi\|_*^2 + \varepsilon^{p^2-3p+2}\|\varphi\|_*^p + \varepsilon^{p^2-p+2}) & \text{if } N \geqslant 7. \end{cases}$$

Proof. Note that $\|\phi\|_* \leqslant C\varepsilon^p$ if $3 \leqslant N \leqslant 6$, $\|\phi\|_* \leqslant C\varepsilon^2$ if $N \geqslant 7$ and $\|\psi\|_* \leqslant C\varepsilon^2$. Since $\|\psi + \varphi\|_* \leqslant \|\psi\|_* + \|\varphi\|_*$, for $\eta = \psi + \varphi$ we have that $\|\eta\|_* < 1$. Also we note that

$$N_\varepsilon(\eta) = C|V + \bar{t}(\eta - \phi)|^{p-2}(\eta - \phi)^2, \tag{3.10}$$

with $\bar{t} \in]0,1[$. Hence, if $3 \leqslant N \leqslant 6$, then

$$|(\omega_1+\omega_2)^{-p/2} N_\varepsilon(\eta)| \leqslant C(\omega_1+\omega_2)^{(p-1)/2}\|\eta - \phi\|_*^2 \leqslant C\|\eta - \phi\|_*^2.$$

On the other hand, for $N \geqslant 7$, if $|\eta| \leqslant \frac{1}{2}(\omega_1+\omega_2)$, we again use (3.10) to obtain

$$\begin{aligned} |(\omega_1+\omega_2)^{-4/(N-2)} N_\varepsilon(\eta)| &\leqslant C(\omega_1+\omega_2)^{(6-N)/(N-2)}\|\eta - \phi\|_*^2 \\ &\leqslant C\varepsilon^{(6-N)/(N-2)}\|\eta - \phi\|_*^2. \end{aligned}$$

In another case we obtain directly from (3.6) that

$$\begin{aligned} |(\omega_1+\omega_2)^{-4/(N-2)} N_\varepsilon(\eta)| &\leqslant C|(\omega_1+\omega_2)^{-4/(N-2)}(\eta - \phi)^p| \\ &\leqslant C\varepsilon^{(6-N)/(N-2)(2/(N-2))}\|\eta - \phi\|_*^p. \end{aligned}$$

The result follows on combining previous estimates. □

We now deal with the following problem:

$$\left.\begin{aligned} \Delta\varphi + pV^{p-1}\varphi = -N_\varepsilon(\eta) + \sum_{i,j} c_{ij}U_i^{p-1}Z_{ij} \quad & \text{in } \Omega_\varepsilon, \\ \varphi = 0 \quad & \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1}Z_{ij}\rangle = 0 \quad & \text{for all } i, j, \end{aligned}\right\} \tag{3.11}$$

where $\eta = \psi + \varphi$ and ψ is the function defined in (3.9).

Proposition 3.3. *Assume that $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}^\varepsilon_\delta$. There then exists $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution $\varphi = \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ to problem (3.11). Moreover, the map $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \mapsto \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ is of class C^1 for the $\|\cdot\|_*$-norm and it satisfies*

$$\|\varphi\|_* \leqslant C\varepsilon^2 \quad \text{and} \quad \|\nabla_{(\boldsymbol{\xi}', \boldsymbol{\Lambda})}\varphi\|_* \leqslant C\varepsilon^2.$$

Proof. Let us set

$$\mathcal{F}_r = \{\varphi \in H_0^1(\Omega_\varepsilon) : \|\varphi\|_* \leqslant r\varepsilon^2\},$$

with $r > 0$ a constant to be fixed later. We define the map $A_\varepsilon : \mathcal{F}_r \to H_0^1(\Omega_\varepsilon)$ as

$$A_\varepsilon(\varphi) = -M_\varepsilon(N_\varepsilon(\psi + \varphi)),$$

where M_ε is the operator defined in proposition 3.1. Since $\psi = -M_\varepsilon(R_\varepsilon)$, solving (3.11) is equivalent to finding a fixed point φ for A_ε. From proposition 3.1 and lemma 3.2, we deduce that if $\varphi \in \mathcal{F}_r$ and $\varepsilon > 0$ is small enough, then

$$\|A_\varepsilon(\varphi)\|_* \leqslant r\varepsilon^2$$

for a suitable choice of $r = r(N)$ which we consider fixed from now on. Note that, for $\varphi_1, \varphi_2 \in \mathcal{F}_r$, from lemma 3.2 we have

$$\|A_\varepsilon(\varphi_1) - A_\varepsilon(\varphi_2)\|_* \leqslant C\|N_\varepsilon(\psi + \varphi_1) - N_\varepsilon(\psi + \varphi_2)\|_{**} \leqslant C\varepsilon^p\|\varphi_1 - \varphi_2\|_*,$$

for all $N \geqslant 3$. It follows that, for $\varepsilon > 0$ small enough, the map A_ε is a contraction $\|\cdot\|_*$ in $\mathcal{F}_r$. Therefore, A_ε has a fixed point in $\mathcal{F}_r$.

Concerning differentiability properties, let us recall that $\eta = \psi + \varphi$ is defined by the relation

$$B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta) \equiv \eta + M_\varepsilon(N_\varepsilon(\psi + \varphi)) = 0.$$

We see that

$$D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)[\theta] = \theta + M_\varepsilon(\theta D_\eta N_\varepsilon(\psi + \varphi)) \equiv \theta + \tilde{M}(\theta)$$

and check that

$$\|\tilde{M}(\theta)\|_* \leqslant C\varepsilon\|\theta\|_*.$$

This implies that, for ε small, the linear operator $D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)$ is invertible in the space of the continuous functions in Ω_ε with bounded $\|\cdot\|_*$-norm, with a uniformly bounded inverse depending continuously on its parameters.

