


PAPER

Jets and differential linear logic

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Abstract

We prove that the category of vector bundles over a fixed smooth manifold and its corresponding category of convenient modules are models for intuitionistic differential linear logic. The exponential modality is modelled by composing the jet comonad, whose Kleisli category has linear differential operators as morphisms, with the more familiar distributional comonad, whose Kleisli category has smooth maps as morphisms. Combining the two comonads gives a new interpretation of the semantics of differential linear logic where the Kleisli morphisms are smooth local functionals, or equivalently, smooth partial differential operators, and the codereliction map induces the functional derivative. This points towards a logic, and hence a computational theory of non-linear partial differential equations and their solutions based on variational calculus.

Keywords: Linear logic; vector bundles; differential operators; differential categories

1. Introduction

In this paper, we study differential linear logic (Ehrhard 2018) through the lens of the category of vector bundles over a smooth manifold. We prove that a number of categories arising from the category of vector bundles are models for intuitionistic differential linear logic. This is part of a larger project aimed at understanding the interaction between differentiable programming, based on differential λ -calculus (Ehrhard and Regnier 2003), and differential linear logic with a view towards extending these concepts to a language of non-linear partial differential equations. Since morphisms come from proofs in differential linear logic, and proofs are identified with programs in differential λ -calculus (via the Curry–Howard correspondence), the denotational semantics here provide tools for differentiable programming.

From a machine learning perspective, the work here suggests the possibility of a non-linear ‘differential equation search’ using gradient-based optimisation given some input and output boundary conditions in analogy with continuous ‘function search’, for example, searching among a subspace of the space of functional programs to find a program satisfying given constraints. A particular case of the latter is neural architecture search given input–output data pairs. This data-driven programming has recently received attention in relation to the optimisation of neural networks with various approaches to making the constituent functional blocks ‘smooth’ (see Graves et al. 2016; Pham et al. 2018; Zoph and Le 2017 for selected works). More recently, such tools have been used to solve certain partial differential equations with initial steps towards equation search (see Lagaris et al. 1998; Long et al. 2019; Weinan et al. 2017 for selected works).

A more foundational approach to these questions, in the spirit of this paper, was recently proposed in Kerjean (2018). With an eye towards non-linear phenomena arising in a diverse range

of scientific applications, we move beyond vector spaces to families of vector spaces parametrised by a smooth manifold.

There exist a number of approaches to the categorical semantics of differential linear logic in the literature. These include Köthe sequence spaces (Ehrhard 2002), finiteness spaces (Ehrhard 2005), convenient vector spaces (Blute et al. 2012), and vector spaces themselves (Clift and Murfet 2017). Our approach begins by considering a smooth generalisation of Clift and Murfet (2017) where our underlying objects are vector spaces parametrised by a fixed base manifold M . More precisely, to a formula A in differential linear logic, we associate the sheaf \mathcal{E} of sections of a vector bundle E on M . When E is the trivial line bundle, then the associated denotation is simply the sheaf \mathcal{C}_M^∞ of smooth functions on M .

We prove that there are two natural comonads on the category of vector bundles to model the exponential modality of linear logic. First, there is the jet comonad $!_j$ introduced in Marvan (1986) which sends a sheaf \mathcal{E} to the sheaf $!_j\mathcal{E}$ of infinite jets of local sections of E . An element of the exponentiation of a formula is an equivalence class of sections of a vector bundle with the same Taylor expansion at each point of M . The idea of a syntactic Taylor expansion in linear logic and λ -calculus through the exponential connective (Ehrhard and Regnier 2003; 2008) is therefore explicitly present here in the semantics of vector bundles. Working in the general setting of infinite jets, as opposed to r -jets for a fixed $r \in \mathbb{N}$, forces us to work in the enlarged category of pro-ind vector bundles (Güneysu and Pflaum 2017). The objects in this category are (co)filtered objects in the category of vector bundles on M .

In fact, to leverage better formal and functional analytic properties of vector bundles, especially in relation to dual objects, we move from pro-ind vector bundles to the category of convenient \mathcal{C}_M^∞ -modules. These are \mathcal{C}_M^∞ -module objects in the category of sheaves of convenient vector spaces on M . Sheaves of convenient vector spaces (see Frölicher and Kriegl 1988; Kriegl and Michor 1997 for the theory of convenient spaces) are a class of sheaves of infinite dimensional vector spaces which are general enough to include sections of an arbitrary vector bundle on a smooth manifold but which also retain excellent formal properties. For example, the category of such objects is complete, cocomplete and closed symmetric monoidal. The jet comonad $!_j$ descends to this category.

The jet construction makes direct contact with the theory of linear differential operators and linear partial differential equations which enables us to understand these concepts within the setting of differential linear logic. The Kleisli category for the jet comonad is the category of convenient vector bundles \mathcal{E} on M and whose morphisms $!_j\mathcal{E} \rightarrow \mathcal{E}'$ are linear differential operators. This category is equivalent to the category of infinitely prolonged linear partial differential equations with M as its manifold of independent variables. Equivalently, as is always the case for the Kleisli category of a comonad, the objects are cofree $!_j$ -coalgebras. We prove that the category of convenient \mathcal{C}_M^∞ -modules with the jet comonad is a symmetric monoidal storage category in the sense of Blute et al. (2019).

The second comonad we consider will be called the distributional comonad. Providing a symmetric monoidal storage category, which is moreover additive in an appropriate sense, with a codereliction map that models the differential in the context of linear logic, defines a model for intuitionistic differential linear logic. It has been shown in Blute et al. (2012) that the category of convenient vector spaces is a model for intuitionistic differential linear logic where the comonad is the map sending a convenient vector space to the Mackey-closure of the linear span of its Dirac distributions. When the convenient vector space is finite dimensional, this is simply the space of distributions with compact support. We extend this result to the category of convenient \mathcal{C}_M^∞ -modules and continuous linear morphisms.

The Kleisli category of the distributional comonad $!_\delta$ is the category of convenient \mathcal{C}_M^∞ -modules and smooth morphisms. The codereliction

$$\vec{d}_\mathcal{E} : \mathcal{E} \rightarrow !_\delta\mathcal{E}$$

sends a section s to $\lim_{h \rightarrow 0} \frac{\delta_{hs} - \delta_0}{h}$. The differential of a smooth functional $F: \mathcal{E} \rightarrow \mathcal{E}'$ is then the linear map

$$dF: \mathcal{E} \multimap (\mathcal{E} \Rightarrow \mathcal{E}')$$

sending (s, t) to the functional derivative $dF(s, t)$ of s in the direction t . A familiar example is when \mathcal{E} and \mathcal{E}' are both the sheaf \mathcal{C}_M^∞ of smooth functions. Then, F is an element of the continuous linear dual $(\mathcal{C}_M^\infty)^\perp$. Using the canonical evaluation pairing, this sheaf is isomorphic to the sheaf of compactly supported distributional densities on M .

Combining the two comonads $!_j$ and $!_\delta$, which are proved to compose in the appropriate sense, becomes quite powerful. We prove that the category of convenient \mathcal{C}_M^∞ -modules with the composite comonad $!_\delta \circ !_j$ is a model for intuitionistic differential linear logic. The codereliction

$$\bar{d}_\mathcal{E}^{j\delta}: \mathcal{E} \rightarrow !_j\delta \mathcal{E}$$

sends a section s to $\lim_{h \rightarrow 0} \frac{\delta_{h(j(s))} - \delta_0}{h}$. In this case, we have a logic of smooth local functionals where a functional is local if and only if the value of its variables at a point x in M depends only on its infinite jet at that point. These functionals are also known as Lagrangians and the functional derivative of a Lagrangian L encodes the Euler–Lagrange equations (plus a total derivative). The functional equation $dL = 0$ then encodes the space of solutions to the equations of motion. Morphisms $!_\delta !_j \mathcal{E} \rightarrow \mathcal{E}'$ from the Kleisli category are interpreted as smooth differential operators. The interaction between these comonads show how to pass between linear and non-linear objects. More work is needed to understand the logic rules underlying this structure. For linear partial differential equations with constant coefficients, this has been explored in Kerjean (2018).

We end the paper by discussing how the above structure arises in the case where our vector bundle denotation is the trivial line bundle and our local functional is the Lagrangian for a free or self-interacting scalar field on an arbitrary Riemannian manifold. In this case, its convenient \mathcal{C}_M^∞ -module is the sheaf of smooth functions on the manifold, and the variational calculus leads to the space of solutions to the scalar field equations.

Remark 1. Most categories arising in linear logic (Girard 1987), objects of which include function spaces, are non-reflexive, i.e., there is no canonical isomorphism between an object and its double dual. This is also the case in our examples. In Girard (1999), Girard explored the denotational semantics of classical linear logic using the notion of coherent Banach space. Adding coherence solves the issue of obtaining a monoidal category of reflexive objects. However, one of the shortcomings of that model is a natural closed structure. Another way of saying this is that coherent Banach spaces do not form a $*$ -autonomous category. Moreover, one cannot take the $*$ -autonomous completion of the category of Banach spaces since they themselves do not form a closed symmetric monoidal category. One can extend our results to the setting of classical differential linear logic by taking the $*$ -autonomous completion of the closed symmetric monoidal category of convenient \mathcal{C}_M^∞ -modules. This completion, whose origins essentially go back to Mackey (1945), is called the Chu-construction (Barr 1991) and is the universal way of overcoming the problem of reflexivity. This remedy pairs each space with a subspace of its continuous linear dual in order to obtain the required canonical isomorphism $\mathcal{E} \simeq \mathcal{E}^{\perp\perp}$.

Relation to other work

We consider vector bundles over a fixed base manifold M . In the case of the distributional comonad with $M = *$, our results correspond to those of Blute et al. (2012), i.e., the category of convenient \mathcal{C}_M^∞ -modules reduces to the category of convenient vector spaces and the models for intuitionistic differential linear logic agree.

When introducing the jet comonad, we have an interpretation of differential operators and linear partial differential equations within the logic. Similar structure in the form of linear partial differential operators with constant coefficients has recently been studied in Kerjean (2018) in the case where $M = *$ and the fibre over $*$ is Euclidean space \mathbb{R}^n .

Combining the two comonads introduces a logical interpretation of non-linear differential operators. A more general extension to non-linear cases, including the full theory of non-linear partial differential equations, involves considering morphisms of fibred manifolds. To obtain a closed symmetric monoidal category in this setting requires moving outside the category of fibred manifolds and taking advantage of topos-theoretic and homotopical methods. We will not consider this extension here. However, see Khavkine and Schreiber (2017) for the theory of non-linear partial differential equations in more general synthetic categories.

2. Models for Intuitionistic Differential Linear Logic

In this section, we recall what it means for a category to be a model for intuitionistic differential linear logic (Ehrhard 2018). To motivate such a definition, we recall some of its basic features. We emphasise that this is not a complete presentation of the logic. We merely highlight some properties in order to orient the reader towards the categorical definition at the end of this section.

The syntax for intuitionistic linear logic involves the connectives $\{\times, \otimes, !, \multimap\}$ with formulas A generated by expressions of the form

$$A ::= \top \mid 1 \mid A \times B \mid A \otimes B \mid A \multimap B \mid !A$$

where \top and 1 are units for \times and \otimes , respectively.

Let Γ and Θ be a (possibly empty) sequence of formulas A_1, \dots, A_n . The connectives satisfy various rules, which can be split between logic rules and structural rules. The logic rules include, among others, the rules

$$\frac{\Gamma, A_1, A_2, \Theta \vdash B}{\Gamma, A_1 \times A_2, \Theta \vdash B} \times \quad \frac{\Gamma, A_1, A_2, \Theta \vdash B}{\Gamma, A_1 \otimes A_2, \Theta \vdash B} \otimes \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \multimap$$

for the additive, multiplicative and implicative connectives. The structural rules are the exchange rule, identity rule, contraction rule, weakening rule and cut rule.

Differential linear logic symmetrises the contraction, weakening and dereliction rules

$$\frac{\Gamma, !A, !A, \Theta \vdash B}{\Gamma, !A, \Theta \vdash B} c \quad \frac{\Gamma, \Theta \vdash B}{\Gamma, !A, \Theta \vdash B} w \quad \frac{\Gamma, A, \Theta \vdash B}{\Gamma, !A, \Theta \vdash B} d$$

for the exponential of linear logic, by adding cocontraction, coweakening and codereliction rules

$$\frac{\Gamma, !A, \Theta \vdash B}{\Gamma, !A, !A, \Theta \vdash B} \bar{c} \quad \frac{\Gamma, !A, \Theta \vdash B}{\Gamma, \Theta \vdash B} \bar{w} \quad \frac{\Gamma, !A, \Theta \vdash B}{\Gamma, A, \Theta \vdash B} \bar{d}$$

respectively. This is a very natural thing to do in light of the symmetry inherent in the full classical linear logic (Girard 1987) of which intuitionistic linear logic is a restriction thereof.

Given a sequent $\Gamma \vdash B$, a proof of $\Gamma \vdash B$ is a series of sequents, beginning with basic axioms, and following various deduction rules which terminate with the sequent $\Gamma \vdash B$. Two proofs are said to be *equivalent* if they are equivalent under cut elimination.

