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Stability of planar flames as gasdynamic discontinuities

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The stability of a steadily propagating planar premixed flame has been the subject of numerous studies since Darrieus and Landau showed that in their model flames are unstable to perturbations of any wavelength. Moreover, the instability was shown to persist even for very small wavelengths, i.e. there was no high-wavenumber cutoff of the instability. In addition to the Darrieus–Landau instability, which results from thermal expansion, analysis of the diffusional thermal model indicates that premixed flames may exhibit cellular and pulsating instabilities as a consequence of preferential diffusion. However, no previous theory captured all the instabilities including a high-wavenumber cutoff for each. In Class, Matkowsky & Klimenko (2003) a unified theory is proposed which, in appropriate limits and under appropriate assumptions, recovers all the relevant previous theories. It also includes additional new terms, not present in previous theories. In the present paper we consider the stability of a uniformly propagating planar flame as a solution of the unified model. The results are then compared to those based on the models of Darrieus–Landau, Sivashinsky and Matalon–Matkowsky. In particular, it is shown that the unified model is the only model to capture the Darrieus–Landau, cellular and pulsating instabilities including a high-wavenumber cutoff for each.

1. Introduction

Darrieus (1938, 1945) and Landau (1944) independently proposed a model which describes a premixed flame as a surface separating the burned and the unburned mixtures. The fluid flow on either side of the flame surface was governed by the non-reactive flow equations. In the Darrieus–Landau model the flame surface propagates normal to itself at a constant speed, i.e. the adiabatic laminar flame speed. The fluid variables on the two sides of the flame surface are related by jump conditions. Specifically, the mass and momentum were assumed to be conserved.

A stability analysis of a uniformly propagating planar flame governed by the Darrieus–Landau model showed that, in contrast to observations, the flame was unconditionally unstable and the model lacks the large-wavenumber cutoff. Indeed, the most unstable perturbations of a flame described by the Darrieus–Landau model correspond to the shortest-wavelength corrugations, e.g. length scales smaller than the

	DL	M	MDKW	E	S	CW	MM	CMK
postulated(p)/derived(d)	p	p	p	p	d	d	d	d
Flame speed relation								
algebraic: stretch(χ)/ curvature(c)/strain(s)	const	c	χ	c, s	–	χ	χ	–
ode/pde in time(t), space($\mathbf{x}$)	–	–	–	–	t	–	–	$t, \mathbf{x}$
Jump conditions								
Darrieus–Landau (DL)	DL	DL	DL	DL	DL	DL	DL	DL
surface compression (σ)							σ	σ
Marangoni ($\nabla_\perp \sigma$)								$\nabla_\perp \sigma$
Instabilities								
Darrieus–Landau (DL)	DL	DL	DL	DL	DL	DL	DL	DL
cutoff for some(+)/ all(++) parameters	–	+	+	+	+	+	+	++
cellular instability (C)	–	–	–	–	C	C	C	C
cutoff for some(+)/ all(++) parameters	–	–	–	–	–	+	+	++
pulsating instability (P)	–	–	–	–	P	–	–	P
cutoff for some(+)/ all(++) parameters	–	–	–	–	–	–	–	++

TABLE 1. Summary of theories of DL, Darrieus (1938, 1945) and Landau (1944); M, Markstein (1951); MDKW, Karlovitz *et al.* (1953); E, Eckhaus (1961) and Markstein (1964); S, Sivashinsky (1976); CW, Clavin & Williams (1982); MM, Matalon & Matkowsky (1982); CMK, Class *et al.* (2003).

thickness of the flame, which is clearly a limit not covered by the model. In addition, the possibility of pulsating instabilities was not even considered. Since then there have been a number of studies which have attempted to improve the Darrieus–Landau model.

In table 1 the various theories are summarized with respect to the form of the flame speed relation, the form of the jump conditions, the instabilities captured and whether or not the high-wavenumber cutoff of the instabilities is exhibited for all or for only some parameter values.

The early phenomenological theories of Markstein (1951), Karlovitz *et al.* (1953), Eckhaus (1961) and Markstein (1964) assumed that there exists a length, the Markstein length, which measures the smallest corrugations of the flame front. The ratio of this length scale to the hydrodynamic length scale is called the Markstein number Mr. The Darrieus–Landau flame speed relation (constant flame speed) was replaced by an algebraic relation, with the flame proportional to either flame curvature, or a combination of curvature and strain, or flame stretch. The jump conditions remained unchanged from those of the Darrieus–Landau model. In order to understand the mechanism of stabilization of short-wave perturbations for $Mr > 0$, consider, for example, a perturbed planar flame in the Markstein model, where $m = 1 - Mr2c$. Here, m is the normal mass flux through the flame and c the mean curvature of the flame. Now, assume that the flame is displaced into the burned (fresh) mixture, i.e. the curvature of the flame is negative (positive). The flame thus has a reduced (enhanced) flame speed so that it returns to its initial planar state. For large perturbation wavelengths this effect is weak, so the Darrieus–Landau instability is observed.