Now, let us consider the differentiability with respect to the $\boldsymbol{\xi}'$ variable; for simplicity we write

$$\frac{\partial}{\partial \xi'_{ij}} = \partial_{\xi'_{ij}}.$$

Then

$$\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta) = \partial_{\xi'_{ij}} M_\varepsilon(N_\varepsilon(\psi + \varphi)) + M_\varepsilon(\partial_{\xi'_{ij}} N_\varepsilon(\psi + \varphi)) \\ + M_\varepsilon(D_\eta N_\varepsilon(\psi + \varphi)\partial_{\xi'_{ij}}\psi).$$

It is clear that all expressions which define to $\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)$ depend continuously on their parameters. Applying the implicit function theorem, we find that $\varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ is a C^1-function in L_*^∞. Additionally, we obtain

$$\partial_{\xi'_{ij}} \varphi = -(D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta))^{-1}(\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta))$$

and, using the first part of this proposition, the estimates in the previous lemmas, proposition 3.1 and the fact that $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$, we conclude that

$$\|\partial_{\xi'_{ij}}\varphi\|_* \leqslant C(\|N_\varepsilon(\psi+\varphi)\|_{**} + \|\partial_{\xi'_{ij}} N_\varepsilon(\psi+\varphi)\|_{**} + \|D_\eta N_\varepsilon(\psi+\varphi)\partial_{\xi'_{ij}}\psi\|_{**}) \leqslant C\varepsilon^2.$$

Similarly, we can analyse the differentiability of B with respect to $\boldsymbol{\Lambda}$. This finishes the proof. □

4. The reduced functional

Now we are ready to solve the full problem. Let us consider $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$ with $\mathcal{M}_\delta^\varepsilon$ defined by (3.1). All the estimates obtained below will be uniform on these points. Let $\varphi = \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ be the unique solution, given by proposition 3.3, of problem (3.8) with $\tilde{\eta} = \psi + \varphi - \phi$, where φ solves (3.9) and ϕ solves (3.5). Note that if $\boldsymbol{\xi} = \varepsilon^{2/(N-2)}\boldsymbol{\xi}' \in \Omega^2$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ so that $c_{ij} = 0$ for all i, j, then a solution of (1.1) is

$$u(x) = -\varepsilon^{-1} v(\varepsilon^{-2/(N-2)} x), \quad x \in \Omega,$$

where $v = V + \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi$. Hence, u will be a critical point of

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \varepsilon \int_\Omega f u,$$

while v will be one of J_ε given by (2.5). Then it is convenient to consider the following functions defined in Ω:

$$\hat{U}_i(x) = \varepsilon^{-1} U_i(\varepsilon^{-2/(N-2)} x) = U_{\lambda_i^\varepsilon, \xi_i}(x), \quad \hat{\psi}(x) = \varepsilon^{-1}\psi(\varepsilon^{-2/(N-2)} x),$$
$$\hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda})(x) = \varepsilon^{-1}\varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})(\varepsilon^{-2/(N-2)} x) \qquad \hat{\phi}(x) = \varepsilon^{-1}\phi(\varepsilon^{-2/(N-2)} x).$$

Note that $\hat{U}_i = U_{\lambda_{i\varepsilon}, \xi_i}$, where $\lambda_{i\varepsilon} = (c_N \Lambda_i^2 \varepsilon)^{2/(N-2)} \in \mathbb{R}_+$ and $\boldsymbol{\xi} = \varepsilon^{2/(N-2)}\boldsymbol{\xi}'$, with $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \mathcal{M}_\delta$ defined by (2.7). Now, let us set $\hat{U} = \hat{U}_1 + \hat{U}_2$. Consider now the functional

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \equiv I_\varepsilon(\hat{U} + \hat{\psi} + \hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - \hat{\phi}). \tag{4.1}$$

It is easy to check that

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\lambda}) = J_\varepsilon(V + \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi).$$

Then, setting $\tilde{\eta} = \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi$, one shows that $DJ_\varepsilon(V + \tilde{\eta})[\vartheta] = 0$ for all $\vartheta \in H_\varepsilon$, where $H_\varepsilon = \{\vartheta \in H_0^1(\Omega_\varepsilon) : \langle \vartheta,\ V_i^{p-1} Z_{ij}\rangle = 0 \text{ for all } i, j\}$. Also one has

$$\frac{\partial V}{\partial \xi'_{lk}} = Z_{lk} + o(1) \quad \text{for all } l, k, \qquad \frac{\partial V}{\partial \Lambda_{l(N+1)}} = Z_{l(N+1)} + o(1) \quad \text{for all } l,$$

with $o(1) \to 0$ in the $\|\cdot\|_*$-norm as $\varepsilon \to 0$. Then from proposition 3.3 we obtain the following basic result.

LEMMA 4.1. *The function $u = \hat{U} + \hat{\psi} + \hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - \hat{\phi}$ is a solution of problem (1.1) if only if $(\boldsymbol{\xi}, \boldsymbol{\Lambda})$ is a critical point of $\mathcal{I}$.*

Next step is then to give an asymptotic estimate for $\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda})$. Set

$$\sigma_f = \int_\Omega f(x) w(x)\, \mathrm{d}x, \tag{4.2}$$

where w is the solution of (2.6). We then have the following proposition.

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PROPOSITION 4.2. *The following expansion holds:*

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = 2C_N + \varepsilon^2\{\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) + \sigma_f\} + o(\varepsilon^2)\theta(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \tag{4.3}$$

uniformly in the C^1-sense with respect to $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \mathcal{M}_\delta$, where θ is a bounded uniformly function independently of $\varepsilon > 0$. Here C_N is the constant given by (2.8) and Φ is the function given by (2.9).

Proof. The first step to achieve our goal is to prove that

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = o(\varepsilon^2) \tag{4.4}$$

and

$$\nabla_{(\boldsymbol{\xi}, \boldsymbol{\Lambda})}(\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi})) = o(\varepsilon^2). \tag{4.5}$$

Let us set $\vartheta = V + \psi - \phi$ and note that

$$\begin{aligned}\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = &-\int_0^1 t\biggl(\int_{\Omega_\varepsilon} N_\varepsilon(\psi + \varphi)\varphi\biggr)\,\mathrm{d}t \\ &+ \int_0^1 t\biggl(\int_{\Omega_\varepsilon} p(|V|^{p-1} - |\vartheta + t\varphi|^{p-1})\varphi^2\biggr)\,\mathrm{d}t.\end{aligned}$$

Now, differentiating with respect to the $\boldsymbol{\xi}$ variable, we obtain

$$\begin{aligned}D_{\boldsymbol{\xi}}(\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - I_\varepsilon(\hat{\vartheta})) = &-\varepsilon^{-2/(N-2)}\int_0^1 t\int_{\Omega_\varepsilon} p\nabla_{\boldsymbol{\xi}'}[|\vartheta + t\varphi|^{p-1}\varphi^2 - |V|^{p-1}\varphi^2]\,\mathrm{d}t \\ &- \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon} \nabla_{\boldsymbol{\xi}'}(N_\varepsilon(\psi + \varphi)\varphi).\end{aligned}$$

Bearing in mind that $\|N_\varepsilon(\psi + \varphi)\|_* + \|\varphi\|_* + \|\psi\|_* + \|\nabla_{\xi_i'}\varphi\|_* + \|\nabla_{\xi_1'}\psi\|_* \leqslant O(\varepsilon^2)$, we see that (4.4) and (4.5) hold.

