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Minimum Degrees and Codegrees of Ramsey-Minimal 3-Uniform Hypergraphs*†

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A uniform hypergraph H is called k-*Ramsey* for a hypergraph F if, no matter how one colours the edges of H with k colours, there is always a monochromatic copy of F. We say that H is k-*Ramsey-minimal for* F if H is k-Ramsey for F but every proper subhypergraph of H is not. Burr, Erdős and Lovasz studied various parameters of Ramsey-minimal graphs. In this paper we initiate the study of minimum degrees and codegrees of Ramsey-minimal 3-uniform hypergraphs. We show that the smallest minimum vertex degree over all k-Ramsey-minimal 3-uniform hypergraphs for $K_t^{(3)}$ is exponential in some polynomial in k and t. We also study the smallest possible minimum codegree over 2-Ramsey-minimal 3-uniform hypergraphs.

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1. Introduction and new results

A graph G is said to be *Ramsey* for a graph F if, no matter how one colours the edges of G with two colours, say red and blue, there is a monochromatic copy of F. A classical result of Ramsey [14] states that for every F there is an integer n such that K_n is Ramsey for F. Moreover, generalizations to more than two colours and to hypergraphs hold as well [14]. If G is Ramsey for F but every proper subgraph of G is not Ramsey for F, then we say that G is *Ramsey-minimal for* F. We denote by $\mathcal{M}_k(F)$ the set of minimal graphs G with the property that no matter how one colours the edges of G with

* After this paper was accepted and processed, we managed to obtain BEL-gadgets for uniformities $r \geqslant 4$. This work will appear elsewhere.

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k colours, there is a monochromatic copy of F in it, and refer to these as *k-Ramsey-minimal graphs for F*. There are many challenging open questions concerning the study of various parameters of k-Ramsey-minimal graphs for various F. The most studied ones are the classical *(vertex) Ramsey number* $r_k(F) := \min_{G\in\mathcal{M}_k(F)} v(G)$ and the *size Ramsey number* $\hat{r}_k(F) := \min_{G\in\mathcal{M}_k(F)} e(G)$, where $v(G)$ is the number of vertices in G and $e(G)$ is its number of edges. To determine the classical Ramsey number $r_2(K_t)$ is a notoriously difficult problem and essentially the best known bounds are $2^{(1+o(1))t/2}$ and $2^{(2+o(1))t}$ due to Spencer [16] and Conlon [4].

Burr, Erdős and Lovász [1] were the first to study other possible parameters of the class $\mathcal{M}_2(K_t)$. In particular they determined the minimum degree

$$s_2(K_t) := \min_{G\in\mathcal{M}_2(K_t)} \delta(G) = (t-1)^2,$$

which looks surprising given the exponential bound on the minimum degree of K_n with $n = r_2(K_t)$ (it is not difficult to see that $K_n \in \mathcal{M}_2(K_t)$). Extending their results, Fox, Grinshpun, Liebenau, Person and Szabó [10] studied the minimum degree

$$s_k(K_t) := \min_{G\in\mathcal{M}_k(K_t)} \delta(G)$$

for more colours showing a general bound on $s_k(K_t) \leqslant 8(t-1)^6k^3$ and proving quasi-quadratic bounds in k on $s_k(K_t)$ for fixed t. Further results concerning Ramsey-minimal graphs were studied in [2, 9, 11, 15, 17].

Here we initiate the study of Ramsey-minimal 3-uniform hypergraphs and provide first bounds on various types of minimum degree for Ramsey-minimal hypergraphs. Generally, an *r-uniform hypergraph H* is a tuple (V,E) with vertex set V and $E \subseteq \binom{V}{r}$ being its edge set. Ramsey's theorem holds for r-uniform hypergraphs as well, as shown originally by Ramsey himself [14]. We write $G \longrightarrow (F)_k$ if G is k-Ramsey for F, that is, if no matter how one colours the edges of the r-uniform hypergraph G, there is a monochromatic copy of F. We denote by $K_t^{(r)}$ the complete r-uniform hypergraph with t vertices, that is, $K_t^{(r)} = ([t], \binom{[t]}{r})$. The hypergraph Ramsey number $r_k(F)$ is the smallest n such that $K_n^{(r)} \longrightarrow (F)_k$. While in the graph case the known bounds on $r_2(K_t)$ are only polynomially far apart, already in the case of 3-uniform hypergraphs the bounds on $r_2(K_t^{(3)})$ differ by one exponent, $2^{c_1t^2} \leqslant r_2(K_t^{(3)}) \leqslant 2^{2^{c_2t}}$ for some absolute positive constants c_1 and c_2, and a similar situation occurs for higher uniformities. For further information on Ramsey numbers we refer the reader to the standard book on Ramsey theory [12] and for newer results to the survey by Conlon, Fox and Sudakov [5].

Given $\ell \in [r-1]$, we define the *degree* $\deg(S)$ of an ℓ-set S in an r-uniform hypergraph $H = (V,E)$ as the number of edges that contain S, and we define the *minimum ℓ-degree* $\delta_\ell(H) := \min_{S\in\binom{V}{\ell}} \deg(S)$. For two vertices u and v we simply write $\deg(u,v)$ for $\deg(\{u,v\})$, sometimes referred to as the *codegree* of u and v. Similar to the graph case, we extend verbatim the notion of Ramsey-minimal graphs to Ramsey-minimal r-uniform hypergraphs in a natural way. That is, $\mathcal{M}_k(F)$ is the set of all *k-Ramsey-minimal r-uniform hypergraphs H*, that is, consisting of those H with $H \longrightarrow (F)_k$ but $H' \not\longrightarrow (F)_k$ for all

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$H' \subsetneq H$. We define

$$s_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \delta_\ell(H), \tag{1.1}$$

which extends the introduced graph parameter $s_k(K_t)$. It will be shown in fact that $s_{2,2}(K_t^{(3)})$ is zero, and thus it makes sense to ask for the second smallest value of the minimum ℓ-degree. This motivates the following parameter $s'_{k,\ell}(K_t^{(r)})$:

$$s'_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \left(\min\left\{ \deg_H(S) : S \in \binom{V(H)}{\ell},\ \deg_H(S) > 0 \right\} \right).$$

We prove the following results on the minimum degree and codegree of Ramsey-minimal 3-uniform hypergraphs for cliques $K_t^{(3)}$.

Theorem 1.1. *The following holds for all $t \geqslant 4$ and $k \geqslant 2$:*

$$\hat{r}_k(K_{t-1}) \leqslant s_{k,1}(K_t^{(3)}) \leqslant k^{20kt^4}. \tag{1.2}$$

The lower bound $\hat{r}_k(K_{t-1})$ is the *size-Ramsey number* for K_{t-1} and it was shown by Erdős, Faudree, Rousseau and Schelp [7] that $\hat{r}_k(K_\ell) = \binom{r_k(K_\ell)}{2}$. Using the lower bound on $r_k(K_\ell) \geqslant 2^{(1-o(1))/4k\ell}$ (see, *e.g.*, [5]) we obtain

$$s_{k,1}(K_t^{(3)}) \geqslant 2^{\frac{1}{2}kt(1-o(1))}.$$

Theorem 1.2. *Let $t \geqslant 4$ be an integer. Then*

$$s_{2,2}(K_t^{(3)}) = 0 \quad \textit{and} \quad s'_{2,2}(K_t^{(3)}) = (t-2)^2.$$

Note that with $s'_{2,2}$ we ask for the smallest *positive* codegree, while for $s_{2,2}$ we also allow the codegree to be zero. This in particular means that in *any* 2-Ramsey-minimal hypergraph H for $K_t^{(3)}$ we have that a pair of vertices u and v are either not contained in a common edge or have codegree at least $(t-2)^2$. This might look surprising at first sight, since taking $K_n^{(3)}$ with $n = r_2(K_t^{(3)})$ and then deleting all edges that contain two distinguished vertices gives a non-Ramsey hypergraph.

Methods. The methods we are going to use are generalizations of signal senders introduced first by Burr, Erdős and Lovász in [1], and generalized later by Burr, Nešetřil and Rödl [2] and by Rödl and Siggers [15], that we combine with probabilistic arguments analysing certain properties of random 3-uniform hypergraphs.

Organization of the paper. In Section 2 we generalize existence results for 'almost' Ramsey graphs, that is, graphs whose edge colourings without a monochromatic copy of some complete graph K_t impose certain colour patterns, first introduced for hypergraphs by Burr, Erdős and Lovász [1]. Then in Section 3 we study the vertex degree for k-Ramsey-minimal 3-uniform hypergraphs for $K_t^{(3)}$, while in Section 4 we look into the case of codegrees in 2-Ramsey-minimal 3-uniform hypergraphs for $K_t^{(3)}$.

2. BEL-gadgets for 3-uniform hypergraphs

For a given hypergraph $H = (V, E)$, the *link* of a vertex $v \in V$, denoted by $\mathrm{link}(v)$, consists of the edges of H that contain v, minus the vertex v itself (thus, these form an $(r-1)$-uniform hypergraph). Formally, the edge set of $\mathrm{link}(v)$ is $\{e \setminus \{v\} : v \in e \in E\}$. In this paper we will be dealing exclusively with 3-uniform hypergraphs, thus the links of their vertices are just the edges of some graph.

