
Expansion in n^{-1} for Percolation Critical Values on the n -cube and \mathbb{Z}^n : the First Three Terms

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Let $p_c(\mathbb{Q}_n)$ and $p_c(\mathbb{Z}^n)$ denote the critical values for nearest-neighbour bond percolation on the n -cube $\mathbb{Q}_n = \{0, 1\}^n$ and on \mathbb{Z}^n , respectively. Let $\Omega = n$ for $\mathbb{G} = \mathbb{Q}_n$ and $\Omega = 2n$ for $\mathbb{G} = \mathbb{Z}^n$ denote the degree of \mathbb{G} . We use the lace expansion to prove that for both $\mathbb{G} = \mathbb{Q}_n$ and $\mathbb{G} = \mathbb{Z}^n$,

$$p_c(\mathbb{G}) = \Omega^{-1} + \Omega^{-2} + \frac{7}{2}\Omega^{-3} + O(\Omega^{-4}).$$

This extends by two terms the result $p_c(\mathbb{Q}_n) = \Omega^{-1} + O(\Omega^{-2})$ of Borgs, Chayes, van der Hofstad, Slade and Spencer, and provides a simplified proof of a previous result of Hara and Slade for \mathbb{Z}^n .

1. Main result

We consider bond percolation on \mathbb{Z}^n with edge set consisting of pairs $\{x, y\}$ of vertices in \mathbb{Z}^n with $\|x - y\|_1 = 1$, where $\|w\|_1 = \sum_{j=1}^n |w_j|$ for $w \in \mathbb{Z}^n$. Bonds (edges) are independently occupied with probability p and vacant with probability $1 - p$. We also consider bond percolation on the n -cube \mathbb{Q}_n , which has vertex set $\{0, 1\}^n$ and edge set consisting of pairs $\{x, y\}$ of vertices in $\{0, 1\}^n$ with $\|x - y\|_1 = 1$, where we regard \mathbb{Q}_n as an additive group with addition component-wise modulo 2. Again bonds are independently occupied with probability p and vacant with probability $1 - p$. We write \mathbb{G} in place of \mathbb{Q}_n and \mathbb{Z}^n when we wish to refer to both models simultaneously. We write Ω for the degree of \mathbb{G} , so that $\Omega = 2n$ for \mathbb{Z}^n and $\Omega = n$ for \mathbb{Q}_n .

Given a vertex x of \mathbb{G} , let $C(x)$ denote the connected component of x , *i.e.*, the set of vertices y such that y is connected to x by a path consisting of occupied bonds. For the case of \mathbb{Z}^n , the critical value is given by

$$p_c(\mathbb{Z}^n) = \inf\{p : \exists \text{ an infinite connected component a.s.}\}. \quad (1.1)$$

Let $|C(x)|$ denote the cardinality of $C(x)$, and let $\chi(p) = \mathbb{E}_p|C(0)|$ denote the expected size of the component containing the origin. Results of [1, 20] imply that

$$p_c(\mathbb{Z}^n) = \sup\{p : \chi(p) < \infty\} \tag{1.2}$$

is an equivalent definition of the critical value.

For percolation on a finite graph \mathbb{G} , such as \mathbb{Q}_n , the above characterizations of $p_c(\mathbb{G})$ are inapplicable: there can be no infinite component and $\chi(p)$ is at finite for all p . In [8, 9, 10] (in particular, see [10]), it was shown that there is a small positive constant λ_0 such that the critical value $p_c(\mathbb{Q}_n) = p_c(\mathbb{Q}_n; \lambda_0)$ for the n -cube is defined implicitly by

$$\chi(p_c(\mathbb{Q}_n)) = \lambda_0 2^{n/3}. \tag{1.3}$$

Given λ_0 , (1.3) uniquely specifies $p_c(\mathbb{Q}_n)$, since $\chi(p)$ is a polynomial in p that increases from $\chi(0) = 1$ to $\chi(1) = 2^n$.

Our main result is the following theorem.

Theorem 1.1.

(i) For $\mathbb{G} = \mathbb{Z}^n$,

$$p_c(\mathbb{Z}^n) = \frac{1}{2n} + \frac{1}{(2n)^2} + \frac{7}{2} \frac{1}{(2n)^3} + O\left(\frac{1}{(2n)^4}\right) \quad \text{as } n \rightarrow \infty. \tag{1.4}$$

(ii) For \mathbb{Q}_n , fix constants c, c' independent of n , and choose p such that $\chi(p) \in [cn^3, c'n^{-6}2^n]$ (e.g., $p = p_c(\mathbb{Q}_n; \lambda_0)$ for large n). Then

$$p = \frac{1}{n} + \frac{1}{n^2} + \frac{7}{2} \frac{1}{n^3} + O\left(\frac{1}{n^4}\right) \quad \text{as } n \rightarrow \infty. \tag{1.5}$$

The constant in the error term depends on c, c' , but does not depend otherwise on p .

By Theorem 1.1, the expansions of $p_c(\mathbb{G})$ in powers of Ω^{-1} are the same for \mathbb{Q}_n and \mathbb{Z}^n , up to and including order Ω^{-3} . Higher-order coefficients could be computed using our methods, but the labour cost increases sharply with each subsequent term. Although we stop short of computing the coefficient of Ω^{-4} , we expect that the coefficients for \mathbb{Q}_n and \mathbb{Z}^n will differ at this order. In [18], for both \mathbb{Q}_n and \mathbb{Z}^n , we prove the existence of asymptotic expansions for $p_c(\mathbb{G})$ to all orders in Ω^{-1} , without computing the numerical values of the coefficients.

For \mathbb{Q}_n , it was shown by Ajtai, Komlós and Szemerédi [3] that $p_c(\mathbb{Q}_n) > n^{-1}(1 + \epsilon)$ for every fixed $\epsilon > 0$ (although the above definition of $p_c(\mathbb{Q}_n)$ did not appear until [8]). Bollobás, Kohayakawa and Łuczak [7] improved this to $p_c(\mathbb{Q}_n) \in [\frac{1-e^{-cn}}{n-1}, \frac{1}{n} + 60\frac{(\log n)^2}{n^2}]$. Theorem 1.1 extends the very recent result $p_c(\mathbb{Q}_n) = n^{-1} + O(n^{-2})$ of [8, 9] by two terms. Bollobás, Kohayakawa and Łuczak [7] raised the question of whether the critical value might be equal to $\frac{1}{n-1}$, but we see from (1.5) that $p_c(\mathbb{Q}_n) = \frac{1}{n-1} + \frac{5}{2}n^{-3} + O(n^{-4})$.

For \mathbb{Z}^n , Theorem 1.1 is identical to a result of Hara and Slade [16, 17]. Earlier, Bollobás and Kohayakawa [6], Gordon [13], Kesten [19] and Hara and Slade [15] obtained the first term in (1.4) for \mathbb{Z}^n with error terms $O((\log n)^2 n^{-2})$, $O(n^{-65/64})$, $O((\log \log n)^2 (n \log n)^{-1})$ and $O(n^{-2})$, respectively. Recently, Alon, Benjamini and Stacey [4] gave an alternate proof

that $p_c(\mathbb{Z}^n)$ is asymptotic to $(2n)^{-1}$ as $n \rightarrow \infty$. The expansion

$$p_c(\mathbb{Z}^n) = \frac{1}{2n} + \frac{1}{(2n)^2} + \frac{7}{2(2n)^3} + \frac{16}{(2n)^4} + \frac{103}{(2n)^5} + \dots \tag{1.6}$$

was reported in [12], but with no rigorous bound on the remainder.

We remark that for *oriented* percolation on \mathbb{Z}^n , defined in such a way that the forward degree is n , it was proved in [11] that the critical value obeys the bounds

$$\frac{1}{n} + \frac{1}{2n^3} + o\left(\frac{1}{n^3}\right) \leq p_c(\text{oriented } \mathbb{Z}^n) \leq \frac{1}{n} + \frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{1.7}$$

Our method is based on the lace expansion and applies the general approach of [16, 17] that was used to prove Theorem 1.1(i) for \mathbb{Z}^n , but our method here is simpler and applies to \mathbb{Z}^n and \mathbb{Q}_n simultaneously.