The goal of denotational semantics is to construct a category, a categorical semantics of differential linear logic in our case, faithfully reflecting this structure. See Melliès (2009) for an overview of the subject. More generally, we should allow multicategories which, like categories, consist of a collection of objects, but allow multimorphisms from a finite sequence of objects to a single target object. If we denote by $\llbracket A \rrbracket$ the denotation of a formula A , then $\llbracket A \rrbracket$ is an object of the multicategory whilst $\llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket$ are multimorphisms for some collection of formulas Γ (Hyland and De Paiva 1993). More precisely, the equivalence class of proofs of the sequent $\Gamma \vdash B$ under cut elimination is assigned to the morphism.

The logical rules for each operation should follow from universal properties in the multicategory. Working in a multicategory ensures that the connectives, together with their complete coherence data, satisfy a universal property. For example, the tensor product in a general monoidal category is not universal. It also illuminates the interpretation of the structural rules categorically. Therefore, the discussion above should suggest, at minimum, the structure of a multicategory where the additive connective corresponds to the product and the multiplicative connective to the tensor product.

Forgoing some generality, we will work directly in a symmetric monoidal category \mathcal{C} with finite products where the coherence data are contained in explicit proofs. Note that we deliberately stay close to the notation used for connectives in linear algebra and make no distinction between the connectives in logic and those in the semantic model. It should be clear from the context if we are referring to a multiplicative or additive product of formulas in logic or of objects in \mathcal{C} .

Before defining what we mean by a model for differential linear logic \mathcal{C} , we give an informal motivation for some of the various other structures on \mathcal{C} which make up the definition. The denotation of implication in differential linear logic will correspond to an internal hom object

$$\llbracket A \multimap B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket := \underline{\text{Hom}}(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

making \mathcal{C} a *closed* symmetric monoidal category with finite products. The contraction and weakening rules show that exponentiated objects $\llbracket !A \rrbracket$ in our category should satisfy coalgebraic rules, whilst the cocontraction and coweakening rules show that algebraic rules should be satisfied. In other words, we enter the realm of bialgebras in the categorical semantics of the exponentiated formulas.

Finally, if we think of maps $\llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ as ‘linear’, then the map

$$i : \text{Hom}(\llbracket A \rrbracket, \llbracket B \rrbracket) \rightarrow \text{Hom}(\llbracket !A \rrbracket, \llbracket B \rrbracket)$$

given by composition with the dereliction map $d : \llbracket !A \rrbracket \rightarrow \llbracket A \rrbracket$ should be thought of as the inclusion of linear maps into ‘non-linear’ maps. Correspondingly, composition with the codereliction $\bar{d} : \llbracket A \rrbracket \rightarrow \llbracket !A \rrbracket$ induces a map

$$D : \text{Hom}(\llbracket !A \rrbracket, \llbracket B \rrbracket) \rightarrow \text{Hom}(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

which is interpreted as a linearisation map, i.e., a differential operator. Then, $D[f]$ is the linear ‘Jacobian’ transformation. The codereliction should satisfy conditions for D to act like a differential operator, for example, satisfying the chain rule.

We now make this discussion more formal. In doing so, we drop the bracket notation for the denotation of a formula for the remainder of this section. The following is just a rewriting of the conditions found in Section 7 of Blute et al. (2019) for a symmetric monoidal category with finite products to be a monoidal storage category (also called a (new) Seely category in the literature) (Bierman 1995; Blute et al. 2009; Melliès 2009).

Definition 2. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category with finite products $(\times, *)$. A storage comonad on \mathcal{C} is a comonad $! = (!, \mu, \varepsilon)$ on \mathcal{C} , where $\mu : ! \rightarrow !!$ is the comultiplication and $\varepsilon : ! \rightarrow \text{id}_{\mathcal{C}}$ is the counit, such that:

- (1) For all $A \in \mathcal{C}$, the object $!A$ is a cocommutative comonoid object in \mathcal{C} with comultiplication $c_A : !A \rightarrow !A \otimes !A$ and counit $e_A : !A \rightarrow 1$ which are both natural transformations.
- (2) For all $A \in \mathcal{C}$, the map $\mu_A : !A \rightarrow !!A$ is a morphism of comonoid objects in \mathcal{C} .
- (3) For all $A, B \in \mathcal{C}$, the induced maps $e_* : !(*) \rightarrow 1$ and

$$(!(\pi_1) \otimes !(\pi_2)) \circ c_{A \times B} : !(A \times B) \rightarrow !A \otimes !B$$

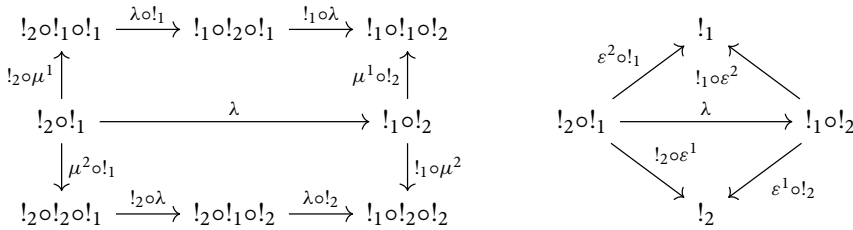
are isomorphisms for the projection maps $(\pi_i)_{i \in \{1,2\}}$.

The isomorphisms in Condition 3 of Definition 2 are called *Seely isomorphisms*.

Definition 3. A symmetric monoidal storage category is a symmetric monoidal category with finite products and a storage comonad.

Some models of logic can be endowed with n linear exponential comonads for some $n \in \mathbb{N}$. To compose comonads, we require extra structure for the composite to remain a comonad. When $n = 2$, we have the following definition from Beck (1969).

Definition 4. Let $!_1 = (!_1, \mu^1, \varepsilon^1)$ and $!_2 = (!_2, \mu^2, \varepsilon^2)$ be two comonads on a category C . Then, a distributive law of $!_1$ over $!_2$ is a natural transformation $\lambda : !_2 \circ !_1 \rightarrow !_1 \circ !_2$ such that the following diagrams



commute in C .

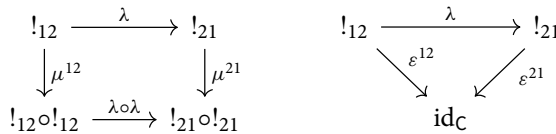
Let $!_1$ and $!_2$ be endofunctors on C . To simplify notation, set $!_{12} = !_2 \circ !_1$. If $!_1$ and $!_2$ are comonads and λ a distributive law of $!_1$ over $!_2$, then there exists a unique composite comonad $!_{12} = (!_{12}, \mu^{12}, \varepsilon^{12})$ where

$$\mu^{12} : !_12 \xrightarrow{\mu^2 \circ \mu^1} !_2 \circ !_1 \circ !_1 \xrightarrow{!_2 \circ \lambda \circ !_1} !_12 \circ !_12$$

is the comultiplication and

$$\varepsilon^{12} : !_12 \xrightarrow{\varepsilon^2 \circ \varepsilon^1} \text{id}_C$$

the counit. Note that the map λ is left implicit in the notation for the composite comonad $!_{12}$. The distributive law λ lifts to a morphism of comonads if the diagrams



commute in C . If the distributive law is an isomorphism, then it is an isomorphism of comonads.

The following gives us the conditions on a category C for a composite comonad on C to be a storage comonad.

Lemma 5. Let C be a symmetric monoidal category with finite products. If $!_1$ is a product preserving comonad, $!_2$ a storage comonad and λ a distributive law of $!_1$ over $!_2$ on C , then the composite comonad $!_{12}$ is a storage comonad on C .

Proof. From the conditions in the lemma, let $!_{12}$ be the unique composite composite comonad. Consider the storage comonad $!_2$ with comonoid comultiplication c_A^2 and counit e_A^2 . Then the object $!_{12}A := !_2(!_1A)$ is clearly a cocommutative comonoid object in C with comultiplication $c_A^{12} := c_{!_1A}^2$:

$!_{12}A \rightarrow !_{12}A \otimes !_{12}A$ and counit $e_A^{12} := e_{!_{12}A}^2 : !_{12}A \rightarrow 1$, and $\mu_A^{12} : !_{12}A \rightarrow !_{12}o!_{12}A$ is a morphism of comonoids in C . The Seely isomorphisms follow from the commutative diagrams

$$\begin{array}{ccc}
 !_{12}(A \times B) & \xrightarrow{c_{A \times B}^{12}} & !_{12}(A \times B) \otimes !_{12}(A \times B) \\
 \downarrow \sim & & \downarrow !_{12}(\pi_1) \otimes !_{12}(\pi_2) \\
 !_2(!_1A \times !_1B) & \xrightarrow{\sim} & !_{12}A \otimes !_{12}B
 \end{array}
 \qquad
 \begin{array}{ccc}
 !_{12}(*) & \xrightarrow{e_*^{12}} & 1 \\
 \searrow \sim & & \swarrow \sim \\
 & & !_2(*)
 \end{array}$$

owing to the fact that $!_1$ preserves finite products, and $!_2$ is a storage comonad satisfying the Seely isomorphisms, respectively. □

Since many of the categories arising in applications are locally presentable (Adámek and Rosický 1994), including our own, it is useful to include here another characterisation of a symmetric monoidal storage category.

Proposition 6. *Let C be a locally presentable strong symmetric monoidal category with finite products. Then, C is endowed with a storage comonad if and only if there exists a locally presentable cartesian category D and a colimit preserving symmetric monoidal functor $L : D \rightarrow C$ which is bijective on objects.*

Proof. (\Leftarrow) By the adjoint functor theorem, the functor L admits a right adjoint $R : C \rightarrow D$ and since L is strong symmetric monoidal then R is lax symmetric monoidal. Thus, $L \dashv R$ defines a linear–non-linear adjunction in the sense of Melliès (2009). Since L is moreover bijective on objects, by Proposition 25 of *loc.cit.*, the pair $(C, ! := L \circ R)$ define a symmetric monoidal storage category. (\Rightarrow) This follows from Proposition 24 of *loc.cit.*. □

The category D in Proposition 6 is isomorphic to the Kleisli category $C!$. In the parlance of Benton (1994), Melliès (2009), we have a linear–non-linear adjunction which takes the form

$$\begin{array}{ccc}
 C! & \xrightarrow{L} & C \\
 & \xleftarrow{R} & \\
 & & C
 \end{array}$$

for the comonad $! = L \circ R$ on C .

We need to introduce one more piece of structure which is key to interpreting a model for differential linear logic as a differential category (Blute et al. 2006), i.e., a structure enabling one to ‘differentiate’ morphisms. We will call a symmetric monoidal category (C, \otimes) a *CMon-enriched symmetric monoidal category* if it is enriched over the monoidal category $(CMon, +)$ of commutative monoids such that the products are compatible in the sense that $(f + g) \otimes h = f \otimes h + g \otimes h$ and $0 \otimes h = 0$ for zero morphisms 0 .

Definition 7. *A CMon-enriched symmetric monoidal storage category¹ is a symmetric monoidal storage category which is also a CMon-enriched symmetric monoidal category.*

By Theorem 7.4 of Blute et al. (2019), CMon-enriched symmetric monoidal storage categories have finite biproducts and an additive bialgebra modality. So in addition to the cocommutative coalgebra $(!A, c_A, e_A)$, we have a commutative monoid object $(!A, \bar{c}_A, \bar{e}_A)$ for all $A \in C$ with multiplication $\bar{c}_A : !A \otimes !A \rightarrow !A$ and unit $\bar{e}_A : 1 \rightarrow !A$. The categorical analogue of the codereliction rule is then the following.

Definition 8. Let $(C, !)$ be a symmetric monoidal storage category which is also a CMon-enriched category. A natural transformation

$$\bar{d} : \text{id}_C \rightarrow !$$

is called a codereliction if it satisfies the rules :

- (Constant rule) $e_A \circ \bar{d}_A = 0 : A \rightarrow 1$.
- (Linear rule) $\varepsilon_A \circ \bar{d}_A = \text{id}_A : A \rightarrow A$.
- (Product rule) $c_A \circ \bar{d}_A = \bar{d}_A \otimes \bar{e}_A + \bar{e}_A \otimes \bar{d}_A : A \rightarrow !A \otimes !A$.
- (Chain rule) $\mu_A \circ \bar{c}_A \circ (\bar{d}_A \otimes \text{id}_{!A}) = \bar{c}_{!A} \circ (\bar{d}_{!A} \otimes \mu_A) \circ (\bar{c}_A \otimes \text{id}_{!A}) \circ (\bar{d}_A \otimes c_A) : A \otimes !A \rightarrow !!A$.

We now state the main overarching definition of this paper.

Definition 9. A model for intuitionistic differential linear logic is a CMon-enriched symmetric monoidal storage category with a codereliction which is also a closed symmetric monoidal category.

One often finds the notion of a *deriving transformation* (Blute et al. 2006; Ehrhard 2002) in place of a codereliction map in studies of differential categories. Every codereliction induces a deriving transformation. Furthermore, these two structures are equivalent on a CMon-enriched symmetric monoidal storage category by combining Theorem 6 and Theorem 3 of Blute et al. (2019). The deriving transformation associated to the codereliction \bar{d}_A is given by the composition

$$\bar{\partial}_A : !A \otimes A \xrightarrow{\text{id}_{!A} \otimes \bar{d}_A} !A \otimes !A \xrightarrow{\bar{c}_A} !A$$

in C . Then for any morphism $f : !A \rightarrow B$ in C , the composite map

$$df := f \circ \bar{\partial}_A : !A \otimes A \rightarrow B$$

will represent the *derivative* of f in C .