The early models are phenomenological, as they ignore the flame structure. Sivashinsky (1976) derived a flame speed relation for a Darrieus–Landau type model for general Lewis number Le. He used the method of matched asymptotic expansions, where the flame zone is a thin inner region embedded within a constant-density outer flow. The reaction zone is a yet thinner region within the flame structure. By solving the inner equations, i.e. the flame structure equations, and matching them to the outer flow, he recovered the Darrieus–Landau jump conditions and derived a flame speed relation which is an ordinary differential equation in time. The Sivashinsky model exhibits the Darrieus–Landau instability, the cellular instability and the pulsating instability. The cellular and pulsating instabilities are also present in the diffusional thermal theory of Barenblatt, Zeldovich & Istratov (1962) (see also Zeldovich *et al.* 1985). However, it lacks a high-wavenumber cutoff. The value $Le = 1$ is a sharp boundary between two completely different predicted qualitative behaviours. For all $Le < 1$ the cellular instability is exhibited, while for all $Le > 1$ the pulsating instability is exhibited. This is in contrast to diffusional thermal theory which predicts that there is a band of values about $Le = 1$ for which stability is observed. Specifically, the cellular instability is only observed for Le below a critical value which is below $Le = 1$, and the pulsating instability is only observed for Le above a critical value which exceeds $Le = 1$. Due to stability considerations Sivashinsky considered his theory to be valid for stationary flames only.

Matalon & Matkowsky (1982) and Clavin & Williams (1982) derived a model of flames as gasdynamic discontinuities for near-equidiffusional flames, i.e. $Le \approx 1$. They employed the method of matched asymptotic expansions to derive a flame speed relation which is algebraic, similar to the phenomenological relation proposed in Markstein (1964). The flame speed relation of Clavin & Williams (1982) corresponds to the assumption of infinitesimal perturbations of planar flames in nearly uniform flows and is thus a linearization of that of Matalon & Matkowsky (1982) who considered $O(1)$ perturbations in general flow fields. The Matalon–Matkowsky model also includes jump conditions for the flow field.

The stability analysis in Matalon & Matkowsky (1982) shows that for Lewis numbers Le above a critical value $Le_c < 1$, the flame is unstable to long-wave perturbations while it is stable to short-wave perturbations. The oscillatory instability is not captured in this model since Le is restricted to be too close to 1. For $Le < Le_c$, perturbations of any wavelength are unstable. This corresponds to the parameter regime where the cellular instability is observed in the diffusional thermal theory.

In a companion paper (Class, Matkowsky & Klimenko 2003) we derive a unified model which reduces to the previous models if appropriate limits are taken and similar assumptions are made, e.g. ignoring the effects of short wavelengths. The model also includes terms which are not present in earlier theories. The flame speed relation in Class *et al.* (2003) includes the time derivative and the nonlinearity of Sivashinsky (1976), the transverse diffusion terms implicitly contained in the Kuramoto–Sivashinsky equation (Sivashinsky 1980) and the perturbative correction terms for the flame speed relation and the jump conditions in Matalon & Matkowsky (1982) and Clavin & Williams (1982). Thus, we expect to find that all the instabilities described by previous theories are fully contained in the unified theory. Below we will show that this is indeed the case. The unified theory also includes new transverse diffusion terms. These terms will be shown to stabilize high-wavenumber perturbations. They originate from transverse diffusion which stabilizes short-wave perturbations, just as in diffusional thermal theory (Barenblatt *et al.* 1962).

In the present paper we analyse the stability of a uniformly propagating planar flame which is governed by the unified model for premixed flames as gasdynamic discontinuities in Class *et al.* (2003). We show that the unified model exhibits the

Darrieus–Landau instability and both the cellular and pulsating instability of the diffusional thermal theory. Moreover, we will show that the model exhibits the short-wavenumber cutoff for all instabilities.

The structure of the paper is as follows. In § 2 we review the unified theory. In § 3 we analyse the stability of a uniformly propagating planar flame subject to small perturbations for the Darrieus–Landau model, the Matalon–Matkowsky model, the Sivashinsky model and the unified model and compare the results. Finally, in § 4 we summarize our results.

2. Governing equations

The unified model for premixed flames as gasdynamic discontinuities which is derived in Class *et al.* (2003) generalizes the original model of Darrieus and Landau to take into account the flame structure.

The model consists of (*a*) conservation equations for mass and momentum in the fresh and burned gas mixtures, (*b*) jump conditions for mass and momentum, and (*c*) a flame speed relation describing the propagation of the discontinuity surface.

(*a*) In the model the non-reactive flow on either side of the flame is described by the Navier–Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v}) = -\nabla p + \nabla\cdot\boldsymbol{\sigma}, \tag{2.2}$$

where $\rho = \tilde{\rho}/\tilde{\rho}_f$, $\boldsymbol{v} = \tilde{\boldsymbol{v}}/\tilde{s}_F^0$, $p = \tilde{p}/(\tilde{\rho}_f(\tilde{s}_F^0)^2)$ are the non-dimensional density, velocity, and dynamic pressure. Tildes denote dimensional quantities and the index f denotes reference values in the fresh mixture. The adiabatic flame speed $\tilde{s}_F^0$, i.e. the speed of a uniformly propagating planar adiabatic flame relative to the fresh mixture, is the reference velocity. The independent variables are the time $t = \tilde{t}\tilde{s}_F^0/\tilde{l}$ and the Cartesian spatial variables $\eta_i = \tilde{\eta}_i/\tilde{l}$ $(i = 1, 2, 3)$, where $\tilde{l}$ represents a characteristic hydrodynamic length scale of the flow. The nabla operator is $\nabla = (\partial/\partial\eta_1, \partial/\partial\eta_2, \partial/\partial\eta_3)$ and the operator $\otimes$ denotes the dyadic product.