Remark. For \mathbb{Q}_n , it is a direct consequence of [18, Proposition 1.2] that if there is some sequence p (depending on n) with $\chi(p) \in [cn^3, c'n^{-6}2^n]$ such that $p = n^{-1} + n^{-2} + \frac{7}{2}n^{-3} + O(n^{-4})$, then the same asymptotic formula holds for *all* such p . Thus it suffices to prove (1.5) for a single such sequence p . We fix some sequence f_n such that $\lim_{n \rightarrow \infty} f_n n^{-M} = \infty$ for every positive integer M and such that $\lim_{n \rightarrow \infty} f_n e^{-2n} = 0$ for every $\alpha > 0$. We define \bar{p} by $\chi(\bar{p}) = f_n$, and observe that eventually $\chi(\bar{p}) \in [cn^3, c'n^{-6}2^n]$. For $\mathbb{G} = \mathbb{Q}_n$, it therefore suffices to prove that \bar{p} has the expansion (1.5). We will use the notation

$$\bar{p}_c = \bar{p}_c(\mathbb{G}) = \begin{cases} \bar{p} & (\mathbb{G} = \mathbb{Q}_n), \\ p_c(\mathbb{Z}^n) & (\mathbb{G} = \mathbb{Z}^n). \end{cases} \tag{1.8}$$

2. Application of the lace expansion

For \mathbb{Q}_n or \mathbb{Z}^n with n large, the lace expansion [15] gives rise to an identity

$$\chi(p) = \frac{1 + \hat{\Pi}_p}{1 - \Omega p [1 + \hat{\Pi}_p]}, \tag{2.1}$$

where $\hat{\Pi}_p$ is a function that is finite for $p \leq p_c(\mathbb{G})$. Although we do not display the dependence explicitly in the notation, $\hat{\Pi}_p$ does depend on the graph \mathbb{Q}_n or \mathbb{Z}^n . The identity (2.1) is valid for $p \leq p_c(\mathbb{G})$. For a derivation of the lace expansion, see, e.g., [9, Section 3]. It follows from (2.1) that

$$\Omega p = \frac{1}{1 + \hat{\Pi}_p} - \chi(p)^{-1}. \tag{2.2}$$

The function $\hat{\Pi}_p$ has the form

$$\hat{\Pi}_p = \sum_{N=0}^{\infty} (-1)^N \hat{\Pi}_p^{(N)}, \tag{2.3}$$

with

$$|\hat{\Pi}_p^{(N)}| \leq \left(\frac{C}{\Omega}\right)^{N \vee 1} \text{ uniformly in } p \leq \bar{p}_c. \tag{2.4}$$

The bounds (2.4) have different proofs for \mathbb{Q}_n and \mathbb{Z}^n , and we first discuss \mathbb{Q}_n .

For \mathbb{Q}_n , the formula (2.1) and the bounds (2.4) are given in [9, (6.1)] and [9, Lemma 5.4], respectively (with our $\hat{\Pi}_p$ written as $\hat{\Pi}_p(0)$). In more detail, we define the parameter λ in [9, Lemma 5.4] by $\lambda = f_n 2^{-n/3}$, which is exponentially small in n by definition of f_n . Then [9, Lemma 5.4] states that $\hat{\Pi}_p^{(N)} \leq [\text{const}(\lambda^3 \vee \beta)]^{N \vee 1}$ for $p \leq \bar{p}_c(\mathbb{Q}_n)$, where β can be chosen proportional to n^{-1} by [9, Proposition 2.1]. (Note that the hypothesis $f(p) \leq K$ in [9, Lemma 5.4] is proved in [9, Section 5] for $p \leq p_c(\mathbb{Q}_n)$ with $K = 3$.) It follows from (2.2) that

$$n\bar{p}_c(\mathbb{Q}_n) = \frac{1}{1 + \hat{\Pi}_{\bar{p}_c(\mathbb{Q}_n)}} + O(f_n^{-1}). \tag{2.5}$$

The second term on the right-hand side of (2.5) can be neglected in the proof of Theorem 1.1. Equations (2.3)–(2.5) give $\bar{p}_c(\mathbb{Q}_n) = n^{-1} + O(n^{-2})$.

For \mathbb{Z}^n , (2.1) and (2.4) follow from results in [15, Section 4.3.2]. (Note the notational difference that, in [15], our $\hat{\Pi}_p^{(N)}$ is called $\hat{g}_N(0)$ and that $\hat{\Pi}_p^{(N)}$ in [15] is something different.) Since $\chi(p_c(\mathbb{Z}^n)) = \infty$, it follows from (2.2) that

$$2np_c(\mathbb{Z}^n) = \frac{1}{1 + \hat{\Pi}_{p_c(\mathbb{Z}^n)}}. \tag{2.6}$$

With (2.3)–(2.4), this implies that $p_c(\mathbb{Z}^n) = (2n)^{-1} + O(n^{-2})$.

The identities (2.5) and (2.6) give implicit equations for \bar{p}_c which we analyse recursively. To prove Theorem 1.1 using these equations, we will apply the following proposition. In its statement, we write

$$\Omega' = \begin{cases} n - 1 & \text{for } \mathbb{Q}_n, \\ 2n - 2 & \text{for } \mathbb{Z}^n. \end{cases} \tag{2.7}$$

Proposition 2.1. *For $\mathbb{G} = \mathbb{Z}^n$ and $\mathbb{G} = \mathbb{Q}_n$, uniformly in $p \leq \bar{p}_c(\mathbb{G})$,*

$$\hat{\Pi}_p^{(0)} = \frac{3}{2}\Omega\Omega'p^4 + O(\Omega^{-3}), \tag{2.8}$$

$$\hat{\Pi}_p^{(1)} = \Omega p^2 + 4\Omega\Omega'p^4 + O(\Omega^{-3}), \tag{2.9}$$

$$\hat{\Pi}_p^{(2)} = \Omega p^3 + \Omega(\Omega - 1)p^4 + O(\Omega^{-3}), \tag{2.10}$$

$$\sum_{N=3}^{\infty} \hat{\Pi}_p^{(N)} = O(\Omega^{-3}). \tag{2.11}$$

□

We show now that Proposition 2.1 implies Theorem 1.1. It follows from $\Omega\bar{p}_c(\mathbb{G}) = 1 + O(\Omega^{-1})$ (as noted below (2.5) and (2.6)), (2.3), and Proposition 2.1 that

$$\hat{\Pi}_{\bar{p}_c(\mathbb{G})} = -\frac{1}{\Omega} + O(\Omega^{-2}). \tag{2.12}$$

With (2.5)–(2.6), this implies that

$$\Omega\bar{p}_c(\mathbb{G}) = 1 + \frac{1}{\Omega} + O(\Omega^{-2}). \tag{2.13}$$

Using this in the bounds of Proposition 2.1, along with (2.3), gives

$$\begin{aligned} \hat{\Pi}_{\bar{p}_c(\mathbb{G})} &= \frac{3}{2\Omega^2} - \Omega \left(\frac{1}{\Omega} + \frac{1}{\Omega^2} \right)^2 - \frac{4}{\Omega^2} + \frac{1}{\Omega^2} + \frac{1}{\Omega^2} + O(\Omega^{-3}) \\ &= -\frac{1}{\Omega} - \frac{5}{2\Omega^2} + O(\Omega^{-3}). \end{aligned} \tag{2.14}$$

Substitution of this improvement of (2.12) into (2.5)–(2.6) then gives

$$\Omega \bar{p}_c(\mathbb{G}) = 1 + \frac{1}{\Omega} + \frac{7}{2\Omega^2} + O(\Omega^{-3}). \tag{2.15}$$

Thus, to prove Theorem 1.1 it suffices to prove Proposition 2.1. Since (2.11) is a consequence of (2.4), we must prove (2.8)–(2.10). Precise definitions of $\hat{\Pi}_p^{(N)}$, for $N = 0, 1, 2$, will be given in Section 4.

3. Preliminaries

Before proving Proposition 2.1, we recall and extend some estimates from [9, 15].

