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ASYMPTOTIC EXPANSION OF THE DENSITY FOR HYPOELLIPTIC ROUGH DIFFERENTIAL EQUATION

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Abstract. We study a rough differential equation driven by fractional Brownian motion with Hurst parameter H $(1/4 < H \leqslant 1/2)$. Under Hörmander's condition on the coefficient vector fields, the solution has a smooth density for each fixed time. Using Watanabe's distributional Malliavin calculus, we obtain a short time full asymptotic expansion of the density under quite natural assumptions. Our main result can be regarded as a "fractional version" of Ben Arous' famous work on the off-diagonal asymptotics.

CONTENTS

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§1. Introduction

In this paper we study from the viewpoint of Malliavin calculus the following rough differential equation (RDE) driven by d-dimensional fractional Brownian motion (fBm) (w_t) with Hurst parameter $H \in (1/4, 1/2]$.

$$dy_t = \sum_{i=1}^{d} V_i(y_t) dw_t^i + V_0(y_t)\, dt \quad \text{with } y_0 = a \in \mathbf{R}^n.$$

Here, V_i $(0 \leqslant i \leqslant d)$ is a sufficiently regular vector field on $\mathbf{R}^n$. When $H = 1/2$, fBm is the usual Brownian motion and this RDE coincides with the usual stochastic differential equation (SDE) of Stratonovich type.

Malliavin calculus for RDEs driven by fractional Brownian rough path or Gaussian rough path is a quite active topic now and a number of papers were published on it recently. See [BNOT16, BOZ15, BOZ16, BGQ16, CF10, CFV09, CHLT15, Dri13, HP13, HT13, Ina14, Ina16b] among others.

Due to [CHLT15, Ina14] and the general theory of Malliavin calculus, under Hörmander's bracket generating condition on V_i $(0 \leqslant i \leqslant d)$ at the starting point a, the solution y_t has a smooth density $p_t(a, a')$ with respect to the Lebesgue measure for every $t > 0$, that is, the function $a' \mapsto p_t(a, a')$ is smooth and $\boldsymbol{P}(y_t \in A) = \int_A p_t(a, a')\, da'$ holds for every Borel subset $A \subset \mathbf{R}^n$.

In this paper we are interested in short time asymptotics of this density function. We will prove in Theorem 2.3 a full asymptotic expansion of $p_t(a, a')$ as $t \searrow 0$ for $a \neq a'$ under, loosely speaking, Hörmander's condition at a and the "unique minimizer" condition (see (A1) and (A2) below, respectively).

This kind of short time asymptotic expansion of the density (under the unique minimizer condition) was first shown for $H \in (1/2, 1)$ in the framework of Young integration theory by [BO11, Ina16a], then for $H \in (1/3, 1/2]$ by [Ina16b] in the framework of rough path theory. These results are not completely satisfactory, however, for the following two reasons. First, from the viewpoint of rough path theory, the condition on Hurst parameter should be $H > 1/4$. Second, in these papers the ellipticity assumption on the vector fields is assumed. From the viewpoint of Malliavin calculus, it should be replaced by Hörmander's condition.

The purpose of the present paper is to generalize this kind of off-diagonal asymptotic expansion to a satisfactory form by refining the arguments in [Ina16b]. Hence, this is a continuation of [Ina16b] and our proof, just like the one in [Ina16b], is based on Watanabe's distributional Malliavin calculus [Wat87, IW89].

Due to the generalizations, however, many parts of our proof become more complicated than their counterparts in [Ina16b]. Examples are as follows. In § 3.5 we calculate the Young translation of Besov rough path for the third level case. Since we work under Hörmander's condition, we must calculate Malliavin covariance matrices more carefully in Section 5. In particular, we prove Kusuoka–Stroock type estimate (Proposition 5.3) and the uniform non-degeneracy for the scaled–shifted RDE (Proposition 5.4).

When $H = 1/2$, $p_t(a, a')$ can also be viewed as the heat kernel of the corresponding parabolic equation and has been extensively studied by the analytic and the probabilistic methods. When it comes to off-diagonal asymptotic expansion under the unique minimizer condition, Ben Arous [BA88] seems to be the most famous. In the special case $H = 1/2$, our main result (Theorem 2.3) recovers the main result in [BA88]. Therefore, Theorem 2.3 can be regarded as an "fBm-version" of [BA88].

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The organization of this paper is as follows. In Section 2 we introduce the setting and assumptions and then state our main result. In Section 3 we gather basic results in rough path theory for later use, including many probabilistic properties of fractional Brownian rough path. In Section 4, a Taylor-like expansion of Lyons–Itô map, both in the deterministic and probabilistic senses, is given. In Section 5 we apply Malliavin calculus to an RDE driven by fractional Brownian rough path with Hurst parameter $H \in (1/4, 1/2]$. Following [BOZ15], [CHLT15] and [GOT17], we carefully prove a few important propositions. Those are used in Section 6 to prove our main theorem. In Section 7 we apply our main result to concrete examples. Section 8 is devoted to showing a technical lemma.

In what follows, we use the following notation. For $p \geqslant 0$, $\lfloor p \rfloor$ denotes the integer part of p. For $1 < p < \infty$, $0 < \alpha < 1$ and a metric space E, $C^{p\text{-var}}([0,1]; E)$ and $C^{\alpha\text{-Höl}}([0,1]; E)$ stand for the space of all continuous paths of bounded p-variation and the space of all α-Hölder paths from $[0,1]$ to E, respectively. We denote by $C_o^{p\text{-var}}([0,1]; E)$ and $C_o^{\alpha\text{-Höl}}([0,1]; E)$ the subsets of $C^{p\text{-var}}([0,1]; E)$ and $C^{\alpha\text{-Höl}}([0,1]; E)$, respectively, of the paths starting from $o \in E$. For a real symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of A, respectively. A smooth vector field $F = \sum_{i=1}^{n} F^i (\partial/\partial\eta^i)$ on $\mathbf{R}^n$ is often identified with the corresponding $\mathbf{R}^n$-valued smooth function $(F^i(\eta_1, \ldots, \eta_n))_{1 \leqslant i \leqslant n}$ on $\mathbf{R}^n$. Its Jacobi matrix is denoted by ∇F, that is, $(\nabla F)_{ij} = \partial F^i/\partial\eta_j = \partial F^i/\partial\eta_j (1 \leqslant i, j \leqslant n)$.

§2. Setting and main result

2.1 Setting

In this subsection, we introduce a stochastic process that will play a main role in this paper. From now on we denote by $(w_t)_{t \geqslant 0}$ the d-dimensional fBm with Hurst parameter H. Throughout this paper we assume $1/4 < H \leqslant 1/2$. It is a unique d-dimensional, mean-zero, continuous Gaussian process with covariance $\boldsymbol{E}[w_s^i w_t^j] = \delta_{ij} R(s,t)$ $(s, t \geqslant 0)$, where

$$R(s,t) = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\}.$$

Note that, for any $c > 0$, $(w_{ct})_{t \geqslant 0}$ and $(c^H w_t)_{t \geqslant 0}$ have the same law. This property is called self-similarity or scale invariance. When $H = 1/2$, it is the usual Brownian motion. In what follows the time interval will always be $[0,1]$. It is well known that $(w_t)_{0 \leqslant t \leqslant 1}$ admits a canonical rough path lift $\boldsymbol{w} = (\boldsymbol{w}^1, \ldots, \boldsymbol{w}^{\lfloor 1/H \rfloor})$, which is called fractional Brownian rough path [CQ02].

Let $V_i : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞, that is, V_i is a bounded smooth function with bounded derivatives of all order $(0 \leqslant i \leqslant d)$. We shall identify V_i with its corresponding vector field and denote the vector field by the same symbol. We consider the following RDE:

$$dy_t = \sum_{i=1}^{d} V_i(y_t)\, dw_t^i + V_0(y_t)\, dt \quad \text{with } y_0 = a, \tag{2.1}$$

where $a \in \mathbf{R}^n$ is a deterministic initial point. This RDE is driven by the Young pairing $(\boldsymbol{w}, \boldsymbol{\lambda})$, where $\lambda_t = t$. The unique solution is denoted by $\boldsymbol{y} = (\boldsymbol{y}^1, \ldots, \boldsymbol{y}^{\lfloor 1/H \rfloor})$ and we set $y_t := a + \boldsymbol{y}_{0,t}^1$ as usual. We will sometimes write $y_t = y_t(a) = y_t(a, \boldsymbol{w})$ etc. to make explicit the dependence on a and $\boldsymbol{w}$. We often use a matrix notation for (2.1), that is, we set $b = V_0$ and $\sigma = [V_1, \ldots, V_d]$, which is $n \times d$ matrix-valued, and write (2.1) as

$$dy_t = \sigma(y_t)\, dw_t + b(y_t)\, dt \quad \text{with } y_0 = a.$$

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2.2 Assumptions

In this subsection, we introduce assumptions of the main theorem. First, we introduce the Hörmander condition:

DEFINITION 2.1. (Hörmander condition) Set

$$\mathcal{V}_m = \begin{cases} \{V_i \mid 1 \leqslant i \leqslant d\}, & m = 0, \\ \{[V_i, U] \mid U \in \mathcal{V}_{m-1}, 0 \leqslant i \leqslant d\}, & m \geqslant 1, \end{cases} \qquad \mathcal{V} = \bigcup_{m=0}^{\infty} \mathcal{V}_m.$$

For $x \in \mathbf{R}^n$, $\mathcal{V}_m(x)$ (resp. $\mathcal{V}(x)$) stands for the subset of $\mathbf{R}^n$ obtained by plugging x into the vector field of $\mathcal{V}_m$ (resp. $\mathcal{V}$). We say that the vector fields $V_0, V_1, \dots, V_d$ satisfy the Hörmander condition at $x \in \mathbf{R}^n$ if $\mathcal{V}(x)$ linearly spans $\mathbf{R}^n$.

We assume the following:

(A1) $V_0, V_1, \dots, V_d$ satisfy the Hörmander condition at the initial point $a \in \mathbf{R}^n$.

It is known that, under (A1), the law of the solution y_t has a density $p_t(a, a')$ with respect to the Lebesgue measure on $\mathbf{R}^n$ for any $t > 0$ (see [HP13, CHLT15]). Hence, for any Borel subset $A \subset \mathbf{R}^n$, $\boldsymbol{P}(y_t(a) \in A) = \int_A p_t(a, a')\, da'$.

Let $\mathfrak{H} = \mathfrak{H}^H$ be the Cameron–Martin space of fBm on the time interval $[0, 1]$. For every $H \in (1/4, 1/2]$, there exists $q \in [1, 2)$ such that every $\gamma \in \mathfrak{H}$ is continuous and of finite q-variation. For $\gamma \in \mathfrak{H}$, we denote by $\phi_t^0 = \phi_t^0(\gamma)$ the solution of the following Young ordinary differential equation (ODE):

$$d\phi_t^0 = \sum_{i=1}^{d} V_i(\phi_t^0)\, d\gamma_t^i \quad \text{with } \phi_0^0 = a. \tag{2.2}$$

Set, for $a' \neq a$, $K_a^{a'} = \{\gamma \in \mathfrak{H} \mid \phi_1^0(\gamma) = a'\}$. We only consider the case where $K_a^{a'}$ is not empty. From goodness of the rate function in Schilder-type large deviation for fractional Brownian rough path, it follows that $\inf\{\|\gamma\|_{\mathfrak{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathfrak{H}} \mid \gamma \in K_a^{a'}\}$. Now we introduce the following assumption:

(A2) $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathfrak{H}$-norm exists uniquely.

In what follows, $\bar{\gamma}$ denotes the minimizer in (A2). Note that the map $\gamma \in \mathfrak{H} \hookrightarrow C^{q\text{-var}}([0, 1]; \mathbf{R}^d) \mapsto \phi_1^0(\gamma) \in \mathbf{R}^n$ is Fréchet differentiable; $D\phi_1^0(\gamma)$ stands for the Fréchet derivative and is expressed by

$$\langle D[\phi_1^0(\gamma)]^k, h\rangle_{\mathfrak{H}} = \sum_{i=1}^{d} \int_0^1 [J_1(\gamma) K_s(\gamma) \sigma(\phi_s^0(\gamma))]_i^k\, dh_s^i, \tag{2.3}$$

where $[\bullet]^k$ and $[\bullet]_i^k$ are the kth component of the vector and the (k, i)-component of the matrix, respectively, and $J(\gamma)$ and $K(\gamma)$ are the Jacobi process of $\phi^0(\gamma)$ and its inverse, respectively. We define the deterministic Malliavin covariance matrix $Q(\gamma) = (Q(\gamma)_{kl})_{1 \leqslant k, l \leqslant n}$ by

$$Q(\gamma)_{kl} = \langle D[\phi_1^0(\gamma)]^k, D[\phi_1^0(\gamma)]^l\rangle_{\mathfrak{H}^*}. \tag{2.4}$$

We assume that Q is nondegenerate at the minimizer $\bar{\gamma}$:

(A3) There is a positive constant c such that $Q(\bar{\gamma}) \geqslant cI$, where I stands for the identity matrix.

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Finally, we assume that $\| \bullet \|_{\mathfrak{H}}^2/2$ is not so degenerate at $\bar{\gamma}$ in the following sense:
(A4) At $\bar{\gamma}$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathfrak{H}}^2/2$ is strictly positive in the quadratic form sense. More precisely, if $(-\epsilon_0, \epsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$, then $(d/du)^2|_{u=0}\|f(u)\|_{\mathfrak{H}}^2/2 > 0$.

REMARK 2.2. By a standard argument, (A3) is equivalent to the surjectivity of the tangent map $D\phi_1^0(\bar{\gamma})\colon \mathfrak{H} \to \mathbf{R}^n$. By the implicit function theorem, this implies that $K_a^{a'}$ has a Hilbert-manifold structure near $\bar{\gamma}$. Therefore, "Hessian at $\bar{\gamma}$" and "a smooth curve near $\bar{\gamma}$" make sense.

2.3 Index sets

In this subsection we introduce several index sets for the exponent of the small time parameter $t > 0$, which will be used in the asymptotic expansion. Unfortunately, these index sets are not subsets of $\mathbf{N} = \{0, 1, 2, \ldots\}$ and are rather complicated in general. (However, they are discrete subsets of $(\mathbf{Z} + H^{-1}\mathbf{Z}) \cap [0, \infty)$ with the minimum 0.)