A second step is to prove that

$$I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi}) = o(\varepsilon^2) \tag{4.6}$$

and

$$\nabla_{(\boldsymbol{\xi}, \boldsymbol{\Lambda})}(I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi})) = o(\varepsilon^2). \tag{4.7}$$

Put $\eta = V - \phi$ and, by the fundamental calculus theorem, note that

$$\begin{aligned}I_\varepsilon(\hat{\eta} + \hat{\psi}) - I_\varepsilon(\hat{\eta}) = &\int_0^1 (1 - t)\biggl(\int_{\Omega_\varepsilon} p|\eta + t\psi|^{p-1}\psi^2 - \int_{\Omega_\varepsilon} |\nabla\psi|^2\biggr)\,\mathrm{d}t \\ &+ \int_{\Omega_\varepsilon} (|V|^p - |\eta|^p - p|V|^{p-1}\phi)\psi + \int_{\Omega_\varepsilon} R_\varepsilon\psi.\end{aligned} \tag{4.8}$$

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Now, differentiating with respect to $\boldsymbol{\xi}$ variables, we obtain

$$
\begin{aligned}
D_{\boldsymbol{\xi}}(I_\varepsilon(\hat\eta+\hat\psi)-I_\varepsilon(\hat\eta)) = \varepsilon^{-2/(N-2)}\int_0^1(1-t)\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(p|\eta+t\psi|^{p-1}\psi^2-|\nabla\psi|^2)\,\mathrm{d}t \\
+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(|V|^p-|\eta|^p-p|V|^{p-1}\phi)\psi \\
+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}(|V|^p-|\eta|^p-p|V|^{p-1}\phi)\nabla_{\boldsymbol{\xi}'}\psi \\
+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}R_\varepsilon\psi+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}R_\varepsilon\nabla_{\boldsymbol{\xi}'}\psi.
\end{aligned}
$$

Since $\|R_\varepsilon\|_{**}+\|\nabla_{\xi_i'}R_\varepsilon\|_{**}+\|\phi\|_\infty+\|\psi\|_*+\|\nabla_{\xi_i'}\psi\|_*\leqslant O(\varepsilon^2)$ and $\|\phi\|_*\leqslant O(\varepsilon^p)$ if $3\leqslant N\leqslant 6$, $\|\phi\|_*\leqslant O(\varepsilon^2)$ if $N\geqslant 7$, we obtain the result that (4.6) and (4.7) hold.

Finally, we need only the following two estimates to hold:

$$
I_\varepsilon(\hat V-\hat\phi)-I_\varepsilon(\hat V)=\varepsilon^2\sigma_f+o(\varepsilon^2), \tag{4.9}
$$

where σ_f is given by (4.2), and

$$
D_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}(I_\varepsilon(\hat V-\hat\phi)-I_\varepsilon(\hat V))=o(\varepsilon^2). \tag{4.10}
$$

Now, we have

$$
\begin{aligned}
I_\varepsilon(\hat V-\hat\phi)-I_\varepsilon(\hat V)=\int_0^1\biggl(\int_{\Omega_\varepsilon}|\nabla\phi|^2-\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\biggr)\,\mathrm{d}t \\
+\int_{\Omega_\varepsilon}(\bar U_1^p+\bar U_2^p-|V-t\phi|^p)\phi. \qquad (4.11)
\end{aligned}
$$

Note that

$$
\int_0^1 t\int_{\Omega_\varepsilon}|\nabla\phi|^2\,\mathrm{d}t=\int_{\Omega_\varepsilon}|\nabla\phi|^2=\varepsilon^{p+1}\int_{\Omega_\varepsilon}\tilde f\phi=\varepsilon^2\int_\Omega fw=\varepsilon^2\sigma_f,
$$

and since $\|\phi\|_\infty\leqslant O(\varepsilon^{p+1})$, we have that

$$
\biggl|\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\biggr|\leqslant C\varepsilon^4\int_{\Omega_\varepsilon}(\omega_1+\omega_2)^{p-1}\leqslant o(\varepsilon^2).
$$

On the other hand, it is not difficult to check that

$$
\begin{aligned}
\biggl|\int_{\Omega_\varepsilon}\biggl(\sum_{i=1}^2\bar U_i^p-|V-t\phi|^p\biggr)\phi\biggr| = \biggl|\int_{\Omega_\varepsilon}R_\varepsilon\phi+\int_{\Omega_\varepsilon}(|V|^p-|V-t\phi|^p-p|V|^{p-1}\phi)\phi\biggr| \\
\leqslant o(\varepsilon^2).
\end{aligned}
$$

The above estimates yield (4.9). Now, from (4.11) we obtain

$$
\begin{aligned}
D_{\boldsymbol{\xi}}(I_\varepsilon(\hat V-\hat\phi)-I_\varepsilon(\hat V))=\varepsilon^{-2/(N-2)}\int_0^1 t\int_{\Omega_\varepsilon}p|V-t\phi|^{p-2}\nabla_{\boldsymbol{\xi}'}V\phi^2\,\mathrm{d}t \\
+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(\bar U_1^p+\bar U_2^p-|V-t\phi|^p)\phi, \qquad (4.12)
\end{aligned}
$$

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but since $\|\phi\|_\infty \leqslant O(\varepsilon^{p+1})$, it is easy to check that (4.10) holds. Similarly, our results hold for differentiability with respect to $\boldsymbol{\Lambda}$. □

REMARK 4.3. Lemma 2.1 and the previous proposition yield

$$\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) = \varepsilon^2\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) + o(\varepsilon^2)\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\theta(\boldsymbol{\xi},\boldsymbol{\Lambda}), \tag{4.13}$$

uniformly with respect to $(\boldsymbol{\xi},\boldsymbol{\Lambda}) \in \mathcal{M}_\delta$, where θ and $\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\theta$ are bounded uniformly functions, independently of all $\varepsilon > 0$ small.

5. An auxiliary function on the exterior domain

In this section we consider the domain Ω defined in (1.2) with $P = 0$, $\mu > 0$ small and fixed and we assume that $f \in C^{0,\gamma}(\bar{\Omega})$, for some $0 < \gamma < 1$, with $\min_{x\in\Omega} f(x) = \alpha > 0$. Let w be the unique solution in $C^{2,\gamma}(\bar{\Omega})$ of problem (2.6). It is then easy to check that $w_\mu(x) = \mu^{-2}w(\mu x)$ is the unique $C^{2,\gamma}(\overline{\mu^{-1}\Omega})$ solution of the problem

$$\begin{aligned} -\Delta w_\mu &= \hat{f} \quad \text{in } \mu^{-1}\Omega, \\ w_\mu &= 0 \quad \text{on } \partial(\mu^{-1}\Omega), \end{aligned}$$

where $\hat{f}(x) = f(\mu x)$ for $x \in (\mu^{-1}\Omega)$.