Let $D(x) = \Omega^{-1}$ if x is adjacent to 0, and $D(x) = 0$ otherwise. Thus $D(y - x)$ is the transition probability for simple random walk on \mathbb{G} to make a step from x to y . Let $\tau_p(y - x) = \mathbb{P}_p(x \leftrightarrow y)$ denote the two-point function. For $i \geq 0$, we denote by

$$\{x \overset{i}{\leftrightarrow} y\} \tag{3.1}$$

the event that x is connected to y by an occupied (self-avoiding) path of length at least i , and define

$$\tau_p^{(i)}(x, y) = \mathbb{P}(x \overset{i}{\leftrightarrow} y). \tag{3.2}$$

We define the Fourier transform of an absolutely summable function f on the vertex set \mathbb{V} of \mathbb{G} by

$$\hat{f}(k) = \sum_{x \in \mathbb{V}} f(x) e^{ik \cdot x} \quad (k \in \mathbb{V}^*), \tag{3.3}$$

where $\mathbb{V}^* = \{0, \pi\}^n$ for \mathbb{Q}_n and $\mathbb{V}^* = [-\pi, \pi]^n$ for \mathbb{Z}^n . We write the inverse Fourier transform as

$$f(x) = \int \hat{f}(k) e^{-ik \cdot x}, \tag{3.4}$$

where we use the convenient notation

$$\hat{g}(k) = \begin{cases} 2^{-n} \sum_{k \in \{0, \pi\}^n} \hat{g}(k) & (\mathbb{G} = \mathbb{Q}_n), \\ \int_{[-\pi, \pi]^n} \hat{g}(k) \frac{d^n k}{(2\pi)^n} & (\mathbb{G} = \mathbb{Z}^n). \end{cases} \tag{3.5}$$

Let

$$(f * g)(x) = \sum_{y \in \mathbb{V}} f(y) g(x - y) \tag{3.6}$$

denote convolution, and let f^{*i} denote the convolution of i factors of f .

Recall from [2] that $\hat{\tau}_p(k) \geq 0$ for all k . For i, j nonnegative integers, let

$$T_p^{(i,j)} = \int |\hat{D}(k)|^i \hat{\tau}_p(k)^j, \tag{3.7}$$

$$T_p = \sup_x (p\Omega)(D * \tau_p^{*3})(x). \tag{3.8}$$

We will use the following lemma, which provides minor extensions of results of [9, 15]. The lemma will also be useful in [18].

Lemma 3.1. *For $\mathbb{G} = \mathbb{Z}^n$ and $\mathbb{G} = \mathbb{Q}_n$, there are constants $K_{i,j}$ and K (independent of Ω) such that, for all $p \leq \bar{p}_c(\mathbb{G})$,*

$$T_p^{(i,j)} \leq K_{i,j} \Omega^{-i/2} \quad (i, j \geq 0), \tag{3.9}$$

$$T_p \leq K \Omega^{-1}, \tag{3.10}$$

$$\sup_x \tau_p^{(i)}(x) \leq \begin{cases} K \Omega^{-1} & (i = 1), \\ 2^i K_{i,1} \Omega^{-i/2} & (i \geq 2). \end{cases} \tag{3.11}$$

The above bounds are valid for $n \geq 1$ for \mathbb{Q}_n , and for n larger than an absolute constant for \mathbb{Z}^n , except that (3.9) also requires $n \geq 2j + 1$ for \mathbb{Z}^n .

Proof. We prove the bounds (3.9)–(3.11) in sequence.

Proof of (3.9). We first prove that for \mathbb{Z}^n and \mathbb{Q}_n , and for positive integers i , there is a positive a_i such that

$$\int \hat{D}(k)^{2i} \leq \frac{a_i}{\Omega^i}. \tag{3.12}$$

The left-hand side is equal to the probability that a random walk that starts at the origin returns to the origin after $2i$ steps, and is therefore equal to Ω^{-2i} times the number of walks that make the transition from 0 to 0 in $2i$ steps. Each such walk must take an even number of steps in each coordinate direction, so it must lie within a subspace of dimension $\ell \leq \min\{i, n\}$. If we fix the subspace, then each step in the subspace can be chosen from at most 2ℓ different directions (for \mathbb{Q}_n , from ℓ directions). Thus, there are at most $(2\ell)^{2i}$ walks in the subspace. Since the number of subspaces of fixed dimension ℓ is given by $\binom{n}{\ell} \leq n^\ell / \ell!$, we obtain the bound

$$\sum_{\ell=1}^i \frac{1}{\ell!} n^\ell (2\ell)^{2i} \leq n^{i;2i} \sum_{\ell=1}^i \frac{1}{\ell!} 2^{2i} \tag{3.13}$$

for the number of walks that make the transition from 0 to 0 in $2i$ steps. Multiplying by Ω^{-2i} to convert the number of walks into a probability leads to (3.12). This proves (3.9) for $j = 0$ and i even. For $j = 0$ and i odd, we apply the Cauchy–Schwarz inequality in the form $\int |\hat{D}(k)|^i \leq (\int |\hat{D}(k)|^{2i})^{1/2}$, and (3.9) then follows also for i odd. This proves (3.9) for the case $j = 0$, so we now assume $j \geq 1$.

Fix an even integer $s = s(j)$ such that $t = s/(s - 1)$ obeys $jt < j + \frac{1}{2}$. By Hölder's inequality,

$$T_p^{(i,j)} \leq \left(\int \hat{D}(k)^{is} \right)^{1/s} \left(\int \hat{\tau}_p(k)^{jt} \right)^{1/t}. \tag{3.14}$$

By (3.12), it suffices to show that $\int \hat{\tau}_p(k)^{jt}$ is bounded by a constant depending on j . We give separate arguments for this, for \mathbb{Z}^n and \mathbb{Q}_n .

For \mathbb{Z}^n , the infrared bound [15, (4.7)] implies that $\hat{\tau}_p(k) \leq 2[1 - \hat{D}(k)]^{-1}$ for sufficiently large n , uniformly in $p \leq p_c(\mathbb{Z}^n)$. Thus,

$$\int \hat{\tau}_p(k)^{jt} \leq 2^{jt} \int \frac{1}{[1 - \hat{D}(k)]^{jt}}. \tag{3.15}$$

For $A > 0$ and $m > 0$,

$$\frac{1}{A^m} = \frac{1}{\Gamma(m)} \int_0^\infty u^{m-1} e^{-uA} du, \tag{3.16}$$

so that

$$\int \frac{1}{[1 - \hat{D}(k)]^{jt}} = \frac{1}{\Gamma(jt)} \int_0^\infty du u^{jt-1} \left(\int_{-\pi}^\pi e^{-um^{-1}(1-\cos\theta)} \frac{d\theta}{2\pi} \right)^n. \tag{3.17}$$

The right-hand side is non-increasing in n , since $\|f\|_p \leq \|f\|_q$ for $0 < p \leq q \leq \infty$ on a probability space. Since

$$1 - \hat{D}(k) = \sum_{j=1}^n (1 - \cos k_j) \geq \frac{2}{\pi^2} \frac{|k|^2}{n}, \tag{3.18}$$

and since $2jt < 2j + 1$, the integral on the left-hand side of (3.17) is finite when $n = 2j + 1$. This completes the proof for \mathbb{Z}^n .

For \mathbb{Q}_n , we use the fact that $\hat{\tau}_p(0) = \chi(p)$ to see that

$$\int \hat{\tau}_p(k)^{jt} = 2^{-n} \chi(p)^{jt} + 2^{-n} \sum_{k \in \{0, \pi\}^n : k \neq 0} \hat{\tau}_p(k)^{jt}. \tag{3.19}$$

The first term on the right-hand side is at most $2^{-n} \chi(\bar{p}_c(\mathbb{Q}_n))^{jt} = 2^{-n} f_n^{jt}$, which is exponentially small. For the second term, we recall from [9, Theorem 6.1] that $\hat{\tau}_p(k) \leq [1 + O(n^{-1})][1 - \hat{D}(k)]^{-1}$, so it suffices to prove that

$$2^{-n} \sum_{k \in \{0, \pi\}^n : k \neq 0} \frac{1}{[1 - \hat{D}(k)]^{jt}} \tag{3.20}$$

is bounded uniformly in $n \geq 1$.