Set

$$\Lambda_1 = \left\{ n_1 + \frac{n_2}{H} \,\middle|\, n_1, n_2 \in \mathbf{N} \right\}.$$

We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ all the elements of Λ_1 in increasing order. Several smallest elements are explicitly given as follows: for $1/3 < H < 1/2$,

$$\kappa_1 = 1, \qquad \kappa_2 = 2, \qquad \kappa_3 = \frac{1}{H}, \qquad \kappa_4 = 3, \qquad \kappa_5 = 1 + \frac{1}{H}, \qquad \kappa_6 = 4, \ldots$$

and, for $1/4 < H < 1/3$,

$$\kappa_1 = 1, \qquad \kappa_2 = 2, \qquad \kappa_3 = 3, \qquad \kappa_4 = \frac{1}{H}, \qquad \kappa_5 = 4, \qquad \kappa_6 = 1 + \frac{1}{H}, \ldots.$$

We also set

$$\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{\kappa_0\}\} = \left\{0, 1, \frac{1}{H} - 1, 2, \frac{1}{H}, 3, \ldots\right\},$$
$$\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{\kappa_0, \kappa_1\}\} = \left\{0, \frac{1}{H} - 2, 1, \frac{1}{H} - 1, 2, \ldots\right\}.$$

Note that in the above explicit expression of these two index sets the elements are not sorted in increasing order when $1/4 < H \leqslant 1/3$. Next we set

$$\Lambda_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2\},$$
$$\Lambda_3' = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2'\},$$

where $\mathbf{N}_+ = \{1, 2, \ldots\}$. In the sequel, $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$ and $0 = \rho_0 < \rho_1 < \rho_2 < \cdots$ stand for all the elements of Λ_3 and Λ_3' in increasing order, respectively. Finally,

$$\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}.$$

We denote by $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ all the elements of Λ_4 in increasing order.

We remark that when $H = 1/2$ and $H = 1/3$, all these index sets Λ_i, Λ_j' above are just $\mathbf{N}$.

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2.4 Statement of the main result

Now we state our main theorem. This is basically analogous to many preceding works on the standard Brownian motion such as [BA88, Wat87]. However, when $H \neq 1/2, 1/3$ and the drift term exists, there are some differences. First, the exponents of t are not (a constant multiple of) natural numbers. Second, cancelation of "odd terms" (see e.g., [Wat87, pp. 20 and 34]) does not occur in general. (These phenomena were already observed in [Ina16a] and [Ina16b] when $H \in (1/3, 1)$.) Our proof uses the Watanabe distribution theory. Therefore, the asymptotic expansion is actually obtained at the level of Watanabe distributions.

THEOREM 2.3. *Assume $a \neq a'$ and* (A1)–(A4). *Then, we have the following asymptotic expansion as $t \searrow 0$:*

$$p_t(a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2t^{2H}}\right)\frac{1}{t^{nH}}\{\alpha_0 + \alpha_{\lambda_1}t^{\lambda_1 H} + \alpha_{\lambda_2}t^{\lambda_2 H} + \cdots\}$$

for a certain positive constant α_0 and certain real constants α_{λ_j} $(j = 1, 2, \ldots)$. Here, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ are all the elements of Λ_4 in increasing order.

The above result improves that in [Ina16b]. The reason is as follows. First, we work under Hörmander condition instead of the ellipticity assumption. Second, the case $1/4 < H \leqslant 1/3$ is also treated. Hence, we must calculate the third level rough paths.

REMARK 2.4. If RDE (2.1) has no drift term, that is, $V_0 \equiv 0$, then we can replace Λ_4 by $2\mathbf{N}$ in Theorem 2.3, namely,

$$p_t(a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2t^{2H}}\right)\frac{1}{t^{nH}}\{\alpha_0 + \alpha_2 t^{2H} + \alpha_4 t^{4H} + \cdots\} \tag{2.5}$$

as $t \searrow 0$. The reason is as follows. First, if $V_0 \equiv 0$, the index sets Λ_i, Λ'_j $(1 \leqslant i \leqslant 4, 2 \leqslant j \leqslant 3)$ in the sequel can be replaced by $\mathbf{N}$. Second, the odd-numbered terms in the asymptotics of the generalized Wiener functional are also odd as generalized Wiener functionals. Hence, their generalized expectations all vanish.

When $H = 1/2$, (2.5) holds for the same reason. Thus, we reprove the main result of Ben Arous [BA88] via rough path theory.

§3. Preliminaries

3.1 Rough path analysis

In this paper we basically work in Lyons' original framework of rough path theory. (The only exception is § 5.2, where the controlled path theory is used.) We borrow most of notations and terminologies from [LQ02, LCL07, FV10]. Let $p \in [2, 4)$ be the roughness constant and let $q \in [1, 2)$ be such that $1/p + 1/q > 1$. Set $\alpha = 1/p$ and let $m \in \mathbf{N}_+$ satisfy $\alpha - 1/m > 0$. We denote by $G\Omega_p(\mathbf{R}^d)$, $G\Omega^{\mathrm{H}}_{\alpha}(\mathbf{R}^d)$, and $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)$ the geometric rough path spaces with p-variation, α-Hölder and (α, m)-Besov topologies, respectively. In this paper, $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{\lfloor p \rfloor})$ stands for a generic element in these rough path spaces. Recall that the p-variation, α-Hölder and (α, m)-Besov topologies are induced by

$$\|\boldsymbol{x}^i\|_{p/i\text{-var}} = \sup_{0=t_0<\cdots<t_K=1}\left(\sum_{k=1}^K |\boldsymbol{x}^i_{t_{k-1},t_k}|^{p/i}\right)^{i/p},$$

$$\|\boldsymbol{x}^i\|_{i\alpha\text{-Höl}} = \sup_{0\leqslant s<t\leqslant 1}\frac{|\boldsymbol{x}^i_{s,t}|}{|t-s|^{i\alpha}},$$

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$$\|\boldsymbol{x}^i\|_{(i\alpha,m/i)\text{-Bes}} = \left(\iint_{0\leqslant s<t\leqslant 1} \frac{|\boldsymbol{x}^i_{s,t}|^{m/i}}{|t-s|^{1+i\alpha\cdot m/i}}\, dsdt \right)^{i/m} \quad (i=1,\dots,\lfloor p \rfloor).$$

Next we introduce homogeneous norms to $G\Omega_p(\mathbf{R}^d)$, $G\Omega^{\mathrm{H}}_\alpha(\mathbf{R}^d)$, and $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)$ which are consistent with their topology. More explicitly, they are given by

$$\|\boldsymbol{x}\|_{p\text{-var}} = \sum_{i=1}^{\lfloor p \rfloor} \|\boldsymbol{x}^i\|^{1/i}_{p/i\text{-var}}, \qquad \|\boldsymbol{x}\|_{\alpha\text{-Höl}} = \sum_{i=1}^{\lfloor 1/\alpha \rfloor} \|\boldsymbol{x}^i\|^{1/i}_{i\alpha\text{-Höl}},$$

$$\|\boldsymbol{x}\|_{(\alpha,m)\text{-Bes}} = \sum_{i=1}^{\lfloor 1/\alpha \rfloor} \|\boldsymbol{x}^i\|^{1/i}_{(i\alpha,m/i)\text{-Bes}}.$$

We denote by $\|\boldsymbol{x}^i\|_{p/i\text{-var};[s,t]}$, $\|\boldsymbol{x}\|_{p\text{-var};[s,t]}$, etc. the norms restricted on the subinterval $[s,t]$.

For a rough path $\boldsymbol{x}$, we write $x_t = \boldsymbol{x}^1_{0,t}$ as usual. For $x \in C_0^{r\text{-var}}([0,1];\mathbf{R}^d)$ with $r\in[1,2)$, we denote the natural lift of x (i.e., the smooth rough path lying above x) by the corresponding boldface letter $\boldsymbol{x}$. Note that, for $x \in C_0^{r\text{-var}}([0,1];\mathbf{R}^d)$ and $y \in C_0^{r\text{-var}}([0,1];\mathbf{R}^e)$, $(\boldsymbol{x},\boldsymbol{y}) \in G\Omega_p(\mathbf{R}^{d+e})$ stands for the natural lift of $(x,y) \in C_0^{r\text{-var}}([0,1];\mathbf{R}^{d+e})$, not for the pair $(\boldsymbol{x},\boldsymbol{y}) \in G\Omega_p(\mathbf{R}^d)\times G\Omega_p(\mathbf{R}^e)$. In a similar way, for $\boldsymbol{x}\in G\Omega_p(\mathbf{R}^d)$ and $h \in C_0^{q\text{-var}}(\mathbf{R}^e)$ with $1/p+1/q>1$, $(\boldsymbol{x},\boldsymbol{h}) \in G\Omega_p(\mathbf{R}^{d+e})$ stands for the Young pairing. For $\boldsymbol{x}\in G\Omega_p(\mathbf{R}^d)$ and $h\in C_0^{q\text{-var}}(\mathbf{R}^d)$ with $1/p+1/q>1$, $\boldsymbol{x}+\boldsymbol{h} = \tau_h(\boldsymbol{x}) \in G\Omega_p(\mathbf{R}^d)$ stands for the Young translation. (See [FV10, Section 9.4].) We denote by $c\boldsymbol{x}$ the dilation of a rough path $\boldsymbol{x}$ by $c\in\mathbf{R}$. These notations may be somewhat misleading. But, they make many operations intuitively clear and easy to understand when we treat rough paths over a direct sum of many vector spaces.

Note that the spaces $G\Omega_p(\mathbf{R}^d)$, $G\Omega^{\mathrm{H}}_\alpha(\mathbf{R}^d)$, and $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)$ enjoy the next continuous embeddings. We have $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d) \hookrightarrow G\Omega_p(\mathbf{R}^d)$ and

$$\|\boldsymbol{x}\|_{p\text{-var};[s,t]} \leqslant c\|\boldsymbol{x}\|_{(\alpha,m)\text{-Bes}}(t-s)^{\alpha-(1/m)} \quad (\boldsymbol{x}\in G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)) \tag{3.1}$$

from [FV10, Corollary A.3]. (The continuity of this embedding is not explicitly written in [FV10], but can be shown by standard and slightly lengthy argument.) For $\alpha<\beta\leqslant 1/\lfloor 1/\alpha\rfloor$, a direct computation implies Besov–Hölder embedding $G\Omega^{\mathrm{H}}_\beta(\mathbf{R}^d) \hookrightarrow G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)$.

For a control function ω in the sense of [LQ02, p. 16], we write $\bar\omega := \omega(0,1)$. For any $\boldsymbol{x}\in G\Omega_p(\mathbf{R}^d)$,

$$\omega_{\boldsymbol{x}}(s,t) := \sum_{i=1}^{\lfloor p\rfloor} \|\boldsymbol{x}^i\|^{p/i}_{p/i\text{-var};[s,t]} \quad (0\leqslant s\leqslant t\leqslant 1) \tag{3.2}$$

defines a control function. (This control function is equivalent to the one defined by Carnot–Caratheodory metric.) Similarly, we set $\omega_h(s,t) := \|h\|^q_{q\text{-var};[s,t]}$ for $h\in C_0^{q\text{-var}}([0,1];\mathbf{R}^e)$.

For $\delta>0$ and $\boldsymbol{x}\in G\Omega_p(\mathbf{R}^d)$, set $\tau_0(\delta)=0$ and

$$\tau_{i+1}(\delta) = \inf\{t\in(\tau_i(\delta),1] \mid \omega_{\boldsymbol{x}}(\tau_i(\delta),t)\geqslant\delta\}\wedge 1 \quad (i=1,2,\dots).$$

Define

$$N_\delta(\boldsymbol{x}) = \sup\{i\in\mathbf{N} \mid \tau_i(\delta)<1\}. \tag{3.3}$$

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Superadditivity of $\omega_{\boldsymbol{x}}$ yields $\delta N_\delta(\boldsymbol{x}) \leqslant \overline{\omega_{\boldsymbol{x}}}$. This quantity (3.3) was first studied by Cass, Litterer, and Lyons [CLL13].

Let $\boldsymbol{x} \in G\Omega_p(\mathbf{R}^{d+1})$. We consider an RDE

$$dy_t = \sum_{i=0}^{d} V_i(y_t)\, dx_t^i \quad \text{with } y_0 = a. \tag{3.4}$$

It is well known that the solution map $G\Omega_p(\mathbf{R}^{d+1}) \ni \boldsymbol{x} \mapsto \boldsymbol{y} = (\boldsymbol{y}^1, \ldots, \boldsymbol{y}^{\lfloor p \rfloor}) \in G\Omega_p(\mathbf{R}^n)$ is continuous. Set $\Phi(\boldsymbol{x}) = \boldsymbol{y}$ and $y_t = a + \boldsymbol{y}^1_{0,t}$. The Jacobi process $J = ((J_t^{kl} = \frac{dy_t^k}{da^l})_{t\in[0,1]})_{1\leqslant k,l\leqslant n}$ and its inverse $K = (K^{kl})_{1\leqslant k,l\leqslant n}$ exist and satisfy

$$dJ_t = +\sum_{i=0}^{d} \nabla V_i(y_t) J_t\, dx_t^i \quad \text{with } J_0 = I, \tag{3.5}$$

$$dK_t = -\sum_{i=0}^{d} K_t \nabla V_i(y_t)\, dx_t^i \quad \text{with } K_0 = I. \tag{3.6}$$

Here, I stands for the identity matrix with size n. It is well known that the system of RDEs (3.4)–(3.6) is globally well-posed. In particular, the map $\boldsymbol{x} \mapsto (\boldsymbol{y}, \boldsymbol{J}, \boldsymbol{K})$ is continuous.

Here, we make a remark on continuity of the solution map. Let λ be a one-dimensional path defined by $\lambda_t = t$. The solution map $\boldsymbol{x} \mapsto (\boldsymbol{y}, \boldsymbol{J}, \boldsymbol{K})$ is continuous on $G\Omega_p(\mathbf{R}^{d+1})$ as stated above. In addition, $G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$ is embedded in $G\Omega_p(\mathbf{R}^{d+1})$ continuously. Hence, the composition $(\boldsymbol{x}, c\lambda) \mapsto (\boldsymbol{x}, c\boldsymbol{\lambda}) \mapsto (\boldsymbol{y}, \boldsymbol{J}, \boldsymbol{K})$ of the two maps is continuous on $G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$. This continuity plays an important role in §§ 6.1 and 5.3, in which we will use results on large deviation for fractional Brownian rough path.

3.2 RDEs driven by fBm

In this paper, we consider the case $x = (w, \lambda)$, where w is an fBm with the Hurst parameter $1/4 < H \leqslant 1/2$ and λ is a one-dimensional path defined by $\lambda_t = t$. Let p be larger than but sufficiently close to $1/H$. Since w admits a canonical lift $\boldsymbol{w} \in G\Omega_p(\mathbf{R}^d)$ (see [CQ02]), we can deal with (2.1) in the framework of rough path analysis and obtain $\boldsymbol{y} = \Phi(\boldsymbol{w}, \boldsymbol{\lambda})$. Next we introduce a scaled (and shifted) RDE of (2.1). For every $0 < \epsilon < 1$ and $\gamma \in \mathfrak{H}$, we consider a scaled RDE

$$dy_t^\epsilon = \sum_{i=1}^{d} V_i(y_t^\epsilon)\, d(\epsilon w)_t^i + V_0(y_t^\epsilon)\, d(\epsilon^{1/H} t) \quad \text{with } y_0 = a, \tag{3.7}$$

and a scaled–shifted RDE

$$d\tilde{y}_t^\epsilon = \sum_{i=1}^{d} V_i(\tilde{y}_t^\epsilon)\, d(\epsilon w + \gamma)_t^i + V_0(\tilde{y}_t^\epsilon)\, d(\epsilon^{1/H} t) \quad \text{with } y_0 = a. \tag{3.8}$$

The solutions are respectively given by

$$\boldsymbol{y}^\epsilon = \Phi(\epsilon \boldsymbol{w}, \epsilon^{1/H} \boldsymbol{\lambda}), \qquad \tilde{\boldsymbol{y}}^\epsilon = \Phi(\epsilon \boldsymbol{w} + \boldsymbol{\gamma}, \epsilon^{1/H} \boldsymbol{\lambda}).$$

It is known that the processes (y_t^ϵ) and $(y_{\epsilon^{1/H} t})$ have the same law. We denote the Jacobi processes and its inverses of y^ϵ and $\tilde{y}^\epsilon$ by J^ϵ, K^ϵ, $\tilde{J}^\epsilon$ and $\tilde{K}^\epsilon$. (In the next subsection, a brief summary of the Cameron–Martin space $\mathfrak{H}$ and the Young translation by $\gamma \in \mathfrak{H}$ will be given.)