Now, we consider the exterior domain

$$E = \mathbb{R}^N \setminus \overline{B(0,1)}$$

and we denote by G_E and H_E, respectively, Green's function on E and its regular part. By convenience, in the set

$$\boldsymbol{V} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : G_E(x,y) - H_E^{1/2}(x,x)H_E^{1/2}(y,y) > 0\} \cap (\mu^{-1}\Omega)$$

we define the function

$$\Phi_E(x,y) = \frac{1}{2}\left\{\frac{H_E(x,x)w_\mu^2(y) + 2G_E(x,y)w_\mu(x)w_\mu(y) + H_E(y,y)w_\mu^2(x)}{G_E^2(x,y) - H_E(x,x)H_E(y,y)}\right\}.$$

Then, if x and y are variable vectors whose magnitudes remain constant and we differentiate Φ_E with respect to the angle θ formed between them, we obtain

$$\frac{\partial}{\partial\theta}\Phi_E(x,y) = F(x,y,\theta)\sin\theta$$

for $0 < \theta < \pi$. Since $F(x,y,\theta) > 0$ for all $\theta \in]0,\pi[$, $(x,y) \in \boldsymbol{V}$, we have that for given magnitudes $|x|$ and $|y|$, Φ_E maximizes its value when $\theta = \pi$, is to say when x and y have opposite directions. In the rest of this section we assume that this is the situation.

5.1. A first step to the auxiliary function: a radial case

In this subsection we consider a fixed constant $T > 0$ and the domain

$$\Omega := \mathcal{A}_\mu = \{x \in \mathbb{R}^N : 1 < |x| < \mu^{-1}\} \quad \text{and} \quad f \equiv 1.$$

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We write $R := R(\mu, T) = \mu^{-1}T$ so that $w_\mu \in C^{2,\gamma}(\bar{\mathcal{A}}_\mu)$ is defined by

$$w_\mu(x) := W_R(x) = \frac{1}{2N}\left\{\frac{R^2-1}{R^{2-N}-1}|x|^{2-N} - |x|^2 + R^{2-N}\frac{1-R^N}{R^{2-N}-1}\right\}.$$

From the maximum principle we have that W_R is strictly positive in $\mathcal{A}_\mu$. Additionally, it achieves its maximum value in

$$x_\mu^* \in \mathbb{R}^N \quad \text{such that } |x_\mu^*| = R_\mu^* = \left(\frac{(N-2)R^{N-2}(R^2-1)}{2(R^{N-2}-1)}\right)^{1/N}. \tag{5.1}$$

Note that $R_\mu^* \to +\infty$ as $\mu \to 0$. Now we consider an unitary vector $\boldsymbol{e}$ and we set $x = s\boldsymbol{e}$, $y = -t\boldsymbol{e}$ with $s, t > 1$. Then

$$\begin{aligned}
&2\beta_N \Phi_E(x,y)\\
&\quad := 2\beta_N \Phi_R(x,y)\\
&\quad = 2\beta_N \tilde{\Phi}_R(s,t)\\
&\quad = \left(\frac{\tilde{W}_R^2(t)}{(s^2-1)^{N-2}} + 2\left\{\frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}}\right\}\tilde{W}_R^2(s)\tilde{W}_R^2(t) + \frac{\tilde{W}_R^2(s)}{(t^2-1)^{N-2}}\right)\\
&\qquad \times \left(\left(\frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}}\right)^2 - \frac{1}{[(s^2-1)(t^2-1)]^{N-2}}\right)^{-1},
\end{aligned}$$

where $\tilde{W}_R(r) = W_R(r\boldsymbol{e})$, for $1 < r < R$.

Remark 5.1. We define in $]1, +\infty[\, \times \,]1, +\infty[$ the following function:

$$\tilde{\Psi}(s,t) = \frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} - \frac{1}{[(s^2-1)(t^2-1)]^{(N-2)/2}}. \tag{5.2}$$

From (5.1), it is easy to check that we can choose μ_0 small enough such that for all $0 < \mu < \mu_0$ there exist $1 < k_* < K < R_{\mu_0}^*$ independent of μ, verifying $\tilde{\Psi}(k_*, k_*) = 0$, $\tilde{\Psi}(K,K) = \max_{(x,y)\in E}\tilde{\Psi}(|x|,|y|)$. Moreover, k_* is the unique solution in $]1, +\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2} + (s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and K is the unique solution in $]1, +\infty[$ of

$$\frac{2^{1-N}}{s^N} = \frac{(s^2+1)^{N-1} + (s^2-1)^{N-1}}{(s^4-1)^{N-1}}.$$

Now, it is not difficult to prove the following lemma.

Lemma 5.2. *The function $\tilde{\Phi}_R$ achieves only one minimum value at a critical point of the form $(\rho_R, \rho_R) \in\,]k_*, K[$.*

5.2. General case

Let Ω be the domain defined in (1.2), with $P = 0$. In this subsection we consider the values m, M as follows: m is the radius of the biggest ball centred at the origin contained in $\mathcal{D}$ and M is the radius of the smallest ball centred at the origin

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containing to $\mathcal{D}$. Let w be the unique solution $C^{2,\gamma}(\bar{\Omega})$ of problem (2.6). By the maximum principle, we check that

$$z_m(x) \leqslant w(x) \leqslant z_M(x) \quad \text{for all } \mu < |x| < m,$$

where $z_m(x) = \alpha\mu^2 W_{R_1}(\mu^{-1}x)$ and $z_M(x) = \beta\mu^2 W_{R_2}(\mu^{-1}x)$, with $R_1 = \mu^{-1}m$ and $R_2 = \mu^{-1}M$. Hence,

$$\Phi_{R_1}(\mu^{-1}x, \mu^{-1}y) \leqslant \Phi_E(\mu^{-1}x, \mu^{-1}y) \leqslant \Phi_{R_2}(\mu^{-1}x, \mu^{-1}y) \quad \text{for all } \mu < |x|, |y| < m.$$

Since the function $\tilde{\Psi}(s,s)$ defined in (5.2) is decreasing in its diagonal for values of s greater that K and goes to 0, then is not difficult to show that the system

$$\frac{\tilde{\Phi}_{R_1}(s,s)}{\tilde{\Phi}_{R_2}(K,K)} \geqslant 1, \quad s \geqslant K,$$

possesses a solution, we say k^*, when we have chosen $\mu > 0$ sufficiently small but fixed. Indeed, if we set $\beta = \max_{x\in\Omega} f(x)$ and $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 \neq 0$, then, in the limit for μ, we can choose

$$k^* = \max\left\{K, \left\{\left(\frac{\alpha m^2 K^{N-2}}{(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2}\right)_+\right\}^{1/(N-2)}\right\}.$$

If $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 = 0$, we change K by a value a few greater than K in the definition of k^*. Then the following lemma is obtained.