We define the n -fold derivative by induction : we set $\bar{\partial}_A^0 = \text{id}_{!A}$ and

$$\bar{\partial}_A^{n+1} := \bar{\partial}_A \circ (\bar{\partial}_A^n \otimes \text{id}_A) : !A \otimes A^{\otimes n} \rightarrow !A$$

and define

$$d^n f := f \circ \bar{\partial}_A^n : !A \otimes A^{\otimes n} \rightarrow B$$

in C . The notation for intuitionistic implication $A \Rightarrow B := !A \multimap B$ is now revealing since, by adjunction, the linear differential operator d^n is given by

$$d^n : (A \Rightarrow B) \rightarrow (A^{\otimes n} \multimap (A \Rightarrow B))$$

and should be thought of as sending a ‘non-linear’ morphism f to a ‘multi-linear’ morphism $d^n f : A^{\otimes n} \rightarrow (A \Rightarrow B)$. The basic example is the following.

Example 10. Given a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ in the category of vector spaces over \mathbb{R} , we have the linear morphism

$$df := f \circ \bar{\partial}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow (\mathbb{R}^n \Rightarrow \mathbb{R}^m)$$

given by $df(x)(y) = d_x f(y) = (J_x f)y$ where $J_x f$ is the Jacobian of f at x , ie. $df(x)(y)$ is the derivative of f at x in the direction y . This satisfies the chain rule

$$d_x(g \circ f) = d_{f(x)}(g) \circ d_x(f)$$

in addition to other basic properties of differentiation contained in the definition of a codereliction.

In the following, Example 10 will be generalised to the case where f is a section of an arbitrary vector bundle over a smooth manifold M from the point of view of differential linear logic. When M is a point $*$ and the fibres are all of the form \mathbb{R}^n for some $n \in \mathbb{N}$, we recover the simple example above.

3. Vector Bundles and the Jet Comonad

We will henceforth work over the field \mathbb{R} of real numbers and fix a smooth n -dimensional manifold M over \mathbb{R} for the remainder of the article.

We consider the geometric approach to the theory of partial differential equations which begins with the study of jet bundles (Saunders 1989). The r -jet of a function $f : M \rightarrow \mathbb{R}$ at a point x of M can be thought of as the coordinate-free Taylor polynomial

$$(j_x^r f)(z) : f(x) + f'(x)z + \dots + \frac{1}{r!} f^{(r)}(x)z^r$$

for a formal variable z . We go beyond functions f , interpreted as sections of the trivial vector bundle $M \times \mathbb{R} \rightarrow M$, and consider local sections of a general vector bundle in this paper. The jet bundle associated to a vector bundle is itself a vector bundle whose coordinates represent the derivatives of the fibre coordinates.

More precisely, let $\text{VBun}(M)$ denote the category of finite rank vector bundles and $\pi : E \rightarrow M$ an object in $\text{VBun}(M)$. To the vector bundle E , we associate its vector bundle $\pi_r : J^r(E) \rightarrow M$ of r -jets of local sections. For a local section s at $x \in M$, its r -jet is denoted $j_x^r(s)$. Two sections are in the equivalence class $j_x^r(s)$ if they have the same r th order Taylor expansion at x .

Consider the sheaf \mathcal{C}_M^∞ of smooth functions on M and its category $\text{Mod}(\mathcal{C}_M^\infty)$ of modules. Then, the functor $\text{VBun}(M) \rightarrow \text{Mod}(\mathcal{C}_M^\infty)$ sending a vector bundle E to its sheaf of sections $\mathcal{E} := \Gamma(E)$ is fully faithful with essential image the category $\mathcal{V}(M)$ of locally free sheaves of finite rank. We also refer to objects in this equivalent category as vector bundles on M .

The category $\mathcal{V}(M)$ is a symmetric monoidal category in two ways. First, via the direct sum $\mathcal{E} \oplus \mathcal{E}'$ of sheaves, and second, via the tensor product $\mathcal{E} \otimes_{\mathcal{C}_M^\infty} \mathcal{E}'$ of sheaves (Serre 1955). We have canonical isomorphisms

$$\mathcal{E} \otimes_{\mathcal{C}_M^\infty} \bigoplus_i \mathcal{E}'_i \simeq \bigoplus_i (\mathcal{E} \otimes_{\mathcal{C}_M^\infty} \mathcal{E}'_i)$$

showing that the tensor product distributes over coproducts. If we denote by $\underline{\text{Hom}}(\mathcal{E}, \mathcal{E}')$, the sheaf of morphisms between \mathcal{E} and \mathcal{E}' which sends U to $\text{Hom}_{\mathcal{C}_M^\infty|_U}(\mathcal{E}|_U, \mathcal{E}'|_U)$, then there exists an isomorphism

$$\text{Hom}_{\mathcal{C}_M^\infty}(\mathcal{F} \otimes_{\mathcal{C}_M^\infty} \mathcal{E}, \mathcal{E}') \simeq \text{Hom}_{\mathcal{C}_M^\infty}(\mathcal{F}, \underline{\text{Hom}}(\mathcal{E}, \mathcal{E}'))$$

which makes the category of vector bundles $(\mathcal{V}(M), \otimes, \mathcal{C}_M^\infty)$ a closed additive symmetric monoidal category for the tensor product. We will also be concerned with the cocartesian monoidal structure $(\mathcal{V}(M), \oplus, 0)$ where 0 is the constant sheaf with value $\{0\}$ which is a zero object of $\mathcal{V}(M)$. We have an isomorphism

$$\mathcal{E} \oplus \mathcal{E}' \simeq \mathcal{E} \times \mathcal{E}'$$

of sheaves.

Let $\mathcal{J}^r(\mathcal{E})$ denote the sheaf of sections of $J^r(E)$ on M . Given a section s of $\pi_r|_U$, the r -jet prolongation of s is the smooth section

$$j^r(s) : U \rightarrow J^r(E)$$

of π_r such that $j^r(s)(x) = j_x^r(s)$ for all x in $U \subseteq M$. Then $j^r : \mathcal{E} \rightarrow \mathcal{J}^r(\mathcal{E})$ is a morphism of sheaves of sets. Consider the endofunctor $!_j^r : \mathcal{V}(M) \rightarrow \mathcal{V}(M)$ sending \mathcal{E} to $\mathcal{J}^r(\mathcal{E})$ and a morphism $f :$

$\mathcal{E} \rightarrow \mathcal{E}'$ to its r -jet prolongation $\mathcal{J}^r(f) : \mathcal{J}^r(\mathcal{E}) \rightarrow \mathcal{J}^r(\mathcal{E}')$ which elementwise sends $j^r(s)$ to $j^r(f \circ s)$. We will often make the abuse of writing $s \in \mathcal{E}$ for a local section in $\mathcal{E}|_U$.

Recall that a category I is said to be *cofiltered* (Artin et al. 1972) if it is non-empty; for any pair of objects i and j in I , there exists an object k together with morphisms $k \rightarrow i$ and $k \rightarrow j$; and for every pair of morphisms f and g with the same source and target, there exists a morphism h such that $f \circ h = g \circ h$. A cofiltered diagram in a category \mathcal{C} is a functor $X : I \rightarrow \mathcal{C}$ indexed by a cofiltered category. The category $\text{Pro}(\mathcal{C})$ of *pro-objects* in \mathcal{C} has cofiltered diagrams in \mathcal{C} as objects, and for two objects $X : I \rightarrow \mathcal{C}$ and $Y : I' \rightarrow \mathcal{C}$, morphisms defined by $\text{Hom}_{\text{Pro}(\mathcal{C})}(X, Y) := \lim_{i' \in I'} \text{colim}_{i \in I} \text{Hom}(X_i, Y_{i'})$.

Let \mathcal{E} be a vector bundle on M . We denote by

$$\mathcal{J}(\mathcal{E}) = \text{“lim”}_{r \in \mathbb{N}}(\mathcal{J}^r(\mathcal{E}))$$

the pro-object

$$\dots \rightarrow \mathcal{J}^{r+1}(\mathcal{E}) \xrightarrow{\pi_{r+1,r}} \mathcal{J}^r(\mathcal{E}) \rightarrow \dots \rightarrow \mathcal{J}^1(\mathcal{E}) \xrightarrow{\pi_{1,0}} \mathcal{J}^0(\mathcal{E}) = \mathcal{E}$$

in the category $\mathcal{V}(M)$ of vector bundles on M . Here $\pi_{r+1,r} : \mathcal{J}^{r+1}(\mathcal{E}) \rightarrow \mathcal{J}^r(\mathcal{E})$ is the canonical projection.

Given a pro-vector bundle $\mathcal{E} : I \rightarrow \mathcal{V}(M)$, the infinite jet bundle of \mathcal{E} is the pro-object $\mathcal{J}(\mathcal{E}) : \mathbb{N}^{op} \times I \rightarrow \mathcal{V}(M)$ given by “lim” $_{r,i}(\mathcal{J}^r(\mathcal{E}_i))$ in $\mathcal{V}(M)$. Then the infinite jet prolongation $j : \mathcal{E} \rightarrow \mathcal{J}(\mathcal{E})$ lifts to a morphism of pro-sheaves. We have an induced endofunctor $!_j : \text{Pro}(\mathcal{V}(M)) \rightarrow \text{Pro}(\mathcal{V}(M))$ sending \mathcal{E} to $\mathcal{J}(\mathcal{E})$ induced from infinite prolongation.

The dual of a pro-object in the category $\mathcal{V}(M)$ of vector bundles is an *ind-object*. The category of ind-objects in $\mathcal{V}(M)$ will be denoted $\text{Ind}(\mathcal{V}(M)) = \text{Pro}(\mathcal{V}(M)^{op})^{op}$. If \mathcal{E} is a vector bundle, then the dual $\mathcal{J}(\mathcal{E})^\perp := \underline{\text{Hom}}_{\mathcal{C}_M^\infty}(\mathcal{J}(\mathcal{E}), \mathcal{C}_M^\infty)$ is an ind-object in $\mathcal{V}(M)$. Here $\underline{\text{Hom}}_{\mathcal{C}_M^\infty}$ denotes the sheaf of continuous linear maps. When \mathcal{E} is an ind-object in $\mathcal{V}(M)$, then $\mathcal{J}(\mathcal{E})$ is a pro-ind-object in $\mathcal{V}(M)$.

Definition 11. A pro-ind vector bundle is an object in the category $\text{PI}(M) := \text{Pro}(\text{Ind}(\mathcal{V}(M)))$ of pro-ind-objects in $\mathcal{V}(M)$.

We will often identify a vector bundle with its image under the fully faithful map $i : \mathcal{V}(M) \rightarrow \text{PI}(M)$ where $\mathcal{V}(M) \simeq \text{Pro}(\mathcal{V}(M)) \cap \text{Ind}(\mathcal{V}(M)) \subset \text{PI}(M)$ is an equivalence of categories.

Remark 12. Pro and Ind objects are often used in practice as presentations of infinite dimensional objects (Artin et al. 1972; Grothendieck 1960). For example, the category of vector spaces Vect is equivalent to the category $\text{Ind}(\text{Vect}^{\text{fin}})$ of ind-objects in the category Vect^{fin} of finite dimensional vector spaces. Alternatively, the category $\text{Pro}(\text{Aff}_S^{\text{fin}})$ of pro-objects in the category $\text{Aff}_S^{\text{fin}}$ of affine schemes of finite type over a quasi-separated base scheme S is equivalent to the category Aff_S of all affine schemes.

We have an endofunctor

$$!_j : \text{PI}(M) \rightarrow \text{PI}(M)$$

on the category $\text{PI}(M)$ given by infinite prolongation.

Lemma 13. The endofunctor $!_j$ is a comonad on the category $\text{PI}(M)$.

Proof. We have a natural comultiplication map $\mu^j : !_j \rightarrow !_j !_j$ which object-wise $\mu^j_{\mathcal{E}} : !_j \mathcal{E} \rightarrow !_j !_j \mathcal{E}$ sends $j_x(s)$ to $j_x(j(s))$ and a natural counit map $\varepsilon^j : !_j \rightarrow \text{id}$ which object-wise $\varepsilon^j_{\mathcal{E}} : !_j \mathcal{E} \rightarrow \mathcal{E}$ sends $j_x(s)$ to $s(x)$. The commutativity of the relevant diagrams can be easily verified. \square

Remark 14. Currently $!_j \mathcal{E}$ and its continuous linear dual $!_j \mathcal{E}^{\perp}$ are defined as formal filtered limits and colimits. This will be remedied in Section 4 by introducing functional analytic tools.

Remark 15. The observation that the infinite jet functor defines a comonad in the smooth setting goes back to Marvan (1986). A far reaching generalisation, encompassing many examples, is contained in Khavkine and Schreiber (2017).

Let $\text{PI}(M)_{!_j}$ denote the Kleisli category of the comonad $!_j$. We have a linear–non-linear adjunction

$$\begin{array}{ccc} & X & \\ & \curvearrowright & \\ \text{PI}(M)_{!_j} & & \text{PI}(M) \\ & \curvearrowleft & \\ & U & \end{array}$$

where $!_j = X \circ U$. The left adjoint X sends a pro-ind-vector bundle \mathcal{E} to $!_j \mathcal{E}$ and a morphism $F : !_j \mathcal{E} \rightarrow \mathcal{E}'$ to $!_j(F) \circ \mu^j_{\mathcal{E}} : !_j \mathcal{E} \rightarrow !_j \mathcal{E}'$. The right adjoint is an identity on objects and sends a morphism $G : \mathcal{E} \rightarrow \mathcal{E}'$ to $G \circ \varepsilon^j_{\mathcal{E}} = \varepsilon^j_{\mathcal{E}'} \circ !_j(G) : !_j \mathcal{E} \rightarrow \mathcal{E}'$ as a morphism in $\text{PI}(M)$. The unit of the adjunction on \mathcal{E} is simply the morphism $\eta_{\mathcal{E}} = \text{id}_{!_j \mathcal{E}} : \mathcal{E} \rightarrow !_j \mathcal{E}$, and the counit is given by $\varepsilon^j_{\mathcal{E}}$.