We remark that everything in this subsection still holds if p-variation topology is replaced by $1/p$-Hölder topology. See [FV10, FV06].

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3.3 The Cameron–Martin space and its isometric space

Next we introduce the Cameron–Martin space $\mathfrak{H}$ associated to a d-dimensional fBm on the time interval $[0,1]$. Let $\mathfrak{E}_1$ be the linear span of $\{R(t,\bullet)\}_{t\in[0,1]}$ and define the inner product

$$\left\langle \sum_{k=1}^m a_k R(s_k,\bullet), \sum_{l=1}^n b_l R(t_l,\bullet)\right\rangle_{\mathfrak{E}_1} = \sum_{k=1}^m\sum_{l=1}^n a_k b_l R(s_k,t_l).$$

The Cameron–Martin space $\mathfrak{H}$ associated to a d-dimensional fBm is given by the completion of $\mathfrak{E}=\mathfrak{E}_1^d$ with respect to the norm $\|\bullet\|_{\mathfrak{H}}=\langle\bullet,\bullet\rangle_{\mathfrak{H}}$, where $\langle f,g\rangle_{\mathfrak{H}}=\sum_{i=1}^d\langle f^i,g^i\rangle_{\mathfrak{E}_1}$ for $f=(f^1,\dots,f^d)^\top$, $g=(g^1,\dots,g^d)^\top\in\mathfrak{E}$. Note that $\mathfrak{H}$ is unitarily isometric to the first Wiener chaos $\mathcal{C}_1$, which is the Hilbert space defined by the $\|\bullet\|_{L^2(\Omega;\mathbf{R})}$-completion of the linear span of $(w_t)_{t\in[0,1]}$. From [FV06, Corollary 1], we see the continuous embedding $\mathfrak{H}\hookrightarrow C_0^{q\text{-var}}([0,1];\mathbf{R}^d)$ for $(H+1/2)^{-1}<q<2$. According to [FGGR16], the embedding holds for $q=(H+1/2)^{-1}$. (Note that if $p>1/H$ is sufficiently close to $1/H$, then $1/p+1/q>1$ holds since $H>1/4$. Therefore, the Young translation by $\gamma\in\mathfrak{H}$ is well-defined on $G\Omega_p(\mathbf{R}^d)$.)

Next we introduce another Hilbert space $\tilde{\mathfrak{H}}$, which is isometric to $\mathfrak{H}$. Let $\tilde{\mathfrak{E}}_1$ be the linear span of $\{\mathbb{1}_{[0,t]}\}_{t\in[0,1]}$ and define an inner product by

$$\left\langle \sum_{k=1}^m a_k \mathbb{1}_{[0,s_k]}, \sum_{l=1}^n b_l \mathbb{1}_{[0,t_l]}\right\rangle_{\tilde{\mathfrak{E}}_1} = \sum_{k=1}^m\sum_{l=1}^n a_k b_l R(s_k,t_l).$$

We define $\tilde{\mathfrak{H}}$ by the completion of $\tilde{\mathfrak{E}}=\tilde{\mathfrak{E}}_1^d$ with respect to the norm $\|\bullet\|_{\tilde{\mathfrak{H}}}=\langle\bullet,\bullet\rangle_{\tilde{\mathfrak{H}}}$, where $\langle f,g\rangle_{\tilde{\mathfrak{H}}}=\sum_{i=1}^d\langle f^i,g^i\rangle_{\tilde{\mathfrak{E}}_1}$ for $f=(f^1,\dots,f^d)^\top$, $g=(g^1,\dots,g^d)^\top\in\tilde{\mathfrak{E}}$. Note that $\tilde{\mathfrak{H}}=I_{1-}^{1/2-H}(L^2([0,1];\mathbf{R}^d))$ from [DÜ99] and [Nua06, p. 284]. Here, $I_{1-}^{1/2-H}$ denotes the fractional integral of order $1/2-H$. Hence, the inclusions $C^{(H-\delta)\text{-Höl}}([0,1];\mathbf{R}^d)\subset\tilde{\mathfrak{H}}\subset L^2([0,1];\mathbf{R}^d)$ hold for $0<\delta<2(H-1/4)$. The linear map $\mathcal{R}=(\mathcal{R}^1,\dots,\mathcal{R}^d)^\top:\tilde{\mathcal{E}}\to\mathfrak{H}$ defined by $\mathcal{R}^i\mathbb{1}_{[0,t]}=R(t,\bullet)$ extends to a unitary isometry from $\tilde{\mathfrak{H}}$ to $\mathfrak{H}$.

REMARK 3.1. Under the condition $1/4<H\leqslant 1/2$, we can choose two numbers $1/H<p<1+\lfloor 1/H\rfloor$ and $(H+1/2)^{-1}<q<2$ with $1/p+1/q>1$. For every $A=(A_1,\dots,A_d)\in C^{p\text{-var}}([0,1];\mathbf{R}^d)$, we can define $\phi_A\in\mathfrak{H}^*$ by

$$\phi_A(h)=\sum_{i=1}^d\int_0^1 A_i(s)\,dh_s^i,$$

where the right-hand side is the Young integral since $h\in\mathfrak{H}\subset C_0^{q\text{-var}}([0,1];\mathbf{R}^d)$. By using the isometry between $\mathfrak{H}^*$ and $\mathcal{C}_1$, the definition of $\tilde{\mathfrak{H}}$ and the inequality $\|\bullet\|_{L^2([0,1];\mathbf{R}^d)}\leqslant c\|\bullet\|_{\tilde{\mathfrak{H}}}$ for some positive constant c, we have

$$\|\phi_A\|_{\mathfrak{H}^*}^2=\boldsymbol{E}\left[\left(\sum_{i=1}^d\int_0^1 A_i(s)\,dw_s^i\right)^2\right]=\|A\|_{\tilde{\mathfrak{H}}}^2\geqslant c^{-2}\|A\|^2_{L^2([0,1];\mathbf{R}^d)}.$$

3.4 Large deviations

Let $(1+\lfloor 1/H\rfloor)^{-1}<\alpha<\beta<H$ and $m\in\mathbf{N}_+$ satisfy $\alpha-1/m>0$. Recall that fBm (w_t) admits a natural lift almost surely via the dyadic piecewise linear approximation and the lift $\boldsymbol{w}$ is a random variable taking values in $G\Omega_\beta^{\mathrm{H}}(\mathbf{R}^d)$. For $0<\epsilon<1$, the law of $\epsilon\boldsymbol{w}$ on this space is denoted by $\nu_\epsilon=\nu_\epsilon^H$. Note that the lift of Cameron–Martin space $\mathfrak{H}$ is contained in

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$G\Omega^{\mathrm{H}}_{\beta}(\mathbf{R}^d)$. Moreover, as $\epsilon \searrow 0$, Schilder-type large deviations hold for $\{\nu_\epsilon\}_{\epsilon>0}$. (See [FV07] and [FV10, Sections 13.6 and 15.7].) Because of the Besov–Hölder embedding mentioned above, these properties also hold in $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d)$. As usual, the good rate function $\mathcal{I}$ is given as follows: $\mathcal{I}(\boldsymbol{x}) = \|h\|^2_{\mathfrak{H}}/2$ if $\boldsymbol{x}$ is the lift of some $h \in \mathfrak{H}$ and $\mathcal{I}(\boldsymbol{x}) = \infty$ if otherwise.

Next, set $\hat{\nu}_\epsilon = \nu_\epsilon \otimes \delta_{\epsilon^{1/H}\lambda}$, where λ is a one-dimensional path defined by $\lambda_t = t$ and $\otimes$ stands for the product of probability measures. This measure is supported on $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$. The Young pairing map $G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle \to G\Omega^{\mathrm{B}}_{\alpha,m}(\mathbf{R}^{d+1})$ is continuous. The law of this map under $\hat{\nu}_\epsilon$ is the law of $(\epsilon\boldsymbol{w}, \epsilon^{1/H}\boldsymbol{\lambda})$, the Young pairing of $\epsilon\boldsymbol{w}$ and $\epsilon^{1/H}\lambda$. Define $\hat{\mathcal{I}}(\boldsymbol{x}, l) = \|h\|^2_{\mathfrak{H}}/2$ if $\boldsymbol{x}$ is the lift of some $h \in \mathfrak{H}$ and $l_t \equiv 0$ and define $\hat{\mathcal{I}}(\boldsymbol{x}, l) = \infty$ if otherwise. Here, l is a one-dimensional path. We can easily show that $\{\hat{\nu}_\epsilon\}_{\epsilon>0}$ also satisfies a large deviation principle as $\epsilon \searrow 0$ with a good rate function $\hat{\mathcal{I}}$. We will use this in Lemma 6.1 to show that we may localize on a neighborhood of the minimizer $\bar{\gamma}$ in order to obtain the asymptotic expansion. (The existence of $\bar{\gamma}$ is assumed in (A2).)

3.5 The Young translation

In this subsection, we show that the Young translation is well-defined and continuous. In this subsection, we fix $1/4 < H \leqslant 1/2$. Let $(1 + \lfloor 1/H \rfloor)^{-1} < \alpha < H$ and choose $m \in \mathbf{N}_+$ such that $H - \alpha > 2/m$. Set $p = 1/\alpha$ and $q = (H + 1/2 - 1/m)^{-1}$. Then we have $1/p + 1/q > 1$ and $1/q - 1/2 - \alpha > 1/m$. Since $(H + 1/2)^{-1} \leqslant q < 2$, [FV06, Corollary 1] implies

$$\|\gamma\|_{q\text{-var};[s,t]} \leqslant c\|\gamma\|_{W^{1/q,2}}(t-s)^{(1/q-1/2)} \leqslant c\|\gamma\|_{\mathfrak{H}}(t-s)^{(1/q-1/2)} \quad (\gamma \in \mathfrak{H}).$$

Throughout this subsection, c denotes a positive constant and may change from line to line.

We define the Young translation $\tau = (\tau^1, \ldots, \tau^{\lfloor 1/\alpha \rfloor})$ from $G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d) \times \mathfrak{H}$ to $G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d)$; for $\boldsymbol{x} \in G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d)$ and $\gamma \in \mathfrak{H}$, define

$$\begin{aligned}
\tau_\gamma(\boldsymbol{x})^1_{s,t} &= \boldsymbol{x}^1_{s,t} + \boldsymbol{\gamma}^1_{s,t},\\
\tau_\gamma(\boldsymbol{x})^2_{s,t} &= \boldsymbol{x}^2_{s,t} + A^1_{s,t} + A^2_{s,t} + \boldsymbol{\gamma}^2_{s,t},\\
\tau_\gamma(\boldsymbol{x})^3_{s,t} &= \boldsymbol{x}^3_{s,t} + B^1_{s,t} + B^2_{s,t} + B^3_{s,t} + C^1_{s,t} + C^2_{s,t} + C^3_{s,t} + \boldsymbol{\gamma}^3_{s,t},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\gamma}^1_{s,t} &= \gamma_t - \gamma_s, & \boldsymbol{\gamma}^i_{s,t} &= \int_s^t \boldsymbol{\gamma}^{i-1}_{s,u} \otimes d\gamma_u \quad (i = 2, 3),\\
A^1_{s,t} &= \int_s^t \boldsymbol{x}^1_{s,u} \otimes d\gamma_u, & A^2_{s,t} &= \int_s^t \boldsymbol{\gamma}^1_{s,u} \otimes dx_u,\\
B^1_{s,t} &= \int_s^t \boldsymbol{x}^2_{s,u} \otimes d\gamma_u, & B^2_{s,t} &= \int_s^t \int_s^v \boldsymbol{x}^1_{s,u} \otimes d\gamma_u \otimes dx_v,\\
B^3_{s,t} &= -\int_s^t \boldsymbol{\gamma}^1_{s,u} \otimes d\boldsymbol{x}^2_{u,t}, & C^1_{s,t} &= -\int_s^t \boldsymbol{x}^1_{s,u} \otimes d\boldsymbol{\gamma}^2_{u,t},\\
C^2_{s,t} &= \int_s^t \int_s^v \boldsymbol{\gamma}^1_{s,u} \otimes dx_u \otimes d\gamma_v, & C^3_{s,t} &= \int_s^t \boldsymbol{\gamma}^2_{s,u} \otimes dx_u.
\end{aligned}$$

We should understand the integrals in the Young sense since $1/p + 1/q > 1$. At first glance, B^3 and C^1 look strange. However, if x and γ are smooth and $\boldsymbol{x}$ is the natural lift of x, we have

$$B^3_{s,t} = \int_s^t \boldsymbol{\gamma}^1_{s,u} \otimes dx_u \otimes (x_t - x_u) = \int_s^t \int_s^v \boldsymbol{\gamma}^1_{s,u} \otimes dx_u \otimes dx_v.$$

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In the first equality, we used $\frac{d\boldsymbol{x}^2_{u,t}}{du} = -\frac{dx_u}{du} \otimes (x_t - x_u)$. Since a similar equality for $C^3_{s,t}$ holds, we see B^3 and C^1 are defined appropriately. Then we see the next proposition.

PROPOSITION 3.2. *Let H, α and m be as above. Then, the Young translation $\tau\colon G\Omega^B_{\alpha,12m}(\mathbf{R}^d) \times \mathfrak{H} \to G\Omega^B_{\alpha,12m}(\mathbf{R}^d)$ is well-defined and continuous. Moreover, restricted to every bounded subset of the domain, τ is Lipschitz continuous.*

Proof. Let $0 \leqslant s \leqslant u < v \leqslant t \leqslant 1$. Recall that $\boldsymbol{x}^i_{s,\cdot}$ and $\boldsymbol{x}^i_{\cdot,t}$ have finite p-variation and that $\|\boldsymbol{x}^i_{s,\cdot}\|_{p\text{-var};[u,v]}$ and $\|\boldsymbol{x}^i_{\cdot,t}\|_{p\text{-var};[u,v]}$ are bounded above by $c\|\boldsymbol{x}\|^{i-1}_{p\text{-var};[s,t]}\|\boldsymbol{x}\|_{p\text{-var};[u,v]}$ for $i = 1, 2, 3$.