Lemma 5.3. *The function $\Phi_E(x,y)$ achieves a relative minimum value in a critical point (x_μ, y_μ) with x_μ and y_μ having opposite directions, and $(|x_\mu|, |y_\mu|) \in]k_*, k^*[$. Moreover, $|x_\mu|$ and $|y_\mu|$ belong to a compact region fully contained in $]k_*, k^*[$, which is independent of all sufficiently small $\mu > 0$.*

Let

$$\mathbb{Q} = \{(x,y) \in \boldsymbol{V} \times \boldsymbol{V} : k_* < |x|, |y| < k^*\}.$$

We then define the following value:

$$c_\mu = \Phi_E(x_\mu, y_\mu) = \min_{(x,y)\in\mathbb{Q}} \Phi_E(x,y). \tag{5.3}$$

Let $\delta_\mu > 0$ be a suitable small value such that the level set

$$\{(x,y) \in \mathbb{Q} : \Phi_E(x,y) = \delta_\mu\}$$

is a closed curve and that $\nabla\Phi_E(x,y)$ does not vanish on it. Let us set

$$\Upsilon_\mu = \{(x,y) \in \mathbb{Q} : \Phi_E(x,y) < \delta_\mu\}. \tag{5.4}$$

Thus, on this region we have that $\Phi_E(x,y) < \delta_\mu$ and if $(x,y) \in \partial\Upsilon_\mu$, then one of the following situations happens: either there is a tangential direction τ to $\partial\Upsilon_\mu$ such that $\nabla\Phi_E(x,y)\cdot\tau \neq 0$, or x and y lie in opposite directions, where $\Phi_E(x,y) = \delta_\mu$ and $\nabla\Phi_E(x,y) \neq 0$ points orthogonally outwards to Υ_μ. Moreover, for fixed sufficiently small $\mu_0 > 0$,

$$\Upsilon_{\hat{\mu}} \subset\subset \Upsilon_\mu \subset\subset \mathbb{Q} \quad \text{for all } 0 < \hat{\mu} < \mu < \mu_0. \tag{5.5}$$

Let us now consider the exterior domain

$$E_\mu = \mathbb{R}^N \setminus \overline{B(0,\mu)},$$

and we denote by G_μ and H_μ, respectively, Green's function on E_μ and its regular part. Then

$$G_\mu(x,y) = \mu^{2-N} G_E(\mu^{-1}x, \mu^{-1}y) \quad \text{and} \quad H_\mu(x,y) = \mu^{2-N} H_E(\mu^{-1}x, \mu^{-1}y).$$

In particular, if we set

$$\Sigma_\Omega^\mu = \mu \Upsilon_\mu, \tag{5.6}$$

with Υ_μ defined by (5.4), then Σ_Ω^μ corresponds precisely to the set where

$$\Phi_E(\mu^{-1}x, \mu^{-1}y) < \delta_\mu,$$

with δ_μ defined by (5.4). Moreover, since

$$G(x,y) = G_\mu(x,y) + O(1) \quad \text{for all } (x,y) \in \mu\mathbb{Q},$$

where the quantity $O(1)$ is bounded independently of all small μ, in the C^1-sense, and the same is true for the function H, we have that, in the region $\mu\mathbb{Q}$, the function

$$\Phi_\Omega(x,y) = \frac{1}{2}\left\{\frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}\right\} \tag{5.7}$$

satisfies the following relation:

$$\Phi_\Omega(x,y) = \mu^{N+2}\Phi_E(\mu^{-1}x, \mu^{-1}y) + o(1), \tag{5.8}$$

where the quantity $o(1)$ is bounded independently of all small numbers $\mu > 0$ in the C^1-sense. Additionally, $o(1) \to 0$ as $\mu \to 0$.

6. The min–max scheme and proof of the main result

In this section $\mu > 0$ is a fixed sufficiently small number and Ω is the domain given in (1.2) with $P = 0$. According to the results (4.1) and (4.13), obtained above, our problem reduces to that of finding a critical point for

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \frac{1}{2}\left\{\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right\} + \sum_{i=1}^{2} \Lambda_i w(\xi_i), \tag{6.1}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2) \in \mathbb{R}_+^2$. Here we consider the function Φ defined over the class $\Sigma_\Omega^\mu \times \mathbb{R}_+^2$, where Σ_Ω^μ is defined by (5.6). Indeed Φ has some singularities on this class which we can avoid by replacing the term $G(\xi_1, \xi_2)$ in (6.1) by

$$G|_M(\xi_1, \xi_2) = \begin{cases} G(\xi_1, \xi_2) & \text{if } G(\xi_1, \xi_2) \leqslant M, \\ M & \text{if } G(\xi_1, \xi_2) > M, \end{cases} \tag{6.2}$$

where M is a big number. Hence, we can work with the modified functional, which, for simplicity, we still denote by Φ.

For every $\boldsymbol{\xi} \in \Sigma_\Omega^\mu$ we choose $d(\boldsymbol{\xi}) = (d_1(\boldsymbol{\xi}), d_2(\boldsymbol{\xi})) \in \mathbb{R}^2$, which is a vector defining the negative direction of the associated quadratic form with Φ. Such a direction exists since $G^2(x, y) - H(x, x)H(y, y) > 0$ over Σ_Ω^μ. More precisely, for fixed $\boldsymbol{\xi}_0 \in \Sigma_\Omega^\mu$, the function

$$\Phi(\boldsymbol{\xi}_0, \boldsymbol{d}) = \frac{1}{2}\left\{\sum_{i=1}^{2} d_i^2 H(\xi_{0,i}, \xi_{0,i}) - 2d_1 d_2 G(\xi_{0,1}, \xi_{0,2})\right\} + \sum_{i=1}^{2} d_i w(\xi_{0,i}),$$

regarded as a function of $\boldsymbol{d} = (d_1, d_2)$ only, with $d_1, d_2 > 0$, has a unique critical point $\bar{\boldsymbol{d}}(\boldsymbol{\xi}_0) = (\bar{d}_1(\boldsymbol{\xi}_0), \bar{d}_2(\boldsymbol{\xi}_0))$ given by

$$\bar{d}_i(\boldsymbol{\xi}_0) = \frac{H(\xi_{0,j}, \xi_{0,j})w(\xi_{0,i}) + G(\xi_{0,i}, \xi_{0,j})w(\xi_{0,j})}{G^2(\xi_{0,i}, \xi_{0,j}) - H(\xi_{0,i}, \xi_{0,i})H(\xi_{0,j}, \xi_{0,j})}, \quad i, j = 1, 2, \ i \neq j.$$