First we show a lemma that asserts the existence of a 3-uniform hypergraph H and two edges $f, e \in E(H)$ with $|f \cap e| = 2$ and $e(H[e \cup f]) = 2$ so that H is not k-Ramsey for $K_t^{(3)}$, with the property that any k-colouring of $E(H)$ without a monochromatic $K_t^{(3)}$ colours the edges e and f differently. We will refer to such hypergraphs that impose a certain structure on $K_t^{(3)}$-free colourings as *BEL-gadgets*. Moreover, we occasionally refer in the following to a colouring without a monochromatic copy of F as an *F-free colouring*.

Lemma 2.1. *Let $t \geqslant 4$ and $k \geqslant 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$ and two edges $e_{\mathcal{H}}, f_{\mathcal{H}} \in E(\mathcal{H})$ with $|f_{\mathcal{H}} \cap e_{\mathcal{H}}| = 2$ and $e(\mathcal{H}[e_{\mathcal{H}} \cup f_{\mathcal{H}}]) = 2$ such that the following properties hold:*

(1) $\mathcal{H} \not\rightarrow (K_t^{(3)})_k$,

(2) *for every k-colouring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$, we have that $c(e_{\mathcal{H}}) \neq c(f_{\mathcal{H}})$.*

Proof. Set $m = r_k(K_t^{(3)})$ and define a hypergraph F' on the vertex set $[m]$ as follows. Delete from $K_m^{(3)}$ all edges that contain vertices $m-1$ and m. It is easy to see that then $F' \not\rightarrow (K_t^{(3)})_k$. Indeed, fix a k-colouring of $E(K_{m-1}^{(3)})$ without a monochromatic $K_t^{(3)}$, then extend this colouring to $E(F')$ by colouring each edge (x, y, m) with the colour of $(x, y, m-1)$. Since every copy of $K_t^{(3)}$ in F' may contain at most one of the vertices $m-1$ and m, we see that $F' \not\rightarrow (K_t^{(3)})_k$.

Define

$$F_i := ([m], E(F') \cup \{\{j, m-1, m\} : j \leqslant i\})$$

and set $F := F_\ell$, where ℓ is maximal such that F_ℓ is not k-Ramsey for $K_t^{(3)}$ but $F_{\ell+1}$ is (this is possible since $F_{m-2} = K_m^{(3)}$ is k-Ramsey for $K_t^{(3)}$ by the choice of $m = r_k(K_t^{(3)})$).

For a colouring $\psi : E(F) \to [k]$ without a monochromatic copy of $K_t^{(3)}$ we define an *admissible pattern* $(a_1, \ldots, a_k)$, where a_i denotes the number of edges in the colour i containing both vertices $m-1$ and m. Moreover, we let $\mathcal{P}$ denote the set of all admissible patterns. In particular, by the choice of ℓ we have that $\mathcal{P} \neq \emptyset$.

Note that $\sum_{i \in [k]} a_i = \ell$ for every $(a_1, \ldots, a_k) \in \mathcal{P}$, and $a_c \notin \{0, \ell\}$ for every $c \in [k]$. Indeed if, say, there is a pattern $(a_1, \ldots, a_k) \in \mathcal{P}$ with $a_j = 0$ for some $j \in [k]$, then we could take a corresponding k-colouring of the edges of F_ℓ avoiding monochromatic copies of $K_t^{(3)}$ with pattern $(a_1, \ldots, a_k)$, which we would then extend to a k-colouring of $E(F_{\ell+1})$ without a monochromatic copy of $K_t^{(3)}$ just by colouring the edge $\{\ell+1, m-1, m\}$ with colour j. Indeed, this new edge cannot participate in a monochromatic copy of $K_t^{(3)}$ in this colouring, as its colour is j, while all other edges containing both $m-1$ and m have colours different from j. But this contradicts the definition of ℓ.

Moreover, note that the following holds. If $\varphi\colon [\ell] \to [k]$ is a colouring of the first ℓ vertices of F such that $(|\varphi^{-1}(1)|, \dots, |\varphi^{-1}(k)|) \in \mathcal{P}$, then there exists a colouring $c\colon E(F) \to [k]$ avoiding monochromatic copies of $K_t^{(3)}$ such that $c(i, m-1, m) = \varphi(i)$ for every $i \in [\ell]$.

Now, let H be an ℓ-uniform hypergraph. We say that a colouring $\psi\colon V(H) \to [k]$ is admissible if, for every edge $e \in E(H)$, we have $(c_1, \dots, c_k) \in \mathcal{P}$, where c_i denotes the number of vertices in e coloured i.

Now we proceed analogously to Claim 2 from [1]. We find an ℓ-uniform hypergraph H^* with $\text{girth}(H^*) \geqslant 3$ (this means that any two distinct edges e and f satisfy $|e \cap f| \leqslant 1$) and two vertices $x, y \in V(H^*)$ with $\deg_{H^*}(x, y) = 0$ such that there exist admissible colourings for H^*, and in every such colouring the colour of x differs from the colour of y. For completeness we provide this elegant argument here. We start with an ℓ-uniform hypergraph H with $\text{girth}(H) \geqslant 3$ and chromatic number $\chi(H) \geqslant k+1$. It was shown by Erdős and Hajnal [8] that such hypergraphs exist.

Since every k-colouring of the vertices of H yields a monochromatic edge, while $(\ell, 0, \dots, 0), \dots, (0, \dots, 0, \ell) \notin \mathcal{P}$, H does not have admissible colourings. Now, we can take a subhypergraph H' of H which is minimal (with respect to the number of edges) for the property of not having admissible k-colourings. For an arbitrary edge $f = \{x_1, \dots, x_\ell\} \in H'$ and arbitrary vertices $y_1, \dots, y_\ell \notin V(H')$, we define a sequence of hypergraphs H_i on $V(H') \cup \{y_1, \dots, y_i\}$ with $H_i = H' - f + f_i$, where $f_i = \{y_1, \dots, y_i, x_{i+1}, \dots, x_\ell\}$. By the definition, $H_0 = H'$ does not have admissible colourings while H_ℓ does, so there is a minimal index $i \in [\ell]$ such that H_{i-1} does not have admissible colourings but H_i does. We now set $H^* = H_i$ and $x := x_i$, $y := y_i$. It is clear that $\text{girth}(H^*) \geqslant 3$, $\deg_{H^*}(x, y) = 0$ and that H^* has admissible colourings. Moreover, for any such admissible k-colouring, x and y need to have distinct colours, for otherwise, taking an admissible colouring of H_i with x and y coloured the same and then identifying x with y would yield an admissible colouring of H_{i-1}, a contradiction.

Finally, we define a 3-uniform hypergraph $\mathcal{H}$ as follows. First we introduce for each $e \in E(H^*)$ a set

$$V_e := e \cup \{m-1, m\} \cup (\{e\} \times \{\ell+1, \dots, m-2\}),$$

and then we define a 3-uniform hypergraph F_e which is a copy of $F = F_\ell$ that contains all vertices from e as follows:

$$F_e := \left(V_e, \binom{V_e}{3} \setminus \{\{(e,i), m-1, m\} : i = \ell+1, \dots, m-2\}\right).$$

The hypergraph $\mathcal{H}$ is then the union over all F_e: $\mathcal{H} := \cup_{e \in E(H^*)} F_e$. In other words, we obtain $\mathcal{H}$ by placing F_e, a copy of F, for each edge $e \in E(H^*)$ so that the vertices $\{1, \dots, \ell\}$ of F are identified with e. Further, we set $e_{\mathcal{H}} = \{m-1, m, x\}$ and $f_{\mathcal{H}} = \{m-1, m, y\}$. Before showing that $\mathcal{H}$, $e_{\mathcal{H}}$ and $f_{\mathcal{H}}$ fulfil requirements (1) and (2), we establish the following claim.

Claim 2.2. *Any copy K of $K_t^{(3)}$ in $\mathcal{H}$ is contained in F_e for some $e \in E(H^*)$.*

To prove the claim, we assume first that $V(K) \setminus (\{m-1, m\} \cup V(H^*)) \neq \emptyset$ holds. Thus K contains a vertex of the form (e, s) for some $e \in E(H^*)$ and $s \in [\ell]$. The link of (e, s) is a graph on $m-1$ vertices whose vertex set is $V_e \setminus \{(e, s)\}$, by construction of $\mathcal{H}$. This, with $\mathcal{H}[V_e] = F_e$, then implies that $K \subseteq F_e$.

From now on we may assume that $V(K) \subseteq V(H^*) \cup \{m-1, m\}$. First we assume that $K \cong K_4^{(3)}$ and $m-1, m \in V(K)$. Thus, the remaining two vertices, call them a and b, must lie in some edge $e \in E(H^*)$ (since $\{m, a, b\}$ is an edge in $\mathcal{H}\big[V(H^*) \cup \{m-1, m\}\big]$), which implies $K \subseteq F_e$. Finally, we may assume that $|V(K) \cap V(H^*)| \geqslant 3$, and setting $S := V(K) \cap V(H^*)$ we have $K[S] \cong K_s^{(3)}$, $s \geqslant 3$. Since $\mathcal{H}\big[V(H^*)\big]$ consists of cliques $K_\ell^{(3)}$ that intersect in at most one vertex as $\mathrm{girth}(H^*) \geqslant 3$, this implies that S has to be contained in some $e \in E(H^*)$. Again this yields $K \subseteq F_e$, which completes the proof of the claim.