We show well-definedness of the map τ. Since $1/q + 1/q > 1$, we can define $\boldsymbol{\gamma}^2$ in the Young sense and see that it is of finite q-variation and satisfies

$$\|\boldsymbol{\gamma}^2_{s,\cdot}\|_{q\text{-var};[u,v]},\ \ \|\boldsymbol{\gamma}^2_{\cdot,t}\|_{q\text{-var};[u,v]} \leqslant c\|\gamma\|_{q\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[u,v]}.$$

In this estimate, we used [FV10, Theorem 6.8]. By the same reason, A^1 and A^2 are well-defined in the Young sense and they are of finite q- and p-variation, respectively; more precisely, it holds that

$$\begin{aligned}\|A^1_{s,\cdot}\|_{q\text{-var};[u,v]} &\leqslant c\|x\|_{p\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[u,v]},\\ \|A^2_{s,\cdot}\|_{p\text{-var};[u,v]} &\leqslant c\|\gamma\|_{q\text{-var};[s,t]}\|x\|_{p\text{-var};[u,v]}.\end{aligned}$$

Note that $B^2_{s,t} = \int_s^t A^1_{s,r} \otimes dx_r$ and $C^2_{s,t} = \int_s^t A^2_{s,r} \otimes d\gamma_r$ can be defined and

$$\begin{aligned}\|B^2_{s,\cdot}\|_{p\text{-var};[u,v]} &\leqslant c\|A^1_{s,\cdot}\|_{q\text{-var};[s,t]}\|x\|_{p\text{-var};[u,v]},\\ \|C^2_{s,\cdot}\|_{q\text{-var};[u,v]} &\leqslant c\|A^2_{s,\cdot}\|_{p\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[u,v]}.\end{aligned}$$

By the same reason, we see that $B^1_{s,t}$, $B^3_{s,t}$, $C^1_{s,t}$, and $C^3_{s,t}$ are well-defined and

$$\begin{aligned}\|\boldsymbol{\gamma}^3_{s,\cdot}\|_{q\text{-var};[u,v]} &\leqslant c\|\boldsymbol{\gamma}^2_{s,\cdot}\|_{q\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[u,v]},\\ \|B^1_{s,\cdot}\|_{q\text{-var};[u,v]} &\leqslant c\|\boldsymbol{x}^2_{s,\cdot}\|_{p\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[u,v]},\\ \|B^3_{s,\cdot}\|_{p\text{-var};[u,v]} &\leqslant c\|\gamma\|_{q\text{-var};[s,t]}\|\boldsymbol{x}^2_{\cdot,t}\|_{p\text{-var};[u,v]},\\ \|C^1_{s,\cdot}\|_{q\text{-var};[u,v]} &\leqslant c\|x\|_{p\text{-var};[s,t]}\|\boldsymbol{\gamma}^2_{\cdot,t}\|_{q\text{-var};[u,v]},\\ \|C^3_{s,\cdot}\|_{p\text{-var};[u,v]} &\leqslant c\|\boldsymbol{\gamma}^2_{s,\cdot}\|_{q\text{-var};[s,t]}\|x\|_{p\text{-var};[u,v]}.\end{aligned}$$

By recalling upper bounds of $\|\gamma\|_{q\text{-var};[u,v]}$, $\|\boldsymbol{\gamma}^2_{s,\cdot}\|_{q\text{-var};[u,v]}$, $\|\boldsymbol{\gamma}^2_{\cdot,t}\|_{q\text{-var};[u,v]}$, $\|x\|_{q\text{-var};[u,v]}$, $\|\boldsymbol{x}^2_{s,\cdot}\|_{q\text{-var};[u,v]}$, and $\|\boldsymbol{x}^2_{\cdot,t}\|_{q\text{-var};[u,v]}$, we have

$$\begin{aligned}|\boldsymbol{\gamma}^2_{s,t}| &\leqslant c\|\gamma\|^2_{q\text{-var};[s,t]} \leqslant c\|\gamma\|^2_{\mathfrak{H}}(t-s)^{2(1/q-1/2)},\\ |A^1_{s,t}|, |A^2_{s,t}| &\leqslant c\|\boldsymbol{x}\|_{p\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[s,t]}\\ &\leqslant c\|\boldsymbol{x}\|_{(\alpha,12m)\text{-Bes}}\|\gamma\|_{\mathfrak{H}} \cdot (t-s)^{(\alpha-1/12m)+(1/q-1/2)}.\end{aligned}$$

Here, we used (3.1). Moreover, we obtain

$$\begin{aligned}|B^1_{s,t}|, |B^2_{s,t}|, |B^3_{s,t}| &\leqslant c\|\boldsymbol{x}\|^2_{p\text{-var};[s,t]}\|\gamma\|_{q\text{-var};[s,t]}\\ &\leqslant c\|\boldsymbol{x}\|^2_{(\alpha,12m)\text{-Bes}}\|\gamma\|_{\mathfrak{H}} \cdot (t-s)^{2(\alpha-1/12m)+(1/q-1/2)},\end{aligned}$$

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$$
\begin{aligned}
|C^1_{s,t}|, |C^2_{s,t}|, |C^3_{s,t}| &\leqslant c\|\boldsymbol{x}\|_{p\text{-var};[s,t]}\|\gamma\|^2_{q\text{-var};[s,t]} \\
&\leqslant c\|\boldsymbol{x}\|_{(\alpha,12m)\text{-Bes}}\|\gamma\|^2_{\mathfrak{H}} \cdot (t-s)^{(\alpha-1/12m)+2(1/q-1/2)}.
\end{aligned}
$$

Note

$$
\begin{gathered}
\left(\alpha-\frac{1}{12m}\right)+\left(\frac{1}{q}-\frac{1}{2}\right)=2\alpha+\left(\frac{1}{q}-\frac{1}{2}-\alpha-\frac{1}{12m}\right)>2\alpha, \\
2\left(\alpha-\frac{1}{12m}\right)+\left(\frac{1}{q}-\frac{1}{2}\right)=3\alpha+\left(\frac{1}{q}-\frac{1}{2}-\alpha-\frac{2}{12m}\right)>3\alpha, \\
\left(\alpha-\frac{1}{12m}\right)+2\left(\frac{1}{q}-\frac{1}{2}\right)=3\alpha+2\left(\frac{1}{q}-\frac{1}{2}-\alpha-\frac{1}{2\cdot 12m}\right)>3\alpha, \\
\frac{1}{q}-\frac{1}{2}=\alpha+\left(\frac{1}{q}-\frac{1}{2}-\alpha\right)>\alpha.
\end{gathered}
$$

Hence, A^i and $\boldsymbol{\gamma}^2$ are of finite $2(\alpha+\delta)$-Hölder norm for some $\delta>0$ and therefore they are of finite $(2\alpha, 6m)$-Besov norm. We also see that B^i, C^i and $\boldsymbol{\gamma}^3$ are of finite $(3\alpha, 4m)$-Besov norm. Hence, τ is well-defined.

The continuity follows from the continuous embedding $C^{q\text{-var}}_0([0,1];\mathbf{R}^d)$ and the continuity of the Young integration. ∎

§4. Moment estimate for Taylor expansion of the Lyons–Itô map

In this section, we prove a Taylor-like expansion for $\tilde{y}^\epsilon$, which was defined in (3.8). The base point $\gamma\in\mathfrak{H}$ of the expansion is arbitrary, but fixed. We will show that $\tilde{y}^\epsilon$ admits the next expansion

$$
\tilde{y}^\epsilon \sim \phi^0+\epsilon^{\kappa_1}\phi^{\kappa_1}+\cdots+\epsilon^{\kappa_k}\phi^{\kappa_k}+\cdots \tag{4.1}
$$

as $\epsilon\searrow 0$ in appropriate senses. Here $\kappa_k\in\Lambda_1$ and ϕ^{κ_k} is a nice Wiener functional (for Λ_1 and $0=\kappa_0<\kappa_1<\kappa_2<\cdots$, see § 2.3).

Let us fix some notations for fractional order expansion (4.1). In this section, H is not necessarily the Hurst parameter, but a positive parameter unless otherwise specified. For notational simplicity, however, $H\in(0,1/2]$ is assumed. For such H, we define $0=\kappa_0<\kappa_1<\kappa_2<\cdots$ in the same way as § 2.3. We will work in 1-variation topology for a while. The ODE that corresponds to (3.8) leads

$$
\begin{aligned}
d\tilde{y}^\epsilon_t &= \sigma(\tilde{y}^\epsilon_t)\,(\epsilon dx_t+dh_t)+\epsilon^{1/H}b(\tilde{y}^\epsilon_t)\,dt \\
&= \sigma(\tilde{y}^\epsilon_t)\epsilon\,dx_t+\left[\sigma(\tilde{y}^\epsilon_t)\,dh_t+\epsilon^{1/H}b(\tilde{y}^\epsilon_t)\,d\lambda_t\right] \quad \text{with } \tilde{y}^\epsilon_0=a.
\end{aligned} \tag{4.2}
$$

Here $x,h\in C^{1\text{-var}}_0([0,1];\mathbf{R}^d)$ and $\lambda_t=t$. Next we define ϕ^{κ_m} appeared in (4.1) formally; for intuitive derivation of ϕ^{κ_m}, see [Ina16a, Section 7]. By setting $\epsilon=0$ in (4.2), we can easily see that $\phi^0=\phi^0(h)$ satisfies

$$
d\phi^0_t=\sigma(\phi^0_t)\,dh_t \quad \text{with } \phi^0_0=a. \tag{4.3}
$$

An ODE for $\phi^1=\phi^1(x,h)$ is given by

$$
d\phi^1_t-\nabla\sigma(\phi^0_t)\langle\phi^1_t,dh_t\rangle=\sigma(\phi^0_t)\,dx_t \quad \text{with } \phi^1_0=0. \tag{4.4}
$$

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For $\kappa_m \geqslant 2$, $\phi^{\kappa_m} = \phi^{\kappa_m}(x,h)$ satisfies

$$
\begin{aligned}
& d\phi_t^{\kappa_m} - \nabla\sigma(\phi_t^0)\langle \phi_t^{\kappa_m}, dh_t\rangle \\
&\quad = \sum_{k=1}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m-1} \frac{\nabla^k\sigma(\phi_t^0)}{k!}\langle \phi^{\kappa_{i_1}},\ldots,\phi^{\kappa_{i_k}}; dx_t\rangle \\
&\qquad + \sum_{k=2}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m} \frac{\nabla^k\sigma(\phi_t^0)}{k!}\langle \phi^{\kappa_{i_1}},\ldots,\phi^{\kappa_{i_k}}; dh_t\rangle \\
&\qquad + \sum_{k=1}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m-1/H} \frac{\nabla^k b(\phi_t^0)}{k!}\langle \phi^{\kappa_{i_1}},\ldots,\phi^{\kappa_{i_k}}\rangle\, dt \quad \text{with } \phi_0^{\kappa_m}=0.
\end{aligned}
\tag{4.5}
$$

The summations in the first term on the right-hand side is taken over all $\kappa_{i_1},\ldots,\kappa_{i_k} \in \Lambda_1 \setminus \{0\}$ such that $\kappa_{i_1}+\cdots+\kappa_{i_k} = \kappa_m - 1$ holds. It is not allowed that $\kappa_{i_j}=0$. So, the sum is actually a finite sum. The second and the third terms should be understood in the same way. An important observation is that the right-hand side of (4.5) does not involve ϕ^{κ_m}, but only $\phi^0,\ldots,\phi^{\kappa_{m-1}}$. These ODEs have a rigorous meaning and can actually be solved by variation of constants method. So, we inductively define ϕ^{κ_m} as a unique solution of (4.3), (4.4) and (4.5). Finally, set

$$
r_\epsilon^{\kappa_{k+1}} = \tilde{y}^\epsilon - (\phi^0 + \epsilon^{\kappa_1}\phi^{\kappa_1} + \cdots + \epsilon^{\kappa_k}\phi^{\kappa_k}). \tag{4.6}
$$

Then, $(x,h) \mapsto (x,h,\lambda,\tilde{y}^\epsilon,\phi^{\kappa_0},\ldots,\phi^{\kappa_k},r_\epsilon^{\kappa_{k+1}})$ is continuous from $C_0^{1\text{-var}}([0,1];\mathbf{R}^d)^2$ to $C_0^{1\text{-var}}([0,1];(\mathbf{R}^d)^{\oplus 2}\oplus\mathbf{R}\oplus(\mathbf{R}^n)^{k+3})$.

Furthermore, it is known that this map extends to a continuous map with respect to the rough path topology; more precisely, for $2 \leqslant p < 4$, $1 \leqslant q < 2$ with $1/p + 1/q > 1$,

$$
\begin{aligned}
& G\Omega_p(\mathbf{R}^d) \times C_0^{q\text{-var}}([0,1];\mathbf{R}^d) \ni (\boldsymbol{x},h) \\
&\quad \mapsto (\boldsymbol{x},\boldsymbol{h},\boldsymbol{\lambda},\tilde{\boldsymbol{y}}^\epsilon,\boldsymbol{\phi}^0,\boldsymbol{\phi}^{\kappa_1},\ldots,\boldsymbol{\phi}^{\kappa_k},\boldsymbol{r}_\epsilon^{\kappa_{k+1}}) \in G\Omega_p((\mathbf{R}^d)^{\oplus 2}\oplus\mathbf{R}\oplus(\mathbf{R}^n)^{k+3})
\end{aligned}
$$

is locally Lipschitz for any k. Here, (4.2) is viewed as an RDE driven by $(\epsilon\boldsymbol{x}+\boldsymbol{h},\boldsymbol{\lambda})$ in the same way as in (3.8).

PROPOSITION 4.1. *Assume $2 \leqslant p < 4$, $1 \leqslant q < 2$ and $1/p + 1/q > 1$. Let $(\boldsymbol{x},h) \in G\Omega_p(\mathbf{R}^d) \times C_0^{q\text{-var}}([0,1];\mathbf{R}^d)$ and consider RDE (4.2) and keep the same notations as above. Then, the following hold.*

(1) *For any $\rho > 0$ and $k = 1,2,\ldots$, there exists a positive constant $C = C(\rho,k)$ which satisfies that*
$$\|(\boldsymbol{\phi}^{\kappa_k})^1\|_{p\text{-var}} \leqslant C(1 + \overline{\omega_{\boldsymbol{x}}}^{1/p})^{\kappa_k}$$
for any $\boldsymbol{x} \in G\Omega_p(\mathbf{R}^d)$ and $h \in C_0^{q\text{-var}}([0,1];\mathbf{R}^d)$ with $\|h\|_{q\text{-var}} \leqslant \rho$.