For this, we let $m(k)$ denote the number of nonzero components of k . We fix an $\varepsilon > 0$ and divide the sum according to whether $m(k) \leq \varepsilon n$ or $m(k) > \varepsilon n$. An elementary computation (see [9, Section 2.2.1]) gives $1 - \hat{D}(k) = 2m(k)/n$. Therefore, the contribution to (3.20) due to $m(k) > \varepsilon n$ is bounded by a constant depending only on ε and j . On the

other hand, for $k \neq 0$, we use $1 - \hat{D}(k) = 2m(k)/n \geq 2/n$ to see that

$$\begin{aligned} 2^{-n} \sum_{k \in \{0, \pi\}^n : 0 < m(k) \leq \varepsilon n} \frac{1}{[1 - \hat{D}(k)]^{jt}} &\leq 2^{-jt} n^{jt} 2^{-n} \sum_{k \in \{0, \pi\}^n : 0 < m(k) \leq \varepsilon n} 1 \\ &= 2^{-jt} n^{jt} 2^{-n} \sum_{m=1}^{\varepsilon n} \binom{n}{m} \\ &\leq 2^{-jt} n^{jt} \mathbb{P}(X \leq \varepsilon n), \end{aligned} \tag{3.21}$$

where X is a binomial random variable with parameters $(n, 1/2)$. Since $\mathbb{E}[X] = n/2$, the right-hand side of (3.21) is exponentially small in n as $n \rightarrow \infty$ if we choose $\varepsilon < \frac{1}{2}$, by standard large deviation bounds for the binomial distribution (see, e.g., [5, Theorem A.1.1]). This completes the proof for \mathbb{Q}_n .

Proof of (3.10). We repeat the argument of [9, Lemma 5.5] for \mathbb{Q}_n , which applies verbatim for \mathbb{Z}^n . It follows from the BK (van den Berg-Kesten) inequality that if $x \neq 0$ then

$$\tau_p(x) \leq p\Omega(D * \tau_p)(x). \tag{3.22}$$

Using this, we conclude that

$$p\Omega(D * \tau_p^{*3})(x) \leq p\Omega D(x) + 3(p\Omega)^2(D^{*2} * \tau_p^{*3})(x), \tag{3.23}$$

where the first term is the contribution where each of the three two-point functions $\tau_p(u)$ in τ_p^{*3} is evaluated at $u = 0$, and the second term takes into account the case where at least one of the three displacements is nonzero. Since $p \leq \bar{p}_c = \Omega^{-1} + O(\Omega^{-2}) \leq 2\Omega^{-1}$ for large Ω , this gives

$$T_p \leq 2\Omega^{-1} + 12T_p^{(2,3)} \leq (2 + 12K_{2,3})\Omega^{-1} = K\Omega^{-1}, \tag{3.24}$$

where in the first inequality we used (3.4) to rewrite the second term of (3.23).

Proof of (3.11). For $i \geq 1$, the BK inequality can be applied as in the proof of (3.22) to obtain

$$\tau_p^{(i)}(x) \leq (p\Omega)^i (D^{*i} * \tau_p)(x). \tag{3.25}$$

It follows from (3.4) and (3.25) that

$$\sup_x \tau_p^{(i)}(x) \leq \sup_x (p\Omega)^i \int \hat{D}(k)^i \hat{\tau}_p(k) e^{-ik \cdot x} \leq (p\Omega)^i T_p^{(i,1)} \leq 2^i K_{i,1} \Omega^{-i/2}, \tag{3.26}$$

where we have used the fact that $p\Omega \leq 2$ for Ω sufficiently large. For $i = 1$, this can be improved by observing that, for Ω sufficiently large,

$$\tau_p^{(1)}(x) \leq p\Omega D(x) + \tau_p^{(2)}(x) \leq 2\Omega^{-1} + 2K_{2,1}\Omega^{-1}. \tag{3.27}$$

□

4. Proof of Proposition 2.1

We now complete the proof of Proposition 2.1, by proving (2.8), (2.9), (2.10), in Sections 4.1, 4.2, 4.3, respectively. Throughout this section we fix $p \leq \bar{p}_c(\mathbb{G})$.

4.1. Expansion for $\hat{\Pi}_p^{(0)}$

Given a configuration, we say that x is *doubly connected* to y , and we write $x \Leftrightarrow y$, if $x = y$ or if there are at least two bond-disjoint paths from x to y consisting of occupied bonds. For $\ell \geq 4$, an ℓ -cycle is a set of bonds that can be written as $\{\{v_{i-1}, v_i\}\}_{1 \leq i \leq \ell}$ with $v_\ell = v_0$, and a *cycle* is an ℓ -cycle for some $\ell \geq 4$. By definition,

$$\hat{\Pi}_p^{(0)} = \sum_{x \neq 0} \mathbb{P}_p(0 \Leftrightarrow x) = \sum_{x \neq 0} \mathbb{P}_p(\exists \text{ occupied cycle containing } 0, x). \tag{4.1}$$

We decompose the summand into (a) the probability that there exists an occupied 4-cycle containing $0, x$, plus (b) the probability that there exists an occupied cycle of length at least 6 containing $0, x$ and no occupied 4-cycle containing $0, x$.

The contribution to $\hat{\Pi}_p^{(0)}$ due to (a) is bounded above by summing p^4 over $x \neq 0$ and over 4-cycles containing $0, x$. The number of 4-cycles containing 0 is $\frac{1}{2}\Omega\Omega'$, and each such cycle has three possibilities for x . Therefore,

$$\text{contribution due to (a)} \leq \frac{3}{2}\Omega\Omega'p^4. \tag{4.2}$$

For a lower bound, we apply inclusion-exclusion and subtract from this upper bound the sum of p^7 over $x \neq 0$ and over pairs of 4-cycles, each containing $0, x$. In this case, x must be a neighbour of 0 , and p^7 is the probability of simultaneous occupation of the two 4-cycles. There are order Ω^3 such pairs of 4-cycles. Since we already know that $\bar{p}_c(\mathbb{G}) \leq O(\Omega^{-1})$, this gives

$$\text{contribution due to (a)} = \frac{3}{2}\Omega\Omega'p^4 + O(\Omega^3p^7) = \frac{3}{2}\Omega\Omega'p^4 + O(\Omega^{-4}). \tag{4.3}$$

For the contribution due to (b), we use Lemma 4.1 below. Given increasing events E, F , we use the standard notation $E \circ F$ to denote the event that E and F occur disjointly. Roughly speaking, $E \circ F$ is the set of bond configurations for which there exist two disjoint sets of occupied bonds such that the first set guarantees the occurrence of E and the second guarantees the occurrence of F . The BK inequality asserts that $\mathbb{P}(E \circ F) \leq \mathbb{P}(E)\mathbb{P}(F)$, for increasing events E and F . (See [14, Section 2.3] for a proof, and for a precise definition of $E \circ F$.)

Lemma 4.1. *Let $p \leq \bar{p}_c(\mathbb{G})$. Let $\Pi_p^{(0,\ell)}(x)$ denote the probability that there is an occupied cycle containing $0, x$, of length ℓ or longer. Then for $\ell \geq 4$ and for Ω sufficiently large (not depending on ℓ),*

$$\sum_{x \neq 0} \Pi_p^{(0,\ell)}(x) \leq (\ell - 1)2^\ell K_{\ell,2} \Omega^{-\ell/2}. \tag{4.4}$$

Proof. Let $\ell \geq 4$, and suppose there exists an occupied cycle containing $0, x$, of length ℓ or longer. Then there is a $j \in \{1, \dots, \ell - 1\}$ such that $\{0 \leftrightarrow x\} \circ \{0 \overset{\ell-j}{\leftarrow} x\}$ occurs.

Therefore, by the BK inequality,

$$\Pi_p^{(0,\ell)}(x) \leq \sum_{j=1}^{\ell-1} \tau_p^{(j)}(x) \tau_p^{(\ell-j)}(x). \tag{4.5}$$

By (3.25), by the fact that $p\Omega \leq 2$ for Ω sufficiently large, and by (3.9), it follows that

$$\sum_{x \neq 0} \Pi_p^{(0,\ell)}(x) \leq (\ell - 1)2^\ell (D^{*\ell} * \tau_p^{*2})(0) \leq (\ell - 1)2^\ell T_p^{(\ell,2)} \leq (\ell - 1)2^\ell K_{\ell,2} \Omega^{-\ell/2}, \tag{4.6}$$

as required. □

The contribution due to case (b) is therefore at most $\sum_{x \neq 0} \Pi_p^{(0,6)}(x) \leq O(\Omega^{-3})$, and hence

$$\hat{\Pi}_p^{(0)} = \frac{3}{2} \Omega \Omega' p^4 + O(\Omega^{-3}), \tag{4.7}$$

which proves (2.8).

4.2. Expansion for $\hat{\Pi}_p^{(0)}$

To define $\hat{\Pi}_p^{(0)}$, we need the following definitions.

Definition. (i) Given a bond configuration, vertices x, y , and a set A of vertices of \mathbb{G} , we say x and y are *connected through A* , and write $x \overset{A}{\leftrightarrow} y$, if every occupied path connecting x to y has at least one bond with an endpoint in A .