Continuing with the proof of Lemma 2.1, we first recall that we defined $e_{\mathcal{H}} = \{m-1, m, x\}$ and $f_{\mathcal{H}} = \{m-1, m, y\}$. By construction of $\mathcal{H}$ and since $\deg_{H^*}(x, y) = 0$, it is clear that $\{x, y, m-1\}$ and $\{x, y, m\}$ are non-edges in $\mathcal{H}$. We now prove that this choice of $\mathcal{H}$, $e_{\mathcal{H}}$ and $f_{\mathcal{H}}$ fulfils requirements (1) and (2) of our lemma.

(1) By construction there exists an admissible colouring $c\colon V(H^*) \to [k]$. Note that two hypergraphs F_e and F_f for distinct $e, f \in E(H^*)$ have in common both vertices $m-1$ and m and additionally at most one further vertex v (and if so also the edge $\{v, m-1, m\}$), by construction and since $\mathrm{girth}(H^*) \geqslant 3$. Since $\mathcal{H}$ consists of copies of F that intersect pairwise in at most one edge (containing both vertices $m-1$ and m), we can find colourings of these copies without monochromatic $K_t^{(3)}$ so that these colourings agree on common edges $\{v, m-1, m\}$. Indeed, for every edge $e \in E(H^*)$ we have an admissible colour pattern $(d_1, \ldots, d_k) \in \mathcal{P}$ which depends on c. Thus, there exists a colouring $\varphi_e : E(F_e) \to [k]$ without monochromatic $K_t^{(3)}$ so that $\varphi_e(\{v, m-1, m\}) = c(v)$ for all $v \in e$.

We need to show that the union of φ_e over all $e \in E(H^*)$ gives us a k-colouring φ of $E(\mathcal{H})$ without monochromatic copies of $K_t^{(3)}$. By Claim 2.2, any copy of $K_t^{(3)}$ is contained in F_e for some $e \in E(H^*)$. Since $E(F_e)$ does not contain any monochromatic $K_t^{(3)}$ under φ_e, requirement (1) is verified.

(2) Now, let $c\colon E(\mathcal{H}) \to [k]$ be a colouring on the edge set of $\mathcal{H}$ which avoids monochromatic copies of $K_t^{(3)}$. Define $\varphi : V(H^*) \to [k]$ with $\varphi(v) := c(\{v, m-1, m\})$. Then φ is an admissible colouring of H^* and thus, by the properties of H^* we know that $c(e_{\mathcal{H}}) = \varphi(x) \neq \varphi(y) = c(f_{\mathcal{H}})$. □

Now we prove a lemma that allows us to obtain a 'rainbow star'.

Lemma 2.3. *Let $t \geqslant 4$ and $k \geqslant 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$, a 2-element set $S \subseteq V(\mathcal{H})$ and edges $e_1, \ldots, e_k \in E(\mathcal{H})$ with $e_i \cap e_j = S$ (for all $i \neq j \in [k]$), $|\cup_{i\in[k]} e_i| = k+2$ and $e(\mathcal{H}[\cup_{i\in[k]} e_i]) = k$ such that the following properties hold:*

(1) $\mathcal{H} \not\longrightarrow (K_t^{(3)})_k$,

(2) *for every k-colouring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $\{c(e_i) : i \in [k]\} = [k]$, that is, the colours of the e_i are all distinct.*

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Proof. Take $\binom{k}{2}$ vertex-disjoint copies $(\mathcal{H}_{ij})_{1\leqslant i<j\leqslant k}$ of the hypergraph $\mathcal{H}'$ as guaranteed to us by Lemma 2.1, and let e_{ij} and f_{ij} be the corresponding edges of $\mathcal{H}'$ that satisfy property (2) of Lemma 2.1. We start with the hypergraph H on the vertex set $[k+2]$ and with edge set $\{\{i,k+1,k+2\} : i \in [k]\}$, and we set $S := \{k+1,k+2\}$.

We construct the hypergraph $\mathcal{H}$ as follows. For each $i < j \in [k]$ we identify the vertices $k+1$ and $k+2$ (arbitrarily) with the two vertices from $C_{ij} := e_{ij} \cap f_{ij}$ and the only vertex from $e_{ij} \setminus C_{ij}$ is identified with i while the only vertex from $f_{ij} \setminus C_{ij}$ is identified with j. Otherwise the hypergraphs $\mathcal{H}_{ij}$ do not intersect each other in further vertices. We claim that the properties from Lemma 2.3 are satisfied. Indeed, since $\mathcal{H}_{ij} \not\rightarrow (K_t^{(3)})_k$ and by the symmetry of the colours, we can assume that there is a $K_t^{(3)}$-free colouring φ_{ij} of $\mathcal{H}_{ij}$ such that $\varphi(e_{ij}) = i$ and $\varphi(f_{ij}) = j$ (and $i < j$). We obtain the colouring φ of $\mathcal{H}$ by colouring the corresponding edges according to appropriate φ_{ij}. This is possible since the edge $\{i,k+1,k+2\}$ is identified with e_{ij} and $f_{\ell i}$ for $\ell < i < j$, and these are coloured with the colour i. The colouring φ is $K_t^{(3)}$-free, since each copy of $K_t^{(3)}$ is contained in one of the $\mathcal{H}_{ij}$. To see property (2), we use property (2) of Lemma 2.1, which asserts that in any $K_t^{(3)}$-free colouring of $\mathcal{H}$ the edges $\{i,k+1,k+2\}$ and $\{j,k+1,k+2\}$ are coloured differently (with $i < j$). □

The next lemma allows us to construct a BEL-gadget that colours two edges the same.

Lemma 2.4. *Let $t \geqslant 4$ and $k \geqslant 2$ be integers. Then there exist a 3-uniform hypergraph $\mathcal{H}$ and edges e and f with $|e \cap f| = 2$ and $e(\mathcal{H}[e \cup f]) = 2$ such that the following properties hold:*

(1) $\mathcal{H} \not\rightarrow (K_t^{(3)})_k$,

(2) *for every k-colouring c of $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $c(e) = c(f)$.*

Proof. We take two vertex-disjoint copies of $\mathcal{H}_1$ and $\mathcal{H}_2$ as asserted by Lemma 2.3, along with the corresponding edges $e_{1,1},\dots,e_{1,k}$ for $\mathcal{H}_1$ and $e_{2,1},\dots,e_{2,k}$ for $\mathcal{H}_2$ respectively. Recall that there exist S_1 and S_2 such that $e_{\ell,i} \cap e_{\ell,j} = S_\ell$ for all $i < j \in [k]$ and $\ell \in [2]$. We obtain the hypergraph $\mathcal{H}$ by identifying the edge $e_{1,i}$ with $e_{2,i}$ for all $2 \leqslant i \leqslant k$ such that the vertices from S_1 are identified with those from S_2.

We set $e := e_{1,1}$ and $f := e_{2,1}$ and claim that $\mathcal{H}$ fulfils the requirements. By the symmetry of the colours, we may assume that $e_{\ell,i}$ may be coloured with the colour i for all $i \in [k]$ and $\ell \in [2]$, and then we may extend the colouring by colouring the (otherwise disjoint) copies $\mathcal{H}_1$ and $\mathcal{H}_2$ separately. Since any copy of $K_t^{(3)}$ is contained fully either in $\mathcal{H}_1$ or in $\mathcal{H}_2$, we see $\mathcal{H} \not\rightarrow (K_t^{(3)})_k$. On the other hand, any $K_t^{(3)}$-free colouring φ of $\mathcal{H}$ is a $K_t^{(3)}$-free colouring of $\mathcal{H}_1$ and $\mathcal{H}_2$, and from the properties from Lemma 2.3 we have that the edges $e_{\ell,1},\dots,e_{\ell,k}$ are coloured differently for each $\ell \in [2]$ and, by the construction, $\varphi(e_{1,i}) = \varphi(e_{2,i})$ for all $2 \leqslant i \leqslant k$. Thus, we also have $\varphi(e_{1,1}) = \varphi(e_{2,1})$. □

We introduce the following definition of a path in hypergraphs. In an *r-uniform path* (or *path* for short) with t edges $e_1,\dots,e_t$ the vertices of $\cup_{i\in[t]} e_i$ are ordered linearly and

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the edges are *consecutive* segments with the property that $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in [t-1]$. We will refer to the edges e_1 and e_t as *ends* of such a path. In particular, in our notation the path is a vertex-connected subhypergraph of a so-called tight path on the vertex set $\cup_{i\in[t]} e_i$ (where in a *tight path* it is $|e_i \cap e_{i+1}| = r-1$).

Further, we say that two edges e and f have *distance* $\mathrm{dist}_H(e,f) := s$ in H if any r-uniform path in H with ends e and f contains at least s vertices and there exists at least one such path with exactly s vertices. We call a path from e to f with $\mathrm{dist}_H(e,f)$ vertices a *shortest path*. If no such path exists, we set $\mathrm{dist}_H(e,f) := \infty$.

Finally, we construct BEL-gadgets with monochromatic edges in every $K_t^{(3)}$-free colouring that are 'far' from each other according to our notion of distance.