The temperature T is assumed to be uniform in the fresh mixture $(T_f = 1)$. In the burned mixture small variations about the adiabatic temperature T_b are observed, resulting in small density variations. However, in Class *et al.* (2003), it was shown that these variations are negligible to the order of accuracy considered in the model. Therefore, density is also constant in the burned mixture. Thus, we take the density $\rho = \rho_f = 1$ in the fresh mixture and $\rho = \rho_b = 1/T_b$ in the burned mixture.

The stress tensor is $\boldsymbol{\sigma} = Pr\,Pe^{-1}\lambda(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T - \frac{2}{3}\boldsymbol{I}\nabla\cdot\boldsymbol{v})$ where the superscript T denotes the transpose and $\boldsymbol{I}$ the identity matrix. The Péclet number $Pe = \tilde{l}/\tilde{l}_0$ is the ratio of the hydrodynamic length scale $\tilde{l}$ to the flame thickness $\tilde{l}_0 = \tilde{\lambda}_f/(\tilde{\rho}_f\tilde{c}_{pf}\tilde{s}_F^0)$, where $\tilde{\lambda}_f$ and $\tilde{c}_{pf}$ are the thermal conductivity and specific heat of the fresh mixture, respectively. The Prandtl number Pr is the ratio of the kinematic viscosity $\tilde{\nu}_f$ to the thermal diffusivity $\tilde{\kappa} = \tilde{\lambda}/(\tilde{\rho}_f\tilde{c}_{pf})$. The non-dimensional thermal conductivity $\lambda = \tilde{\lambda}/\tilde{\lambda}_f$ is unity in the fresh mixture and, from kinetic gas theory, it follows that $\lambda \sim T_b^{1/2}$ in the burned mixture.

(*b*) The fluid fields on the two sides of the flame surface are related by the jump conditions for the normal mass flux, normal momentum flux, and tangential momentum flux, respectively:

$$[m] = 0, \tag{2.3}$$

$$[mv_n + p - \sigma_{nn}] = -\frac{1}{Pe} 2cmI_\sigma + o(Pe^{-1}), \tag{2.4}$$

$$[m\boldsymbol{v}_\perp - \boldsymbol{\sigma}_{n\perp}] = -\frac{1}{Pe} \nabla_\perp (mI_\sigma) + o(Pe^{-1}). \tag{2.5}$$

The normal mass flux across the flame surface is $m = \rho(v_n - u_n)$, where $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ is the normal velocity component with respect to the flame surface, and u_n is the normal propagation speed of the flame surface relative to a fixed frame of reference. The normal vector on the flame surface pointing into the burned product is denoted by $\boldsymbol{n}$. The tangential velocity vector on the flame surface is $\boldsymbol{v}_\perp = \boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n}$, and similarly the tangential derivative is $\nabla_\perp(\,) = \boldsymbol{n} \times \nabla(\,) \times \boldsymbol{n}$. The normal and tangential stresses σ_{nn} and $\sigma_{n\perp}$ at the discontinuity surface are $\sigma_{nn} = \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}$ and $\sigma_{n\perp} = \boldsymbol{n} \times (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \times \boldsymbol{n}$, respectively. The jump condition (2.3) states that normal mass flux is continuous across the flame. According to the jump condition (2.4) normal momentum exhibits an $O(Pe^{-1})$ jump which is proportional to the curvature $c = -(1/2)\nabla \cdot \boldsymbol{n}$ of the flame surface. The term $-Pe^{-1}mI_\sigma$ plays a role similar to surface tension but is here referred to as surface compression since it has the opposite sign. Gradients of surface compression result in a jump of tangential momentum (2.5), which is the analogue of Marangoni forces. The quantity $I_\sigma > 0$, in the reaction sheet approximation, is given by

$$I_\sigma = \tfrac{4}{3}(Pr + 1)\left(T_b^{3/2} - 1\right) - 2(T_b - 1), \tag{2.6}$$

and thus grows with thermal expansion.

(*c*) The propagation of the flame normal to itself is associated with the normal mass flux m through the flame, which is governed by the nonlinear partial differential equation

$$C(I_H(\partial/\partial t + \chi)(1/m) - Pe^{-1} I_\Delta \nabla_\perp^2 (1/m) + Pe^{-1} I_{\nabla^2} m (\nabla_\perp (1/m))^2) \\ + m \ln(m + Pe^{-1}((I_Y - I_X)\chi/m + 2cI_X)) = 0, \tag{2.7}$$

which we refer to as the unified flame speed relation.