(2) *For any $\rho_1,\rho_2 > 0$ and $k = 1,2,\ldots$, there exists a positive constant $\tilde{C} = \tilde{C}(\rho_1,\rho_2,k)$, which is independent of ϵ and satisfies that*
$$\|(\boldsymbol{r}_\epsilon^{\kappa_{k+1}})^1\|_{p\text{-var}} \leqslant \tilde{C}(\epsilon + \epsilon\overline{\omega_{\boldsymbol{x}}}^{1/p})^{\kappa_{k+1}}$$
for any $\boldsymbol{x} \in G\Omega_p(\mathbf{R}^d)$ with $\overline{\omega_{\epsilon\boldsymbol{x}}}^{1/p} = \epsilon\overline{\omega_{\boldsymbol{x}}}^{1/p} \leqslant \rho_1$ and any $h \in C_0^{q\text{-var}}([0,1];\mathbf{R}^d)$ with $\|h\|_{q\text{-var}} \leqslant \rho_2$.

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Proof. This is essentially shown in [Ina10]. The only difference is that the index set in [Ina10] is $\mathbf{N}$ while it is Λ_1 here. But, this is a trivial issue. ∎

Now, we provide an L^r-version of the above proposition. By integrability lemmas of $\overline{\omega_{\boldsymbol{x}}}$ and $N_\delta(\boldsymbol{x})$, which are found in [FO10, Proposition 7] and [CLL13, Theorem 6.3 and Remark 6.4], respectively, the following theorem applies to fractional Brownian rough path $\boldsymbol{w}$ and $h \in \mathfrak{H}$ when $H \in (1/4, 1/2]$. More precisely, for every $H \in (1/4, 1/2]$, and $p > 1/H$ that is sufficiently close to $1/H$, we can let H be the Hurst parameter itself and take $\boldsymbol{x} = \boldsymbol{w}$ and $h \in \mathfrak{H}$ in Theorem 4.2.

THEOREM 4.2. *Assume $2 \leqslant p < 4$, $1 \leqslant q < 2$ and $1/p + 1/q > 1$. Let $h \in C_0^{q\text{-}var}([0,1]; \mathbf{R}^d)$ and let $\boldsymbol{x}$ be a $G\Omega_p(\mathbf{R}^d)$-valued random variable such that $\overline{\omega_{\boldsymbol{x}}} = \omega_{\boldsymbol{x}}(0,1) \in \bigcap_{1 \leqslant r < \infty} L^r$ and $\exp(N_\delta(\boldsymbol{x})) \in \bigcap_{1 \leqslant r < \infty} L^r$ for every $\delta > 0$. We consider RDE (4.2).*

Then, for any $\boldsymbol{x}$, h and $k \in \mathbf{N}$, there exist control functions $\eta_k = \eta_{k,\boldsymbol{x},h}$ such that the following hold:

(1) *η_k are nondecreasing in k, that is, $\eta_{k,\boldsymbol{x},h}(s,t) \leqslant \eta_{k+1,\boldsymbol{x},h}(s,t)$ for all $k, \boldsymbol{x}, h, (s,t)$.*
(2) *$\overline{\eta_{k,\boldsymbol{x},h}} \in \bigcap_{1 \leqslant r < \infty} L^r$ for all k, h.*
(3) *For all $\epsilon \in (0,1]$, $k \in \mathbf{N}$, h, $\boldsymbol{x}$, $0 \leqslant s \leqslant t \leqslant 1$, and $1 \leqslant j \leqslant \lfloor p \rfloor$, we have*

$$\left| \left(\boldsymbol{x}, \boldsymbol{h}, \tilde{\boldsymbol{y}}^\epsilon, \boldsymbol{\phi}^0, \boldsymbol{\phi}^{\kappa_1}, \ldots, \boldsymbol{\phi}^{\kappa_k}, \epsilon^{-\kappa_{k+1}} \boldsymbol{r}_\epsilon^{\kappa_{k+1}} \right)^j_{s,t} \right| \leqslant \eta_{k,\boldsymbol{x},h}(s,t)^{j/p}.$$

In particular, for all $k \in \mathbf{N}$ and h, $\|(\boldsymbol{\phi}^{\kappa_k})^1\|_{p\text{-}var} \in \bigcap_{1 \leqslant r < \infty} L^r$ and $\|(\boldsymbol{r}_\epsilon^{\kappa_{k+1}})^1\|_{p\text{-}var} = O(\epsilon^{\kappa_{k+1}})$ in L^r for any $1 \leqslant r < \infty$.

Proof. One can prove this by combining Proposition 4.1 above and computing $N_\delta(\boldsymbol{x})$ in the same way as [Ina16b]. For $k = 0$, (4.2) and (4.3) imply

$$\frac{1}{\epsilon} dr^1_{\epsilon,t} = \sigma(\tilde{y}^\epsilon_t)\, dx_t + \frac{1}{\epsilon}\{\sigma(\tilde{y}^\epsilon_t) - \sigma(\phi^0_t)\}\, dh_t + \epsilon^{1/H-1} b(\tilde{y}^\epsilon_t)\, dt.$$

Since the first term on the right-hand side can be interpreted as a rough integration of one form along $(\boldsymbol{x}, \boldsymbol{h}, \tilde{\boldsymbol{y}}^\epsilon, \boldsymbol{\phi}^0)$, there exists a control function $\eta'_{\boldsymbol{x},h}$ determined by $\omega_{\boldsymbol{x}}$ and ω_h such that

$$\left| \left(\boldsymbol{x}, \boldsymbol{h}, \tilde{\boldsymbol{y}}^\epsilon, \boldsymbol{\phi}^0, \int \sigma(\tilde{\boldsymbol{y}}^\epsilon)\, \boldsymbol{x} \right)^j_{s,t} \right| \leqslant \eta'_{\boldsymbol{x},h}(s,t)^{j/p}$$

for $1 \leqslant j \leqslant \lfloor p \rfloor$. The second and third terms can admit

$$\left| \frac{1}{\epsilon} \int_s^t \{\sigma(\tilde{y}^\epsilon_u) - \sigma(\phi^0_u)\}\, dh_u + \epsilon^{1/H-1} \int_s^t b(\tilde{y}^\epsilon_u)\, du \right| \leqslant \eta''_{\boldsymbol{x},h}(s,t)^{1/q},$$

where $\eta''_{\boldsymbol{x},h}$ is a control function determined by $\omega_{\boldsymbol{x}}$, ω_h and $N_\delta(\boldsymbol{x})$. These three equations correspond to (3.16), (3.17) and (3.20) in [Ina16b]. Hence, the assertion for $k = 0$ holds from these estimates and a property of Young pairing ($\eta_{1,\boldsymbol{x},h}$ appeared in the assertion consists of $\eta'_{\boldsymbol{x},h}$ and $\eta''_{\boldsymbol{x},h}$). For details of the discussion above and the proof of the assertion for general k, we consult on [Ina16b, Section 3]. ∎

§5. Malliavin calculus for the solution of RDE driven by fBm

Our main purpose in this section is to prove that y^ϵ_1 and $\tilde{y}^\epsilon_1$ are nondegenerate uniformly in the sense of Malliavin in ϵ. Here, y^ϵ and $\tilde{y}^\epsilon$ are solutions to (3.7) and (3.8), respectively. Throughout this section, $\gamma \in \mathfrak{H}$ in the definition of $\tilde{y}^\epsilon_t$ in (3.8) will be fixed.

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5.1 Notation and results of this section

We recall Watanabe's theory of generalized Wiener functionals (Watanabe distributions) in Malliavin calculus. We borrow the notations used in this paper from Ikeda and Watanabe [IW89, V.8-V.10]. We also refer to Nualart [Nua06], Shigekawa [Shi04], Matsumoto and Taniguchi [MT17] and Hu [Hu17].

We introduce a measure. Let $\boldsymbol{P}=\boldsymbol{P}^H$ be the law of the fBm with Hurst parameter H. This is a probability measure on $\Omega=\overline{\mathfrak{H}}$, which is the closure of the Cameron–Martin space $\mathfrak{H}=\mathfrak{H}^H$ in $C_0^{p\text{-var}}([0,1];\mathbf{R}^d)$. Then, the triple $(\Omega,\mathfrak{H},\boldsymbol{P})$ is an abstract Wiener space. We denote by $\mathbf{D}_{r,s}(K)$ the K-valued Gaussian–Sobolev space, where $1<r<\infty$, $s\in\mathbf{R}$ and K is a real separable Hilbert space. Note that r and s stand for the integrability index and the differentiability index, respectively. As usual, we set the spaces $\mathbf{D}_\infty(K)=\bigcap_{s=1}^\infty\bigcap_{1<r<\infty}\mathbf{D}_{r,s}(K)$ and $\tilde{\mathbf{D}}_\infty(K)=\bigcap_{s=1}^\infty\bigcup_{1<r<\infty}\mathbf{D}_{r,s}(K)$ of test functions and the spaces $\mathbf{D}_{-\infty}(K)=\bigcup_{s=1}^\infty\bigcup_{1<r<\infty}\mathbf{D}_{r,-s}(K)$ and $\tilde{\mathbf{D}}_{-\infty}(K)=\bigcup_{s=1}^\infty\bigcap_{1<r<\infty}\mathbf{D}_{n,-s}(K)$ of Watanabe distributions. For $F\in\mathbf{D}_{r,s}(K)$, D^kF denotes the kth derivative of F for $k\in\mathbf{N}_+$.

We set $V_0^\epsilon=\epsilon^{1/H}V_0$, $V_i^\epsilon=\epsilon V_i$ $(1\leqslant i\leqslant d)$, and $\sigma^\epsilon=[V_1^\epsilon,\ldots,V_d^\epsilon]$, which is viewed as an $n\times d$ matrix. It holds that $\sigma^\epsilon=\epsilon\sigma$. Set $\tilde{A}^\epsilon(s)=\tilde{J}_1^\epsilon\tilde{K}_s^\epsilon\sigma^\epsilon(\tilde{y}_s^\epsilon)$, whose size is again $n\times d$. Let $\tilde{A}^{\epsilon,k}(s)$ and $\tilde{A}_i^{\epsilon,k}(s)$ denote the kth row and the (k,i)-component of $\tilde{A}^\epsilon(s)$, respectively.

Under this setting, we see that $\tilde{y}_1^\epsilon$ is differentiable in the sense of Malliavin and the derivatives admit good estimate as follows:

PROPOSITION 5.1. *We have* $\tilde{y}_1^\epsilon\in\mathbf{D}_\infty(\mathbf{R}^n)$ *and*

$$\langle D\tilde{y}_1^{\epsilon,k},h\rangle_{\mathfrak{H}}=\phi_{\tilde{A}^{\epsilon,k}}(h)=\sum_{i=1}^d\int_0^1\tilde{A}_i^{\epsilon,k}(s)\,dh_s^i\quad(k=1,\ldots,n).\tag{5.1}$$

In addition, for any $m=0,1,2,\ldots$ *and* $1<r<\infty$, *there exists a positive constant* $c=c_{m,r}$ *such that*

$$\boldsymbol{E}[\|D^m\tilde{y}_1^{\epsilon,k}\|^r_{\mathfrak{H}^{\otimes m}}]^{1/r}\leqslant c\epsilon^m\quad(k=1,\ldots,n).$$

Proof. Malliavin differentiability of solutions of RDEs driven by Gaussian rough path was studied in [Ina14]. A slight modification of that argument proves this proposition. □

PROPOSITION 5.2. *We have the following asymptotic expansion as* $\epsilon\searrow 0$:

$$\tilde{y}_1^\epsilon\sim\phi_1^0+\epsilon^{\kappa_1}\phi_1^{\kappa_1}+\cdots+\epsilon^{\kappa_k}\phi_1^{\kappa_k}+\cdots\quad\textit{in }\mathbf{D}_\infty(\mathbf{R}^n).$$

This means that for each k, *(i)* $\phi_1^{\kappa_k}\in\mathbf{D}_\infty(\mathbf{R}^n)$ *and (ii)* $\mathbf{D}_{r,s}$*-norm of* $r_{\epsilon,1}^{\kappa_{k+1}}$ *is* $O(\epsilon^{\kappa_{k+1}})$ *for any* $1<r<\infty$ *and* $s\geqslant 0$. *Here,* $r_{\epsilon,1}^{\kappa_{k+1}}$ *is defined by (4.6).*

Proof. We can show the assertion in the same way as [Ina16b, Proposition 4.3]. Indeed, it follows from Theorem 4.2, Proposition 5.1, Meyer's inequality and the fact that $\phi_1^{\kappa_k}$ belongs to the $\mathbf{R}^n$-valued inhomogeneous Wiener chaos of order $\lfloor\kappa_k\rfloor$. □

The Malliavin covariance matrices $Q=(Q_{kl})_{1\leqslant l,k\leqslant n}$ of y_1, $Q^\epsilon=(Q_{kl}^\epsilon)_{1\leqslant l,k\leqslant n}$ of y_1^ϵ, and $\tilde{Q}^\epsilon=(\tilde{Q}_{kl}^\epsilon)_{1\leqslant l,k\leqslant n}$ of $\tilde{y}_1^\epsilon$ are defined by

$$Q_{kl}=\langle Dy_1^k,Dy_1^l\rangle_{\mathfrak{H}},\qquad Q_{kl}^\epsilon=\langle Dy_1^{\epsilon,k},Dy_1^{\epsilon,l}\rangle_{\mathfrak{H}},\qquad\tilde{Q}_{kl}^\epsilon=\langle D\tilde{y}_1^{\epsilon,k},D\tilde{y}_1^{\epsilon,l}\rangle_{\mathfrak{H}},\tag{5.2}$$

respectively. In this paper we do not express these covariance matrices as two-parameter Young integrals (cf. [CHLT15, (6.1)]). In this notation, we will state the following two propositions.

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The first one is Kusuoka–Stroock type estimate for the solution of the scaled RDE. Although this is not surprising, there seems to be no literature that actually proves it.

PROPOSITION 5.3. *Suppose that (A1) holds. Then, there are positive constants c and μ such that for every $0<\epsilon<1$ and $1<r<\infty$, we have*

$$\boldsymbol{E}[|\det Q^{\epsilon}|^{-r}]^{1/r} \leqslant c\epsilon^{-\mu}.$$

Here, $c=c(r)$ can be chosen independent of ϵ, while μ is independent of both r and ϵ.

The second one is about the uniform non-degeneracy in the sense of Malliavin calculus of the solution of the scaled–shifted RDE under (A3). The Schilder-type large deviations for fractional Brownian rough path are used in the proof.

PROPOSITION 5.4. *Suppose that (A1) holds. Let γ in the definition of $\tilde{y}^{\epsilon}$ in (3.8) satisfy $Q(\gamma)\geqslant cI$ for some $c>0$. Then, for every $1<r<\infty$, we have*

$$\sup_{0<\epsilon<1} \boldsymbol{E}[|\det \epsilon^{-2}\tilde{Q}^{\epsilon}|^{-r}]^{1/r} < \infty.$$

Let $a'\in\mathbf{R}^n$. Then, we have

$$\epsilon^{-2}\tilde{Q}^{\epsilon}_{kl} = \left\langle D\left(\frac{\tilde{y}^{\epsilon,k}_1-(a')^k}{\epsilon}\right), D\left(\frac{\tilde{y}^{\epsilon,l}_1-(a')^l}{\epsilon}\right)\right\rangle_{\mathfrak{H}}.$$

It follows from Proposition 5.4 and this identity that $(\tilde{y}^{\epsilon}_1-a')/\epsilon$ with $\gamma=\bar{\gamma}$ is uniformly nondegenerate in the sense of Malliavin calculus under (A1), (A2) and (A3).