Lemma 2.5. *Let $s, t \geqslant 4$ and $k \geqslant 2$ be integers. There exist a 3-uniform hypergraph H and two edges $e, f \in E(H)$ such that the following properties hold:*

(1) $H \not\rightarrow\!\!\!\rightarrow (K_t^{(3)})_k$,
(2) *e and f have distance at least s, and*
(3) *for every k-colouring φ on $E(H)$ which avoids monochromatic copies of $K_t^{(3)}$ we have that $\varphi(e) = \varphi(f)$.*

Proof. First we construct a hypergraph $\mathcal{H}$ which is not k-Ramsey for $K_t^{(3)}$, but contains two edges e and f at distance 5 that are coloured the same by any k-colouring of $E(\mathcal{H})$ without monochromatic $K_t^{(3)}$. We apply Lemma 2.4 twice and obtain 3-uniform hypergraphs $\mathcal{H}_1$ with edges $e_{\mathcal{H}_1} = \{a, b, x_1\}$ and $f_{\mathcal{H}_1} = \{a, b, y_1\}$ and $\mathcal{H}_2$ with edges $e_{\mathcal{H}_2} = \{c, d, x_2\}$ and $f_{\mathcal{H}_2} = \{c, d, y_2\}$ respectively. Furthermore, we may assume $V(\mathcal{H}_1) \cap V(\mathcal{H}_2) = \emptyset$. We define a new hypergraph $\mathcal{H}$ by taking both $\mathcal{H}_1$ and $\mathcal{H}_2$ and identifying y_1 with d, b with c, and a with y_2. Observe that in $\mathcal{H}$ any copy of $K_t^{(3)}$ is completely contained within one of the $\mathcal{H}_i$. This implies that $\mathcal{H} \not\rightarrow (K_t^{(3)})_k$. Indeed, according to Lemma 2.1 we can colour $\mathcal{H}_1$ and $\mathcal{H}_2$ without monochromatic $K_t^{(3)}$. Moreover, by swapping the colours appropriately if necessary, we may do so that the edges $f_{\mathcal{H}_1} \in E(\mathcal{H}_1)$ and $f_{\mathcal{H}_2} \in E(\mathcal{H}_2)$ receive the same colour. This gives us a $K_t^{(3)}$-free colouring of $E(\mathcal{H})$.

Next we use property (2) of Lemma 2.4, which asserts that any $K_t^{(3)}$-free colouring colours the edges $\{a, b, x_1\}$ and $\{a, b, y_1\}$ the same, and the colours of $\{c, d, x_2\}$ and $\{c, d, y_2\}$ are the same as well. Since $\{a, b, y_1\} = \{c, d, y_2\}$ in $\mathcal{H}$, the edges $f := \{c, d, x_2\}$ and $e := \{a, b, x_1\}$ are coloured the same through any $K_t^{(3)}$-free colouring of $\mathcal{H}$. We have thus arrived at a hypergraph $\mathcal{H}$ that satisfies the following properties:

(a) there are two edges e and f at distance 5,
(b) $\mathcal{H} \not\rightarrow\!\!\!\rightarrow (K_t^{(3)})_k$,
(c) for every k-colouring c on $E(\mathcal{H})$ which avoids monochromatic copies of $K_t^{(3)}$, we have that $c(e) = c(f)$.

Next we proceed iteratively. We take two isomorphic hypergraphs H_1 and H_2, along with edges e_1, f_1 and e_2, f_2 respectively, which satisfy (b) and (c). Assuming that $\mathrm{dist}_{H_1}(e_1, f_1) = d = \mathrm{dist}_{H_2}(e_2, f_2)$ for some $d \geqslant 5$, we now aim to construct a hypergraph H', along with edges e, f, such that (b) and (c) hold and $\mathrm{dist}_{H'}(e, f) \geqslant d + 1$. For the construction, we

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identify the edge f_1 with e_2 such that none of the vertices of e_1 and f_2 are identified, and we set $e = e_1$ and $f = f_2$. In this way properties (b) and (c) are naturally preserved in H'.

Thus, it remains to show that the distance between e_1 and f_2 is at least $d+1$ in H'. Let $v_1, \ldots, v_\ell$ be the vertices of a shortest path from e_1 to f_2 in H' in linear order, that is,

$$\{v_1, v_2, v_3\} = e_1 \quad \text{and} \quad \{v_{\ell-2}, v_{\ell-1}, v_\ell\} = f_2.$$

Let $i \geqslant 4$ be the smallest index such that $v_i \notin V(H_1)$. If $i < d-1$, then we have $v_{i-1} \in f_1$ and if $\{v_{i-3}, v_{i-2}, v_{i-1}\} \notin E(H_1)$ holds then we additionally have $\{v_{i-4}, v_{i-3}, v_{i-2}\} \in E(H_1)$ and $v_{i-2} \in f_1$. In any case we would obtain a 3-path from e_1 to f_1 with at most $d-1$ vertices, which consists of some edges of P contained in $\{v_1, \ldots, v_{i-1}\}$ and the edge f_1, a contradiction to $\mathrm{dist}_{H_1}(e_1, f_1) = d$. Thus we may assume $i \geqslant d-1$. If additionally $d > 5$, then it follows that none of the vertices from f_2 are among $\{v_1, \ldots, v_{i-1}\}$, resulting in $\mathrm{dist}_{H'}(e_1, f_2) \geqslant d+1$. If $d = 5$, then since none of the vertices of e_1 and f_2 are identified, $\mathrm{dist}_{H'}(e_1, f_2) \geqslant 6 > d$. □

Finally we are in position to build non-Ramsey hypergraphs which assert more structure in any $K_t^{(3)}$-free colouring.

Theorem 2.6. *Let $k \geqslant 2$ and $t \geqslant 4$ be integers. Let H be a 3-uniform hypergraph with $H \not\rightarrow (K_t^{(3)})_k$ and let $c : E(H) \to [k]$ be a k-colouring which avoids monochromatic copies of $K_t^{(3)}$. Then, there exists a 3-uniform hypergraph $\mathcal{H}$ with the following properties:*

(1) $\mathcal{H} \not\rightarrow (K_t^{(3)})_k$,
(2) *$\mathcal{H}$ contains H as an induced subhypergraph, and*
(3) *for every colouring $\varphi : E(\mathcal{H}) \to [k]$ without a monochromatic copy of $K_t^{(3)}$, the colouring of H under φ agrees with the colouring c, up to a permutation of the k colours.*
(4) *If there are two vertices $a, b \in V(H)$ with $\deg_H(a,b) = 0$, then $\deg_{\mathcal{H}}(a,b) = 0$ as well.*
(5) *If $|V(H)| \geqslant 4$ then for every vertex $x \in V(\mathcal{H}) \setminus V(H)$ there exists a vertex $y \in V(H)$ such that $\deg_{\mathcal{H}}(x,y) = 0$.*

Proof. Let a hypergraph H and a $K_t^{(3)}$-free colouring c be given according to the theorem. We take a hypergraph $\mathcal{H}'$ as asserted by Lemma 2.3, along with the edges $e'_1, \ldots, e'_k$, such that $V(H) \cap V(\mathcal{H}') = \emptyset$. Moreover, let H' be given according to Lemma 2.5, along with edges e' and f' of distance at least 7. Then, for every edge $g \in E(H)$, we take a copy H_g of the hypergraph H' on a set of new vertices, along with edges e_g and f_g representing e' and f'. We identify the edge g with e_g, and if g is coloured i under the colouring c then we identify f_g with e'_i. We denote the obtained hypergraph by $\mathcal{H}$.

We verify the desired properties one by one.

(1) It is easily seen that every copy F of $K_t^{(3)}$ is contained either in H or in $\mathcal{H}'$ or in some H_g with $g \in E(H)$. Indeed, if such a copy contains a vertex $x \in V(H_g) \setminus (e_g \cup f_g)$ for some $g \in E(H)$, then every other vertex $v \in V(F)$ needs to share an edge with x, which by construction needs to be part of H_g. Thus, $V(F) \subseteq V(H_g)$ and $F \subseteq \mathcal{H}[V(H_g)] = H_g$. Otherwise, F contains no such vertices x, and therefore $V(F) \subseteq V(H) \cup V(\mathcal{H}')$. By construction of $\mathcal{H}$ we know that $\mathrm{dist}_{H_g}(e_g, f_g) \geqslant 7$ for all $g \in E(H)$ and thus $\deg_{\mathcal{H}[V(H)\cup V(\mathcal{H}')]}(u,v)=0$ for every $u\in V(H)$ and $v\in V(\mathcal{H}')$, which yields $F\subseteq H$ or $F\subseteq\mathcal{H}'$.

Now we colour $E(H)$ according to c. As $V(H)\cap V(\mathcal{H}')=\emptyset$ we can easily extend c to a $K_t^{(3)}$-free colouring of $E(H)\cup E(\mathcal{H}')$ such that e_i' is coloured i for each $i\in[k]$. Here we use that by Lemma 2.3, the edges $e_1',\ldots,e_k'$ have different colours in any $K_t^{(3)}$-free colouring. Moreover, observe that for every $g\in E(H)$ we then have that e_g and f_g receive the same colour.

Next we can extend the above colouring to a $K_t^{(3)}$-free colouring of $E(\mathcal{H})$, by Lemma 2.5 and since the H_g have only already coloured edges from $\{e_1',\ldots,e_k'\}$ in common. Thus, $\mathcal{H}\not\longrightarrow(K_t^{(3)})_k$.

(2) H occurs as an induced subhypergraph in $\mathcal{H}$ since $\mathrm{dist}_{H_g}(e_g,f_g)\geqslant 6$, and thus $e_g\cap f_g=\emptyset$ for all $g\in E(H)$.