Here, $C = Pe^{-1} Ze(1 - Le^{-1})$, where Ze is the Zeldovich number, is a non-dimensional number which measures the combined effect of preferential diffusion and the temperature sensitivity of the flame speed. The flame stretch χ is the relative temporal change in surface area of a flame surface element, where points on the surface move with the local tangential flow speed. Note that the tangential speed is to leading order continuous across the flame, and thus χ may be calculated using either the unburned or the burned flow field. The positive constants I_H, I_Δ, I_{∇^2}, I_Y, and I_X are given by

$$I_H = \frac{1}{1 - Le^{-1}} \int_0^1 ((T_b - 1)\Theta + 1)^{-1/2} (1 - \Theta^{Le-1})\, \mathrm{d}\Theta, \tag{2.8}$$

$$I_\Delta = 1 + Le^{-1} + \frac{Le(3 + Le)}{4(1 + Le)^2}(T_b - 1), \tag{2.9}$$

$$I_{\nabla^2} = \frac{7 + Le(4 + Le)}{8(1 + Le)^3} Le(T_b - 1), \tag{2.10}$$

$$I_Y = \int_0^1 ((T_b - 1)\Theta + 1)^{-1/2}\Theta^{Le-1}\,\mathrm{d}\Theta + 2\big(T_b^{1/2} - 1\big), \tag{2.11}$$

$$I_X = \frac{2T_b}{T_b^{1/2} + 1}. \tag{2.12}$$

The time derivative indicates that the mass flux m needs a definite amount of time to adjust to new conditions. The operator $\nabla_\perp^2(\ldots) = \nabla\cdot(\boldsymbol{n}\times\nabla(\ldots)\times\boldsymbol{n})$ is the surface Laplacian. Thus, neighbouring points on the flame cannot propagate at uncorrelated speeds. A second mechanism coupling neighbouring flame elements is provided by the nonlinear term $(\nabla_\perp(1/m))^2 = (\nabla_\perp(1/m))\cdot(\nabla_\perp(1/m))$, where the operator $\nabla_\perp(\ldots) = (\boldsymbol{n}\times\nabla(\ldots)\times\boldsymbol{n})$ is the surface gradient.

If we ignore the effect of short-wavelength variations, i.e. the terms $\nabla_\perp^2(1/m)$ and $(\nabla_\perp(1/m))^2$, the unified flame speed relation reduces to the flame speed relations previously derived by Sivashinsky (1976) and Matalon & Matkowsky (1982) in the appropriate limits, a fact that is extensively discussed in Class *et al.* (2003).

3. Stability of steadily propagating planar flames

The time-dependent version of the flame speed relation derived in Sivashinsky (1976) for Lewis numbers bounded away from unity was rejected by him due to stability considerations. In particular, the pulsating instability sets in for any Lewis number which exceeds unity. However, from the stability analysis of near-equidiffusional flames in Pelce & Clavin (1982) and Matalon & Matkowsky (1982) it is known that for Lewis numbers which exceed unity by only $O(1/Pe)$ the pulsating instability does not appear. The general flame speed relation (2.7) bridges the results for Lewis numbers close to and bounded away from unity, so that we expect to find that the pulsating instability sets in if the Lewis number exceeds a critical value Le_c slightly greater than unity. Furthermore, our inclusion of short-wave perturbation effects yields a term which, as we will see, is effective in cutting off the high-wavenumber instability. Specifically, we will show that planar flames are stable with respect to high-wavenumber (short-wavelength) perturbations for all parameters. In addition, in agreement with the results of diffusional–thermal theory, we will show that the uniformly propagating planar flame exhibits no instability other than the long-wave Darrieus–Landau instability in a band of Lewis numbers about $Le = 1$.

Consider a uniformly propagating planar flame. In the moving Cartesian coordinate system (y^1, y^2, y^3) attached to the flame, the flame surface appears to be at rest at $y^1 = 0$. In the fresh mixture we have unity density ρ and conductivity λ while in the burned mixture the density $\rho = \rho_b$ and conductivity $\lambda = \lambda_b$ take constant values which in general differ from the fresh values. The planar basic state is given by

$$m = v_f^1 = 1, \qquad v_b^1 = \frac{1}{\rho_b}, \qquad p_b = 1 - \frac{1}{\rho_b}, \qquad v_f^\alpha = v_b^\alpha = p_f = 0. \tag{3.1}$$

Now consider perturbations of the basic state so that the perturbed flame is located at $y^1 = \varepsilon\psi(y^2, y^3, t)$, where ε is a small parameter, much smaller than any other parameter of the problem. As a consequence of the perturbation a perturbed flow is induced ahead of and behind the flame:

$$m = 1 + \varepsilon m', \qquad v_f^1 = 1 + \varepsilon v_f^{1\prime}, \qquad v_b^1 = \frac{1}{\rho_b} + \varepsilon v_b^{1\prime}, \tag{3.2}$$

$$v_f^\alpha = \varepsilon v_f^{\alpha\prime}, \qquad v_b^\alpha = \varepsilon v_b^{\alpha\prime}, \qquad p_f = \varepsilon p_f', \qquad p_b = 1 - \frac{1}{\rho_b} + \varepsilon p_b' \tag{3.3}$$

where primes indicate perturbed quantities governed by the linearized fluid equations, boundary and jump conditions and the flame speed relation. Though the perturbed flame is slightly distorted from the origin $y^1 = 0$, we apply the jump conditions at $y^1 = 0$, by using a Taylor series expansion of all quantities about $y^1 = 0$. In the stability analysis we treat all quantities other than ε as $O(1)$.