Propositions 5.3 and 5.4 will be proved in §§ 5.2 and 5.3, respectively.

5.2 Covariance matrix of the solution of scaled RDE driven by fBm

In this subsection, we show Proposition 5.3. Set

$$C^{\epsilon} = \int_0^1 K^{\epsilon}_s\sigma^{\epsilon}(y^{\epsilon}_s)\{K^{\epsilon}_s\sigma^{\epsilon}(y^{\epsilon}_s)\}^{\top}\, ds.$$

Then we see Proposition 5.3 from the following three lemmas.

LEMMA 5.5. *For every $0<\epsilon<1$, we have*

$$\lambda_{\min}(Q^{\epsilon}) \geqslant c\lambda_{\min}(C^{\epsilon})\lambda_{\min}(J^{\epsilon}_1(J^{\epsilon}_1)^{\top}).$$

Here, c is a positive constant independent of ϵ.

LEMMA 5.6. *Suppose that (A1) holds. Then, there are positive constants c and μ such that for every $0<\epsilon<1$ and $1<r<\infty$, we have*

$$\boldsymbol{E}[\lambda_{\min}(C^{\epsilon})^{-r}]^{1/r} \leqslant c\epsilon^{-\mu}.$$

Here, $c=c(r)$ can be chosen independent of ϵ, while μ is independent of both r and ϵ.

LEMMA 5.7. [CLL13] *Let $p>1/H$. For every $1<r<\infty$, we have*

$$\sup_{0<\epsilon<1} \boldsymbol{E}[\|J^{\epsilon}\|^r_{C^{p\text{-}var}([0,1];\mathbf{R}^{n^2})}]^{1/r} < \infty, \qquad \sup_{0<\epsilon<1} \boldsymbol{E}[\|K^{\epsilon}\|^r_{C^{p\text{-}var}([0,1];\mathbf{R}^{n^2})}]^{1/r} < \infty.$$

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Now we show Proposition 5.3 by using lemmas above.

Proof of Proposition 5.3. From Lemma 5.5, we see

$$\det Q^\epsilon \geqslant \lambda_{\min}(Q^\epsilon)^n \geqslant c^n \lambda_{\min}(C^\epsilon)^n \lambda_{\min}(J_1^\epsilon (J_1^\epsilon)^\top)^n.$$

Hence,

$$\begin{aligned}\boldsymbol{E}[|\det Q^\epsilon|^{-r}]^{1/r} &\leqslant c^{-n}\boldsymbol{E}[\lambda_{\min}(C^\epsilon)^{-nr}\lambda_{\min}(J_1^\epsilon (J_1^\epsilon)^\top)^{-nr}]^{1/r}\\ &\leqslant c^{-n}\boldsymbol{E}[\lambda_{\min}(C^\epsilon)^{-2nr}]^{1/(2r)}\boldsymbol{E}[\lambda_{\min}(J_1^\epsilon (J_1^\epsilon)^\top)^{-2nr}]^{1/(2r)}.\end{aligned}$$

Noting $\lambda_{\min}(J_1^\epsilon (J_1^\epsilon)^\top)^{-1} = \lambda_{\max}(K_1^\epsilon (K_1^\epsilon)^\top)$ and applying Lemmas 5.6 and 5.7, we see the assertion.

Next we show Lemma 5.5.

Proof of Lemma 5.5. Set $A^\epsilon(s) = \mathbf{1}_{[0,1]}(s) J_1^\epsilon K_s^\epsilon \sigma^\epsilon(y_s^\epsilon)$ and let $A^{\epsilon,k}(s)$ denote the kth row of $A^\epsilon(s)$. Then, for every $v=(v_1,\ldots,v_n)^\top \in \mathbf{R}^n$, we have $\langle v, Q^\epsilon v\rangle_{\mathbf{R}^n} = \|\sum_{k=1}^n v_k D y_1^{\epsilon,k}\|_{\mathfrak{H}}^2$. It follows from the Riesz representation theorem and Remark 3.1 that

$$\left\|\sum_{k=1}^n v_k D y_1^{\epsilon,k}\right\|_{\mathfrak{H}}^2 = \left\|\sum_{k=1}^n v_k D y_1^{\epsilon,k}\right\|_{\mathfrak{H}^*}^2 \geqslant c\left\|\sum_{k=1}^n v_k A^{\epsilon,k}\right\|_{L^2([0,1];\mathbf{R}^d)}^2.$$

Here, c is a positive constant independent of v and ϵ. Noting $|\sum_{k=1}^n v_k A^{\epsilon,k}(s)|_{\mathbf{R}^d}^2 = \langle v, A^\epsilon(s)A^\epsilon(s)^\top v\rangle_{\mathbf{R}^n}$, we have

$$\left\|\sum_{k=1}^n v_k A^{\epsilon,k}\right\|_{L^2([0,1];\mathbf{R}^d)}^2 = \int_0^1 \langle v, A^\epsilon(s)A^\epsilon(s)^\top v\rangle_{\mathbf{R}^n}\, ds = \langle v, J_1^\epsilon C_\epsilon (J_1^\epsilon)^\top v\rangle_{\mathbf{R}^n}.$$

These imply

$$\langle v, Q^\epsilon v\rangle_{\mathbf{R}^n} \geqslant c\langle (J_1^\epsilon)^\top v, C^\epsilon (J_1^\epsilon)^\top v\rangle_{\mathbf{R}^n}.$$

Hence,

$$\langle v, Q^\epsilon v\rangle_{\mathbf{R}^n} \geqslant c\lambda_{\min}(C^\epsilon)\langle (J_1^\epsilon)^\top v, (J_1^\epsilon)^\top v\rangle_{\mathbf{R}^n} \geqslant c\lambda_{\min}(C^\epsilon)\lambda_{\min}(J_1^\epsilon (J_1^\epsilon)^\top)|v|^2.$$

The proof has been completed.

In the rest of this subsection we show Lemma 5.6, following [CHLT15] closely. To end this, we regard (3.7) as an RDE driven by $\boldsymbol{w}$ with the coefficients $V_0^\epsilon, V_1^\epsilon, \ldots, V_d^\epsilon$. (Except in the rest of this subsection, we regard (3.7) as an RDE driven by $\epsilon\boldsymbol{w}$ with the coefficients $V_0, V_1, \ldots, V_d$.) Keeping this in mind, we introduce a quantity which plays an important role in a Norris-type lemma. Note that the quantity is defined in the framework of the controlled path theory. Let $(1+\lfloor 1/H\rfloor)^{-1} < \alpha < H$. Fix $a \in \mathbf{R}^n$ and $0<\theta<1$. For every $0<\epsilon<1$, define

$$\mathcal{L}_w^\epsilon(a,\theta,1) = 1 + L_\theta(w)^{-1} + |a| + \|(y^\epsilon, J^\epsilon, K^\epsilon)\|_{Q_{\boldsymbol{w}}^\alpha} + \mathcal{N}_{\boldsymbol{w},\alpha}.$$

Here, $\mathcal{N}_{\boldsymbol{w},\alpha} = \sum_{i=1}^{\lfloor 1/H\rfloor} \|\boldsymbol{w}^i\|_{i\alpha\text{-Höl}}$ and $Q_{\boldsymbol{w}}^\alpha$ stands for the Banach space of controlled paths with respect to $\boldsymbol{w}$. We refer to [HP13, Definition 3] and [CHLT15, Definition 5.2] for $L_\theta(w)$,

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which is called the modulus of θ-Hölder roughness of w, and to [CHLT15, Definition 5.1] for $\|(y^\epsilon, J^\epsilon, K^\epsilon)\|_{Q^\alpha_{\boldsymbol{w}}}$.

Although we do not discuss the details of $\mathcal{L}^\epsilon_w(a, \theta, 1)$ for concise, we note that expectations of rth power of $\mathcal{L}^\epsilon_w(a, \theta, 1)$ are bounded in ϵ (Lemma 5.8) and it gives a good estimate of $\langle v, C^\epsilon v\rangle_{\mathbf{R}^n}$ (Lemma 5.10). Combining these two lemmas, we can prove Lemma 5.6.

Let us start to prove Lemma 5.6 with the next lemma.

LEMMA 5.8. *For every $1 < r < \infty$, we have*

$$\sup_{0<\epsilon<1} \boldsymbol{E}[\mathcal{L}^\epsilon_w(a, \theta, 1)^r] < \infty.$$

Proof. We see $\boldsymbol{E}[L_\theta(w)^{-r}] < \infty$ for all $r > 1$ from [HP13, Lemma 3] and [CHLT15, Corollary 5.10]. We obtain $\sup_{0<\epsilon<1} \boldsymbol{E}[\|(y^\epsilon, J^\epsilon, K^\epsilon)\|^r_{Q^\alpha_{\boldsymbol{w}}}]\infty$ for all $r > 1$ by reading carefully [CHLT15, Corollary 8.1] and using Lemma 5.7. Finally, since $\|\boldsymbol{w}^i\|^{1/i}_{i\alpha\text{-Höl}}$ has a Gaussian tail, we see $\boldsymbol{E}[\mathcal{N}^r_{\boldsymbol{w},\alpha}] < \infty$. The proof is finished. ∎

Before stating the next lemma, we make a remark.

REMARK 5.9. For every $v = (v_1, \ldots, v_n)^\top \in \mathbf{R}^n$, we have

$$\langle v, C^\epsilon v\rangle_{\mathbf{R}^n} = \sum_{i=1}^d \int_0^1 \langle v, K^\epsilon_s V^\epsilon_i(y^\epsilon_s)\rangle^2_{\mathbf{R}^n}\, ds, \tag{5.3}$$

which follows from

$$\langle v, C^\epsilon v\rangle_{\mathbf{R}^n} = \int_0^1 \langle v, K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\{K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\}^\top v\rangle_{\mathbf{R}^n}\, ds$$

and

$$\begin{aligned}\langle v, K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\{K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\}^\top v\rangle_{\mathbf{R}^n} &= \langle \{K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\}^\top v, \{K^\epsilon_s \sigma^\epsilon(y^\epsilon_s)\}^\top v\rangle_{\mathbf{R}^d} \\ &= \sum_{i=1}^d \langle v, K^\epsilon_s V^\epsilon_i(y^\epsilon_s)\rangle^2_{\mathbf{R}^n}.\end{aligned}$$

We denote by $\mathcal{V}^\epsilon_m$ and $\mathcal{V}^\epsilon$ sets of vectors which are defined by replacing V_i by V^ϵ_i in Definition 2.1. Then, we see the relationship of $\mathcal{L}^\epsilon_w(a, \theta, 1)$ and $\langle v, C^\epsilon v\rangle_{\mathbf{R}^n}$ from the following lemma.

LEMMA 5.10. *Let $m \in \mathbf{N}$. For every $0 < \epsilon < 1$, $W \in \mathcal{V}^\epsilon_m$, $v \in \mathbf{R}^n$ with $|v| = 1$ and $0 \leqslant s \leqslant 1$, we have*

$$|\langle v, K^\epsilon_s W(y^\epsilon_s)\rangle_{\mathbf{R}^n}| \leqslant c_m \mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu(m)} \langle v, C^\epsilon v\rangle^{\pi(m)}_{\mathbf{R}^n},$$

where c_m, $\mu(m)$ and $\pi(m)$ are certain positive constants independent of ϵ, W, v and s.

Proof. The proof is done by induction on m. Let $m = 0$. Then, $W = V^\epsilon_i$ for some $1 \leqslant i \leqslant d$. Since $f^\epsilon_i = \langle v, K^\epsilon V^\epsilon_i(y^\epsilon)\rangle_{\mathbf{R}^n}$ is α-Hölder continuous, we can use [HP11, Lemma A.3] to obtain

$$\|f^\epsilon_i\|_\infty \leqslant 2\|f^\epsilon_i\|^{1/(2\alpha+1)}_{\alpha\text{-Höl}} \|f^\epsilon_i\|^{2\alpha/(2\alpha+1)}_{L^2([0,1];\mathbf{R})}.$$

Since

$$\|f^\epsilon_i\|_{\alpha\text{-Höl}} \leqslant c\{1 + \|K^\epsilon\|_{\alpha\text{-Höl}}\}\{|a| + \|y^\epsilon\|_{\alpha\text{-Höl}}\} \leqslant c\mathcal{L}^\epsilon_w(a, \theta, 1)^2$$

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holds and $\|f_i^\epsilon\|_{L^2([0,1];\mathbf{R})} \leqslant \langle v, C^\epsilon v\rangle_{\mathbf{R}^n}^{1/2}$ follows from (5.3), we have

$$\|f_i^\epsilon\|_\infty \leqslant 2c^{1/(2\alpha+1)}\mathcal{L}_w^\epsilon(a,\theta,1)^{2/(2\alpha+1)}\langle v, C^\epsilon v\rangle_{\mathbf{R}^n}^{\alpha/(2\alpha+1)}.$$

This is the conclusion for $m=0$.

Assuming the conclusion to hold for $m-1$, we will prove it for m. Note that for every $W \in \mathcal{V}_m^\epsilon$, there exists $U \in \mathcal{V}_{m-1}^\epsilon$ and $0 \leqslant i \leqslant d$ such that $W = [V_i^\epsilon, U]$. We have

$$\begin{aligned}&\langle v, K_t^\epsilon U(y_t^\epsilon)\rangle_{\mathbf{R}^n} - \langle v, U(a)\rangle_{\mathbf{R}^n}\\ &\quad = \sum_{i=1}^d \int_0^t \langle v, K_s^\epsilon[V_i^\epsilon, U](y_s^\epsilon)\rangle_{\mathbf{R}^n}\, dw_s^i + \int_0^t \langle v, K_s^\epsilon[V_0, U](y_s^\epsilon)\rangle_{\mathbf{R}^n}\, ds.\end{aligned}$$

From a Norris-type lemma ([CHLT15, Theorem 5.6], [HP13, Theorem 3.1]), there exist positive constants Q and R such that

$$\begin{aligned}&\|\langle v, K_\bullet^\epsilon W(y_\bullet^\epsilon)\rangle_{\mathbf{R}^n}\|_\infty\\ &\quad\leqslant M\mathcal{L}_w^\epsilon(a,\theta,1)^Q \|\langle v, K_\bullet^\epsilon U(y_\bullet^\epsilon)\rangle_{\mathbf{R}^n} - \langle v, U(a)\rangle_{\mathbf{R}^n}\|_\infty^R\\ &\quad\leqslant M\mathcal{L}_w^\epsilon(a,\theta,1)^Q (2c_{m-1}\mathcal{L}_w^\epsilon(a,\theta,1)^{\mu(m-1)}\langle v, C^\epsilon v\rangle_{\mathbf{R}^n}^{\pi(m-1)})^R\\ &\quad= M(2c_{m-1})^R \mathcal{L}_w^\epsilon(a,\theta,1)^{Q+\mu(m-1)R}\langle v, C^\epsilon v\rangle_{\mathbf{R}^n}^{\pi(m-1)R}\end{aligned}$$

for some M depending only on d and n. This is the conclusion for m. The proof is finished. ∎

We are now in a position to show Lemma 5.6.