(3) Given any $K_t^{(3)}$-free colouring φ of $\mathcal{H}$, it holds by Lemma 2.3 that $e_1',\ldots,e_k'$ are coloured differently. Moreover, by Lemma 2.5, the edges f_g and e_g are coloured the same (for each $g\in e(H)$) in such a way that the ith colour class of H under c obtains the colour $\varphi(e_i')$ for each $i\in[k]$.

(4) Suppose that $\deg_H(a,b)=0$ for some two distinct vertices $a,b\in V(H)$. By construction, any two of the auxiliary hypergraphs (*i.e.*, $\mathcal{H}'$, H, H_g) overlap only in one edge (if at all). This way it follows that $\deg_{\mathcal{H}}(a,b)=0$.

(5) Finally, take some $x\in V(\mathcal{H})\setminus V(H)$. If $x\in V(\mathcal{H}')\setminus(\cup_{g\in E(H)}V(H_g))$, then $\deg_{\mathcal{H}}(x,y)=0$ for all $y\in V(H)$. If $x\in V(H_g)$ for some $g\in E(H)$, then again, by construction of $\mathcal{H}$, we have that $x\notin g\subseteq V(H)$ and therefore every $y\in V(H)\setminus g$ satisfies $\deg_{\mathcal{H}}(x,y)=0$. □

3. Minimum degrees of Ramsey-minimal 3-uniform hypergraphs

Before we prove Theorem 1.1, we first show the existence of an appropriate BEL-gadget which will be crucial for the upper bound (1.2) in Theorem 1.1. It is useful to think of a monochromatic clique $K_t^{(3)}$ as a structure that consists of two monochromatic cliques on the same vertex set: the graph K_{t-1} and the 3-uniform hypergraph $K_{t-1}^{(3)}$, where the edges of K_{t-1} 'encode' the link of the tth vertex. The following lemma will almost immediately imply the upper bound in Theorem 1.1 in that it asserts the existence of a colour pattern on some 3-uniform hypergraph H, where *any* colouring of the *edges* of the complete graph on the same vertex set $V(H)$ yields a monochromatic graph clique $F=K_{t-1}$, which is 'supported' by the $K_{t-1}^{(3)}$ of the same colour in the coloured H and on the vertex set $V(F)$.

Lemma 3.1. *Let $t\geqslant 4$ and $k\geqslant 2$ be integers. There is a 3-uniform hypergraph H on $n=k^{10kt^4}$ vertices, which can be written as an edge-disjoint union of k 3-uniform hypergraphs $H_1,\ldots,H_k$ with the following properties:*

(a) *for every $i\in[k]$, H_i contains no copies of $K_t^{(3)}$, and*

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(b) *for any colouring c of the edges of the complete graph K_n with k colours, there exists a colour $x \in [k]$ and k sets $S_1, \ldots, S_k$ that induce copies of K_{t-1} in colour x under the colouring c such that $H_1[S_1] \cong \cdots \cong H_k[S_k] \cong K_{t-1}^{(3)}$.*

Before we proceed, we state a simple quantitative version of Ramsey's theorem.

Fact 3.2. *Let $n \geqslant r_k(\ell)$. Then, in any k-colouring of $E(K_n)$ there are at least*

$$\frac{n^\ell}{k(r_k(\ell))^\ell}$$

monochromatic copies of K_ℓ in the same colour.

Proof. Fix an arbitrary red–blue colouring φ of $E(K_n)$. First observe that we find in *any* subset of $r_k(\ell)$ vertices of K_n a monochromatic K_ℓ. We estimate pairs of subsets of $[n]$ of the form (R, L) with $|R| = r_k(\ell)$, $|L| = \ell$ and $L \subseteq R$ such that all edges from $\binom{L}{2}$ are coloured the same. As a lower bound we obtain $\binom{n}{r_k(\ell)}$, while the upper bound is the number of monochromatic copies of K_ℓ under φ times the number of $r_k(\ell)$-sets containing a particular copy (which is $\binom{n-\ell}{r_k(\ell)-\ell}$). This yields that there are at least

$$\binom{n-\ell}{r_k(\ell)-\ell}^{-1}\binom{n}{r_k(\ell)} = \frac{n \cdot \ldots \cdot (n-\ell+1)}{r_k(\ell) \cdot \ldots \cdot (r_k(\ell)-\ell+1)} \geqslant \left(\frac{n}{r_k(\ell)}\right)^\ell$$

monochromatic K_ℓ. At least $1/k$ proportion of them must be in the same colour. Hence the claim follows. □

The random 3-uniform hypergraph $H^{(3)}(n,p)$ is the probability space of all labelled 3-uniform hypergraphs on the vertex set $[n]$ where each edge exists with probability p independently of the other edges. The rough idea of the proof of Lemma 3.1 is to take k random hypergraphs of appropriate density on the same vertex set and then show that even after deleting common edges and edges that lie in copies of $K_t^{(3)}$, we are left with k edge-disjoint hypergraphs that satisfy condition (b). We now turn to the details.

Proof of Lemma 3.1. We choose with foresight

$$p := C \cdot n^{\frac{-6}{(t-1)(t-2)}}, \quad \text{where } C := k^{100k/t} \text{ and } n = k^{10kt^4}. \tag{3.1}$$

We use the simple upper bound $r_k(t) \leqslant k^{kt-2k+1}$ and we define $f(t) := k^{-kt^2}$ so that, with Fact 3.2, there are at least $f(t) \cdot n^{t-1}$ monochromatic copies of K_{t-1} in one of the colours in any k-colouring of the edges of K_n.

We take k independent random 3-uniform hypergraphs $H_1', \ldots, H_k' \sim H^{(3)}(n,p)$, $i \in [k]$, on the vertex set $[n]$, and we observe first that

$$\mathbb{E}(e(H_i' \cap H_j')) = \binom{n}{3}p^2, \quad \mathbb{E}(e(H_i')) = \binom{n}{3}p \quad \text{and}$$
$$\mathbb{E}(\text{number of copies of } K_t^{(3)} \text{ in } H_i') = \binom{n}{t}p^{\binom{t}{3}}$$

for all $i \neq j \in [k]$.

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For $i \in [k]$, we let E_i' denote the (random) set of edges in H_i' that belong either to some copy of $K_t^{(3)}$ in H_i' or to the edge set of some hypergraph H_j', $j \in [k] \setminus \{i\}$. We set $H_i := H_i' \setminus E_i'$. Obviously, $H_1, \dots, H_k$ satisfy (a). To prove the lemma, it thus remains to show that (b) is satisfied with positive probability. This will be immediate from the following two claims, proved below.

Claim 3.3. *With probability larger than 3/5, the following holds. Each H_i' contains at most $0.2f(t)n^{t-1}p^{\binom{t-1}{3}}$ copies of $K_{t-1}^{(3)}$ that contain an edge from E_i'.*

Claim 3.4. *The following holds with probability at least 2/3. For every colouring $\psi: E(K_n) \to [k]$ there is a colour x such that for every $i \in [k]$, there are at least $0.5f(t)n^{t-1}p^{\binom{t-1}{3}}$ monochromatic copies F of K_{t-1} in colour x with $\binom{V(F)}{3} \subseteq E(H_i')$.*

With positive probability the conclusions of Claims 3.3 and 3.4 hold. So fix $H_1', \dots, H_k'$ that satisfy the conclusions of these claims. Recall that $H_i = H_i' \setminus E_i'$ and we only need to verify (b) as $H_1, \dots, H_k$ obviously satisfy (a). Let $\psi: E(K_n) \to [k]$ be an arbitrary colouring. Claim 3.4 asserts that there is a colour x such that for every $i \in [k]$, there are at least $0.5f(t)n^{t-1}p^{\binom{t-1}{3}}$ monochromatic copies F of K_{t-1} in colour x and such that $\binom{V(F)}{3} \subseteq E(H_i')$. By Claim 3.3, for each $i \in [k]$, at most $0.2f(t)n^{t-1}p^{\binom{t-1}{3}}$ of these copies satisfy $\binom{V(F)}{3} \not\subseteq E(H_i)$, and thus condition (b) is satisfied. □

3.1. Proofs of Claims 3.3 and 3.4

Proof of Claim 3.3. Fix an $i \in [k]$. We first consider the number X of copies of $K_{t-1}^{(3)}$ in H_i' that contain an edge e which is part of some copy of $K_t^{(3)}$ in H_i'. For a pair (T_1, T_2) of subsets of $[n]$ with $|T_1| = t-1$ and $|T_2| = t$ we define the indicator variable $I_{(T_1,T_2)}$ by

$$I_{(T_1,T_2)} := \begin{cases} 1 & \text{if } H_i'[T_1] \cong K_{t-1}^{(3)} \text{ and } H_i'[T_2] \cong K_t^{(3)}, \\ 0 & \text{else,} \end{cases}$$

and observe that

$$X \leqslant \sum_{s=3}^{t-1} \sum_{\substack{(T_1,T_2): \\ |T_1 \cap T_2| = s}} I_{(T_1,T_2)}. \tag{3.2}$$

By linearity of expectation it follows that

$$\mathbb{E}(X) \leqslant \sum_{s=3}^{t-1} n^{t-1} \cdot \binom{t-1}{s} \cdot n^{t-s} \cdot p^{\binom{t-1}{3}+\binom{t}{3}-\binom{s}{3}} \leqslant 2^t n^{2t-1} p^{\binom{t-1}{3}+\binom{t}{3}} \sum_{s=3}^{t-1} n^{-s} p^{-\binom{s}{3}}. \tag{3.3}$$