We seek solutions of the linearized equations

$$\frac{\partial v^{i\prime}}{\partial y^i} = 0, \tag{3.4}$$

$$\rho\frac{\partial v^{1\prime}}{\partial t} + \frac{\partial v^{j\prime}}{\partial y^j} + \frac{\partial p'}{\partial y^1} - \frac{Pr}{Pe}\lambda\frac{\partial^2 v^{1\prime}}{(\partial y^j)^2} = 0, \tag{3.5}$$

$$\rho\frac{\partial v^{\alpha\prime}}{\partial t} + \frac{\partial p'}{\partial y^\alpha} - \frac{Pr}{Pe}\lambda\frac{\partial^2 v^{\alpha\prime}}{(\partial y^j)^2} = 0, \qquad \alpha = 2, 3, \tag{3.6}$$

subject to homogeneous boundary conditions as $y^1 \to \pm\infty$ in the form

$$\psi = a^\psi \exp(\mathrm{i}k_2 y^2 + \mathrm{i}k_3 y^3 + \omega t), \tag{3.7}$$

$$m' = a^m \exp(\mathrm{i}k_2 y^2 + \mathrm{i}k_3 y^3 + \omega t), \tag{3.8}$$

$$v^{1\prime} = a^1(y^1)\exp(\mathrm{i}k_2 y^2 + \mathrm{i}k_3 y^3 + \omega t), \tag{3.9}$$

$$v^{\alpha\prime} = a^\alpha(y^1)\exp(\mathrm{i}k_2 y^2 + \mathrm{i}k_3 y^3 + \omega t), \tag{3.10}$$

$$p' = a^p(y^1)\exp(\mathrm{i}k_2 y^2 + \mathrm{i}k_3 y^3 + \omega t), \tag{3.11}$$

where the terms $a^1(y^1)$, $a^\alpha(y^1)$, and $a^p(y^1)$ in the fresh and burned mixtures are related by the linearized jump conditions. Here, k_2 and k_3, with $k = \sqrt{k_2^2 + k_3^2}$, are the wavenumbers in the y^2- and y^3-directions, respectively, and ω is the growth rate of the perturbations. Instability of the planar solution corresponds to $Re\omega > 0$.

The solvability condition for the resulting system of equations yields the dispersion relation for the eigenvalue $\omega = \omega(k)$. The form of the dispersion relation depends on the relative sizes of the parameters. Each of the cases is considered separately in the sections that follow. Note that the effect of transverse diffusion, which stabilizes short waves, appears only in § 3.4 where the unified model is discussed.

3.1. *The Darrieus–Landau model*

If $C = 0$ and $O(Pe^{-1})$ terms are neglected, the dispersion relation reduces to that of Darrieus and Landau:

$$k^2(\rho_b - 1) + 2k\rho_b\omega + \rho_b(1 + \rho_b)\omega^2 = 0. \tag{3.12}$$

Figure 1 shows the stability diagram for the Darrieus–Landau instability, and plots the growth rate versus the wavenumber k of the perturbation. There are two real eigenvalues for any wavenumber k. For planar perturbations ($k = 0$) we have $\omega = 0$. Otherwise, one eigenvalue ω is positive and the other is negative. Positive ω corresponds to instability. We see that ω grows with increasing k, i.e. with decreasing wavelength. Thus, a planar flame is unstable with respect to any perturbation, no matter how short in wavelength it is. This result is in contradiction to experimental observations. We expect a cutoff of short-wavelength wrinkles along the flame, and indeed this will be shown to be the case in § 3.4. below.

3.2. *The Matalon–Matkowsky model* ($C = O(Pe^{-1})$)

If $C = O(Pe^{-1})$, i.e. $PeC = O(1)$, retaining the first two terms in an asymptotic expansion in powers of Pe^{-1} yields the Matalon & Matkowsky (1982) dispersion

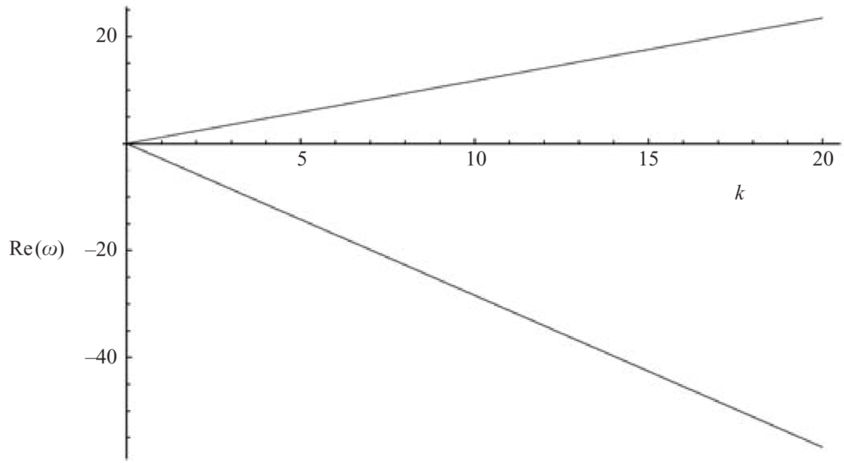

FIGURE 1. Growth rate ω versus wavenumber for the Darrieus–Landau model ($T_b = 5$).