(ii) Given a bond configuration, and a bond b , we define $\tilde{C}^b(x)$ to be the set of vertices connected to x in the new configuration obtained by setting b to be vacant.

(iii) Given a bond configuration and vertices x, y , we say that the directed bond (u, v) is *pivotal* for $x \leftrightarrow y$ if (a) $x \leftrightarrow y$ occurs when the bond $\{u, v\}$ is set occupied, and (b) when $\{u, v\}$ is set vacant $x \leftrightarrow y$ does not occur, but $x \leftrightarrow u$ and $v \leftrightarrow y$ do occur (enforcing use of (u, v) in a particular direction). Note that there is a distinction between the events $\{(u, v)$ is pivotal for $x \leftrightarrow y\}$ and $\{(v, u)$ is pivotal for $x \leftrightarrow y\} = \{(u, v)$ is pivotal for $y \leftrightarrow x\}$.

Let

$$E'(v, x; A) = \{v \overset{A}{\leftrightarrow} x\} \cap \{\nexists \text{ pivotal } (u', v') \text{ for } v \leftrightarrow x \text{ s.t. } v \overset{A}{\leftrightarrow} u'\}. \tag{4.8}$$

We will refer to the ‘no pivotal’ condition of the second event on the right-hand side of (4.8) as the ‘NP’ condition.

By definition,

$$\hat{\Pi}_p^{(0)} = \sum_x p \sum_{(u,v)} \mathbb{E}_0 [I[0 \leftrightarrow u] \mathbb{P}_1(E'(v, x; \tilde{C}_0^{(u,v)}(0)))], \tag{4.9}$$

where the sum over (u, v) is a sum over directed bonds. On the right-hand side, the cluster $\tilde{C}_0^{(u,v)}(0)$ is random with respect to the expectation \mathbb{E}_0 , so that $\tilde{C}_0^{(u,v)}(0)$ should be regarded as a *fixed* set inside the probability \mathbb{P}_1 . The latter introduces a second percolation model which depends on the original percolation model via the set $\tilde{C}_0^{(u,v)}(0)$. We use subscripts

for \tilde{C} and expectations, to indicate to which expectation \tilde{C} belongs, and refer to the bond configuration corresponding to expectation j as the ‘level- j ’ configuration. We also write F_j to indicate an event F at level j . Then (4.9) can be written as

$$\hat{\Pi}_p^{(1)} = \sum_x p \sum_{(u,v)} \mathbb{P}^{(1)}(\{0 \leftrightarrow u\}_0 \cap E'(v, x; \tilde{C}_0^{(u,v)}(0))_1), \tag{4.10}$$

where $\mathbb{P}^{(1)}$ represents the joint expectation of the percolation models at levels 0 and 1.

We begin with a minor extension of a standard estimate for $\hat{\Pi}_p^{(1)}$ (see [9, Section 4] for a related discussion with our present notation). Making the abbreviation $\tilde{C}_0 = \tilde{C}_0^{(u,v)}(0)$, we may insert within the square brackets on the right-hand side of (4.10) the disjoint union

$$(\{u = 0\} \cap \{x \in \tilde{C}_0\}) \dot{\cup} (\{u = 0\} \cap \{x \notin \tilde{C}_0\}) \dot{\cup} \{u \neq 0\}. \tag{4.11}$$

The first term is the leading term and the other two produce error terms.

We first show that the term $\{u \neq 0\}$ produces an error term. We define the events

$$F_0(0, u, w, z) = \{0 \leftrightarrow u\} \circ \{0 \leftrightarrow w\} \circ \{w \leftrightarrow u\} \circ \{w \leftrightarrow z\}, \tag{4.12}$$

$$F_1(v, t, z, x) = \{v \leftrightarrow t\} \circ \{t \leftrightarrow z\} \circ \{t \leftrightarrow x\} \circ \{z \leftrightarrow x\}. \tag{4.13}$$

Note that $F_1(v, t, z, x) = F_0(x, z, t, v)$. Recalling the definition of $\{x \leftrightarrow_j y\}$ from (3.1), we also define

$$F_0^{(j)}(0, u, w, z) = \bigcup_{j_1+j_2+j_3=j} \{0 \leftrightarrow_{j_1} u\} \circ \{0 \leftrightarrow_{j_2} w\} \circ \{w \leftrightarrow_{j_3} u\} \circ \{w \leftrightarrow z\}, \tag{4.14}$$

$$F_1^{(j)}(v, t, z, x) = \bigcup_{j_1+j_2+j_3=j} \{v \leftrightarrow t\} \circ \{t \leftrightarrow_{j_1} z\} \circ \{t \leftrightarrow_{j_2} x\} \circ \{z \leftrightarrow_{j_3} x\}. \tag{4.15}$$

For $u \neq 0$, it can be seen from the fact that u and 0 are in a level-0 cycle of length at least 4 that

$$\{0 \leftrightarrow u \neq 0\}_0 \cap E'(v, x; \tilde{C}_0)_1 \subset \bigcup_{t,w,z} (F_0^{(4)}(0, u, w, z)_0 \cap F_1(v, t, z, x)_1), \tag{4.16}$$

and hence this contribution to $\hat{\Pi}_p^{(1)}$ is at most

$$p \sum_{x,(u,v),t,w,z} \mathbb{P}_p(F_0^{(4)}(0, u, w, z)) \mathbb{P}_p(F_1(v, t, z, x)). \tag{4.17}$$

Let

$$A_3(t, z, x) = \tau_p(x - t) \tau_p(z - t) \tau_p(z - x), \tag{4.18}$$

$$A_3^{(j)}(t, z, x) = \sum_{j_1+j_2+j_3=j} \tau_p^{(j_1)}(x - t) \tau_p^{(j_2)}(z - t) \tau_p^{(j_3)}(z - x), \tag{4.19}$$

$$B_1(w, u, z, t) = (p\Omega D * \tau_p)(t - u) \tau_p(z - w). \tag{4.20}$$

By the BK inequality, (4.17) is at most

$$\sum_{u,w} A_3^{(4)}(0, u, w) \sum_{t,z} B_1(w, u, z, t) \sum_x A_3(t, z, x). \tag{4.21}$$

Replacing w, z, t, x by $w = w' + u, z = z' + u, t = t' + u, x = x' + u$, and using symmetry, this is equal to

$$\sum_{u, w'} A_3^{(4)}(0, u, w') \sum_{t', z'} B_1(w', 0, z', t') \sum_{x'} A_3(t', z', x'). \tag{4.22}$$

We note that $B_1(w', 0, z', t') = B_1(-t', -t', z' - w' - t', 0)$, and set $z'' = z' - t', x'' = x' - t'$ and then $t'' = -t'$ to rewrite (4.22) as

$$\begin{aligned} & \sum_{u, w'} A_3^{(4)}(0, u, w') \sum_{t'', z''} B_1(t'', t'', z'' - w', 0) \sum_{x''} A_3(0, z'', x'') \\ & \leq \left(\sum_{u, w'} A_3^{(4)}(0, u, w') \right) \left(\sup_a \sum_{t''} B_1(t'', t'', a, 0) \right) \left(\sum_{z'', x''} A_3(0, z'', x'') \right). \end{aligned} \tag{4.23}$$

By (3.8) and the fact that $\tau_p(u) \leq (\tau_p * \tau_p)(u)$,

$$\sup_a \sum_{t''} B_1(t'', t'', a, 0) = \sup_a (p\Omega D * \tau_p * \tau_p)(a) \leq T_p. \tag{4.24}$$

Also,

$$\sum_{z'', x''} A_3(0, z'', x'') = (\tau_p * \tau_p * \tau_p)(0) = T_p^{(0,3)}, \tag{4.25}$$

and, using (3.25) and $p\Omega \leq 2$,

$$\sum_{u, w'} A_3^{(4)}(0, u, w') = \sum_{j_1 + j_2 + j_3 = 4} (\tau_p^{(j_1)} * \tau_p^{(j_2)} * \tau_p^{(j_3)})(0) = O(T^{(4,3)}). \tag{4.26}$$

Therefore, (4.23) is bounded above by $O(T_p^{(4,3)} T_p T_p^{(0,3)})$, which is $O(\Omega^{-3})$ by (3.9) and (3.10).