In particular,

$$\Phi(\boldsymbol{\xi}_0, \bar{\boldsymbol{d}}(\boldsymbol{\xi}_0)) = \Phi_\Omega(\boldsymbol{\xi}_0), \tag{6.3}$$

where Φ_Ω is the function given by (5.7). Then we simply choose $d(\boldsymbol{\xi}) = \bar{\boldsymbol{d}}(\boldsymbol{\xi})$. Let x_μ and y_μ the points given by (5.3). From now on we consider $\hat{\rho}_\mu = |x_\mu|$ and $\bar{\rho}_\mu = |y_\mu|$. Set

$$\mathbb{S} = \{(x, y) \in \mathbb{Q}^2 : (|x|, |y|) = (\mu\hat{\rho}_\mu, \mu\bar{\rho}_\mu)\}.$$

Let $\mathcal{K}$ be the class of all continuous functions

$$\kappa : \mathbb{S} \times I_0 \times [0, 1] \to \Sigma_\Omega^\mu \times \mathbb{R}_+^2$$

such that

(i) $\kappa(\boldsymbol{\xi}, \sigma_0, t) = (\boldsymbol{\xi}, \sigma_0 d(\boldsymbol{\xi}))$ and $\kappa(\boldsymbol{\xi}, \sigma_0^{-1}, t) = (\boldsymbol{\xi}, \sigma_0^{-1} d(\boldsymbol{\xi}))$ for all $\boldsymbol{\xi} \in \mathbb{S}$, $t \in [0, 1]$.

(ii) $\kappa(\boldsymbol{\xi}, \sigma, 0) = (\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$ for all $(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0$, where $I_0 = [\sigma_0, \sigma_0^{-1}]$ and σ_0 is a small number to be chosen later.

Then we define the min–max value as

$$c(\Omega) = \inf_{\kappa \in \mathcal{K}} \sup_{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0} \Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)). \tag{6.4}$$

In what follows we prove that $c(\Omega)$ is a critical value of Φ.

Lemma 6.1. *For all sufficiently small $\mu > 0$, the following estimate holds:*

$$c(\Omega) \leqslant \mu^{N+2} c_\mu + o(1),$$

where $o(1) \to 0$ as $\mu \to 0$, and c_μ is the value defined in (5.3).

Proof. For all $t \in [0, 1]$, we consider the test path defined as $\kappa(\boldsymbol{\xi}, \sigma, t) = (\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$. Maximizing $\Phi(\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$ in the variable σ, we note that this maximum value is attained at $\sigma = 1$, because of our choice of the vector $d(\boldsymbol{\xi})$. Hence, from (6.3), we have

$$\max_{\sigma \in I_0} \Phi(\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi})) = \Phi(\boldsymbol{\xi}, d(\boldsymbol{\xi})).$$

On the other hand, by the definition of $\mathbb{S}$, we see that

$$\Phi_E(\mu^{-1}\xi_1, \mu^{-1}\xi_2) = c_\mu.$$

Then the conclusion is immediate from (5.8) and the definition of $c(\Omega)$. □

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In order to prove that $c(\Omega)$ is indeed a critical point of Φ we need an intersection lemma. The idea behind this result is the topological continuation of the set of solutions of an equation (see [15]). For every $(\boldsymbol{\xi}, \sigma, t) \in \mathbb{S} \times I_0 \times [0,1]$ we define

$$\kappa(\boldsymbol{\xi}, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\boldsymbol{\xi}, \sigma, t)) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2,$$

with $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$, $\tilde{\Lambda} = (\tilde{\Lambda}_1, \tilde{\Lambda}_2)$, and we define the set

$$\mathbb{M} = \{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0 : \tilde{\Lambda}_1(\boldsymbol{\xi}, \sigma, 1) \cdot \tilde{\Lambda}_2(\boldsymbol{\xi}, \sigma, 1) = 1\}.$$

The following lemma has been proved by Del Pino *et al.* in [12, lemma 6.2]. Therefore, we omit the proof here.

LEMMA 6.2. *For every open neighbourhood W of $\mathbb{M}$ in $\mathbb{S} \times I_0$, the projection $g : W \to \mathbb{S}$ induces a monomorphism in cohomology, that is*

$$g^* : H^*(\mathbb{S}) \to H^*(W)$$

is injective.

PROPOSITION 6.3. *There exists a constant $A > 0$ such that*

$$\sup_{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0} \Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)) \geqslant -A \quad \textit{for all } \kappa \in \mathcal{K}.$$

Proof. Note that $\boldsymbol{\xi} \in \Sigma_\Omega^\mu$ implies that $\xi_i \in B(0, \mu k^*) \setminus B(0, \mu k_*)$, for $i = 1, 2$, with $\hat{\rho}_\mu, \bar{\rho}_\mu \in]k_*, k^*[$ for any μ sufficiently small. Thus, we can find a number $\delta_0 > 0$ such that if $|\xi_1 - \xi_2| < \delta_0$, then $\xi_1 \cdot \xi_2 > 0$. Let $A_0 > 0$ be such that $G(x, y) \geqslant A_0$ implies $|x - y| < \delta_0$.

We argue by contradiction. Let us assume that, for certain $\kappa \in \mathcal{K}$, we have

$$\Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)) \leqslant -A_0 \quad \text{for all } (\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0.$$

This implies that, for all $(\boldsymbol{\xi}, \sigma) \in \mathbb{M}$, $(\tilde{\xi}, \tilde{\sigma}) = (\tilde{\xi}(\boldsymbol{\xi}, \sigma, 1), \tilde{\Lambda}(\boldsymbol{\xi}, \sigma, 1))$, we have

$$2G(\tilde{\xi}_1, \tilde{\xi}_2) - (\tilde{\Lambda}_1^2 H(\tilde{\xi}_1, \tilde{\xi}_1) + 2\tilde{\Lambda}_1 w(\tilde{\xi}_1) + \tilde{\Lambda}_2^2 H(\tilde{\xi}_2, \tilde{\xi}_2) + 2\tilde{\Lambda}_2 w(\tilde{\xi}_2)) \geqslant 2A_0$$

and since $H(\tilde{\xi}_i, \tilde{\xi}_i) > 0$ and $w(\tilde{\xi}_i) > 0$, we conclude that if we take a small neighbourhood W of $\mathbb{M}$ in $\mathbb{S} \times I_0$, then for every $(\boldsymbol{\xi}, \sigma) \in W$ we have

$$G(\tilde{\xi}(\xi, \sigma, 1)) \geqslant A_0.$$

Hence, $|\tilde{\xi}_1 - \tilde{\xi}_2| < \delta_0$. Let us fix points $\zeta_i \in \mathbb{R}^N$, $i = 1, 2$, such that $|\zeta_1| = \hat{\rho}_\mu$ and $|\zeta_2| = \bar{\rho}_\mu$. Then $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{S}$. Setting $\kappa_1 = \kappa(\cdot, 1)$, we see that, because of the above conclusion, $\kappa_1(W) \subset (\Sigma_\Omega^\mu \setminus T(\boldsymbol{\zeta})) \times \mathbb{R}_+^2$, where $T(\boldsymbol{\zeta}) = \{(t_1\zeta_1, t_2\zeta_2) : t_1, t_2 \in]k, K[\}$.