We now give several interpretations of $\text{PI}(M)_{!_j}$ which includes the theory of linear differential operators, D -modules, $!_j$ -coalgebras and linear partial differential equations. Let \mathcal{E} and \mathcal{E}' be vector bundles on M and $\text{Diff}^r(\mathcal{E}, \mathcal{E}')$ the sheaf of linear partial differential operators. It sends $U \subseteq M$ to the $\mathcal{C}_M^{\infty}(U)$ -module whose elements are morphisms $P_U : \mathcal{E}(U) \rightarrow \mathcal{E}'(U)$ given by $\sum_{|\alpha| \leq r} a_{\alpha} \circ \partial_{\alpha}$ for any trivialisation where $a_{\alpha} \in \text{Hom}_{\mathcal{C}_M^{\infty}(U)}(\mathcal{E}(U), \mathcal{E}'(U))$. The functor

$$\text{Diff}^r(\mathcal{E}, -) : \mathcal{V}(M) \rightarrow \text{Set}$$

is representable by the vector bundle $\mathcal{J}^r(\mathcal{E})$. The isomorphism

$$\text{Hom}_{\mathcal{V}(M)}(\mathcal{J}^r(\mathcal{E}), \mathcal{E}') \simeq \text{Diff}^r(\mathcal{E}, \mathcal{E}')$$

is given by the map $F \mapsto \widehat{F} := F \circ j^r$. The ind-object $\text{Diff}(\mathcal{E}, \mathcal{E}') := \text{“colim”}_{r \in \mathbb{N}} \text{Diff}^r(\mathcal{E}, \mathcal{E}')$ given by the natural inclusions induce an isomorphism

$$\text{Diff}(\mathcal{E}, \mathcal{E}') \simeq \text{“colim”}_{r \in \mathbb{N}} \text{Hom}_{\mathcal{V}(M)}(\mathcal{J}^r(\mathcal{E}), \mathcal{E}') \simeq \text{Hom}_{\text{PI}(M)}(\text{“lim”}_{r \in \mathbb{N}} \mathcal{J}^r(\mathcal{E}), \mathcal{E}')$$

and so $\text{Diff}(\mathcal{E}, \mathcal{E}')$ is represented by $\mathcal{J}(\mathcal{E})$ in $\text{PI}(M)$. The result of this discussion is that we can identify the image of the functor $U : \mathcal{V}(M) \subset \text{PI}(M) \rightarrow \text{PI}(M)_{!_j}$ in the Kleisli category of $!_j$ with the category of vector bundles on M with linear partial differential operators as morphisms.

Let $\widehat{F} : \mathcal{E} \rightarrow \mathcal{E}'$ be a r th order differential operator. We associate to \widehat{F} its corresponding bundle map $F : \mathcal{J}^r(\mathcal{E}) \rightarrow \mathcal{E}'$ and vice versa. Given a q th order differential operator \widehat{G} between \mathcal{E}' and \mathcal{E}'' , composition with \widehat{F} is given by

$$G \circ F : \mathcal{J}^{r+q}(\mathcal{E}) \xrightarrow{\mu_{\mathcal{E}}^{r,q}} \mathcal{J}^q \mathcal{J}^r(\mathcal{E}) \xrightarrow{\mathcal{J}^q \circ F} \mathcal{J}^q(\mathcal{E}') \xrightarrow{G} \mathcal{E}''$$

where $\mu_{\mathcal{E}}^{r,q}$ is the injection sending $j_x^{r+q}(s)$ to $j_x^q(j_x^r(s))$ (and so $\mu = \mu^{\infty, \infty}$). When they are both of infinite order, we obtain a linear differential operator $\widehat{G} \circ \widehat{F} : \mathcal{E} \rightarrow \mathcal{E}''$ and Kleisli composition is well defined.

The Kleisli category of the jet comonad has a natural interpretation in the language of D -modules (Kashiwara 2003). This extension is as follows. Let $\mathcal{D}_M(\mathcal{E}, \mathcal{E}')$ denote the sheaf of linear

differential operators on M and

$$\mathcal{D}_M^\infty := \text{Diff}(\mathcal{C}_M^\infty, \mathcal{C}_M^\infty)$$

the sheaf of linear differential operators between the sheaf of smooth functions. This is a sheaf of non-commutative \mathcal{C}_M^∞ -algebras with product given by composition. We denote the symmetric monoidal category of D -modules by

$$\text{Mod}(\mathcal{D}_M^\infty) := \text{Mod}_{\mathcal{D}_M^\infty}(\text{Mod}(\mathcal{C}_M^\infty))$$

where the symmetric monoidal structure is given by tensoring over \mathcal{C}_M^∞ . If \mathcal{E} is a vector bundle, then endowing \mathcal{E} with a D -module structure is equivalent to the choice of flat connection

$$\nabla : \mathcal{E} \rightarrow \Omega_M^1 \otimes_{\mathcal{C}_M^\infty} \mathcal{E}$$

on \mathcal{E} which characterises D -modules with an underlying locally free \mathcal{C}_M^∞ -module.

The sheaf $\mathcal{J}(\mathcal{E})$ is endowed with a canonical D -module structure, the flat connection given by defining a section ξ in $\mathcal{J}(\mathcal{E})$ to be flat if $\xi = j(s)$ for some $s \in \mathcal{E}$, i.e., horizontal sections of the connection are infinite prolongations of sections of \mathcal{E} . This is also called the Cartan connection. Explicitly, after choosing coordinates x_1, \dots, x_n on $U \subseteq M$ and a trivialisation $U \times E_0$ of E , we have

$$\mathcal{J}(\mathcal{E})(U) = \mathcal{C}_M^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}[[x_1, \dots, x_n]] \otimes_{\mathbb{R}} E_0$$

and the flat connection is given by $\nabla(f \otimes g \otimes v) = df \otimes g \otimes v + \sum_i f dx_i \otimes \frac{\partial}{\partial x_i} g \otimes v$. Alternatively, it is defined through the Cartan distribution of tangent planes to sections of the form $j(s)$. This is the map $\mu_{\mathcal{E}}^{\infty,1} : \mathcal{J}(\mathcal{E}) \rightarrow \mathcal{J}^1 \mathcal{J}(\mathcal{E})$ which is spanned by vector fields of the form

$$D_i = \frac{\partial}{\partial x_i} + \sum_{k,I} u_{Ii}^k \frac{\partial}{\partial u_I^k}$$

for fibre coordinates u^k and a multi-index I . Finally, there exists a bijection

$$\text{Hom}_{\mathcal{D}_M^\infty}(\mathcal{J}(\mathcal{E}), \mathcal{J}(\mathcal{E}')) \simeq \text{Diff}(\mathcal{E}, \mathcal{E}')$$

which induces a fully faithful functor $\mathcal{J} : \text{PI}(M)_{!j} \rightarrow \text{Pro}(\text{Ind}(\text{Mod}(\mathcal{D}_M^\infty)))$ sending \mathcal{E} to $\mathcal{J}(\mathcal{E})$.

Remark 16. For the multicategory interpretation of the Kleisli category, one takes the multicategory of vector bundles and polydifferential operators

$$\text{PolyDiff}(\mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n, \mathcal{E}') := \text{Diff}(\mathcal{E}_1, \mathcal{C}_M^\infty) \otimes_{\mathcal{C}_M^\infty} \dots \otimes_{\mathcal{C}_M^\infty} \text{Diff}(\mathcal{E}_n, \mathcal{C}_M^\infty) \otimes_{\mathcal{C}_M^\infty} \mathcal{E}'$$

where the action of \mathcal{C}_M^∞ on $\text{Diff}(\mathcal{E}_i, \mathcal{C}_M^\infty)$ is given by left multiplication $(fD)(s) := f(Ds)$ for $s \in \mathcal{E}_i$. There exists a bijection

$$\text{Hom}_{\mathcal{D}_M^\infty}(\mathcal{J}(\mathcal{E}_1) \otimes_{\mathcal{C}_M^\infty} \dots \otimes_{\mathcal{C}_M^\infty} \mathcal{J}(\mathcal{E}_n), \mathcal{J}(\mathcal{E}')) \simeq \text{PolyDiff}(\mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n, \mathcal{E}')$$

where the left-hand side denotes morphisms which are continuous.

Another interpretation of $\text{PI}(M)_{!j}$ is as a full subcategory of the Eilenberg–Moore category of $!j$ -coalgebras. A $!j$ -coalgebra for the comonad $!j$ is a pair $(\mathcal{E}, v_{\mathcal{E}})$ where \mathcal{E} is a pro-ind vector bundle and $v_{\mathcal{E}} : \mathcal{E} \rightarrow !j\mathcal{E}$ is a morphism of pro-ind vector bundles such that $\varepsilon_{\mathcal{E}} \circ v_{\mathcal{E}} = \text{id}_{\mathcal{E}}$ and $\mu_{\mathcal{E}} \circ v_{\mathcal{E}} = !j(v_{\mathcal{E}}) \circ v_{\mathcal{E}}$. A morphism between $!j$ -coalgebras $(\mathcal{E}, v_{\mathcal{E}})$ and $(\mathcal{E}', v'_{\mathcal{E}'})$ is a morphism $f : \mathcal{E} \rightarrow \mathcal{E}'$ of pro-ind vector bundles such that $!j(f) \circ v_{\mathcal{E}} = v'_{\mathcal{E}'} \circ f$. The category of $!j$ -coalgebras, often called the Eilenberg–Moore category, will be denoted $\text{PI}(M)^{!j}$.

The Eilenberg–Moore category of $!j$ is equivalent to a certain category of partial differential equations introduced in Vinogradov (1980) (see also Marvan 1986). We first recall some geometric definitions (Pommaret and Lichnerowicz 1978).

Definition 17. Let $\pi : E \rightarrow M$ be a vector bundle. A r th order partial differential equation on E is a fibred submanifold of $\pi_r : J^r(E) \rightarrow M$. An inhomogeneous linear partial differential equation is an affine subbundle of π_r . A homogeneous linear partial differential equation is a vector subbundle of π_r .

Let H^r be a r th order linear partial differential equation. In the homogenous case, there exists a vector bundle $E' = \text{coker}(H^r)$ on M and a morphism of vector bundles $f : J^r(E) \rightarrow E'$ such that $H^r = \ker(f)$. This corresponds to the standard interpretation $f(x_i, u^\alpha, u_i^\alpha) = 0$ where u^α are coordinates in the fibre of E . A linear partial differential equation will be henceforth considered homogenous unless otherwise specified. A (local) solution of a r th order partial differential equation H^r is a section s of $\pi_r|_U$ such that $j^r s(x) \in H^r$ for all $x \in U$.

The q th order prolongation of $h : H^r \subseteq J^r(E)$ is the pullback

$$\begin{array}{ccc} H^{r,q} & \longrightarrow & J^{r+q}(E) \\ \downarrow & & \downarrow \mu_E^{r,q} \\ J^q(H^r) & \xrightarrow{J^q(h)} & J^q J^r(E) \end{array}$$

in the category of vector bundles. The infinite prolongation $H \subseteq J(E)$ of H^r is the pro-object

$$\dots \rightarrow H^{r,k+1} \xrightarrow{\pi_{k+1,k}^r} H^{r,k} \rightarrow \dots \rightarrow H^{r,1} \xrightarrow{\pi_{0,1}^r} H^{r,0} = H^r \subseteq J^r(E)$$

in the category of vector bundles. It can be interpreted as H^r together with its system of total derivatives. A morphism between infinitely prolonged linear equations is a morphism of pro-vector bundles.

These constructions are clearly extended to the case where E itself is a pro-ind-vector bundle. We obtain a category $\text{LPDE}(M)$ of infinitely prolonged linear partial differential equations.

Remark 18. In this geometric formulation of partial differential equations, infinitesimal symmetries are given by tangent vector fields on the jet bundle whose flows preserve this submanifold (Olver 2012; Pommaret and Lichnerowicz 1978).

The sheaf interpretation of this result is as follows. The vector bundle H^r induces a sheaf \mathcal{H}^r of solutions and $H^{r,q}$ a prolonged sheaf $\mathcal{H}^{r,q} \subseteq \mathcal{J}^{q+r}(\mathcal{E})$ of solutions. The infinite prolongation $\mathcal{H} \subseteq \mathcal{J}(\mathcal{E})$ is a pro-object in the category of vector bundles $\mathcal{V}(M)$ over M . If \mathcal{E} is a pro-ind vector bundle, then the same is so for \mathcal{H} . There is an equivalence $\mathcal{H} \simeq \mathcal{H}^r$ of sheaves, i.e., a section of \mathcal{E} is a solution of H^r if and only if it is a solution of the prolonged equation H .