Similarly, an upper bound $O(\Omega^{-3})$ can be obtained for the contribution due to $\{u = 0\} \cap \{x \notin \tilde{C}_0\}$, starting from the observation that

$$\{u = 0\} \cap \{x \notin \tilde{C}_0\} \cap E'(v, x; \tilde{C}_0)_1 \subset \bigcup_{t, w, z: z \neq x} (F_0(0, 0, 0, z)_0 \cap F_1^{(4)}(v, t, z, x)_1). \tag{4.27}$$

The inclusion (4.27) follows from the fact that if $x \notin \tilde{C}_0$, then to obtain a nonzero contribution to $\mathbb{P}_1(E'(v, x; \tilde{C}_0))$, x must be in a level-1 occupied cycle of length at least 4 which contains a vertex $z \in \tilde{C}_0$.

We are left to consider the leading term

$$\sum_x p \sum_{(0, v)} \mathbb{P}^{(1)}(\{x \in \tilde{C}_0^{(0, v)}(0)\} \cap E'(v, x; \tilde{C}_0^{(0, v)}(0))_1). \tag{4.28}$$

See Figure 1 for a depiction of the event appearing in (4.28).

The event in (4.28) is a subset of the event $\{x \in \tilde{C}_0\} \cap \{v \leftrightarrow x\}_1$. Let ℓ_0 be the length of a shortest level-0 occupied path from 0 to x (not using the bond $\{0, v\}_0$), and let ℓ_1 be the length of a shortest level-1 occupied path from v to x . Then either $\ell_0 + \ell_1 \leq 4$, or $\{0 \leftrightarrow x\}_0 \cap \{v \leftrightarrow x\}_1$ occurs with $i_0 + i_1 = 5$. This decomposition is not disjoint, as the latter possibility does not imply that the former does not occur, but this is fine for an upper bound. By (3.25) and (3.9), the contribution due to the latter case is

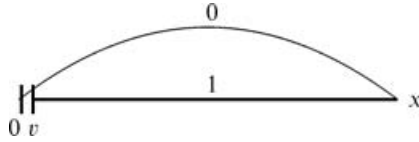


Figure 1. Depiction of the event appearing in (4.28). Line 0 corresponds to a connection in level 0 and line 1 to a connection in level 1.

bounded above by

$$\sum_{i_0=0}^5 \sum_x p \sum_{(0,v)} \tau_p^{(i_0)}(x) \tau_p^{(5-i_0)}(x-v) = \sum_{i_0=0}^5 (p\Omega D * \tau_p^{(i_0)} * \tau_p^{(5-i_0)})(0) \leq 6(p\Omega)^6 T_p^{(6,2)} = O(\Omega^{-3}), \tag{4.29}$$

so this is an error term.

Since v and 0 have opposite parity, if there is a level-0 connection from 0 to x of length ℓ_0 and a level-1 connection from v to x of length ℓ_1 , then $\ell_0 + \ell_1$ must be odd. Thus, we are left to deal with the cases $\ell_0 + \ell_1 = 1$ and $\ell_0 + \ell_1 = 3$, and we consider these separately.

Case of $\ell_0 + \ell_1 = 1$. If $\ell_1 = 0$, then $x = v \in \tilde{C}_0^{(0,v)}(0)$, which forces $\ell_0 \geq 3$. This is inconsistent with $\ell_0 + \ell_1 = 1$ and therefore need not be considered here. We may therefore assume that $\ell_0 = 0$ and $\ell_1 = 1$, so that $x = 0$, $\{0, v\}_1$ is occupied, and, to satisfy the NP condition of (4.8), $v \notin \tilde{C}_0^{(0,v)}(0)$. We use inclusion-exclusion on the latter, writing

$$I[v \notin \tilde{C}_0^{(0,v)}(0)] = 1 - I[v \in \tilde{C}_0^{(0,v)}(0)]. \tag{4.30}$$

The first term contributes

$$p \sum_{(0,v)} p = \Omega p^2. \tag{4.31}$$

The second term requires a level-0 connection from 0 to v of length 3 or more, which has probability $\tau_p^{(3)}(v)$, so that by (3.25) and (3.9), the second term contributes

$$p \sum_{(0,v)} p \tau_p^{(3)}(v) = p(p\Omega)(D * \tau_p^{(3)})(0) \leq p(p\Omega)^4 T^{(4,1)} = O(\Omega^{-3}), \tag{4.32}$$

and hence is an error term. Thus, the case $\ell_0 + \ell_1 = 1$ contributes

$$\Omega p^2 + O(\Omega^{-3}). \tag{4.33}$$

Case of $\ell_0 + \ell_1 = 3$. There are four possibilities: $\ell_1 = 0, 1, 2, 3$. If $\ell_1 = 0$ then $x = v$, the NP condition is trivially satisfied, and there is an occupied level-0 path from 0 to v of length 3 (which is the minimum possible since the bond $\{0, v\}_0$ cannot be used). This contribution is

$$\Omega \Omega' p^4 + O(\Omega^3 p^7) = \Omega \Omega' p^4 + O(\Omega^{-4}), \tag{4.34}$$

where we have used inclusion-exclusion in a manner similar to that of the argument used for (4.2)–(4.3). In more detail, the first term in (4.34) accounts for the sum of the

probability of an occupied level-0 path of length 3 from 0 to v , while the second term accounts for overcounting due to simultaneous occupation of more than one such path.

For $\ell_1 = 1, 2, 3$, we note that

$$\{x \in \tilde{C}_0\} \cap E'(v, x; \tilde{C}_0)_1 = \{x \in \tilde{C}_0\} \cap \{v \leftrightarrow x\}_1 \cap \text{NP}, \tag{4.35}$$

and use $I[\text{NP}] = 1 - I[\text{NP}^c]$, to conclude that

$$I[x \in \tilde{C}_0]I[E'(v, x; \tilde{C}_0)_1] = I[x \in \tilde{C}_0]I[\{v \leftrightarrow x\}_1] - I[x \in \tilde{C}_0]I[\{v \leftrightarrow x\}_1]I[\text{NP}^c]. \tag{4.36}$$

We first consider the first term on the right-hand side of (4.36). In the following, we write e to denote a neighbour of 0 that is not $\pm v$, and which will ultimately be summed over. We again apply an inclusion-exclusion argument similar to that used for (4.34), but do not discuss its details.

The case $\ell_1 = 1$ corresponds to $\ell_0 = 2$, so that $x = v + e$, with the three bonds $\{0, e\}_0, \{e, x\}_0, \{x, v\}_1$ each occupied. This contributes $\Omega\Omega'p^4$. Note that in the related configuration in which $e = v$, so that $\{0, v\}_0, \{v, x\}_0, \{x, v\}_1$ are each occupied, the level-0 path $\{0, v\}_0, \{v, x\}_0$ from 0 to x uses the bond $\{0, v\}_0$, and therefore need not be considered. For \mathbb{Z}^n , the configurations with $x = 2v$ or $x = -v$ contribute $O(\Omega p^4) = O(\Omega^{-3})$ which is an error term.

The case $\ell_1 = 2$ corresponds to $\ell_0 = 1$, so that $x = e$, either with the three bonds $\{0, x\}_0, \{x, x + v\}_1, \{x + v, v\}_1$ each occupied, or with the three bonds $\{0, x\}_0, \{0, x\}_1, \{0, v\}_1$ each occupied. This contributes $2\Omega\Omega'p^4$. For \mathbb{Z}^n , the configuration with $x = -v$ and with $\{0, -v\}_0, \{0, v\}_1, \{0, -v\}_1$ each occupied contributes $O(\Omega p^4) = O(\Omega^{-3})$ and thus is an error term.

The case $\ell_1 = 3$ corresponds to $\ell_0 = 0$, so that $x = 0$, with the three bonds $\{0, e\}_1, \{e, e + v\}_1, \{e + v, v\}_1$ each occupied. This contributes $\Omega\Omega'p^4$.

In summary, the $\ell_1 = 0$ contribution, plus the first term on the right-hand side of (4.36) with $\ell_1 = 1, 2, 3$, contribute

$$5\Omega\Omega'p^4 + O(\Omega^{-3}). \tag{4.37}$$

Next, we consider the effect of the second term in (4.36), for $\ell_1 = 1, 2, 3$.