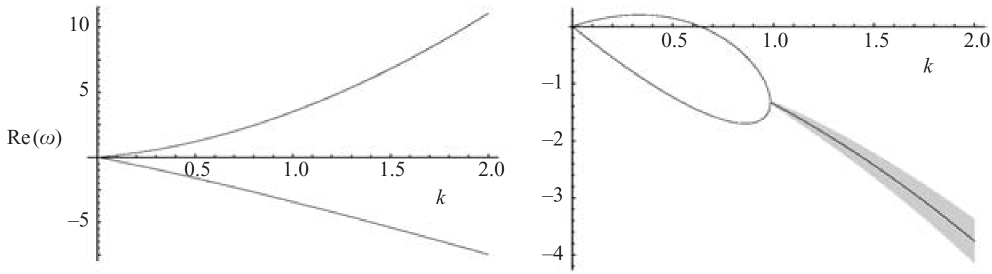

FIGURE 2. Stability diagram for the Matalon-Matkowsky model. Left: case (*b*), destabilizing trend beyond the Darrieus–Landau instability ($Le = 0.5$, $Ze = 8$, $Pe = 100$, $Pr = 0.7$, $T_b = 5$). Right: case (*c*), high-wavenumber cutoff of Darrieus–Landau instability and stable pulsating solution. The frequency Im(ω) of the pulsations is proportional to the width of the shaded region ($Le = 0.62$, $Ze = 8$, $Pe = 100$, $Pr = 0.7$, $T_b = 5$).

relation:

$$k^2(\rho_b - 1) + 2k\rho_b\omega + \rho_b(1+\rho_b)\omega^2 - 2k^2(PeCI_H + I_Y - I_X(1+\rho_b))(k+\rho_b\omega) = 0 \quad (3.13)$$

Figure 2 shows the corresponding stability diagrams for near-equidiffusional flames; the width of the shaded region corresponds to the frequency of the oscillating solutions. As for the Darrieus–Landau instability, there are two real eigenvalues for any wavenumber k. We present results for three cases.

(*a*) If $PeCI_H + I_Y - I_X(1+\rho_b) = 0$, we recover the Darrieus–Landau result. This case corresponds to Lewis numbers slightly below unity.

(*b*) If $PeCI_H + I_Y - I_X(1+\rho_b) > 0$, the instability is enhanced, and we observe no cutoff of high-wavenumber perturbations. This corresponds to Lewis numbers below a critical Lewis number smaller than unity.

(*c*) If $PeCI_H + I_Y - I_X(1+\rho_b) < 0$, the instability is inhibited, i.e. we observe a stabilizing trend. The result represents a perturbative correction to the Darrieus–Landau result. However, for large k the growth rate ω becomes negative and we observe the cutoff of high-wavenumber perturbations. For yet larger k a stable pulsating eigenvalue is found. The pulsating instability which results from the time

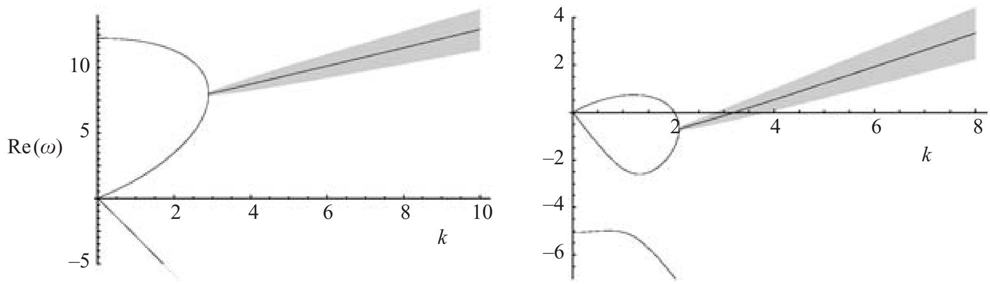

FIGURE 3. Stability diagram for the Sivashinsky model. The frequency Im(ω) of the pulsations is proportional to the width of the shaded regions. Left: case (b), destabilizing trend beyond the Darrieus–Landau and pulsating instability for high wavenumbers ($Le=0.3$, $Ze=8$, $Pe=100$, $Pr=0.7$, $T_b=5$). Right: case (c), high-wavenumber cutoff of Darrieus–Landau instability and pulsating instability for high wavenumbers ($Le=4$, $Ze=8$, $Pe=10$, $Pr=0.7$, $T_b=5$).

derivative in the flame speed relation is not captured by the Matalon–Matkowsky model.

3.3. *The Sivashinsky model* ($C = O(1)$)

If $C = O(1)$ and $O(Pe^{-1})$ terms are neglected, we obtain the dispersion relation of Sivashinsky (1976):

$$(1 - CI_H\omega)(k^2(\rho_b - 1) + 2k\rho_b\omega + \rho_b(1 + \rho_b)\omega^2) - 2k^2CI_H(k + \rho_b\omega) = 0. \qquad (3.14)$$

Figure 3 shows the corresponding stability diagrams. We present results for three cases.

(a) If $C=0$, i.e. if $Le=1$, we recover the Darrieus–Landau dispersion relation.

(b) If $C<0$, corresponding to Lewis numbers below unity, we observe a destabilizing trend so that we obtain instability for any wavenumber. For long-wave perturbations we obtain a positive real eigenvalue, corresponding to a cellular instability of the flame. For large wavenumbers k we find a pair of complex-conjugate eigenvalues with positive growth rate, corresponding to pulsating flames.

(c) $C>0$ corresponds to Lewis numbers above unity. We observe a trend toward stabilization of the Darrieus–Landau instability. For an $O(1)$ wavenumber there is a cutoff of the instability, i.e. ω becomes negative on the upper branch. For yet higher wavenumbers, the curves for the two real eigenvalues emerging from the origin merge to become a pair of complex-conjugate eigenvalues.