We call the map $h : \mathcal{H} \rightarrow !_j \mathcal{E}$ simply the sheaf of solutions. Given two sheaves of solutions $h : \mathcal{H} \rightarrow !_j \mathcal{E}$ and $h' : \mathcal{H}' \rightarrow !_j \mathcal{E}'$, a morphism $\gamma : h \rightarrow h'$ is a commutative diagram

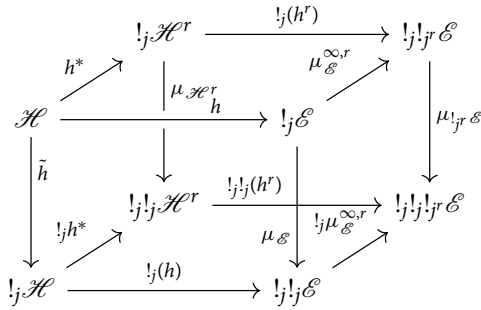
$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h} & !_j \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H}' & \xrightarrow{h'} & !_j \mathcal{E}' \end{array}$$

in $\text{PI}(M)$. We denote by $\text{Soln}(M)$ the category of sheaves of solutions and morphisms between them.

Proposition 19. There exists a chain of equivalences $\text{LPDE}(M) \simeq \text{Soln}(M) \simeq \text{PI}(M)^{\downarrow}$ of categories.

Proof. This can be deduced from Propositions 2.4 and 2.5 of Marvan (1986) so we only sketch the proof. The first equivalence is clear. For the second, consider the sheaf of solutions

$h^r : \mathcal{H}^r \rightarrow !_j{}^r \mathcal{E}$ to a r th order linear partial differential equation $H^r \subseteq J^r(E)$ and its corresponding infinite prolongation $h : \mathcal{H} \rightarrow !_j \mathcal{E}$. Consider the diagram



in $\text{PI}(M)$ where h^* is the morphism making the square in the top face commute and \tilde{h} is the morphism making the resulting full diagram commute. We have a functor $(\tilde{\cdot})$ sending the solution sheaf h to the pair $(\mathcal{H}, \tilde{h} : \mathcal{H} \rightarrow !_j \mathcal{H})$, and this pair can be shown to be a $!_j$ -coalgebra. The right adjoint functor sends a $!_j$ -coalgebra $(\mathcal{E}, \nu : \mathcal{E} \rightarrow !_j \mathcal{E})$ to the solution sheaf $\nu : \mathcal{E} \subset !_j \mathcal{E}$ satisfying $\mu_{\mathcal{E}} = !_j(\nu)$ which is infinitely prolonged. Then composition with $(\tilde{\cdot})$ gives an adjoint equivalence. □

There exists a natural inclusion

$$\text{PI}(M)_j \hookrightarrow \text{PI}(M)^j$$

sending a pro-ind vector bundle \mathcal{E} to $(!_j \mathcal{E}, \mu_{\mathcal{E}}^j)$ and a differential operator $F : !_j \mathcal{E} \rightarrow \mathcal{E}'$ to the composition $!_j(F) \circ \mu_{\mathcal{E}}^j : !_j \mathcal{E} \rightarrow !_j \mathcal{E}'$. The essential image of this inclusion is the full subcategory of $!_j$ -coalgebras spanned by cofree $!_j$ -coalgebras. This follows from the fact that the Kleisli category of any comonad is equivalent to the subcategory of cofree coalgebras of the comonad in the Eilenberg–Moore category. Owing to Proposition 19, objects in $\text{PI}(M)_j$ can be identified with the sheaf of solutions to a cofree infinitely prolonged linear partial differential equation.

The category $\text{PI}(M)$ is not a symmetric monoidal storage category with the monoidal structure given by the tensor product, since the comonad $!_j$ does not satisfy the Seelye isomorphisms. However, for the cocartesian monoidal structure, it is satisfied.

Proposition 20. *The category of pro-ind vector bundles on M with the jet comonad $!_j$ is a symmetric monoidal storage category for the cocartesian monoidal structure.*

Proof. The category $(\mathcal{V}(M), \oplus, 0)$ of smooth vector bundles is a symmetric monoidal category and so we can deduce that the category $\text{PI}(M)$ pro-ind objects in $\mathcal{V}(M)$ is also symmetric monoidal. Let $\mathcal{E}, \mathcal{E}' \in \text{PI}(M)$ and s be a local section of \mathcal{E} . By Lemma 13, $!_j$ is a comonad. Since \oplus is also a product, every object of $\text{PI}(M)$ has a unique comonoid structure given by the diagonal map which is cocommutative. Moreover, any morphism in $\text{PI}(M)$ is automatically a comonoid morphism. Therefore, for a comonoid $(!_j \mathcal{E}, c_{\mathcal{E}}, e_{\mathcal{E}})$, the comultiplication is given by

$$c_{\mathcal{E}} : !_j \mathcal{E} \rightarrow !_j \mathcal{E} \times !_j \mathcal{E},$$

the counit is given by $e_{\mathcal{E}} : !_j \mathcal{E} \rightarrow 0$ which sends $j(s)$ to zero, and $\mu_{\mathcal{E}} : !_j \mathcal{E} \rightarrow !_j !_j \mathcal{E}$ is a morphism of comonoid objects. Furthermore, the morphism

$$(!_j(\pi_0) \oplus !_j(\pi_1)) \circ c_{\mathcal{E} \times \mathcal{E}'} : !_j(\mathcal{E} \times \mathcal{E}') \rightarrow !_j \mathcal{E} \oplus !_j \mathcal{E}'$$

is an isomorphism in $\text{PI}(M)$ since $!_j(\mathcal{E} \times \mathcal{E}') \simeq !_j(\mathcal{E} \oplus \mathcal{E}')$ and $j(s + s') \simeq j(s) + j(s')$. Finally, the morphism

$$e : !_j(*) \rightarrow 0$$

is an isomorphism since the terminal object $*$ in $\text{PI}(M)$ is the pro-ind zero vector bundle 0 and it is clear that $!_j(0) \simeq 0$. As a result, $!_j$ is a storage comonad and $\text{PI}(M)$ is a symmetric monoidal storage category. \square

Example 21. (Connections). In analogy with a codereliction, we introduce a map

$$\Gamma_{\mathcal{E}}^1 : \mathcal{E} \rightarrow !_j^1 \mathcal{E}$$

in $\mathcal{V}(M)$ which is natural in \mathcal{E} , such that the linear rule $\varepsilon_{\mathcal{E}}^j \circ \Gamma_{\mathcal{E}}^1 = \text{id}_{\mathcal{E}}$ is satisfied for the comonad $!_j^1$. This is simply a (linear) *connection*. Indeed, consider the canonical map $j^1 : \mathcal{E} \rightarrow \mathcal{J}^1(\mathcal{E})$ of sheaves. Elements in the kernel of this map can be written as $df \otimes s$. The *covariant derivative* associated to $\Gamma_{\mathcal{E}}^1$ is then the (non \mathcal{C}_M^∞ -linear) map

$$\nabla : \mathcal{E} \rightarrow \Gamma(\Omega^1 \otimes_{\mathbb{R}} E)$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s$$

where Ω^1 is the vector bundle of one forms on M . In local coordinates $(x_i, u^k, u_i^k) \circ \Gamma_{\mathcal{E}}^1 = (x_i, u^k, \Gamma_i^k)$, its local expression is $\nabla = dx^i \otimes (\partial_i + \Gamma_i^k \partial_k)$. More generally, higher-order connections $\Gamma_{\mathcal{E}}^k : !_j^{k-1} \mathcal{E} \rightarrow !_j^k \mathcal{E}$ can be defined (Libermann 1964).

Example 22. (Tangent vector fields). Let $E = TM$ be the tangent bundle and $\mathcal{E} = \mathcal{X}$ the sheaf of vector fields on M . Consider the sequent $!A \vdash B$ in linear logic with denotation $\llbracket - \rrbracket_M$ given by a first-order map $F : \llbracket !A \rrbracket_M = !_j^1 \mathcal{X} \subset !_j \mathcal{X} \rightarrow \llbracket B \rrbracket_M = \mathcal{E}'$. Given a vector field $s : U \rightarrow TU$ on $U \subset M$, we have the first jet

$$j^1(s) : U \rightarrow J^1(TU) \subset J(TU)$$

to s and a commutative diagram

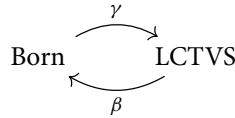
$$\begin{array}{ccc} & !_j \mathcal{X}_U & \\ j^1(s) \nearrow & \downarrow \varepsilon_{\mathcal{X}_U}^j & \searrow F_U \\ * & \xrightarrow{s} \mathcal{X}_U & \xrightarrow{\widehat{F}_U} \mathcal{E}'_U \end{array}$$

in $\text{PI}(M)$. Here $\widehat{F}_U : \mathcal{X}_U \rightarrow \mathcal{E}'_U$, where $\widehat{F}_U(s) \simeq F_U(j^1(s))$, is the first-order linear differential operator associated to F_U .

4. Convenient Sheaves and the Distributional Comonad

Up until now, we have considered the category of pro-ind objects in $\mathcal{V}(M)$. However, there is another approach which takes advantage of functional analytic properties of the space of sections of a vector bundle. In particular, the category of pro-ind-vector bundles has several poor formal properties arising from the category $\mathcal{V}(M)$. This can be remedied by embedding $\text{PI}(M)$ into an appropriate category. We accomplish this by endowing all our function spaces with a complete bornological structure (Hogbe-Nlend 1977), or equivalently, a convenient vector space structure (Frölicher and Kriegl 1988; Kriegl and Michor 1997).

There are a number of equivalent ways one can define the category of convenient vector spaces (Kriegl and Michor 1997). Our choice is the following. Let Born denote the category of (convex) bornological vector spaces and bounded linear morphisms and LCTVS the category of locally convex topological vector spaces and continuous linear morphisms. Consider the adjunction



where γ is left adjoint to the functor β associating to a locally convex topological vector space the bornological vector space with its von-Neumann bornology. The functor γ is fully faithful. Therefore, we have an isomorphism $V \simeq \beta \circ \gamma(V)$ in Born , i.e., every bornological vector space is isomorphic to a vector space whose bornology comes from some locally convex topological vector space. The equivalent category of topological bornological vector spaces will be denoted TBorn . A topological bornological vector space V is said to be c^∞ -complete if a curve $c : \mathbb{R} \rightarrow V$ is smooth if and only if for every bounded linear functional $f : V \rightarrow \mathbb{R}$, the composition $f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is smooth.

We will define the category Conv of convenient vector spaces to be the full subcategory of TBorn spanned by c^∞ -complete objects. The inclusion functor from Conv to the category TBorn has a left adjoint

$$c^\infty : \text{TBorn} \rightarrow \text{Conv}$$

called the c^∞ -completion.

The category Conv is a closed symmetric monoidal category. We will be careful to distinguish the structure on various function spaces. For convenient vector spaces V and W , then $\text{Hom}(V, W)$ will denote the set of morphisms, $\text{Hom}_{\mathbb{R}}(V, W)$ the \mathbb{R} -vector space of \mathbb{R} -linear morphisms and $\underline{\text{Hom}}(V, W)$ the convenient vector space of continuous \mathbb{R} -linear morphisms. We use the notation $V^\vee := \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ for the linear dual of V and $V^\perp := \underline{\text{Hom}}(V, \mathbb{R})$ for the continuous linear dual.

Let $\pi : E \rightarrow M$ be a vector bundle on M . For any $U \subseteq M$, we endow the vector space $\mathcal{E}(U)$ of sections of E with the structure of a convenient vector space induced from the nuclear Fréchet topology of uniform convergence on compact subsets in all derivatives separately. This makes \mathcal{E} a sheaf of convenient vector spaces on M . The same holds for the cosheaf \mathcal{E}_c of compactly supported sections of E . See Lemma 5.1.1 of Costello and Gwilliam (2016) for a formal proof.

In particular, \mathcal{E}_M^∞ is a sheaf of convenient vector spaces and moreover a sheaf of convenient algebras. An algebra is said to be *convenient* if it is a commutative monoid object in the symmetric monoidal category Conv . This makes \mathcal{E} a \mathcal{E}_M^∞ -module object in the category $\text{Sh}_{\text{Conv}_k}(M)$ of sheaves of convenient vector spaces. The category of convenient \mathcal{E}_M^∞ -modules will be denoted by

$$\text{ConMod}(\mathcal{E}_M^\infty) := \text{Mod}_{\mathcal{E}_M^\infty}(\text{Sh}_{\text{Conv}_k}(M)).$$

We have a fully faithful inclusion

$$i : \text{PI}(M) \rightarrow \text{ConMod}(\mathcal{E}_M^\infty)$$

of categories. The inclusion sends a pro-ind vector bundle $\text{‘lim’}_{r \in \mathbb{N}} \text{‘colim’}_{q \in \mathbb{N}} \mathcal{E}^r$ to the genuine limit $\text{lim}_{r \in \mathbb{N}} \text{colim}_{q \in \mathbb{N}} \mathcal{E}^r$ in $\text{ConMod}(\mathcal{E}_M^\infty)$. This limit is well defined since the category of convenient \mathcal{E}_M^∞ -modules is complete and cocomplete.

The category $\text{ConMod}(\mathcal{E}_M^\infty)$ is a closed symmetric monoidal category with tensor product $\otimes_{\mathcal{E}_M^\infty}$ which we simply denote by \otimes . The \mathcal{E}_M^∞ -module of continuous linear morphisms between two \mathcal{E}_M^∞ -modules \mathcal{E} and \mathcal{E}' will be denoted $\underline{\text{Hom}}_{\mathcal{E}_M^\infty}(\mathcal{E}, \mathcal{E}')$.