Proof of Lemma 5.6. Let ν be a positive constant specified later. We will show that, for every $1 < r < \infty$, there exist positive constants $c_{r,1}$ and $c_{r,2}$ such that

$$\boldsymbol{P}(|C^\epsilon| > 1/\xi) \leqslant c_{r,1}\xi^r, \tag{5.4}$$

$$\sup_{|v|=1} \boldsymbol{P}(\langle v, C^\epsilon v\rangle_{\mathbf{R}^n} < \xi) \leqslant c_{r,2}(\epsilon^{-\nu})^r \xi^r \tag{5.5}$$

for any $0 < \epsilon < 1$ and $0 < \xi < 1$. Due to Lemma 8.1, these two estimates are sufficient for Lemma 5.6. Indeed the assertion holds with $\mu = \nu + 1$.

Because (5.4) follows from Lemma 5.7, we show (5.5) in the rest of the proof. Since the vector fields $V_0, V_1, \ldots, V_d$ satisfy the Hörmander condition at a, we can choose $W_1, \ldots, W_n \in \mathcal{V}$ so that $W_1(a), \ldots, W_n(a)$ linearly spans $\mathbf{R}^n$. Then, for every $1 \leqslant k \leqslant n$, there exists a non-negative integer $i_k \in \mathbf{N}$ such that $W_k \in \mathcal{V}_{i_k}$. From the definition of $\mathcal{V}_{i_k}^\epsilon$, we can choose a positive constant ρ_k such that $W_k^\epsilon = \epsilon^{\rho_k} W_k \in \mathcal{V}_{i_k}^\epsilon$. Set $\rho = \max\{\rho_1, \ldots, \rho_n\}$. We define

$$\phi(u) = \max_{1\leqslant k\leqslant n} |\langle u, W_k(a)\rangle_{\mathbf{R}^n}|, \qquad \phi^\epsilon(u) = \max_{1\leqslant k\leqslant n} |\langle u, W_k^\epsilon(a)\rangle_{\mathbf{R}^n}|$$

for every $u \in \mathbf{R}^n$ with $|u| = 1$. Then, $\phi^\epsilon(u) \geqslant \epsilon^\rho \phi(u)$ for any $|u| = 1$. Since $W_1^\epsilon(a), \ldots, W_n^\epsilon(a)$ spans linearly $\mathbf{R}^n$ and ϕ^ϵ is continuous, ϕ^ϵ attains the positive minimum. Let c_m, $\mu(m)$ and $\pi(m)$ be the same symbols as Lemma 5.10 and set $c = \max\{c_{i_1}, \ldots, c_{i_n}\}$, $\mu = \max\{\mu_{i_1}, \ldots, \mu_{i_n}\}$, and $\pi = \min\{\pi_{i_1}, \ldots, \pi_{i_n}\}$. Set $\nu = \rho/\pi$.

We choose $v \in \mathbf{R}^n$ with $|v| = 1$ and $\epsilon > 0$ arbitrarily. For v and ϵ, there exists $1 \leqslant k_0 \equiv k_0(v, \epsilon) \leqslant n$ such that $\phi^\epsilon(v) = |\langle v, W^\epsilon_{k_0}(a)\rangle_{\mathbf{R}^n}|$. From Lemma 5.10 and the above, we have

$$\boldsymbol{P}(\langle v, C^\epsilon v\rangle_{\mathbf{R}^n} < \xi) \leqslant \boldsymbol{P}(|\langle v, W^\epsilon_{k_0}(a)\rangle_{\mathbf{R}^n}| < c_{i_{k_0}} \mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu(i_{k_0})} \xi^{\pi(i_{k_0})})$$

and

$$\begin{aligned}\{|\langle v, W^\epsilon_{k_0}(a)\rangle_{\mathbf{R}^n}| &< c_{i_{k_0}} \mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu(i_{k_0})} \xi^{\pi(i_{k_0})}\} \\ &\subset \{\epsilon^\rho \phi(v) < c\mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu} \xi^{\pi}\} \\ &= \{\phi(v) < c\mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu} \epsilon^{-\rho} \xi^{\pi}\} \\ &\subset \{\min_{|v|=1} \phi(v) < c\mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu} \epsilon^{-\rho} \xi^{\pi}\}.\end{aligned}$$

From the Markov inequality, we see

$$\boldsymbol{P}(\langle v, C^\epsilon v\rangle_{\mathbf{R}^n} < \xi) \leqslant \frac{1}{\{\min_{|v|=1} \phi(v)\}^{r/\pi}} c^{p/\pi} \boldsymbol{E}[\mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu r/\pi}](\epsilon^{-\nu})^r \xi^r.$$

Noting $\boldsymbol{E}[\mathcal{L}^\epsilon_w(a, \theta, 1)^{\mu r/\pi}]$ are bounded from above in ϵ, we obtain (5.5).

5.3 Covariance matrix of the solution of scaled–shifted RDE driven by fBm

In this subsection, we will show Proposition 5.4. Let $1/H < p < \lfloor 1/H \rfloor + 1$ and $(H + 1/2)^{-1} < q < 2$ satisfy $1/p + 1/q > 1$. We start our discussion with making a remark on continuity of Q on $G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$, where λ is a one-dimensional path defined by $\lambda_t = t$. Recall that the solution maps $(\boldsymbol{x}, c\lambda) \mapsto y, J, K$ are continuous on $G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$, where y, J and K are solutions to (3.4), (3.5) and (3.6) driven by $(x, c\lambda)$, respectively. Furthermore, Dy_1 is continuous in y, J and K since the right-hand side of (5.1) is Young integration and Young integration is continuous in both integrands and integrators. Note that the embedding $\mathfrak{H} \subset C_0^{q\text{-var}}([0, 1]; \mathbf{R}^d)$ with $1/p + 1/q > 1$ holds. Hence, we see that Q is continuous on $G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$. We are now in a position to prove Proposition 5.4.

Proof of Proposition 5.4. From the continuity of Q at $(\boldsymbol{\gamma}, 0) \in G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$ and the assumption on γ, there exists an open set $O \subset G\Omega_p(\mathbf{R}^d) \times \mathbf{R}\langle\lambda\rangle$ such that

$$Q(\tau_\gamma(\boldsymbol{x}), \boldsymbol{k}) \geqslant c_1 I \tag{5.6}$$

for any $(\boldsymbol{x}, k) \in O$. Note that O contains $(\mathbf{0}, 0)$. (Here, $(\tau_\gamma(\boldsymbol{x}), \boldsymbol{k})$ is the Young pairing and $(\boldsymbol{x}, k)$ is a pair. In this proof, both of them will appear.)

We decompose $\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-r}]$ into expectations on $U_\epsilon = \{w \in \Omega \mid (\epsilon\boldsymbol{w}, \epsilon^{1/H}\lambda) \in O\}$ and $U_\epsilon^\complement$. First, we study the expectation on U_ϵ. Since $\tilde{Q}^\epsilon = \epsilon^2 Q(\tau_\gamma(\epsilon\boldsymbol{w}), \epsilon^{1/H}\boldsymbol{\lambda})$ and (5.6), we see

$$\det \tilde{Q}^\epsilon = \epsilon^{2n} \det Q(\tau_\gamma(\epsilon\boldsymbol{w}), \epsilon^{1/H}\boldsymbol{\lambda}) \geqslant \epsilon^{2n} c_1^n,$$

which implies

$$\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-r}; U_\epsilon] \leqslant (c_1\epsilon^2)^{-nr}. \tag{5.7}$$

Next we consider the expectation on $U_\epsilon^\complement$. From the Hölder inequality, we have

$$\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-r}; U_\epsilon^\complement] \leqslant \boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-2r}]^{1/2} \boldsymbol{P}(U_\epsilon^\complement)^{1/2}.$$

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Since the rate function of the Schilder-type large deviation principle is good, we see that there exists a positive constant c_2 such that

$$\boldsymbol{P}(U_\epsilon^\complement) = \boldsymbol{P}((\epsilon \boldsymbol{w}, \epsilon^{1/H}\lambda) \in O^\complement) = \hat{\nu}_\epsilon(O^\complement) \leqslant \exp\left(-\frac{c_2}{2\epsilon^2}\right)$$

for small $\epsilon > 0$. Here, we choose $1 < q < \infty$ so that $(q-1)\|\gamma\|_{\mathfrak{H}}^2 < c_2$ and set $1 < q' < \infty$ as the Hölder conjugate of q. The Girsanov theorem and Proposition 5.3 imply

$$\begin{aligned}\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-2r}]^{1/2} &= \boldsymbol{E}\left[|\det Q^\epsilon|^{-2r} \exp\left(\left\langle w, \frac{\gamma}{\epsilon}\right\rangle - \frac{\|\gamma\|_{\mathfrak{H}}^2}{2\epsilon^2}\right)\right]^{1/2} \\ &\leqslant \boldsymbol{E}[|\det Q^\epsilon|^{-2rq'}]^{1/2q'} \boldsymbol{E}\left[\exp\left(q\left\{\left\langle w, \frac{\gamma}{\epsilon}\right\rangle - \frac{\|\gamma\|_{\mathfrak{H}}^2}{2\epsilon^2}\right\}\right)\right]^{1/2q} \\ &\leqslant (c_3\epsilon^{-\mu})^r \exp\left(\frac{(q-1)\|\gamma\|_{\mathfrak{H}}^2}{4\epsilon^2}\right).\end{aligned}$$

From the above, we see

$$\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-r}; U_\epsilon^\complement] \leqslant c_3^r \epsilon^{-r\mu} \exp\left(\frac{(q-1)\|\gamma\|_{\mathfrak{H}}^2}{4\epsilon^2}\right) \exp\left(-\frac{c_2}{4\epsilon^2}\right). \tag{5.8}$$

The estimates (5.7) and (5.8) imply

$$\begin{aligned}\boldsymbol{E}[|\det \epsilon^{-2}\tilde{Q}^\epsilon|^{-r}] &= \epsilon^{2nr}\boldsymbol{E}[|\det \tilde{Q}^\epsilon|^{-r}] \\ &\leqslant \epsilon^{2nr}\left\{(c_1\epsilon^2)^{-nr} + c_3^r\epsilon^{-r\mu}\exp\left(-\frac{c_2-(q-1)\|\gamma\|_{\mathfrak{H}}^2}{4\epsilon^2}\right)\right\} \\ &= c_1^{-nr} + c_3^r\epsilon^{(2n-\mu)r}\exp\left(-\frac{c_2-(q-1)\|\gamma\|_{\mathfrak{H}}^2}{4\epsilon^2}\right).\end{aligned}$$

The right-hand side is bounded as $\epsilon \searrow 0$. The proof has finished. ∎

§6. Off-diagonal short time asymptotics

In this section, following Watanabe [Wat87], we prove the short time asymptotics of kernel function $p_t(a, a')$ when $a \neq a'$ and $1/4 < H \leqslant 1/2$. Unlike in [Wat87], we can localize around the energy minimizing path in the geometric rough path space in this paper, since the Lyons–Itô map is continuous in this setting. (The case $H > 1/2$ was done in [Ina16a] and the case $1/3 < H \leqslant 1/2$ was done in [Ina16b].)

Hereafter in this section, we fix $1/4 < H \leqslant 1/2$. Let $(1 + \lfloor 1/H \rfloor)^{-1} < \alpha < H$ and choose $m \in \mathbf{N}_+$ such that $H - \alpha > 2/m$. Set $p = 1/\alpha$ and $q = (H + 1/2 - 1/m)^{-1}$.

6.1 Localization around energy minimizing path

Let us introduce a cut-off function for the localization. Let $\bar{\gamma} \in \mathfrak{H}$ be as in (A2) and $\epsilon > 0$. Since $\|\boldsymbol{w}^i\|_{(i\alpha, 12m/i)\text{-Bes}}^{12m/i}$ is an element of an inhomogeneous Wiener chaos of order $12m$, so is its Cameron–Martin shift $\|\tau_{-\bar{\gamma}}(\epsilon\boldsymbol{w})^i\|_{(i\alpha, 12m/i)\text{-Bes}}^{12m/i}$. Here, $\tau_{-\bar{\gamma}}$ is the Young translation by $-\bar{\gamma}$. It is a continuous map from $G\Omega_{\alpha,12m}^{\mathrm{B}}(\mathbf{R}^d)$ to itself. So, this Wiener functional is defined for almost all $w \in \Omega$. For any $r \in (1, \infty)$, L^r-norm of this Wiener functional is bounded in ϵ. Hence, so is its $\mathbf{D}_{r,k}$-norm for any r, k. Due to this fact, the localization is allowed even in

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the framework of Watanabe distribution theory. This is the reason why we use this Besov-type norm on the geometric rough path space. Let $\psi : \mathbf{R} \to [0,1]$ be a smooth function such that $\psi(u) = 1$ if $|u| \leqslant 1/2$ and $\psi(u) = 0$ if $|u| \geqslant 1$. For each $\eta > 0$ and $\epsilon > 0$, we set

$$\chi_\eta(\epsilon, w) = \prod_{i=1}^{\lfloor 1/H \rfloor} \psi\biggl(\frac{\|\tau_{-\bar\gamma}(\epsilon \boldsymbol{w})^i\|^{12m/i}_{(i\alpha,12m/i)\text{-Bes}}}{\eta^{12m}}\biggr).$$

The following lemma states that only rough paths sufficiently close to the lift of the minimizer $\bar\gamma$ contribute to the asymptotics. The keys of the proof are (i) the Schilder-type large deviations for fractional Brownian rough path and (ii) the Kusuoka–Stroock type estimate of the Malliavin covariance of y_1^ϵ (Proposition 5.3).

LEMMA 6.1. *Suppose that* (A1) *and* (A2) *hold. Then, for any* $\eta > 0$, *there exists* $c = c_\eta > 0$ *such that*

$$0 \leqslant \boldsymbol{E}[(1 - \chi_\eta(\epsilon, w)) \cdot \delta_{a'}(y_1^\epsilon)] = O\biggl(\exp\biggl\{-\frac{\|\bar\gamma\|^2_{\mathfrak{H}} + c}{2\epsilon^2}\biggr\}\biggr) \quad \text{as } \epsilon \searrow 0.$$

Proof. We show the assertion for $1/4 < H \leqslant 1/3$. We can show it for $1/3 < H \leqslant 1/2$ more easily.