Consider the map $s : \Sigma_\Omega^\mu \times \mathbb{R}_+^2 \to \mathbb{S}$ defined componentwise as

$$s(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \mu\left(\frac{\hat{\rho}_\mu \xi_1}{|\xi_1|}, \frac{\bar{\rho}_\mu \xi_2}{|\xi_2|}\right).$$

Then $\kappa_0^* \circ s^* : H^*(\mathbb{S}) \to H^*(\mathbb{S} \times I_0)$, where $\kappa_0 = \kappa(\cdot, 0)$ is an isomorphism. By the homotopy axiom we then deduce that $\kappa_1^* \circ s^*$ is also an isomorphism. We consider the following commutative diagram:

$$\begin{array}{ccccc}
H^*(\mathbb{S}\times I_0) & \xleftarrow{\kappa_1^*} & H^*(\Sigma_\Omega^\mu \times \mathbb{R}_+^2) & \xleftarrow{\kappa^*} & H^*(\mathbb{S}) \\
\Big\downarrow{\scriptstyle i_1^*} & & \Big\downarrow{\scriptstyle i_2^*} & & \Big\downarrow{\scriptstyle i_3^*} \\
H^*(W) & \xleftarrow{\tilde{\kappa}_1^*} & H^*(\kappa_1(W)) & \xleftarrow{\tilde{s}^*} & H^*(\mathbb{S}\setminus\{\boldsymbol{\zeta}\}),
\end{array}$$

where i_1, i_2 and i_3 are inclusion maps, $\tilde{\kappa}_1 = \kappa_1|_W$ and $\tilde{s} = s|_{\kappa_1(W)}$. From lemma 6.2 we have that i_1^* is a monomorphism, which is a contradiction of the fact that $H^{2N}(\mathbb{S}\setminus\{\boldsymbol{\zeta}\}) = 0$. Thus, the result follows. □

In order to prove that the min–max number (6.4) is a critical value of Φ, we need to take into consideration the fact that the domain in which Φ is defined is not necessarily closed for the gradient flow of Φ. The following lemma is given towards this aim.

Lemma 6.4. *Assume that $\mu > 0$ is a sufficiently small number. Let $(\xi^n, \Lambda^n) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2$ be a sequence such that*

$$\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0. \tag{6.5}$$

Then each component of $\boldsymbol{\Lambda}_n$ is bounded above and below by positive constants.

Proof. Note that $\bar{\Sigma}_\Omega^\mu \subset\subset \Omega$. Hence, $w(\xi_i) > 0$, $i = 1, 2$, for all $\boldsymbol{\xi} \in \bar{\Sigma}_\Omega^\mu$. We set $\boldsymbol{\xi}_n = (\xi_{1,n}, \xi_{2,n})$ and $\boldsymbol{\Lambda}_n = (\Lambda_{1,n}, \Lambda_{2,n})$. Then (6.5) is equivalent to

$$\Lambda_{i,n} H(\xi_{i,n}, \xi_{i,n}) - \Lambda_{j,n} G(\xi_{i,n}, \xi_{j,n}) + w(\xi_{i,n}) \to 0, \quad i, j = 1, 2, \ i \neq j.$$

It is clear that $|\boldsymbol{\Lambda}_n| \to 0$ or $\Lambda_{i,n} \to 0$ and $\Lambda_{j,n} \to C$, with non-zero C and with $i \neq j$, cannot happen. Hence, we can suppose that $|\boldsymbol{\Lambda}_n| \to +\infty$. Since H and G remain uniformly controlled (μ is fixed), we easily see that $\Lambda_{1,n} \to +\infty$ and $\Lambda_{2,n} \to +\infty$. We set $\tilde{\Lambda}_{i,n} = \Lambda_{i,n}/|\boldsymbol{\Lambda}_n|$, for $i = 1, 2$, and passing to a subsequence, if necessary, we may assume that this sequence it approaches a non-zero vector $(\hat{\Lambda}_1, \hat{\Lambda}_2)$ with $\hat{\Lambda}_i \neq 0$ for $i = 1, 2$. It follows that

$$\tilde{\Lambda}_{i,n} H(\xi_{i,n}, \xi_{i,n}) - \tilde{\Lambda}_{j,n} G(\xi_{1,n}, \xi_{2,n}) + \frac{w(\xi_{i,n})}{|\boldsymbol{\Lambda}_n|} \to 0, \quad i, j = 1, 2, \ i \neq j.$$

For a suitable subsequence, for some $(\bar{\xi}_1, \bar{\xi}_2) \in \bar{\Sigma}_\Omega^\mu$, we obtain the system

$$\frac{\hat{\Lambda}_1}{\hat{\Lambda}_2} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_1, \bar{\xi}_1)} \quad \text{and} \quad \frac{\hat{\Lambda}_2}{\hat{\Lambda}_1} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_2, \bar{\xi}_2)}.$$

Hence,

$$G^2(\bar{\xi}_1, \bar{\xi}_2) - H(\bar{\xi}_1, \bar{\xi}_1) H(\bar{\xi}_2, \bar{\xi}_2) = 0,$$

which is a contradiction, since the quantity on the left-hand side in the previous equality is strictly positive when $\mu > 0$ is chosen sufficiently small. This finishes the proof. □

PROPOSITION 6.5. *Let us assume that $\mu > 0$ is a sufficiently small number. Then the functional Φ satisfies the (PS) condition in the region $\Sigma_\Omega^\mu \times \mathbb{R}_+^2$ at the level $c(\Omega)$ given in (6.4).*