The growth rate ω for the complex eigenvalues increases with growing k, so that beyond a critical wavenumber k_{cr} the amplification rate is positive, with no high-wavenumber cutoff. With growing wavenumbers k the frequency of the eigenvalue grows rapidly. The solutions for high wavenumbers correspond to an unstable pulsating flame, which oscillates at high frequency.

The behaviour for low wavenumbers is in agreement with physical observations, i.e. there is a cutoff of the Darrieus–Landau instability. The behaviour for high wavenumbers is non-physical, as we expect to find a cutoff of the pulsating instability as well. Indeed, this is the case, as will be seen in § 3.4. below.

3.4. *The unified model*

Employing the unified flame speed relation (2.7), and the jump conditions (2.3)–(2.5), and treating all the parameters as $O(1)$ quantities yields the dispersion relation of the

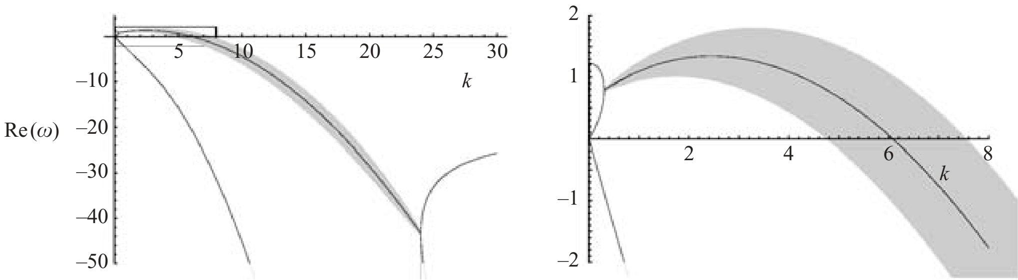

FIGURE 4. Stability diagram for the unified model for case (*a*), i.e. small *Le*. The figure on the right is a blowup of the boxed region in the figure on the left. The figure exhibits the cutoff of the pulsating instability for large wavenumbers ($Le=0.3$, $Ze=8$, $Pe=10$, $Pr=0.7$, $T_b=5$).

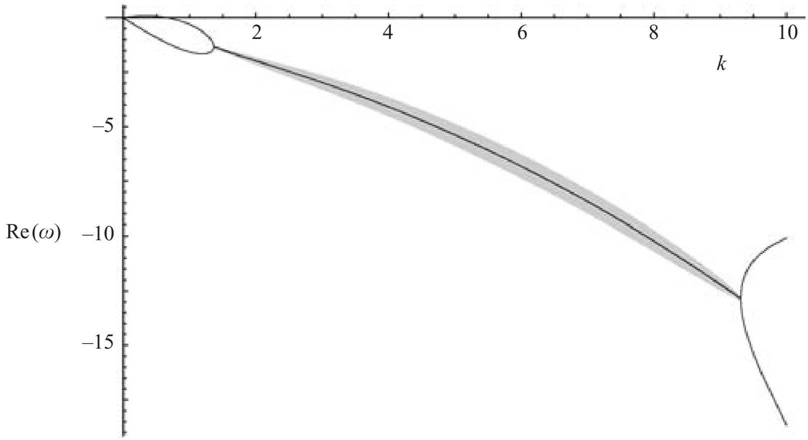

FIGURE 5. Stability diagram for the unified model for case (*b*), i.e. intermediate *Le*. The Darrieus–Landau instability and its cutoff is observed. The pulsating solution branch is stable ($Le=2$, $Ze=8$, $Pe=10$, $Pr=0.7$, $T_b=2.5$).

present model:

$$k^2\left(1+2\lambda_b k\frac{Pr}{Pe}-\sqrt{1+4\lambda_b\frac{Pr}{Pe}\left(\lambda_b k^2\frac{Pr}{Pe}+\rho_b\omega\right)}\right)D_1-(k-\rho_b\omega)D_2=0, \qquad (3.15)$$

where

$$\begin{aligned}D_1 &= k^2\left(CI_H+\frac{I_X-I_Y+I_X\rho_b}{Pe}\right)\left(1+\rho_b+\frac{k}{Pe}(2\lambda_b Pr-(I_\sigma+2Pr)\rho_b)\right)\\ &\quad+\left(1+C\left(I_H\omega+\frac{I_\Delta}{Pe}k^2\right)\right)\left(2\rho_b\omega+k\left(\rho_b-1+2(\lambda_b-1)\frac{Pr}{Pe}\rho_b\omega\right)\right), \qquad (3.16)\end{aligned}$$

$$\begin{aligned}D_2 &= k^2\left(CI_H+\frac{I_X-I_Y+I_X\rho_b}{Pe}\right)\left((2(\rho_b\omega+k)+\frac{k^2}{Pe}(4Pr\lambda_b-I_\sigma\rho_b)\right)\\ &\quad+\left(1+C\left(I_H\omega+\frac{I_\Delta}{Pe}k^2\right)\right)\left(2k\rho_b\omega+\rho_b(1+\rho_b)\omega^2+k^2\left(\rho_b-1+4\lambda_b\frac{Pr}{Pe}\rho_b\omega\right)\right). \qquad (3.17)\end{aligned}$$

Figures 4–6 show the corresponding stability diagram for three cases.