For $\ell_1 = 1$, we have seen above that, to leading order, $\{0, e\}_0, \{e, x\}_0, \{x, v\}_1$ are each occupied. The only possible pivotal bond for the level-1 connection from v to x is therefore $(v, x)_1$, and thus the failure of NP requires $v \in \tilde{C}_0$. This requires a level-0 connection, disjoint from the bonds $\{0, e\}_0$ and $\{e, x\}_0$, which joins either 0 to v , e to v , or x to v . This adds an additional factor $O(\Omega^{-1})$ and hence produces an error term.

For $\ell_1 = 2$, we have seen above that there are two cases to consider. Suppose first that $x = e$ and $\{0, x\}_0, \{x, x + v\}_1, \{x + v, v\}_1$ are each occupied. The only possible pivotal bonds for the level-1 connection from v to x are $(v, x + v)_1$ and $(x + v, x)_1$. Violation of NP therefore requires that either $(v, x + v)_1$ is pivotal and $v \in \tilde{C}_0$, or $(x + v, x)_1$ is pivotal and $x + v \in \tilde{C}_0$. In either of these cases, the condition that \tilde{C}_0 contain an additional vertex is a higher-order effect and leads to an error term $O(\Omega^{-3})$.

The remaining case for $\ell_1 = 2$ has $x = e$ and $\{0, x\}_0, \{0, x\}_1, \{0, v\}_1$ each occupied. The only possible pivotal bonds for the level-1 connection from v to x are $(v, 0)_1$ and $(0, x)_1$. Violation of NP therefore requires that either $(v, 0)_1$ is pivotal and $v \in \tilde{C}_0$, or $(0, x)_1$ is

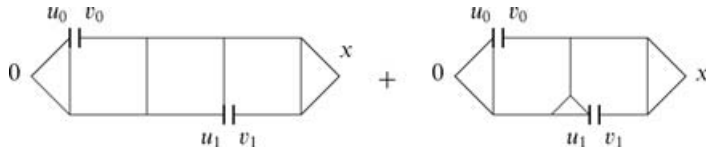


Figure 2. The standard diagrams bounding $\Pi_p^{(2)}$. All vertices other than 0 are summed over the vertex set \mathbb{V} of \mathbb{G} , lines represent factors of τ_p , and vertical bars represent factors $p\Omega D$.

pivotal and $0 \in \tilde{C}_0$. The first of these cases leads to an error term as above. For the second case, $0 \in \tilde{C}_0$ is automatic, and inclusion-exclusion applied to the requirement that $(0, v)_1$ is pivotal leads to a net contribution for $\ell_1 = 2$ of $-\Omega\Omega'p^4 + O(\Omega^{-3})$.

Finally, we consider $\ell_1 = 3$. In this case, $x = 0$, and $\{0, e\}_1, \{e, e + v\}_1, \{e + v, v\}_1$ are each occupied. The only possible violations of NP are: $(v, v + e)_1$ is pivotal for the connection from v to $x = 0$ and $v \in \tilde{C}_0$, or $(v + e, e)_1$ is pivotal and $v + e \in \tilde{C}_0$, or $(e, 0)_1$ is pivotal and $e \in \tilde{C}_0$. In any of these three cases, the condition that \tilde{C}_0 must contain the additional vertex requires extra connections that produce an error term $O(\Omega^{-3})$ overall, using reasoning analogous to that employed above.

We have thus shown that the case $\ell_0 + \ell_1 = 3$ yields a net contribution

$$5\Omega\Omega'p^4 - \Omega\Omega'p^4 + O(\Omega^{-3}) = 4\Omega\Omega'p^4 + O(\Omega^{-3}). \tag{4.38}$$

In summary, combining (4.33) and (4.38), we have proved (2.9), namely

$$\hat{\Pi}_p^{(1)} = \Omega p^2 + 4\Omega\Omega'p^4 + O(\Omega^{-3}). \tag{4.39}$$

4.3. Expansion for $\hat{\Pi}_p^{(2)}$

By definition,

$$\hat{\Pi}_p^{(2)} = \sum_x \sum_{(u_0, v_0)} \sum_{(u_1, v_1)} p^2 \mathbb{E}_0 \left[I[0 \Leftrightarrow u_0] \mathbb{E}_1 \left[I[E'(v_0, u_1; \tilde{C}_0)] \mathbb{E}_2 I[E'(v_1, x; \tilde{C}_1)] \right] \right], \tag{4.40}$$

where we have made the abbreviations $\tilde{C}_0 = \tilde{C}_0^{(u_0, v_0)}(0)$ and $\tilde{C}_1 = \tilde{C}_1^{(u_1, v_1)}(v_0)$. A standard estimate for $\hat{\Pi}_p^{(2)}$ is

$$0 \leq \hat{\Pi}_p^{(2)} \leq 2T_p^{(0,3)}(T_p T_p^{(0,3)})^2 \tag{4.41}$$

(see, e.g., [9, Section 4.2]; one factor 2 in [9, Proposition 4.1] is easily dropped for $N = 2$). This estimate arises from the upper bound for $\hat{\Pi}_p^{(2)}$ depicted in Figure 2. The factor 2 is due to the fact that there are two terms in the upper bound. The two factors T_p in each term arise from the two diagram loops containing lines with vertical bars, and the three factors $T_p^{(0,3)}$ arise from the other three diagram loops.

We claim that contributions to $\hat{\Pi}_p^{(2)}$ in which $u_0 \neq 0$, or $u_1 \notin \tilde{C}_0$, or $x \notin \tilde{C}_1$ produce an error term of order $O(\Omega^{-4})$. This follows from routine estimates, along the lines of those used in Section 4.2, to conclude that we could assume there that $u = 0$ and $x \in \tilde{C}_0$. These estimates, which we do not write down here in detail, show for example that if $u_0 \neq 0$, then the factor $T_p^{(0,3)}$, arising from the leftmost diagram loop can be replaced by a constant multiple of $T_p^{(4,3)}$. By (3.9) and (3.10), this leads to a bound $O(\Omega^{-4})$, which is an error

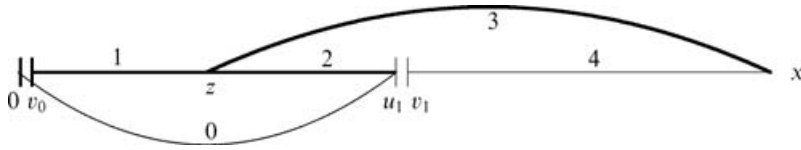


Figure 3. Diagrammatic representation of the event (4.43). Line 0 corresponds to a connection in level 0, lines 1, 2, 3 correspond to connections in level 1 and line 4 to a connection in level 2.

term. Similarly, if $u_1 \notin \tilde{C}_0^{(0,v_0)}(0)$, then the event $E'(v_0, u_1; \tilde{C}_0^{(0,v_0)}(0))$ requires that u_1 must be in an occupied level-1 cycle of length at least 4. In this case, we may again use standard estimates to replace a factor $T_p^{(0,3)}$ in (4.41), arising from the leftmost diagram loop in Figure 2 containing u_1 , by a constant multiple of $T_p^{(4,3)}$, and again this contribution is $O(\Omega^{-4})$. Finally, the same situation arises when $x \notin \tilde{C}_0^{(u_1,v_1)}(v_0)$, in which case we can replace the factor $T_p^{(0,3)}$ arising from the rightmost diagram loop by a constant multiple of $T_p^{(4,3)}$, and again this contribution is $O(\Omega^{-4})$. Thus, we are now left to analyse

$$\sum_x \sum_{(0,v_0)} \sum_{(u_1,v_1)} p^2 \mathbb{P}^{(2)}(\{u_1 \in \tilde{C}_0\} \cap \{x \in \tilde{C}_1\} \cap E'(v_0, u_1; \tilde{C}_0)_1 \cap E'(v_1, x; \tilde{C}_1)_2), \tag{4.42}$$

where we write $\mathbb{P}^{(2)}$ for the joint probability of levels 0, 1 and 2. Let G denote the intersection of events on the right-hand side of (4.42).