Proof. Let us consider a sequence $(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2$ such that

$$\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0 \quad \text{and} \quad \nabla_{\boldsymbol{\xi}}^\tau \Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0,$$

where $\nabla_{\boldsymbol{\xi}}^\tau \Phi$ corresponds to the tangential gradient of Φ to $\partial \Sigma_\Omega^\mu \times \mathbb{R}_+^2$ in the case when $\boldsymbol{\xi}_n$ approaches $\partial \Sigma_\Omega^\mu$ or the full gradient otherwise. From the previous lemma, the components of $\boldsymbol{\Lambda}_n$ are bounded above and below by positive constants, so that we may assume, passing to a subsequence if necessary, that $(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to (\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \in \bar{\Sigma}_\Omega^\mu \times \mathbb{R}_+^2$ and $\Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to c(\Omega)$. Then

$$\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = 0.$$

Observe that if $\boldsymbol{\xi}_0 \in \operatorname{Int}(\Sigma_\Omega^\mu)$, then $\boldsymbol{\xi}_0$ is a critical point of Φ. We assume the opposite, i.e. that $\boldsymbol{\xi}_0 \in \partial \Sigma_\Omega^\mu$. Then

$$\Phi_E(\mu^{-1}\xi_{0,1}, \mu^{-1}\xi_{0,2}) = \delta_\mu.$$

Firstly, we note that $\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = 0$. Then $\boldsymbol{\Lambda}_0$ satisfies

$$\Lambda_{0,i} = \frac{H(\xi_{0,j}, \xi_{0,j}) w(\xi_{0,i}) + G(\xi_{0,i}, \xi_{0,j}) w(\xi_{0,j})}{G^2(\xi_{0,i}, \xi_{0,j}) - H(\xi_{0,i}, \xi_{0,i}) H(\xi_{0,j}, \xi_{0,j})}, \quad i, j = 1, 2, \ i \neq j.$$

Substituting these values in Φ, from (6.3) we obtain

$$c(\Omega) = \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = \Phi_\Omega(\boldsymbol{\xi}_0)$$

and from (5.8) we deduce that

$$c(\Omega) = \mu^{N+2} \Phi_E(\mu^{-1}\xi_{0,1}, \mu^{-1}\xi_{0,2}) + \theta(\boldsymbol{\xi}_0),$$

where $\theta(\boldsymbol{\xi}_0)$ is small in the C^1 sense, as $\mu > 0$ becomes smaller. Hence, $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \cdot \tau \sim 0$ for any direction τ tangential to $\partial \Sigma_\Omega^\mu$. Thus, from the analysis in the previous section, we have that $\xi_{0,1}$ and $\xi_{0,2}$ are in opposite directions; $\Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \sim \mu^{N+2} \delta_\mu$ and $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0)$ must be away from 0. Then choosing τ parallel to $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0)$ we obtain that $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \cdot \tau$ must be away from 0, which is a contradiction. Then, the point $\boldsymbol{\xi}_0 \in \operatorname{Int}(\Sigma_\Omega^\mu)$, which implies that the (PS) condition holds and the results follows. □

Now we are ready to complete the proof of theorem 1.1.

Proof of theorem 1.1. Let us consider the domain $\boldsymbol{\Sigma}_a^b = \Sigma_\Omega^\mu \times [\boldsymbol{a}, \boldsymbol{b}]^2$ with $\boldsymbol{a}, \boldsymbol{b}$ to be chosen later. Then the functional $\mathcal{I}$ given by (4.1) is well defined on $\boldsymbol{\Sigma}_a^b$ except on the set

$$\Delta_\rho = \{(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \boldsymbol{\Sigma}_a^b : |\xi_1 - \xi_2| < \rho\}.$$

From (4.3) we can extend $\mathcal{I}$ to all $\boldsymbol{\Sigma}_a^b$ by extending Φ as in (6.2), and keep relations (4.3) and (4.13) over $\boldsymbol{\Sigma}_a^b$.

From proposition 6.5, Φ satisfies the (PS) condition. There then exist constants $\boldsymbol{b} > 0$, $c > 0$ and $\varrho_0 > 0$, such that if $0 < \varrho < \varrho_0$, and $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \Sigma_\Omega^\mu$ satisfying $|\boldsymbol{\Lambda}| \geqslant \boldsymbol{b}$ and $c(\Omega) - 2\varrho \leqslant \Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \leqslant c(\Omega) + 2\varrho$, then $|\nabla \Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda})| \geqslant c$.

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We now use the min–max characterization of $c(\Omega)$ to choose $\kappa \in \mathcal{K}$ so that

$$c(\Omega) \leqslant \sup_{(\boldsymbol{\xi},\sigma)\in\mathbb{S}\times I_0} \Phi(\kappa(\boldsymbol{\xi},\sigma,1)) \leqslant c(\Omega) + \varrho.$$

By making $\boldsymbol{a}$ small and $\boldsymbol{b}$ large if necessary, we can assume that $\kappa(\boldsymbol{\xi},\sigma,1) \in \boldsymbol{\Sigma}_{2\boldsymbol{a}}^{\boldsymbol{b}/2} \subset \boldsymbol{\Sigma}_a^b$ for all $(\boldsymbol{\xi},\sigma) \in \mathbb{S}\times I_0$.

Consider now $\eta : \boldsymbol{\Sigma}_a^b \times [0,+\infty] \to \boldsymbol{\Sigma}_a^b$, the solution of the equation

$$\dot{\eta} = -h(\eta)\nabla\mathcal{I}(\eta)$$

with initial condition $\eta(\boldsymbol{\xi},\boldsymbol{\Lambda},0) = (\boldsymbol{\xi},\boldsymbol{\Lambda})$. Here the function h is defined in $\boldsymbol{\Sigma}_a^b$ so that $h(\boldsymbol{\xi},\boldsymbol{\Lambda}) = 0$ for all $(\boldsymbol{\xi},\boldsymbol{\Lambda})$ with $\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) \leqslant c(\Omega) - 2\varrho$ and $h(\boldsymbol{\xi},\boldsymbol{\Lambda}) = 1$ if $\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) \geqslant c(\Omega) - \varrho$, satisfying $0 \leqslant h \leqslant 1$.

Hence, by the choice of $\boldsymbol{a}$ and $\boldsymbol{b}$, and bearing in mind (4.3) and (4.13), we have that $\eta(\boldsymbol{\xi},\boldsymbol{\Lambda},t) \in \boldsymbol{\Sigma}_a^b$ for all $t \geqslant 0$. Then the min–max value

$$C(\Omega) = \inf_{t\geqslant 0} \sup_{(\boldsymbol{\xi},\sigma)\in\mathbb{S}\times I_0} \mathcal{I}(\eta(\kappa(\boldsymbol{\xi},\sigma,1),t))$$

is a critical value for $\mathcal{I}$. We always assume that ε is sufficiently small, in order to make the errors in (4.1) sufficiently small. Theorem 1.1 is thus proven. □

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