Each term above is dominated by the sum of its first and last summand. Indeed, let $g(s) := n^{-s}p^{-\binom{s}{3}}$; then for $3 \leqslant s \leqslant t-2$ we have

$$\frac{g(3)}{g(s)} = n^{s-3} \cdot p^{\binom{s}{3}-1} = \left[np^{\frac{s^2+2}{6}}\right]^{s-3} \geqslant \left[np^{\frac{s(s+1)}{6}}\right]^{s-3} \geqslant \left[np^{\frac{(t-1)(t-2)}{6}}\right]^{s-3} \geqslant 1.$$

Thus, we obtain

$$\mathbb{E}(X) \leqslant 2^t n^{2t-1} p^{\binom{t-1}{3}+\binom{t}{3}} \cdot t \cdot (g(3) + g(t-1)).$$

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And we further upper-bound $\mathbb{E}(X)$ with (3.1) by

$$\begin{aligned}\mathbb{E}(X) &\leqslant t2^t n^{t-1} p^{\binom{t-1}{3}} \big(n^t p^{\binom{t}{3}} n^{-3} p^{-1} + n^t p^{\binom{t}{3}} n^{-t+1} p^{-\binom{t-1}{3}}\big) \\ &\overset{(3.1)}{=} t2^t n^{t-1} p^{\binom{t-1}{3}} \big(C^{\binom{t}{3}} n^{-3} p^{-1} + n^{-2} C^{\binom{t-1}{2}}\big) \\ &\overset{(3.1)}{\leqslant} t2^t n^{t-1} p^{\binom{t-1}{3}} \big(k^{50kt^2/3} + k^{50kt}\big) n^{-2} \\ &\overset{(3.1)}{\leqslant} 2^{t+\log_2 t+1} k^{50kt^2/3} k^{-20kt^4} n^{t-1} p^{\binom{t-1}{3}} \leqslant \frac{1}{50k} f(t) n^{t-1} p^{\binom{t-1}{3}}.\end{aligned} \tag{3.4}$$

So, by Markov's inequality, with probability at least $1 - \frac{1}{5k}$ we have

$$X \leqslant 0.1 f(t) n^{t-1} p^{\binom{t-1}{3}}.$$

Next, consider the number Y of copies of $K_{t-1}^{(3)}$ in H_i' that contain an edge e from the intersection $E(H_i') \cap E(H_j')$ for a fixed $j \neq i$. For a subset $S \in \binom{[n]}{t-1}$ and an edge $e \in \binom{S}{3}$, let

$$I_{(S,e)} := \begin{cases} 1 & \text{if } H_i'[S] \cong K_{t-1}^{(3)} \text{ and } e \in E(H_j'), \\ 0 & \text{else,} \end{cases}$$

so that $Y \leqslant \sum_{(S,e)} I_{(S,e)}$. Then

$$\begin{aligned}\mathbb{E}(Y) &\leqslant n^{t-1} \binom{t-1}{3} \cdot p^{\binom{t-1}{3}+1} \overset{(3.1)}{=} n^{t-1} p^{\binom{t-1}{3}} \binom{t-1}{3} k^{100k/t} k^{-\frac{60kt^4}{(t-1)(t-2)}} \\ &\leqslant n^{t-1} p^{\binom{t-1}{3}} t^3 k^{25k} k^{-60kt^2} \leqslant \frac{1}{50k^3} f(t) n^{t-1} p^{\binom{t-1}{3}}.\end{aligned}$$

By Markov's inequality, with probability at least $1 - \frac{1}{5k^2}$ we then have

$$Y \leqslant \frac{1}{10k} f(t) n^{t-1} p^{\binom{t-1}{3}}.$$

In particular, with probability at least $3/5$ it holds for all $i \in [k]$ that H_i' contains at most $0.2 \cdot f(t) \cdot n^{t-1} p^{\binom{t-1}{3}}$ copies of $K_{t-1}^{(3)}$ that contain an edge from E_i'. Therefore the claim follows. □

Proof of Claim 3.4. Fix an $i \in [k]$. Let $\psi : E(K_n) \to [k]$ be an arbitrary colouring. Then there is a colour x such that there are at least $f(t)n^{t-1}$ monochromatic copies of K_{t-1} under colouring ψ which all have the same colour x (by Fact 3.2). We fix a family $\mathcal{F} = \{F_1, \ldots, F_m\}$ of exactly $m = f(t)n^{t-1}$ such copies (say lexicographically smallest ones). Now, let $X_{\mathcal{F},i}$ denote the number of such $F_j \in \mathcal{F}$ with $\binom{V(F_j)}{3} \subseteq E(H_i')$. For every $F_j \in \mathcal{F}$ let

$$X_{F_j,i} = \begin{cases} 1 & \text{if } \binom{V(F_j)}{3} \subseteq E(H_i'), \\ 0 & \text{else,} \end{cases}$$

and observe that $X_{\mathcal{F},i} = \sum_{F \in \mathcal{F}} X_{F,i}$. We define

$$\lambda := \mathbb{E}(X_{\mathcal{F},i}) = f(t) n^{t-1} \cdot p^{\binom{t-1}{3}}.$$

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Observe that by exploiting the choice of p and n in (3.1) we obtain

$$\lambda = k^{-kt^2} n^{t-1} C^{\binom{t-1}{3}} n^{-t+3} = k^{-kt^2} k^{50k(t-1)(t-2)(t-3)/(3t)} n^2. \tag{3.5}$$

Let

$$\overline{\Delta}_i := \sum_{\substack{F,F'\in\mathcal{F} \\ \binom{V(F)}{3}\cap\binom{V(F')}{3}\neq\emptyset}} \mathbb{E}(X_{F,i}X_{F',i}).$$

Next we estimate $\overline{\Delta}_i$ as follows (since each $X_{F,i}$ counts a copy of the complete 3-uniform hypergraph on the vertex set $V(F)$, we can classify pairs of these copies according to the number s of common vertices):

$$\overline{\Delta}_i \leqslant |\mathcal{F}| \sum_{s=3}^{t-1} \binom{t-1}{s} n^{t-1-s} p^{2\binom{t-1}{3}-\binom{s}{3}} \leqslant f(t)\cdot n^{2t-2} p^{2\binom{t-1}{3}} 2^t \sum_{s=3}^{t-1} n^{-s} p^{-\binom{s}{3}},$$

and thus exactly as in the previous claim, Claim 3.3, we estimate the sum by

$$t\big(n^{-3}p^{-1} + n^{-t+1}p^{-\binom{t-1}{3}}\big),$$

which leads to the upper bound

$$\begin{aligned} \overline{\Delta}_i &\leqslant t2^t\lambda\big(n^{t-1}p^{\binom{t-1}{3}}n^{-3}p^{-1} + n^{t-1}p^{\binom{t-1}{3}}n^{-t+1}p^{-\binom{t-1}{3}}\big) \\ &= t2^t\lambda\big(C^{\binom{t-1}{3}}(pn)^{-1}+1\big) \overset{(3.1)}{=} 2^{t+\log_2 t}\lambda\big(k^{\frac{100k}{t}\left[\binom{t-1}{3}-1\right]}k^{-10kt^4+\frac{60kt^4}{(t-1)(t-2)}}+1\big) \leqslant 2^{2t}\lambda. \end{aligned} \tag{3.6}$$

Now with Janson's inequality (see, *e.g.*, Theorem 2.14 in [13]) we obtain

$$\begin{aligned} \mathbb{P}(X_{\mathcal{F},i} \leqslant 0.5\lambda) &\leqslant \exp(-\lambda^2/(8\overline{\Delta}_i)) \overset{(3.6)}{\leqslant} \exp(-2^{-2t-3}\lambda) \\ &\overset{(3.5)}{\leqslant} \exp(-2^{-2t-3}k^{-kt^2+50k(t-1)(t-2)(t-3)/(3t)}n^2) \\ &\leqslant \exp(-2^{-2t-3}k^{-kt^2+50kt^2/32}n^2) \leqslant \exp(-k^{-2t-3+9t^2/8}n^2) \\ &\leqslant \exp(-k^7n^2). \end{aligned}$$

This tells us that for the colour x with probability at least $1-k\exp(-k^7n^2)$ all graphs H_i', $i\in[k]$, contain at least

$$0.5\cdot f(t)\cdot n^{t-1}p^{\binom{t-1}{3}}$$

copies F of K_{t-1} in colour x and with

$$\binom{V(F)}{3} \subseteq E(H_i').$$

Since there are $k^{\binom{n}{2}}$ different colourings of $E(K_n)$, we may apply the union bound to see that the probability that there is a colouring $\psi : E(K_n) \to \{\text{red}, \text{blue}\}$ not satisfying the claim is at most $k^{\binom{n}{2}}\cdot k\exp(-k^7n^2) < 1/3$. □

3.2. Proof of Theorem 1.1

A lower bound on $s_{k,1}(K_t^{(3)})$. The proof of the lower bound is easy. In fact, it follows from the bound on the Ramsey number $r_k(K_t) \geqslant 2^{\frac{1}{4}kt(1-o(1))}$, and is as follows. Take a

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k-Ramsey-minimal hypergraph $\mathcal{H}$ for $K_t^{(3)}$ such that $\delta(\mathcal{H}) = s_{k,1}(K_t^{(3)})$ and let $v \in V(\mathcal{H})$ be a vertex of minimum degree. By minimality of $\mathcal{H}$, we have $\mathcal{H} \setminus \{v\} \not\longrightarrow (K_t^{(3)})_k$ and fix an edge colouring φ that certifies this. Since $\mathcal{H} \longrightarrow (K_t^{(3)})_k$ it follows that the link graph $\mathrm{link}_{\mathcal{H}}(v)$ is Ramsey: $\mathrm{link}_{\mathcal{H}}(v) \longrightarrow (K_{t-1})_k$. Therefore,

$$s_{k,1}(K_t^{(3)}) = \deg(v) \geqslant \hat{r}_k(K_{t-1}) = \binom{r_k(K_{t-1})}{2} \geqslant 2^{\frac{1}{2}kt(1-o(1))},$$

where $\hat{r}_k(K_\ell)$ is the *size-Ramsey number* for K_ℓ, and it was shown by Erdős, Faudree, Rousseau and Schelp [7] that $\hat{r}_k(K_\ell) = \binom{r_k(K_\ell)}{2}$.