We now describe some important examples of (co)sheaves of convenient spaces. Let \mathcal{T}^∞ be the convenient sheaf of distributions on M and denote by

$$\overline{\mathcal{E}} := \mathcal{E} \otimes_{\mathcal{E}_M^\infty} \mathcal{T}^\infty$$

the convenient sheaf of distributional sections of E on M . Let \mathcal{T}_c^∞ be the convenient cosheaf of compactly supported distributions on M and

$$\overline{\mathcal{E}}_c := \mathcal{E}_c \otimes_{\mathcal{C}_c^\infty} \mathcal{T}_c^\infty$$

the convenient cosheaf of compactly supported distributional sections of E on M . We let $\text{Dens}(M) := \wedge^n T^*M \otimes \mathfrak{o}_M$ denote the vector bundle of densities on M , where \mathfrak{o}_M is the orientation line bundle and \mathcal{Dens}_M the convenient sheaf of sections of $\text{Dens}(M)$.

Let \mathcal{E}^\vee denote the convenient sheaf of sections of the vector bundle $E^\vee = E^\vee \otimes \text{Dens}(M)$ on M , where E^\vee is the fibrewise linear dual. Likewise, let \mathcal{E}_c^\vee denote the convenient cosheaf of compactly supported sections of $E^\vee \otimes \text{Dens}(M)$ on M . We define $\mathcal{E}^\perp := \underline{\text{Hom}}_{\mathcal{C}_M^\infty}(\mathcal{E}, \mathcal{C}_M^\infty)$ and $\mathcal{E}_c^\perp := \underline{\text{Hom}}_{\mathcal{C}_M^\infty}(\mathcal{E}_c, \mathcal{C}_M^\infty)$ to be the continuous linear duals endowed with the strong topology of uniform convergence on bounded subsets.

The fibrewise evaluation pairing between E and E^\vee induces a morphism $\text{fib}(-, -) : E^\vee \otimes E \rightarrow \text{Dens}(M)$ of vector bundles which extends to a pairing

$$\text{ev}_U : \mathcal{E}_c^\vee(U) \times \mathcal{E}(U) \rightarrow \mathcal{C}_M^\infty(U)$$

of convenient $\mathcal{C}_M^\infty(U)$ -modules given by sending a pair (ω, s) on $U \subseteq M$ to the integral $\int_U \text{fib}(\omega, s)$. This construction induces isomorphisms

$$\mathcal{E}^\perp(U) \simeq \overline{\mathcal{E}_c^\vee}(U) \quad \mathcal{E}_c^\perp(U) \simeq \overline{\mathcal{E}^\vee}(U)$$

of convenient $\mathcal{C}_M^\infty(U)$ -modules.

Let V be a convenient vector space. A curve $c : \mathbb{R} \rightarrow V$ is said to be *smooth* if all derivatives of c exist in the underlying topological space of V . The set of smooth curves in V is denoted \mathcal{C}_V . A morphism $f : V \rightarrow W$ of convenient vector spaces is said to be *smooth* if $f(\mathcal{C}_V) \subseteq \mathcal{C}_W$. Finally, a morphism $f : \mathcal{E} \rightarrow \mathcal{E}'$ between convenient \mathcal{C}_M^∞ -modules is *smooth* if $\mathcal{E}(U) \rightarrow \mathcal{E}'(U)$ is smooth for all $U \subseteq M$. We denote by $\underline{\text{Hom}}_{\mathcal{C}_M^\infty}^{\text{sm}}(\mathcal{E}, \mathcal{E}')$ the \mathcal{C}_M^∞ -module of smooth morphisms and

$$\mathcal{E}^* := \underline{\text{Hom}}_{\mathcal{C}_M^\infty}^{\text{sm}}(\mathcal{E}, \mathcal{C}_M^\infty)$$

the smooth dual.

Let $\text{ConMod}^{\text{sm}}(\mathcal{C}_M^\infty)$ denote the closed symmetric monoidal category of convenient \mathcal{C}_M^∞ -modules and smooth morphisms. We deduce from Corollary 2.11 of Kriegl and Michor (1997) that a linear map between convenient \mathcal{C}_M^∞ -modules is smooth if and only if it is a bornological morphism. Therefore, we have a natural forgetful functor

$$U : \text{ConMod}(\mathcal{C}_M^\infty) \rightarrow \text{ConMod}^{\text{sm}}(\mathcal{C}_M^\infty)$$

which is the identity on objects and forgets the linear structure.

We now define a number of different functionals on the space of sections of a vector bundle.

Definition 23. Let E be a vector bundle on M . A linear functional on \mathcal{E} is an element of the continuous linear dual \mathcal{E}^\perp . A smooth functional on \mathcal{E} is an element of the smooth dual \mathcal{E}^* .

Example 24. (Polynomial functions). An intermediate class of smooth functionals are polynomials. The algebra of polynomial functions on \mathcal{E} is given by

$$\mathcal{O}_{\mathcal{E}} := \text{Sym}_{\mathcal{C}_M^\infty}(\mathcal{E}^\perp) = \bigoplus_{n=0}^{\infty} ((\mathcal{E}^\perp)^{\otimes n})_{S_n} \simeq \bigoplus_{n=0}^{\infty} ((\overline{\mathcal{E}_c^\vee})^{\otimes n})_{S_n}$$

where the subscript S_n refers to taking coinvariants with respect to the action of the symmetric group on the n -fold tensor product. The algebra of polynomial functions on \mathcal{E}_c is given by

$$\mathcal{O}_{\mathcal{E}_c} := \text{Sym}_{\mathcal{C}_M^\infty}(\mathcal{E}_c^\perp) = \bigoplus_{n=0}^\infty ((\mathcal{E}_c^\perp)^{\otimes n})_{S_n} \simeq \bigoplus_{n=0}^\infty ((\overline{\mathcal{E}_c^\vee})^{\otimes n})_{S_n}.$$

Example 25 (Formal power series). A larger class of smooth functionals are those given by formal power series. That is, the completed symmetric algebra

$$\widehat{\mathcal{O}}_{\mathcal{E}} := \widehat{\text{Sym}}_{\mathcal{C}_M^\infty}(\mathcal{E}^\perp) = \prod_{n=0}^\infty ((\mathcal{E}^\perp)^{\otimes n})_{S_n} \simeq \prod_{n=0}^\infty ((\overline{\mathcal{E}^\vee})^{\otimes n})_{S_n}$$

and that on compactly supported sections

$$\widehat{\mathcal{O}}_{\mathcal{E}_c} := \widehat{\text{Sym}}_{\mathcal{C}_M^\infty}(\mathcal{E}_c^\perp) = \prod_{n=0}^\infty ((\mathcal{E}_c^\perp)^{\otimes n})_{S_n} \simeq \prod_{n=0}^\infty ((\overline{\mathcal{E}_c^\vee})^{\otimes n})_{S_n}.$$

This leads to natural inclusions $\mathcal{E}^\perp \subset \mathcal{O}_{\mathcal{E}} \subset \widehat{\mathcal{O}}_{\mathcal{E}} \subset \mathcal{E}^*$ of sheaves and similarly for compactly supported sections. See Kerjean and Tasson (2018) for a detailed discussion of polynomials and power series in a similar context.

We now describe a comonad which we call the distributional comonad which is a generalisation of that contained in Blute et al. (2012) to the setting of \mathcal{C}_M^∞ -modules. Consider the Dirac distributional density map

$$\delta : \mathcal{E} \rightarrow (\mathcal{E}^*)^\perp$$

sending a section s to $\delta_s : F \mapsto F(s)$ where F is a smooth functional. We denote by $!_\delta \mathcal{E}$ the c^∞ -closure of the linear span of $\delta(\mathcal{E})$ in $(\mathcal{E}^*)^\perp$.

Lemma 26. *The endofunctor $!_\delta$ induces a comonad on $\text{ConMod}(\mathcal{C}_M^\infty)$.*

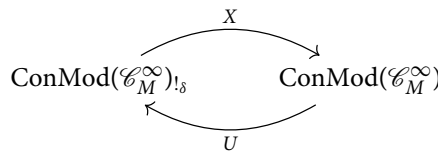
Proof. We have an inclusion $\text{Conv} \rightarrow \text{TBorn}$ of closed symmetric monoidal categories which induces an inclusion $\text{ConMod}(\mathcal{C}_M^\infty) \rightarrow \text{TBMod}(\mathcal{C}_M^\infty)$ of \mathcal{C}_M^∞ -modules where

$$\text{TBMod}(\mathcal{C}_M^\infty) := \text{Mod}_{\mathcal{C}_M^\infty}(\text{Sh}_{\text{TBorn}}(M)).$$

The left adjoint $\gamma : \text{TBMod}(\mathcal{C}_M^\infty) \rightarrow \text{ConMod}(\mathcal{C}_M^\infty)$ of this inclusion is a composition of separation and completion functors.

We have a natural comultiplication map $\mu^\delta : !_\delta \rightarrow !_\delta !_\delta$ which object-wise $\mu_{\mathcal{E}}^\delta : !_\delta \mathcal{E} \rightarrow !_\delta !_\delta \mathcal{E}$ extends linearly the map $\delta_s \mapsto \delta_{\delta_s}$ and applies the separation and completion functor γ . The counit map $\varepsilon^\delta : !_\delta \rightarrow \text{id}$ object-wise $\varepsilon_{\mathcal{E}}^\delta : !_\delta \mathcal{E} \rightarrow \mathcal{E}$ extends linearly the map $\delta_s \mapsto s$ and applies the functor γ . The commutativity of the relevant diagrams can be easily verified. \square

We have a linear–non-linear adjunction



and a symmetric monoidal comonad $!_\delta = X \circ U$ which we call the *distributional comonad*. The functor X sends a \mathcal{C}_M^∞ -module \mathcal{E} to the c^∞ -closure of the linear span of $\delta(\mathcal{E})$, and U is a bijection on objects.

Proposition 27. *There exists an equivalence*

$$\text{ConMod}(\mathcal{C}_M^\infty)_{!_\delta} \simeq \text{ConMod}^{\text{sm}}(\mathcal{C}_M^\infty)$$

of categories.

Proof. The Dirac distributional density map is smooth. It suffices to check the condition object-wise and so the result follows from Lemma 5.1 of Blute et al. (2012). \square

Consider the sequent $!A \vdash B$ in differential linear logic with denotation $\llbracket - \rrbracket_M$ given by the functional $F : \llbracket !A \rrbracket_M = !_\delta \mathcal{E} \rightarrow \llbracket B \rrbracket_M = \mathcal{E}'$ and the diagram

$$\begin{array}{ccc} & !_\delta \mathcal{E} & \\ \delta_s \nearrow & \downarrow \varepsilon_\delta^\mathcal{E} & \searrow F \\ * \xrightarrow{s} & \mathcal{E} & \xrightarrow{F^{\text{sm}}} \mathcal{E}' \end{array}$$

of convenient \mathcal{C}_M^∞ -modules. From Proposition 27, we have the smooth functional $F^{\text{sm}} : \mathcal{E} \rightarrow \mathcal{E}'$ with $F^{\text{sm}}(s) \simeq F(\delta_s)$ associated to F . We also define a map $\vec{d}_\mathcal{E}^\delta : \mathcal{E} \rightarrow !_\delta \mathcal{E}$ for the distributional comonad, following (Blute et al. 2012) by

$$\vec{d}_\mathcal{E}^\delta(s) = \lim_{h \rightarrow 0} \frac{\delta_{hs} - \delta_0}{h}$$

where $s \in \mathcal{E}$, 0 is the zero section and h the constant sheaf.

Theorem. *The category of convenient \mathcal{C}_M^∞ -modules with the distributional comonad $!_\delta$ and map \vec{d}^δ is a model for intuitionistic differential linear logic.*

Proof. The category of convenient vector spaces is locally presentable (Wallbridge 2015) and closed symmetric monoidal (Kriegel and Michor 1997). Sheaves with values in a locally presentable closed symmetric monoidal category themselves form a locally presentable closed symmetric monoidal category, as do modules over a commutative monoid object in such a category of sheaves (Mesablishvili 2014). Therefore, the category of convenient \mathcal{C}_M^∞ -modules is locally presentable closed symmetric monoidal. It is moreover an additive and therefore CMon-enriched, symmetric monoidal category.

By Lemma 26, the functor $!_\delta$ is a comonad. For each object \mathcal{E} in $\text{ConMod}(\mathcal{C}_M^\infty)$, we define a cocommutative comonoid object $(!_\delta \mathcal{E}, c_\mathcal{E}, e_\mathcal{E})$ using the maps $e_\mathcal{E} : \delta_s \mapsto 1$ and

$$c_\mathcal{E} : \delta_s \mapsto \delta_s \otimes \delta_s,$$

and then extending linearly and applying the separation and completion functor γ (see the proof of Lemma 26). Also, since the diagrams

$$\begin{array}{ccc} \delta_s & \xrightarrow{c_\mathcal{E}} & \delta_s \otimes \delta_s \\ \downarrow \mu_\mathcal{E} & & \downarrow \mu_\mathcal{E} \otimes \mu_\mathcal{E} \\ \delta_{\delta_s} & \xrightarrow{c_{!_\delta \mathcal{E}}} & \delta_{\delta_s} \otimes \delta_{\delta_s} \end{array} \qquad \begin{array}{ccc} \delta_s & \xrightarrow{\mu_\mathcal{E}} & \delta_{\delta_s} \\ \swarrow e_\mathcal{E} & & \downarrow e_{!_\delta \mathcal{E}} \\ & & 1 \end{array}$$

commute, $\mu_\mathcal{E} : !_\delta \mathcal{E} \rightarrow !_\delta !_\delta \mathcal{E}$ a morphism of comonoid objects in $\text{ConMod}(\mathcal{C}_M^\infty)$. Let \mathcal{E} and \mathcal{E}' be convenient \mathcal{C}_M^∞ -modules. Then,

$$!_\delta(\mathcal{E} \times \mathcal{E}') \simeq !_\delta \mathcal{E} \otimes !_\delta \mathcal{E}'$$

is an isomorphism of sheaves by extending the fibrewise statement of Proposition 5.2.4 of Frölicher and Kriegel (1988) and Proposition 5.6 of Blute et al. (2012).