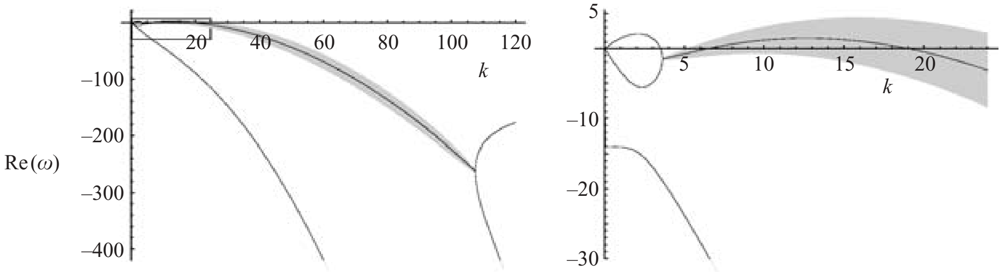

FIGURE 6. Stability diagram for the unified model for case (*c*), i.e. large *Le*. The figure on the right is a blowup of the boxed region in the figure on the left. The figure exhibits the cutoff of the Darrieus–Landau instability and of the pulsating instability ($Le = 4$, $Ze = 35$, $Pe = 100$, $Pr = 0.7$, $T_b = 9$).

(*a*) If $C < C_{cr} < 0$, we find a destabilizing trend beyond the Darrieus–Landau instability. Furthermore, the flame is unstable with respect to a second instability for long-wave perturbations including planar perturbations. This branch merges with the Darrieus–Landau branch and an unstable oscillatory branch emerges. For high wavenumbers this branch becomes stable. For yet higher wavenumbers the oscillatory branch, corresponding to a pair of complex-conjugate eigenvalues, becomes real and splits into two stationary branches, which are both stable. The pulsating instability disappears due to the presence of the transverse diffusion term, which forces neighbouring flame elements to propagate at similar flame speeds and damps pulsations. If the transverse diffusion term were not present, then the oscillatory branch would remain unstable for high wavenumbers, with no cutoff, and the two stationary high-wavenumber branches would not be present.

(*b*) If $C_{cr} < C < C_o$ where $C_o > 0$, we find a stabilization of the Darrieus–Landau instability. Furthermore, there is an additional real branch which is stable for any wavenumber. There is a high-wavenumber cutoff of the Darrieus–Landau instability. For yet higher wavenumbers the stable and unstable Darrieus–Landau instability branches merge and a stable branch corresponding to a pair of complex conjugate eigenvalues emerges. For yet higher wavenumbers the stable oscillatory branch splits into two stationary branches which are both stable. If the transverse diffusion term were omitted, then the oscillatory branch would approach a constant negative value ω for $k \to \infty$. The two stationary branches for high wavenumbers again would not exist. In addition, C_o would become smaller.

(*c*) If $C > C_o > 0$, we find behaviour similar to the case $C_{cr} < C < C_o$. However, the growth rate of the oscillatory branch now becomes positive in a window of wavenumbers k, i.e. an oscillatory instability is observed. If the transverse diffusion term were omitted the cutoff of the high-wavenumber oscillatory branch would no longer exist, i.e. short-wave perturbations with $k \to \infty$ would be unstable, which is not acceptable.

4. Conclusions

The unified model of premixed flames as gasdynamic discontinuities of Class *et al.* (2003) contains elements found in all previous models, i.e. the dependence on flame stretch and curvature suggested earlier in Markstein (1951), Karlovitz *et al.* (1953), Eckhaus (1961) and Markstein (1964), the analogue of the transverse diffusion terms which in the diffusional thermal theory of Barenblatt *et al.* (1962) stabilize short-wave perturbations, the time derivative and the nonlinearity derived by Sivashinsky (1976), a nonlinear coupling of neighbouring flame elements similar to that in the Kuramoto–Sivashinsky theory of Sivashinsky (1980), and a generalization of the fluid flow jump conditions derived in Matalon & Matkowsky (1982).

In the present paper the stability of a uniformly propagating planar flame as a solution of the unified model of flames as gasdynamic discontinuities was considered and compared to stability results based on previous models of premixed flame propagation which account for thermal expansion.

The Darrieus–Landau instability, due to thermal expansion, is captured by all models which include thermal expansion, i.e. the models of Darrieus–Landau, Sivashinsky, Matalon & Matkowsky and the unified model. The high-wavenumber cutoff of the Darrieus–Landau instability is captured by the unified model for all parameters, by the Sivashinsky model if $Le > 1$ and by the Matalon–Matkowsky model if $Le > Le_c$ ($Le_c < 1$).

The cellular instability due to preferential diffusion is captured by the models of Sivashinsky, Matalon & Matkowsky, the diffusional thermal model and the unified model. The high-wavenumber cutoff of the cellular instability is captured by the diffusional thermal model and the unified model.

The pulsating instability due to preferential diffusion is captured by the Sivashinsky, diffusional thermal and unified models. The high-wavenumber cutoff of the pulsating instability is captured by the diffusional thermal and unified models.

The Sivashinsky model and the unified model capture all three instabilities. However, only the unified model exhibits the cutoff of all instabilities. In contrast to previous theories, for the unified model in Class *et al.* (2003) it is not necessary to restrict the range of parameters in order to exhibit the high-wavenumber cutoff.

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