An upper bound on $s_{k,1}(K_t^{(3)})$. Let H be the 3-uniform hypergraph as asserted by Lemma 3.1 along with the hypergraphs $H_1, \ldots, H_k$ that satisfy conditions (a) and (b). We fix the following $K_t^{(3)}$-free k-colouring c of $E(H)$: we colour all edges from H_i with colour $i \in [k]$. Further, let $\mathcal{H}'$ be the hypergraph guaranteed by Theorem 2.6 for given H and c. We define the hypergraph $\mathcal{H}$ by adding to $\mathcal{H}'$ a new vertex v whose link is

$$\mathrm{link}_{\mathcal{H}}(v) := \binom{V(H)}{2}.$$

So $\deg_{\mathcal{H}}(v) = \binom{n}{2} < k^{20kt^4}$ as asserted by Lemma 3.1. In the following we argue that $\mathcal{H}' \not\longrightarrow (K_t^{(3)})_k$ but $\mathcal{H} \longrightarrow (K_t^{(3)})_k$. It then follows immediately that every Ramsey subhypergraph of $\mathcal{H}$ (in particular Ramsey-minimal subhypergraph of $\mathcal{H}$) for $K_t^{(3)}$ needs to contain the vertex v, whose degree is less than k^{20kt^4}. Thus, once these two properties are proved, the upper bound follows.

In fact, $\mathcal{H}' \not\longrightarrow (K_t^{(3)})_k$ is asserted by Theorem 2.6. So, we only need to focus on showing that $\mathcal{H} \longrightarrow (K_t^{(3)})_k$. For contradiction, suppose that there is a colouring $\varphi\colon E(\mathcal{H}) \to [k]$ without monochromatic copies of $K_t^{(3)}$. We then know by property (3) of Theorem 2.6 that $E(H_1), \ldots, E(H_k)$ are all coloured monochromatically, but in different colours. Without loss of generality we may assume that, for each $i \in [k]$, H_i is coloured with the colour i. Now, we define a colouring

$$\psi\colon \binom{V(H)}{2} \to [k]$$

with $\psi(\{u_1, u_2\}) = \varphi(\{u_1, u_2, v\})$. Then, according to Lemma 3.1 there is a colour x and the sets

$$S_1, \ldots, S_k \in \binom{V(H)}{t-1}$$

such that $\binom{S_1}{2}, \ldots, \binom{S_k}{k}$ are monochromatic under ψ in colour x, while for every $i \in [k]$ we have that $H[S_i] \cong K_{t-1}^{(3)}$ is coloured i. But this implies immediately that we have found a monochromatic clique $\mathcal{H}[S_x \cup \{v\}] \cong K_t^{(3)}$ in colour x, a contradiction. □

4. Minimum codegrees of Ramsey-minimal 3-uniform hypergraphs

In this section we prove Theorem 1.2 by showing that $s_{2,2}(K_t^{(3)}) = 0$ and that $s'_{2,2}(K_t^{(3)}) = (t-2)^2$. Our proof strategy is similar to that of [1, 10]: for the lower bound we provide

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an *ad hoc* argument, while for the upper bound we employ BEL-gadgets, Theorem 2.6, combined with a natural construction that we 'plant' via a BEL-gadget (which is an almost Ramsey hypergraph).

Proof of Theorem 1.2

Lower bound argument for $s'_{2,2}$. We first prove that $s'_{2,2}(K_t^{(3)}) \geqslant (t-2)^2$. Take a minimal 2-Ramsey hypergraph H for $K_t^{(3)}$. Fix any two vertices u and $v \in V(H)$ with $\deg_H(u,v) > 0$. We aim to show that $\deg_H(u,v) \geqslant (t-2)^2$. So, assume the opposite, that is, $\deg_H(u,v) < (t-2)^2$.

Let H' be the subhypergraph obtained from H by deleting all edges containing both vertices u and v. Since H is Ramsey-minimal, $H' \not\longrightarrow (K_t^{(3)})_2$. Thus, there is a colouring c with red and blue of $E(H')$ which does not create a monochromatic copy of $K_t^{(3)}$. Define

$$N(u,v) := \{w \in V(H) : \ \{u,v,w\} \in E(H)\},$$

and thus $\deg_H(u,v) = |N(u,v)|$. Take a longest sequence $B_1,\dots,B_k$ of vertex disjoint sets of size $t-2$ in $N(u,v)$, such that both $B_i \cup \{u\}$ and $B_i \cup \{v\}$ span only blue edges under the colouring c in H. By assumption on the codegree $\deg_H(u,v)$, we know that $k < t-2$.

Next we can extend the colouring c as follows. For each edge $e = \{u,v,w\} \in E(H)$ with $w \in \bigcup B_i$ we set $c(e) = \text{red}$, while for all other edges $e = \{u,v,w\} \in E(H)$ we set $c(e) = \text{blue}$. We claim that under this colouring there is no monochromatic copy of $K_t^{(3)}$ in H. Indeed, if there were a monochromatic subgraph F isomorphic to $K_t^{(3)}$, then necessarily $u,v \in V(F)$ (since $E(H')$ were coloured without monochromatic $K_t^{(3)}$). If F is red, then by construction F can have at most one vertex from each of the sets B_i and no vertex from $N(u,v) \setminus \bigcup B_i$, so $|V(F)| < t$, a contradiction. If F is blue, then it cannot contain vertices from $\bigcup B_i$, and therefore

$$V(F) \subseteq (N(u,v) \setminus \bigcup B_i) \cup \{u,v\}.$$

But then we could extend the sequence of the B_i by the set $V(F) \setminus \{u,v\}$, in contradiction to its maximality. Therefore, under the assumption $\deg_H(u,v) < (t-2)^2$ we conclude that $H \not\rightarrow (K_t^{(3)})_2$, a contradiction. Thus, we need to have $\deg_H(u,v) \geqslant (t-2)^2$ for every $u,v \in V$ with $\deg_H(u,v) > 0$. Therefore, $s'_{2,2}(K_t^{(3)}) \geqslant (t-2)^2$.

Upper bound argument for $s'_{2,2}$. First we provide a hypergraph H with a prescribed colouring of $E(H)$ without a monochromatic $K_t^{(3)}$. We set $V(H) := [(t-2)^2] \cup \{a,b\}$ and we further partition the vertices of $[(t-2)^2]$ into $(t-2)$ equal-sized sets $V_1,\dots,V_{t-2}$. Next we choose the edges for H as follows:

$$\begin{aligned} E(H) := & \bigcup_{i=1}^{t-2} \binom{V_i}{3} \cup \left\{ e \cup \{w\} : e \in \binom{V_i}{2} \quad \text{for some } i \in [t-2], w \in \{a,b\} \right\} \\ & \cup \left\{ f : f \in \binom{[(t-2)^2]}{3}, |f \cap V_i| \leqslant 1 \ \forall i \in [t-2] \right\} \\ & \cup \left\{ e \cup \{w\} : e \in \binom{[(t-2)^2]}{2}, |e \cap V_i| \leqslant 1 \ \forall i \in [t-2], w \in \{a,b\} \right\}. \end{aligned} \tag{4.1}$$

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Thus, H is obtained from the clique $K^{(3)}_{(t-2)^2+2}$ on the vertex set $\bigcup V_i \cup \{a,b\}$, where we delete all edges that contain both a and b and moreover we delete all edges that cross exactly two different V_i and contain neither a nor b. Next we provide a red–blue colouring c of the edges of H as follows: the edges contained in $V_i \cup \{a\}$ and in $V_i \cup \{b\}$ for $i \in [t-2]$ are coloured *blue*, while the other edges of H are coloured *red* – thus the edges in the first line of (4.1) are coloured blue, while the edges defined in the second and third line of (4.1) are coloured red. It is immediate that such a colouring does not yield a monochromatic copy of $K^{(3)}_t$. Indeed, a blue copy of $K^{(3)}_s$ cannot use vertices from different sets V_i and, since $\deg_H(a,b) = 0$, it also cannot contain both vertices a, b, which gives $s \leqslant t-1$. Similarly, a red copy of $K^{(3)}_s$ can use at most one vertex from each V_i and, as $\deg_H(a,b) = 0$, it also cannot contain both vertices a, b, which again gives $s \leqslant t-1$.