It remains to show that the map \bar{d}^δ satisfies the conditions to be a codereliction. First, the diagram

$$\begin{CD} \mathcal{E} @>\bar{d}^\delta>> !_\delta \mathcal{E} \\ @V F VV @VV !_\delta(F) V \\ \mathcal{E}' @>\bar{d}^{\delta'}>> !_\delta \mathcal{E}' \end{CD}$$

commutes since $\lim_{h \rightarrow 0} \frac{\delta_{hF(s)} - \delta_0}{h} = \lim_{h \rightarrow 0} \frac{\delta_{F(hs)} - \delta_{F(0)}}{h}$ owing to the property that F is a morphism of \mathcal{C}_M^∞ -modules (explicitly, $hs(x) = h(x)s(x)$ and $F(hs)(x) = h(x)F(s)(x)$). Therefore, \bar{d}^δ is a natural transformation. By Theorem 6 and Corollary 4 of Blute et al. (2019), it now suffices to show that the linear and chain rules of Definition 8 are satisfied. The left-hand side of the linear rule $\varepsilon_{\mathcal{E}} \circ \bar{d}^\delta$ given by

$$s \mapsto \lim_{h \rightarrow 0} \frac{\delta_{hs} - \delta_0}{h} \mapsto \lim_{h \rightarrow 0} \left(\frac{1}{h}(hs - 0) \right) = s$$

coincides with the identity due to continuity of $\varepsilon_{\mathcal{E}}$. The multiplication map of the monoid object in the bialgebra structure is given by $\bar{c}_{\mathcal{E}} : \delta_s \otimes \delta_t \mapsto \delta_{s+t}$ and then extending linearly and applying γ . Therefore, the left-hand side $\mu_{\mathcal{E}} \circ \bar{c}_{\mathcal{E}} \circ (\bar{d}_{\mathcal{E}} \otimes \text{id}_{!_\delta \mathcal{E}})$ of the chain rule gives

$$s \otimes \delta_t \mapsto \left(\lim_{h \rightarrow 0} \frac{\delta_{hs} - \delta_0}{h} \right) \otimes \delta_t \mapsto \lim_{h \rightarrow 0} \frac{\delta_{hs+t} - \delta_t}{h} \mapsto \lim_{h \rightarrow 0} \frac{\delta_{hs+t} - \delta_{\delta_t}}{h}$$

which corresponds to the right-hand side $\bar{c}_{!_\delta \mathcal{E}} \circ (\bar{d}_{!_\delta \mathcal{E}} \otimes \mu_{!_\delta \mathcal{E}}) \circ (\bar{c}_{\mathcal{E}} \otimes \text{id}_{!_\delta \mathcal{E}}) \circ (\bar{d}_{\mathcal{E}} \otimes c_{\mathcal{E}})$ by

$$\begin{aligned} s \otimes \delta_t &\mapsto \left(\lim_{h \rightarrow 0} \frac{\delta_{hs} - \delta_0}{h} \right) \otimes (\delta_t \otimes \delta_t) \mapsto \left(\lim_{h \rightarrow 0} \frac{\delta_{hs+t} - \delta_t}{h} \right) \otimes \delta_t \mapsto \\ &\left(\lim_{h', h \rightarrow 0} \frac{\delta_{\left(\frac{h'}{h}(\delta_{hs+t} - \delta_t)\right) - \delta_{\delta_0}}}{h'} \right) \otimes \delta_{\delta_t} \mapsto \lim_{h', h \rightarrow 0} \frac{\delta_{\left(\frac{h'}{h}(\delta_{hs+t} - \delta_t) + \delta_t\right) - \delta_{\delta_t}}}{h'} \end{aligned}$$

using associativity of the tensor product and then taking the limit $h = h' \rightarrow 0$ along the diagonal. □

We will call \bar{d}^δ the *distributional codereliction*. Let $F : !_\delta \mathcal{E} \rightarrow \mathcal{E}'$ be a morphism in $\text{ConMod}(\mathcal{C}_M^\infty)$. The deriving transformation $\bar{\partial}_{\mathcal{E}} : !_\delta \mathcal{E} \otimes \mathcal{E} \rightarrow !_\delta \mathcal{E}$ is given by

$$\bar{\partial}_{\mathcal{E}} : \delta_s \otimes t \xrightarrow{(\text{id} \otimes \bar{d}_{\mathcal{E}}^\delta)} \delta_s \otimes \left(\lim_{h \rightarrow 0} \frac{\delta_{ht} - \delta_0}{h} \right) \xrightarrow{\bar{c}_{\mathcal{E}}} \lim_{h \rightarrow 0} \frac{\delta_{s+ht} - \delta_s}{h}$$

and the derivative $dF := F \circ \bar{\partial}_{\mathcal{E}} : !_\delta \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}'$ of F in $\text{ConMod}(\mathcal{C}_M^\infty)$ is

$$dF : \delta_s \otimes t \mapsto \lim_{h \rightarrow 0} \frac{F(\delta_{s+ht}) - F(\delta_s)}{h}$$

for local sections $s, t \in \mathcal{E}$. Using the adjunction of Proposition 27, we have, by abuse of notation, an operator

$$d : \text{Hom}_{\mathcal{C}_M^\infty}^{\text{sm}}(\mathcal{E}, \mathcal{E}') \rightarrow \text{Hom}_{\mathcal{C}_M^\infty}(\mathcal{E}, \underline{\text{Hom}}_{\mathcal{C}_M^\infty}(\mathcal{E}, \mathcal{E}'))$$

defined by

$$dF^{\text{sm}}(s, t) = \lim_{h \rightarrow 0} \frac{F^{\text{sm}}(s + ht) - F^{\text{sm}}(s)}{h} = \left. \frac{d}{dh} \right|_{h=0} F^{\text{sm}}(s + ht).$$

This derivative operator is linear and bounded, and $dF^{sm}(s, t)$ is the functional derivative at the section s of \mathcal{E} in the direction of the section t . When $\mathcal{E}' = \mathcal{C}_M^\infty$, another common notation for $dF^{sm}(s, t)$ is

$$dF^{sm}(s, t) = \int_U \frac{\delta F^{sm}}{\delta s}(x)t(x)dx$$

for $U \subseteq M$.

5. Comonad Composition and Non-linearity

In Section 4, we have shown that the category of convenient \mathcal{C}_M^∞ -modules is a model for intuitionistic differential linear logic using the distributional comonad $!_\delta$. Combining this result with the extension of the model in Section 3 to this same category, we obtain a compatible model based on composition with the infinite jet comonad, i.e., these two comonads interact in a natural way so that their composition induces a model for intuitionistic differential linear logic.

First, we update the finite jet functor by lifting it to an endofunctor $!_j : \text{ConMod}(\mathcal{C}_M^\infty) \rightarrow \text{ConMod}(\mathcal{C}_M^\infty)$ and leverage the convenient structure to define

$$\mathcal{J}(\mathcal{E}) := \lim_{r \in \mathbb{N}} (\mathcal{J}^r(\mathcal{E}))$$

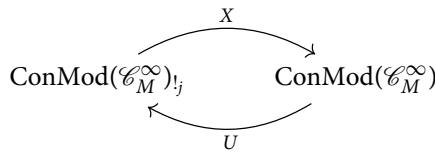
as a genuine limit in $\text{ConMod}(\mathcal{C}_M^\infty)$. The infinite prolongation thus induces an endofunctor

$$!_j : \text{ConMod}(\mathcal{C}_M^\infty) \rightarrow \text{ConMod}(\mathcal{C}_M^\infty)$$

on the category of convenient \mathcal{C}_M^∞ -modules. The following result is clear from Lemma 13.

Corollary 28. *The endofunctor $!_j$ is a comonad on $\text{ConMod}(\mathcal{C}_M^\infty)$.*

The category $\text{ConMod}(\mathcal{C}_M^\infty)$ is endowed with a cocartesian monoidal structure with monoidal product \oplus and unit 0. We have a linear–non-linear adjunction



and a symmetric monoidal comonad $!_j = X \circ U$ which we call the jet comonad. Here X sends a \mathcal{C}_M^∞ -module \mathcal{E} to $\mathcal{J}(\mathcal{E})$ and the right adjoint U is an object bijection. The jet codereliction \bar{d}^j extends to a natural transformation on $\text{ConMod}(\mathcal{C}_M^\infty)$. A corollary of Theorem 20 is now the following.

Corollary 29. *The category of convenient \mathcal{C}_M^∞ -modules with the jet comonad $!_j$ is a symmetric monoidal storage category for the cocartesian monoidal structure.*

Owing to the discussion in Section 3, we have an isomorphism

$$\text{Hom}_{\text{ConMod}^{sm}(\mathcal{C}_M^\infty)}(\mathcal{J}(\mathcal{E}), \mathcal{E}') \simeq \text{Diff}^{sm}(\mathcal{E}, \mathcal{E}')$$

where the right-hand side denotes the set of smooth partial differential operators.

We now define a number of different functionals on the space of jets of sections of a vector bundle.

Definition 30. Let E be a vector bundle on M . A local linear functional on \mathcal{E} is an element of the continuous linear dual $(!_j\mathcal{E})^\perp$. A local smooth functional on \mathcal{E} is an element of the smooth dual $(!_j\mathcal{E})^*$.

Local smooth functionals are also called *Lagrangians* in certain applications. Lagrangians given by formal power series are particularly important in the study of perturbative classical and quantum field theories. This is demonstrated in the following example.

Example 31. Building on Example 25, the algebra of formal power series of local linear functionals is given by

$$\mathcal{O}_{\mathcal{E}}^{\text{loc}} := \widehat{\text{Sym}}_{\mathcal{C}_M^\infty}(!_j\mathcal{E}^\perp)$$

elements of which will be called *Lagrangian densities*. More explicitly, we identify the n th component of a Lagrangian density on M with a compactly supported distributional section of the bundle $(J(E)^\vee)^{\boxtimes n}$ on M^n . Since local linear functionals depend only on the local nature of a section s at each point, i.e., its jet, then we can interpret its n th component as a finite sum of densities of the form $(D_1s)(D_2s) \dots (D_ns)d\Omega$ where each $D_i : \mathcal{E} \rightarrow \mathcal{C}_M^\infty$ is a differential operator. The natural inclusion

$$\iota_U : \mathcal{O}_{\mathcal{E}}^{\text{loc}}(U) \rightarrow \widehat{\mathcal{O}}_{\mathcal{E}}(U)$$

given by integration $\iota_U(\mathcal{L}) : s \mapsto \int_U \mathcal{L}(s)$ defines the *action* $S_U := \iota_U(\mathcal{L}) : \mathcal{E}(U) \rightarrow \mathbb{R}$ of the Lagrangian distributional density \mathcal{L} .

Remark 32. Note that the section s in Example 31 should be nilpotent since in most cases the infinite sum will not converge. Alternatively, we could define a Lagrangian density to be an element \mathcal{L} in $\mathcal{O}_{\mathcal{E}}^{\text{loc}}$ which factors through $\prod_{n=0}^r (!_j\mathcal{E}^\perp)^{\otimes n}_{S_n}$ for some finite r .

From Example 31, a Lagrangian sends a section s in $\mathcal{E}(U)$ to a formal power series in these variables, a density which, when evaluated on a point in U depends only on the infinite jet at that point.

We endow $!_j\mathcal{E}$ with its canonical \mathcal{D}_M^∞ -module structure. This is the canonical flat connection given by the Cartan distribution of Section 3. Then, the convenient \mathcal{C}_M^∞ -module $!_j\mathcal{E}^\perp = \underline{\text{Hom}}_{\mathcal{C}_M^\infty}(!_j\mathcal{E}, \mathcal{C}_M^\infty)$ has a canonical \mathcal{D}_M^∞ -module structure. Therefore, a local functional is a \mathcal{D}_M^∞ -module. Again, every element L of this module takes a section s of $\mathcal{E}(U)$ and returns a smooth function $L(s)$ in $\mathcal{C}_M^\infty(U)$ with the property that $L(s)(x)$ depends only on the ∞ -jet of s at $x \in U$.

Now that the jet comonad is understood as a comonad on the category of convenient \mathcal{C}_M^∞ -modules, we combine this result with the distributional comonad of Section 4. Pre-composition with $!_j$ gives the module $!_j\delta\mathcal{E} := !_j\delta \circ !_j\mathcal{E}$. So $!_j\delta\mathcal{E}$ is the c^∞ -closure of the linear span of $\delta(!_j\mathcal{E})$ in $(!_j\mathcal{E}^*)^\perp$. The two comonads interact in the expected way.