The event G on the right-hand side of (4.42) is contained in the event

$$\{0 \leftrightarrow u_1\}_0 \cap \bigcup_{z \in \mathbb{G}} \{ \{v_0 \leftrightarrow z\} \circ \{z \leftrightarrow u_1\} \circ \{z \leftrightarrow x\} \}_1 \cap \{v_1 \leftrightarrow x\}_2, \tag{4.43}$$

which is depicted in Figure 3. For any choice of j_0, j_1, j_2, j_3, j_4 , a subset of (4.43) is the event

$$\{0 \leftrightarrow u_1\}_{j_0} \cap \bigcup_{z \in \mathbb{G}} \{ \{v_0 \leftrightarrow z\}_{j_1} \circ \{z \leftrightarrow u_1\}_{j_2} \circ \{z \leftrightarrow x\}_{j_3} \}_1 \cap \{v_1 \leftrightarrow x\}_{j_4}. \tag{4.44}$$

Since v_0 has odd parity, and since u_1 and v_1 have opposite parity, we may assume that $j_0 + j_1 + j_2$ and $j_2 + j_3 + j_4$ are both odd. If $j_0 + j_1 + j_2 \geq 3$, then a standard diagrammatic estimate gives $O(T_p^{(4,3)} T_p) = O(\Omega^{-3})$ for the contribution of (4.44). Similarly, if $j_2 + j_3 + j_4 \geq 3$, then again a standard diagrammatic estimate gives an upper bound $O(T_p T_p^{(4,3)}) = O(\Omega^{-3})$. Note that if $u_1 \neq 0$, then we may assume that $j_0 + j_1 + j_2 \geq 3$, which gives an error term.

Thus, we may assume that G occurs, that $u_1 = 0$, and that there is a z such that the connections of Figure 3 occur with lines of length $\ell_0 = 0, \ell_1, \ell_2, \ell_3, \ell_4$, where

$$\ell_1 + \ell_2 = 1, \tag{4.45}$$

$$\ell_2 + \ell_3 + \ell_4 = 1. \tag{4.46}$$

This gives three possibilities for $(\ell_0, \ell_1, \ell_2, \ell_3, \ell_4)$, namely

$$(0, 0, 1, 0, 0), \quad (0, 1, 0, 1, 0), \quad (0, 1, 0, 0, 1), \tag{4.47}$$

and it suffices to compute the contribution from each of these cases. As will be apparent in the following, these three possibilities correspond to distinct configurations.

Case of $(0, 0, 1, 0, 0)$. In this case, $u_0 = u_1 = 0$, $z = x = v_1 = v_0$, and the bond $\{v_0, u_1\}_1$ is occupied. We examine the constraints imposed by the event G of (4.42). The events $\{u_1 \in \tilde{C}_0\}$ and $\{x \in \tilde{C}_1\}$ occur trivially. For the event $E'(v_0, u_1; \tilde{C}_0)_1$, we note that $\{v_0 \overset{\tilde{C}_0}{\leftrightarrow} u_1\}_1 = \{v_0 \overset{\tilde{C}_0}{\leftrightarrow} 0\}_1$ occurs. Violation of the NP condition requires $\{0 \overset{3}{\leftrightarrow} v_0\}_0$, and this contributes at most $\sum_{(0, v_0)} p^3 \tau_p^{(3)}(v_0) \leq p^2 (p\Omega)^4 T^{(4,1)} \leq O(\Omega^{-4})$. Thus, up to an error term, we may assume that $E'(v_0, u_1; \tilde{C}_0)_1$ occurs. Finally, the event $E'(v_1, x; \tilde{C}_1)_2$ occurs trivially, since $v_1 = x \in \tilde{C}_1$. This case contributes

$$\Omega p^3 + O(\Omega^{-3}). \tag{4.48}$$

Case of $(0, 1, 0, 1, 0)$. In this case, $u_1 = 0$, $z = u_1$, $v_1 = x$. Also, the fact that $x \in \tilde{C}_1$ implies that there must be an occupied level-1 path from v_0 to $z = u_1$ to $x = v_1$ that does not use the bond $(u_1, v_1)_1$. This implies that the event $\{u_1 \overset{3}{\leftrightarrow} v_1\}_1$ occurs, and hence this case contributes an error term because it corresponds to (4.44) with $j_3 = 3$.

Case of $(0, 1, 0, 0, 1)$. In this case, $x = z = u_1 = u_0 = 0$, and the bonds $\{0, v_0\}_1, \{0, v_1\}_2$ are occupied. We denote the neighbours of 0 by e_l ($l = 1, \dots, \Omega$), so that $v_0 = e_i$ and $v_1 = e_j$ for some i, j . We examine the constraints imposed by the event G of (4.42). The event $\{u_1 \in \tilde{C}_0\}$ is satisfied trivially, since $u_1 = 0$. For the event $\{x \in \tilde{C}_1\}$, we consider separately the cases $i = j$ (i.e., $v_0 = v_1$) and $i \neq j$ (i.e., $v_0 \neq v_1$). If $i = j$, then $\{x \in \tilde{C}_1\}$ requires that $\{v_0 \overset{3}{\leftrightarrow} x\}_1$, so this is an error term of order $O(\Omega^2 p^7) = O(\Omega^{-5})$. If $i \neq j$, then $\{x \in \tilde{C}_1\}$ is achieved by the bond $\{x, v_0\}_1 = \{0, v_0\}_1$. Thus, we assume henceforth that $i \neq j$.

For the E' events, we first note that $\{v_0 \overset{\tilde{C}_0}{\leftrightarrow} u_1\}_1$ occurs, since $u_1 = 0$, $\{0, v_0\}_1$ is occupied, and $0 \in \tilde{C}_0$. Also, $\{v_1 \overset{\tilde{C}_1}{\leftrightarrow} x\}_2$ occurs, since $x = 0$, $\{0, v_1\}_2$ is occupied, and $0 \in \tilde{C}_1$ (when $i \neq j$). We will argue below that the NP condition in each E' event can be neglected, up to an error term. Assuming this, this case contributes

$$p \sum_{(0, v_0)} p \sum_{(0, v_1): v_1 \neq v_0} p^2 + O(\Omega^{-3}) = \Omega(\Omega - 1)p^4 + O(\Omega^{-3}). \tag{4.49}$$

Next, we investigate the effect of the NP condition neglected above. If the NP condition is violated for $E'(v_0, u_1; \tilde{C}_0)_1 = E'(v_0, 0; \tilde{C}_0)_1$, then the bond $(v_0, 0)_1$ must be pivotal for the level-1 connection from v_0 to 0, and moreover $v_0 \in \tilde{C}_0 = \tilde{C}_0^{(0, v_0)}(0)$ must occur. The latter gives an additional factor $\tau_p^{(3)}(v_0) \leq O(\Omega^{-3/2})$, and hence this contributes an error term.

Finally, if the NP condition is violated for $E'(v_1, x; \tilde{C}_1)_2 = E'(v_1, 0; \tilde{C}_1)_2$, then the bond $(v_1, 0)_2$ must be pivotal for the level-2 connection from v_1 to 0, and also $v_1 \in \tilde{C}_1 = \tilde{C}_1^{(0, v_1)}(v_0)$ must occur. The latter gives an additional factor $\tau_p^{(2)}(v_1 - v_0) \leq O(\Omega^{-1})$, and hence this contributes an error term.

Combining (4.48)–(4.49), we have

$$\hat{\Pi}_p^{(2)} = \Omega p^3 + \Omega(\Omega - 1)p^4 + O(\Omega^{-3}), \tag{4.50}$$

which is (2.10).

5. Conclusions

We have used the lace expansion to prove that $p_c(\mathbb{G}) = \Omega^{-1} + \Omega^{-2} + \frac{7}{2}\Omega^{-3} + O(\Omega^{-4})$ for $\mathbb{G} = \mathbb{Z}^n$ and $\mathbb{G} = \mathbb{Q}_n$. This extends by two terms the result $p_c(\mathbb{Q}_n) = n^{-1} + O(n^{-2})$ of [9], and gives a simplified proof of a result of [16, 17] for \mathbb{Z}^n .

Our proof is essentially mechanical, and with sufficient labour could be directly extended to compute higher coefficients. In particular, it would be interesting to compute the coefficient of Ω^{-4} , which we expect will be different for \mathbb{Z}^n and \mathbb{Q}_n .

We expect that our method can also be applied to other finite graphs for which the lace expansion has been proved to converge in [9]. A specific example is the Hamming cube, which has vertex set $\{0, 1, \dots, s\}^n$ with $s \geq 1$ fixed, and edge set consisting of pairs of vertices which differ in exactly one component. For $s = 1$, the Hamming cube is the n -cube. For $s \geq 2$, the Hamming cube contains cycles of length 3 (in contrast to \mathbb{Z}^n and \mathbb{Q}_n), and it would be interesting to study their effect on the expansion coefficients.

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