Applying Theorem 2.6 to the coloured hypergraph H for this colouring c, we obtain a 3-uniform hypergraph $\mathcal{H}$ which contains H as an induced hypergraph, which is not 2-Ramsey for $K^{(3)}_t$ and such that any red–blue $K^{(3)}_t$-free colouring φ of $E(\mathcal{H})$ agrees on $E(H)$ with the colouring c up to permutation of the two colours. Also, Theorem 2.6 asserts that $\deg_{\mathcal{H}}(a,b) = 0$. Next we define $\mathcal{H}'$ by adding to $\mathcal{H}$ all $(t-2)^2$ edges $\{a,b,u\}$ where $u \in [(t-2)^2]$.

Let us see why $\mathcal{H}' \longrightarrow (K^{(3)}_t)_2$. Fix any colouring φ of $E(\mathcal{H}')$ and assume that no copy of $K^{(3)}_t$ is monochromatic in $\mathcal{H}'$ under φ. Since $\mathcal{H} \subseteq \mathcal{H}'$, it follows that the colour pattern c as described above (up to permutation) is enforced in H. Assume without loss of generality that $E(H)$ is coloured according to c. Then if there is a set V_i such that all edges $\{v,a,b\}$ are coloured blue for all $v \in V_i$ this would yield a blue copy of $K^{(3)}_t$. So, assume that for every V_i there is at least one edge $\{v_i,a,b\}$ which is coloured red for some $v_i \in V_i$. Then $\{a,b,v_1,\ldots,v_{t-2}\}$ forms a red clique $K^{(3)}_t$. Thus, in any case, we find a monochromatic copy of $K^{(3)}_t$, that is, $\mathcal{H} \longrightarrow (K^{(3)}_t)_2$. Moreover, since $\mathcal{H}$ is not 2-Ramsey for $K^{(3)}_t$, any minimal 2-Ramsey subhypergraph of $\mathcal{H}'$ must contain edges that contain both a and b. This shows $s'_{2,2}(K^{(3)}_t) \leqslant (t-2)^2$.

In fact, note that by the previous discussion of the lower bound on $s'_{2,2}$, any such minimal 2-Ramsey subhypergraph of $\mathcal{H}'$ must contain all the $(t-2)^2$ edges that contain both a and b. This will be important in the following proof.

Showing $s_{2,2}(K^{(3)}_t) = 0$. This looks surprising at first sight since taking $K^{(3)}_n$ with $n = r_2(K^{(3)}_t)$ and then deleting all edges that contain two distinguished vertices gives a non-Ramsey hypergraph (which suggests $s_{2,2}(K^{(3)}_t) > 0$). However, this is not the case, and it will follow from the above construction of the hypergraph $\mathcal{H}'$.

As argued above, *any* Ramsey-minimal subhypergraph of $\mathcal{H}'$ for $K^{(3)}_t$ has to contain *all* $(t-2)^2$ edges that contain a and b. Thus, any such minimal hypergraph $\mathcal{H}''$ contains all vertices of H. Next we argue that $\mathcal{H}''[V(H)] \not\longrightarrow (K^{(3)}_t)_2$. Indeed, by construction of $\mathcal{H}'$, we observe that $\mathcal{H}'[V(H)] \supseteq \mathcal{H}''[V(H)]$ contains exactly $(t-2)+(t-2)^{t-2}$ copies of $K^{(3)}_t$, namely exactly $(t-2)$ ones that are induced on $V_i \cup \{a,b\}$ for some $i \in [t-2]$, and $(t-2)^{t-2}$ ones that contain one vertex from each of the V_i and additionally a and b. There are no further copies of $K^{(3)}_t$ since $H[\bigcup V_i]$ contains only copies of $K^{(3)}_{t-2}$ which either cross all the V_i or are equal to some $H[V_i]$. It is now easy to see that $\mathcal{H}'[V(H)] \not\longrightarrow (K^{(3)}_t)_2$, as follows. We can colour the edges of $\mathcal{H}''[V(H)]$ uniformly at random with colours red and blue. Then, the expected number of monochromatic copies of $K_t^{(3)}$ is

$$[(t-2)+(t-2)^{t-2}]\cdot 2^{1-\binom{t}{3}} < 1,$$

as $t \geqslant 4$, that is, there exists a 2-colouring which avoids monochromatic copies of $K_t^{(3)}$.

Thus, $\mathcal{H}''$ has to contain at least one further vertex $x \notin V(H)$. Then, since $|V(H)| = (t-2)^2+2 \geqslant 6$, it follows by property (5) of Theorem 2.6 that there exists a vertex $y \in V(H)$ such that $0 = \deg_{\mathcal{H}'}(x,y) \geqslant \deg_{\mathcal{H}''}(x,y)$. Therefore, $s_{2,2}(K_t^{(3)}) = 0$. □

5. Concluding remarks

In this paper we studied the smallest minimum degree and codegree of Ramsey-minimal 3-uniform hypergraphs for complete hypergraphs $K_t^{(3)}$, $t \geqslant 4$. In particular we showed that the smallest minimum degree $s_{2,1}(K_t^{(3)})$ of minimal 2-Ramsey 3-uniform hypergraph lies between 2^t and 2^{40t^4}. It would be interesting to determine the right order of the exponent. We leave the study of Ramsey-minimal r-uniform hypergraphs for $r \geqslant 4$ to future work and confine ourselves in the following discussion to some speculations about possible values of $s_{k,\ell}(K_t^{(r)})$ and of $s'_{k,\ell}(K_t^{(r)})$.

5.1. BEL-gadgets for uniformities $r \geqslant 4$

It does not seem that the BEL-gadgets for 3-uniform hypergraphs are easily generalizable to higher uniformities. Very roughly speaking, while proving a key lemma, Lemma 2.1, about the existence of hypergraphs all of whose $K_t^{(3)}$-free colourings colour two particular edges differently, we heavily relied on the fact that the hypergraph H, which we obtain from $K_m^{(3)}$ where $m = r_k(K_t^{(3)})$ and where we remove all the edges of the form $\{i, m-1, m\}$, is not k-Ramsey for $K_t^{(3)}$, that is, $H \not\rightarrow (K_t^{(3)})_k$. This allowed us to add some edges back and concentrate our attention on the colour of the edges $\{i, m-1, m\}$, bringing us back to the similar case for graphs. If we would like to carry out a similar argument, then it would be helpful to know whether the hypergraph H that we obtain from $K_m^{(r)}$ with $m = r_k(K_t^{(r)})$ by deleting all edges that contain fixed $(r-1)$ vertices, say, $m-r+2, m-r+3, \ldots, m$, is k-Ramsey. However, this seems hopeless at the moment, as we do not even know if the hypergraph $K_m^{(r)}$ is k-Ramsey-minimal for $K_t^{(r)}$, where $m = r_k(K_t^{(r)})$. However, we believe that gadgets similar to those in Lemma 2.1 should be possible to obtain by other means.

5.2. Possible values of $s_{k,\ell}(K_t^{(r)})$ and of $s'_{k,\ell}(K_t^{(r)})$

As we already mentioned, it appears surprising that $s_{2,2}(K_t^{(3)}) = 0$, that is, there exists a 2-Ramsey-minimal hypergraph H for $K_t^{(3)}$ with two vertices whose codegree is zero. We believe that the same should be the case for more colours and higher uniformity.

Problem 5.1. *Let $r \geqslant 3$, $t \geqslant r+1$ and $2 \leqslant \ell \leqslant r-1$. Show that $s_{k,\ell}(K_t^{(r)}) = 0$.*

Note that the case $r = 3$, $t \geqslant 4$ and $\ell = 2$ is our Theorem 1.2. Also, given that in the graph case $s_{2,1}(K_t) = (t-1)^2$ and the smallest non-zero codegree $s'_{2,2}(K_t^{(3)}) = (t-2)^2$, we believe that it should be generalizable to higher uniformities as follows.

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Problem 5.2. *Let $r \geqslant 4$, $t \geqslant r+1$. Show that $s'_{2,r-1}(K_t^{(r)}) = (t+1-r)^2$.*

Note that this clearly generalizes the cases $s_{2,1}(K_t) = (t-1)^2$ and $s'_{2,2}(K_t^{(3)}) = (t-2)^2$. Moreover, if a BEL-gadget similar to that in Theorem 2.6 exists, then Problem 5.2 follows by a similar construction and the argument given in the proof of Theorem 1.2.

Finally, we think that it is interesting to generalize the vertex degree case to higher uniformities. We believe that the following should hold.

Problem 5.3. *Let $r \geqslant 4$. Show that $s_{k,1}(K_t^{(r)}) = r_k\left(K_{t-1}^{(r-1)}\right)^{c(k,t)}$, where $c(k,t)$ is some polynomial in k and t.*

Theorem 1.1 shows that $c(k,t)$ is a polynomial in k and t. Further, by a similar lower bound argument, we have that $s_{k,1}(K_t^{(r)}) \geqslant \hat{r}_k(K_{t-1}^{(r-1)})$, the corresponding size-Ramsey number, which can be easily shown to be at least $\binom{r_k(K_{t-1}^{(r-1)})}{2}$. For a slightly better lower bound in the case of 3-uniform hypergraphs we refer to a recent paper by Dudek, La Fleur, Mubayi and Rödl [6].

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