Lemma 33. *The composite comonad $!_j\delta$ is a storage comonad.*

Proof. There is a canonical distributive law of $!_j$ over $!_j\delta$ since, by definition, operators act on distributions as $\langle s, j(\delta) \rangle := \langle j(s), \delta \rangle$ and therefore $!_j\delta \simeq !_j\delta$ is an isomorphism of comonads. The result now follows from Lemma 5. □

The comonad $!_j\delta$ will be called the *jet-distributional comonad*. Now consider the map $\bar{d}^{j\delta} : \text{id} \rightarrow !_j\delta$ given by

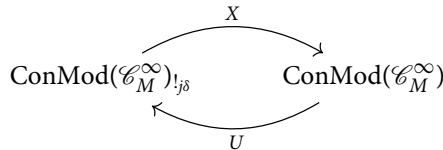
$$\bar{d}_{\mathcal{E}}^{j\delta}(s) = \lim_{h \rightarrow 0} \frac{\delta_{h(j(s))} - \delta_0}{h}$$

for $s \in \mathcal{E}$.

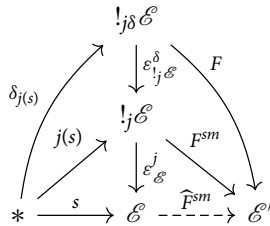
Theorem. The category of convenient \mathcal{C}_M^∞ -modules with the jet-distributional comonad $!_{j\delta}$ and map $\bar{d}^{j\delta}$ is a model for intuitionistic differential linear logic.

Proof. By Lemma 33, the jet-distributional comonad is a storage comonad. The remainder of the proof is obtained by applying the corresponding proof in Theorem 4 to the convenient \mathcal{C}_M^∞ -module $!_{j\mathcal{E}}$. \square

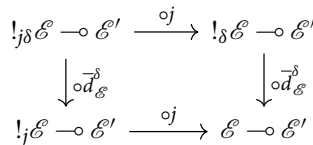
The map $\bar{d}^{j\delta} : \text{id} \rightarrow !_{j\delta}$ will be called the *jet-distributional codereliction*. We have a linear–non-linear adjunction



where the functor X sends an object \mathcal{E} to the c^∞ -closure of the linear span of $\delta(\mathcal{J}(\mathcal{E}))$ and the functor U is a bijection on objects. Objects on the left-hand side are convenient vector bundles and whose morphisms, owing to Proposition 27, include *non-linear* partial differential operators $\widehat{F}^{sm} : \mathcal{E} \rightarrow \mathcal{E}'$. Indeed, let $F : !_{j\delta}\mathcal{E} \rightarrow \mathcal{E}'$ be a morphism of \mathcal{C}_M^∞ -modules and consider the diagram



in $\text{ConMod}(\mathcal{C}_M^\infty)$. We have $F(\delta_{j(s)}) \simeq F^{sm}(j(s)) \simeq \widehat{F}^{sm}(s)$. Moreover, taking advantage of the closed structure and using the notation of linear logic, we have a commutative diagram



sending the convenient \mathcal{C}_M^∞ -module of smooth local functionals to the convenient \mathcal{C}_M^∞ -module of linear functionals.

The deriving transformation $\bar{\partial}_{\mathcal{E}} : !_{j\delta}\mathcal{E} \otimes \mathcal{E} \rightarrow !_{j\delta}\mathcal{E}$ is defined as the composite

$$\bar{\partial}_{\mathcal{E}} : \delta_{j(s)} \otimes t \xrightarrow{(\text{id} \otimes \bar{d}_{\mathcal{E}}^{j\delta})} \delta_{j(s)} \otimes \left(\lim_{h \rightarrow 0} \frac{\delta_{h(j(t))} - \delta_0}{h} \right) \xrightarrow{\bar{c}_{\mathcal{E}}} \lim_{h \rightarrow 0} \frac{\delta_{(j(s)+h(j(t)))} - \delta_{j(s)}}{h}$$

and the derivative $dF := F \circ \bar{\partial}_{\mathcal{E}} : !_{j\delta}\mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}'$ of $F : !_{j\delta}\mathcal{E} \rightarrow \mathcal{E}'$ in $\text{ConMod}(\mathcal{C}_M^\infty)$ is given as

$$dF : \delta_{j(s)} \otimes t \mapsto \lim_{h \rightarrow 0} \frac{F(\delta_{j(s)+hj(t)}) - F(\delta_{j(s)})}{h}.$$

By abuse of notation, we have an operator on smooth differential operators

$$d : \text{Diff}^{sm}(\mathcal{E}, \mathcal{E}') \rightarrow \text{Hom}(\mathcal{E}, \mathcal{D}_M(\mathcal{E}, \mathcal{E}'))$$

defined by

$$d\widehat{F}^{sm}(s, t) = \lim_{h \rightarrow 0} \frac{\widehat{F}^{sm}(s + ht) - \widehat{F}^{sm}(s)}{h} = \left. \frac{d}{dh} \right|_{h=0} \widehat{F}^{sm}(s + ht)$$

which is linear and bounded, i.e., $d\widehat{F}^{sm}(s, t)$ is the derivative of the smooth local functional \widehat{F}^{sm} at s in the direction $t = ds$.

When our sheaf is finite dimensional, we have the following more explicit description of non-linear local functionals.

Example 34. Let \mathcal{E} be a finite dimensional convenient \mathcal{C}_M^∞ -module. Then, there exists an isomorphism

$$!_j \delta \mathcal{E} \simeq (!_j \mathcal{E}^*)^\perp$$

of convenient \mathcal{C}_M^∞ -modules. This can be deduced from Corollary 5.1.8 of Frölicher and Kriegl (1988).

When our smooth functionals are given by formal power series, we also have a more explicit description. We endow $\mathcal{D}ens_M$ with its right \mathcal{D}_M^∞ -module structure. Then, a local density on $U \subseteq M$ with respect to \mathcal{E} is an element $\omega_U \otimes L$ in the space

$$\mathcal{O}_{\mathcal{E}(U)}^{loc} \simeq \mathcal{D}ens_M(U) \otimes_{\mathcal{C}_M^\infty(U)} \widehat{\text{Sym}}_{\mathbb{R}}(!_j \overline{\mathcal{E}}(U)^\vee)$$

with its canonical $\mathcal{D}_M^\infty(U)$ -module structure where $!_j \mathcal{E}(U)^\vee = \underline{\text{Hom}}(!_j \mathcal{E}(U), \mathcal{C}_M^\infty(U))$. In other words, the element $\omega_U \otimes L$ sends a section s in $\mathcal{E}(U)$ to a distributional density $\omega_U \otimes L(s)$ on U such that $(\omega_U \otimes L(s))(x)$ depends only on the ∞ -jet of s at $x \in U$.

We obtain a sheaf $\mathcal{O}_{\mathcal{E}}^{loc}$ on M which is moreover a \mathcal{C}_M^∞ -module. It is a subsheaf $\mathcal{O}_{\mathcal{E}}^{loc} \subset !_j \mathcal{E}^*$ and the c^∞ -closure of the linear span of $\delta(!_j \mathcal{E})$ in $(!_j \mathcal{E}^*)^\perp$ factors through $(\mathcal{O}_{\mathcal{E}}^{loc})^\perp$. This is a subspace of $!_j \delta \mathcal{E}(U)$. This restricted delta distribution δ sends $j(s)$ to

$$\delta_{j(s)} : \omega_U \otimes L \mapsto \int_U L(j(s)) \omega_U$$

where L is a formal power series. In local coordinates on $U \subseteq M$, and using integration by parts, we have

$$dS(s) = d \int_U L(j(s)) \omega_U = \int_U e l_\alpha(L) ds^\alpha \wedge \omega_U + D_\alpha V^\alpha$$

for some total derivative $D_\alpha V^\alpha$ where $e l_\alpha = \sum_I (-D)_I \frac{\partial}{\partial u_I^\alpha}$ is the Euler–Lagrange operator. Here $(-D)_I = (-D_{i_1}) \dots (-D_{i_n})$ for the multi-index I .

Therefore, a Lagrangian is only defined up to a total derivative for compactly supported sections. This can be exploited by forming the tensor product

$$\mathcal{O}_{\mathcal{E}}^{red} \simeq \mathcal{D}ens_M \otimes_{\mathcal{D}_M^\infty} \widehat{\text{Sym}}_{\mathbb{R}}(!_j \overline{\mathcal{E}}^\vee)$$

over \mathcal{D}_M^∞ . Therefore, $dS(s) = 0$ if and only if the section s satisfies the Euler–Lagrange equations $e l_\alpha(L(j(s))) = 0$. Symmetries of the action can also be interpreted as vector fields on the jet bundle (cf. Remark 18).

We end by giving a concrete application of this construction.

Example 35 (Free and interacting scalar fields). Fix a n -dimensional compact Riemannian manifold (M, g) . Consider the sheaf \mathcal{E} of sections of the trivial bundle $\pi : E := M \times \mathbb{R} \rightarrow M$, i.e., \mathcal{E} is simply the sheaf \mathcal{C}_M^∞ of smooth functions on M . There exists an isomorphism $(!_j \mathcal{C}_M^\infty)^\perp \simeq \mathcal{D}_M^\infty$ of

(left) \mathcal{D}_M^∞ -modules. The elements \widehat{L} in $(!_j \mathcal{C}_M^\infty)^\perp$ are spanned by elements of the form $\phi(x_i)\partial_I$ for $x_i \in M$ and a partial differential operator ∂_I depending on a multi-index I .

Let $\phi \in \mathcal{C}_M^\infty(M)$ be a scalar field. We consider the special forms of \widehat{L} given by

$$\widehat{L}(j(\phi)) = \phi D\phi, \quad \widehat{L}(j(\phi)) = \phi D\phi + \eta\phi, \quad L(\delta_{j(\phi)}) = \phi D\phi + V(\phi)$$

where the Laplacian D is the differential operator $D : \mathcal{C}_M^\infty(M) \rightarrow \mathcal{C}_M^\infty(M)$ sending ϕ to $\Delta_g\phi$, and the density is the canonical volume form. The first two functionals are linear local functionals, whereas the last functional is merely smooth in general. The functional derivative of the local action functional S associated to L is

$$dS(\phi) = \int_M d\widehat{L}^{sm}(\phi)\omega_M = \int_M el(\widehat{L}^{sm}(\phi))d\phi \text{ vol}_g$$

where, for local coordinates (x^i, ϕ, ϕ_i) , the Euler–Lagrange equations are

$$el(\widehat{L}^{sm}(\phi)) = \frac{\partial \widehat{L}^{sm}}{\partial \phi} - \frac{\partial}{\partial x^i} \left(\frac{\partial \widehat{L}^{sm}}{\partial \phi_i} \right).$$

The principle of least action $dS = 0$, or equivalently $el(\widehat{L}^{sm}(\phi)) = 0$, leads to the partial differential equations

$$\Delta_g\phi = 0, \quad \Delta_g\phi = \eta, \quad \Delta_g\phi = -V'(\phi)$$

which are the Laplace, Poisson and non-linear Poisson equations, respectively. These define a vector subbundle, affine subbundle and fibred submanifold of $J^2(M \times \mathbb{R})$, respectively.

To see this, let $(x^i, \phi, \phi_i, \phi_{ij})$ be coordinates on $J^2(M \times \mathbb{R})$ and consider the function $f(x^i, \phi, \phi_i, \phi_{ij}) = \sum_{1 \leq i \leq n} \phi_{ii}$ on $J^2(M \times \mathbb{R})$. The preimage of $0, \eta(x)$, and $-V'(\phi)$ with respect to f define a fibred submanifold $H^2 \subseteq J^2(M \times \mathbb{R})$. Taking the infinite prolongation of the equation H^2 , we obtain the equation H which, assuming H^2 is regular, is a pro-ind vector subbundle, pro-ind affine subbundle and pro-ind fibred submanifold of $J(M \times \mathbb{R})$, respectively. A local section ϕ of $\pi : M \times \mathbb{R} \rightarrow M$ is a solution of these equations if and only if $j^2\phi(x_i) \in \mathcal{H}^2 \simeq \mathcal{H}$.

6. Conclusion

We have shown that the category of convenient sheaves is a model for intuitionistic differential linear logic. Using the jet comonad for the exponential modality gives an interpretation of linear differential operators, and hence linear partial differential equations, in linear logic. Alternatively, using the distributional comonad for the exponential gives an interpretation of smooth morphisms between objects in these categories. Composing these comonads provides an interpretation of non-linear differential operators and the variational calculus of smooth local functionals within linear logic.

Some interesting questions remain open. The most pressing item is to elucidate the internal logic of the model in order to provide a computational interpretation of its structure within differential λ -calculus. Indeed, the Kleisli category of a model for intuitionistic differential linear logic is a cartesian closed differential category, and it is these categories, introduced in Bucciarelli et al. (2010) as *differential λ -categories*, that are models of the simply typed differential λ -calculus. See Manzonetto (2012), Blute et al. (2015) for more details.

Other interesting questions include the extension to classical differential linear logic (Girard 1987), the exploration of antiderivatives and integration from the perspective of Ehrhard (2018), Cockett and Lemay (2019) and the application of reverse-mode differentiation from Cockett et al. (2020). Finally, we would like an expansion of the category of vector bundles to include ‘non-smooth’ structures. This requires the introduction of tools from synthetic and derived differential

geometry. These more elaborate structures are needed to make sense of non-linear partial differential equations and their moduli space of solutions within the context of models of differential linear logic.

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Note

1 These categories are called *additive monoidal storage categories* in Blute et al. (2006; 2019). We have decided to use the above more descriptive terminology and retain the standard use of additivity (Mac Lane 1971). In particular, an additive category is enriched over the monoidal category Ab of abelian groups.

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