Set $p_\epsilon = \boldsymbol{E}[(1 - \chi_\eta(\epsilon, w))\delta_{a'}(y_1^\epsilon)]$. We take $\eta' > 0$ arbitrarily and fix it for a while. It is obvious that

$$0 \leqslant p_\epsilon = \boldsymbol{E}\biggl[\{1 - \chi_\eta(\epsilon, w)\}\psi\biggl(\frac{|y_1^\epsilon - a'|^2}{\eta'^2}\biggr)\delta_{a'}(y_1^\epsilon)\biggr].$$

Set $A(\xi_1, \xi_2, \xi_3) = 1 - \psi(\xi_1)\psi(\xi_2)\psi(\xi_3)$ for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ and write $(\xi_i; i = 1, 2, 3) = (\xi_1, \xi_2, \xi_3)$. Set $g(u) = u \vee 0$ for $u \in \mathbf{R}$. Then, in the sense of distributional derivative, $g'' = \delta_0$. Take a bounded continuous function $C : \mathbf{R}^n \to \mathbf{R}$ such that $C(u_1, \ldots, u_n) = g(u_1 - a'_1)g(u_2 - a'_2)\cdots g(u_n - a'_n)$ if $|u - a'| \leqslant 2\eta'$. Then,

$$p_\epsilon = \boldsymbol{E}\biggl[A\biggl(\frac{\|\tau_{-\bar\gamma}(\epsilon \boldsymbol{w})^i\|^{12m}_{(i\alpha,12m/i)\text{-Bes}}}{\eta^{12m}}; i = 1, 2, 3\biggr)\psi\biggl(\frac{|y_1^\epsilon - a'|^2}{\eta'^2}\biggr)(\partial_1^2 \cdots \partial_n^2 C)(y_1^\epsilon)\biggr].$$

Now, we use integration by parts formula for generalized expectations as in [Wat87, IW89] to see that p_ϵ is equal to a finite sum of the following form;

$$p_\epsilon = \sum_{j,k} \boldsymbol{E}\biggl[F_{j,k}(\epsilon, w)\nabla^j A\biggl(\frac{\|\tau_{-\bar\gamma}(\epsilon \boldsymbol{w})^i\|^{12m}_{(i\alpha,12m/i)\text{-Bes}}}{\eta^{12m}}; i = 1, 2, 3\biggr)\psi^{(k)}\biggl(\frac{|y_1^\epsilon - a'|^2}{\eta'^2}\biggr)C(y_1^\epsilon)\biggr].$$

Here $j = (j_1, j_2, j_3)$ and k run over finite subsets of $\mathbf{N}^3$ and $\mathbf{N}$, respectively, $\nabla^j A = \partial_1^{j_1}\partial_2^{j_2}\partial_3^{j_3} A$ and $F_{j,k}(\epsilon, w)$ is a polynomial in components of the following (i)–(iv): (i) y_1^ϵ and its derivatives, (ii) $\|\tau_{-\bar\gamma}(\epsilon \boldsymbol{w})^i\|^{12m/i}_{(i\alpha,12m/i)\text{-Bes}}$ and its derivatives, (iii) Q^ϵ, which is Malliavin covariance matrix of y_1^ϵ and its derivatives, and (iv) $(Q^\epsilon)^{-1}$. Note that the derivatives of $(Q^\epsilon)^{-1}$ do not appear.

From Propositions 5.1 and 5.3, there exists $\rho > 0$ such that $|(Q^\epsilon)^{-1}| = O(\epsilon^{-\rho})$ in L^r as $\epsilon \searrow 0$ for all $1 < r < \infty$. (Recall a well-known formula to obtain the inverse matrix A^{-1} with the adjugate matrix of A divided by $\det A$.) Therefore, there exists $\rho > 0$ such that $|F_{j,k}(\epsilon)| = O(\epsilon^{-\rho})$ in any L^r-norm. ($\rho = \rho(r) > 0$ may change from line to line.) By Hölder's

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inequality, we have

$$p_\epsilon \leqslant \frac{c}{\epsilon^\rho}\sum_{j,k}\boldsymbol{E}\Big[\Big|\nabla^j A\Big(\frac{\|\tau_{-\bar\gamma}(\epsilon\boldsymbol{w})^i\|^{12m}_{(i\alpha,12m/i)\text{-Bes}}}{\eta^{12m}}; i=1,2,3\Big)\Big|^{r'}\Big|\psi^{(k)}\Big(\frac{|y_1^\epsilon - a'|^2}{\eta'^2}\Big)\Big|^{r'}\Big]^{1/r'}$$

$$\text{(6.1)}\qquad \leqslant \frac{c}{\epsilon^\rho}\boldsymbol{P}\Big[\bigcup_{i=1}^{3}\Big\{\|\tau_{-\bar\gamma}(\epsilon\boldsymbol{w})^i\|^{1/i}_{(i\alpha,12m/i)\text{-Bes}} \geqslant \frac{\eta}{2^{1/(12m)}}\Big\}\cap\{|y_1^\epsilon - a'|\leqslant \eta'\}\Big]^{1/r'}.$$

Here, $1/r + 1/r' = 1$ and $c = c(r, r', \eta, \eta')$ is a positive constant, which may change from line to line. Set $U_{\eta''} = \bigcap_{i=1}^3\{\boldsymbol{x}\in G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d) \mid \|\boldsymbol{x}^i\|^{1/i}_{(i\alpha,12m/i)\text{-Bes}} < \eta''\}$ for $\eta'' > 0$. Then this forms a fundamental system of open neighborhoods around $(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3) \equiv (0, 0, 0)$ with respect to $(\alpha, 12m)$-Besov topology. By Proposition 3.2, $\tau_{\bar\gamma}^{-1}(U_{\eta''}) = \{\boldsymbol{x}\in G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d) \mid \tau_{\bar\gamma}(\boldsymbol{x})\in U_{\eta''}\}$ is an open neighborhood of $\bar{\boldsymbol{\gamma}}$ in $(\alpha, 12m)$-geometric rough path space. The first set on the most right-hand side of (6.1) can be written as $\{\epsilon\boldsymbol{w}\notin \tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta})\}$.

First taking $\limsup_{\epsilon\searrow 0}\epsilon^2\log$ and then letting $r'\searrow 1$, we obtain

$$\begin{aligned}
&\limsup_{\epsilon\searrow 0}\epsilon^2\log p_\epsilon\\
&\quad\leqslant \limsup_{\epsilon\searrow 0}\epsilon^2\log \boldsymbol{P}[w\in\Omega \mid \epsilon\boldsymbol{w}\notin\tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta}), |y_1^\epsilon - a'|\leqslant\eta']\\
&\quad= \limsup_{\epsilon\searrow 0}\epsilon^2\log\hat\nu^\epsilon\big[\big\{(\boldsymbol{x}, l)\in G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d)\times\mathbf{R}\langle\lambda\rangle \;\big|\\
&\qquad\qquad \boldsymbol{x}\in\tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta})^{\complement}, |a+\Phi(\boldsymbol{x},\boldsymbol{l})^1_{0,1} - a'|\leqslant\eta'\big\}\big]
\end{aligned}$$

$$\text{(6.2)}\qquad \leqslant -\inf\Big\{\frac{\|\gamma\|^2_{\mathfrak{H}}}{2}\;\Big|\;\gamma\in\mathfrak{H}, \boldsymbol{\gamma}\in\tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta})^c, |a+\Phi(\boldsymbol{\gamma},\boldsymbol{0})^1_{0,1} - a'|\leqslant\eta'\Big\}.$$

Here, $\Phi : G\Omega_p(\mathbf{R}^{d+1}) \to G\Omega_p(\mathbf{R}^n)$ denotes the Lyons–Itô map that corresponds to the coefficient $[\sigma, b]$ and we used the embeddings $G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d)\times\mathbf{R}\langle\lambda\rangle\hookrightarrow G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^{d+1})\hookrightarrow G\Omega_p(\mathbf{R}^{d+1})$ implicitly. In the last inequality we used large deviation upper estimate for a closed set. Notice also that $a + \Phi(\boldsymbol{\gamma}, 0)^1 = \phi^0(\gamma)$.

Now let η' tend to 0. As η' decreases, the most right-hand side of (6.2) decreases. The proof is finished if the limit is strictly smaller than $-\|\bar\gamma\|^2_{\mathfrak{H}}/2$. Assume otherwise. Then, there exists $\{\gamma_k\}_{k=1}^\infty\subset\mathfrak{H}$ such that

$$\boldsymbol{\gamma}_k\in\tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta})^c,\quad |a+\Phi(\boldsymbol{\gamma}_k, 0)^1_{0,1} - a'|\leqslant\frac{1}{k},$$
$$\liminf_{k\to\infty}\Big(-\frac{\|\gamma_k\|^2_{\mathfrak{H}}}{2}\Big)\geqslant -\frac{\|\bar\gamma\|^2_{\mathfrak{H}}}{2}.$$

In particular, $\{\gamma_k\}$ is bounded in $\mathfrak{H}$. Hence, by goodness of the rate function, the lift $\{\boldsymbol{\gamma}_k\}$ is precompact in $G\Omega^{\mathrm{B}}_{\alpha,12m}(\mathbf{R}^d)$. By taking a subsequence if necessary, we may assume $\{\gamma_k\}$ converges to some $\boldsymbol{z}$ in $(\alpha, 12m)$-Besov topology. By the continuity of Φ, we have $a+\Phi(\boldsymbol{z}, 0)^1_{0,1} = a'$. Since $\boldsymbol{z}\in\tau_{\bar\gamma}^{-1}(U_{2^{-1/(12m)}\eta})^c$, $\boldsymbol{z}\neq\bar{\boldsymbol{\gamma}}$. From the lower semicontinuity of the rate function, we see that $\boldsymbol{z}$ is the lift of some $z\in\mathfrak{H}$ and $\|z\|^2_{\mathfrak{H}}/2\leqslant\|\bar\gamma\|^2_{\mathfrak{H}}/2$. This clearly contradicts (A2). ∎

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6.2 Proof of Theorem 2.3

Now, let us calculate the kernel $p_t(a, a')$. We see that $p_{\epsilon^{1/H}}(a, a') = \boldsymbol{E}[\delta_{a'}(y_1^\epsilon)] = I_1(\epsilon) + I_2(\epsilon)$, where

$$I_1(\epsilon) = \boldsymbol{E}[\delta_{a'}(y_1^\epsilon)\chi_\eta(\epsilon, w)], \qquad I_2(\epsilon) = \boldsymbol{E}[\delta_{a'}(y_1^\epsilon)\{1 - \chi_\eta(\epsilon, w)\}].$$

As we have shown in Lemma 6.1, the second term $I_2(\epsilon)$ on the right-hand side does not contribute to the asymptotic expansion for any $\eta > 0$. So, we have only to calculate the first term $I_1(\epsilon)$ for some $\eta > 0$.

However, the proof of the asymptotic expansion of $I_1(\epsilon)$ in the case $H \in (1/4, 1/3]$ is essentially the same as in the case $H \in (1/3, 1/2]$ (see [Ina16b, Subsections 5.2–5.3]). Therefore, for the sake of brevity, we will give a sketch of the proof only.

Sketch of the proof of Theorem 2.3. By the Cameron–Martin formula, we have

$$I_1(\epsilon) = \boldsymbol{E}\left[\exp\left(-\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2} - \frac{1}{\epsilon}\langle\bar{\gamma}, w\rangle\right)\delta_{a'}(\tilde{y}_1^\epsilon)\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right)\right].$$

Moreover, there exists $\bar{\nu} \in \mathbf{R}^n$ such that $\langle\bar{\gamma}, w\rangle = \langle\bar{\nu}, \phi_1^1(w, \bar{\gamma})\rangle$ for all w, where the inner product on the right-hand side is a standard one on $\mathbf{R}^n$. (In fact, $\bar{\nu}$ is a covector that appears in the Lagrange multiplier method for ϕ_1^0 and $\gamma \mapsto \|\gamma\|_{\mathfrak{H}}^2/2$ at $\bar{\gamma}$. Note that ϕ_1^1 is a continuous extension of $\gamma \mapsto D_\gamma\phi_1^0(\bar{\gamma})$.) Hence, we have

$$\begin{aligned}
&\epsilon^n \exp\left(\frac{\|\bar{\gamma}\|_{\mathfrak{H}}^2}{2\epsilon^2}\right) I_1(\epsilon) \\
&\quad = \epsilon^n \boldsymbol{E}\left[\exp\left(-\frac{1}{\epsilon}\langle\bar{\nu}, \phi_1^1\rangle\right)\delta_{a'}(a' + \epsilon\phi_1^1 + r_{\epsilon,1}^2)\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right)\right] \\
&\quad = \boldsymbol{E}\left[\exp\left(-\frac{1}{\epsilon}\langle\bar{\nu}, \phi_1^1\rangle\right)\delta_0(\phi_1^1 + \epsilon^{-1}r_{\epsilon,1}^2)\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right)\right] \\
&\quad = \boldsymbol{E}\left[\exp\left(\frac{\langle\bar{\nu}, r_{\epsilon,1}^2\rangle}{\epsilon^2}\right)\delta_0(\phi_1^1 + \epsilon^{-1}r_{\epsilon,1}^2)\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right)\right] \\
&\quad = \boldsymbol{E}\left[F(\epsilon, w)\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right)\right],
\end{aligned}$$

where

$$F(\epsilon, w) = \exp\left(\frac{\langle\bar{\nu}, r_{\epsilon,1}^2\rangle}{\epsilon^2}\right)\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right)\psi\left(\frac{1}{\eta'^2}\left|\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right|^2\right)$$

for any positive constant η'. Here we have used $\delta_0(\bullet) = \psi(|\bullet|^2/\eta'^2)\delta_0(\bullet)$.

By a slight modification of Watanabe's asymptotic expansion theory and uniform non-degeneracy (Proposition 5.4), $\delta_0((\tilde{y}_1^\epsilon - a')/\epsilon)$ admits the following asymptotic expansion as $\epsilon \searrow 0$ in $\tilde{\mathbf{D}}_{-\infty}$-topology as follows: for some $\Phi_{\nu_j} \in \tilde{\mathbf{D}}_{-\infty}$ $(j = 1, 2, \ldots)$,

$$\delta_0\left(\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right) \sim \delta_0(\phi_1^1) + \epsilon^{\nu_1}\Phi_{\nu_1} + \epsilon^{\nu_2}\Phi_{\nu_2} + \cdots \tag{6.3}$$

(see [IW89, Theorem 9.3, p. 387]). Recall that $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$ are all the elements of Λ_3 in increasing order. Moreover, since ϕ_1^1 is a Gaussian random vector whose covariance

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matrix equals $Q(\bar{\gamma})$, $\delta_0(\phi_1^1)$ is actually a nontrivial finite measure with its total mass $\boldsymbol{E}[\phi_1^1] = (2\pi)^{-n/2}(\det Q(\bar{\gamma}))^{-1/2} > 0$.

Therefore, our problem reduces to showing $F(\epsilon, \bullet)$ belongs to $\tilde{\mathbf{D}}_\infty$ and admits asymptotic expansion in $\tilde{\mathbf{D}}_\infty$-topology with the index set Λ_3'. However, this is highly nontrivial since $\exp(\langle \bar{\nu}, r_{\epsilon,1}^2\rangle/\epsilon^2)$ itself does not have nice integrability. This is why the two technical factors are involved in the definition of $F(\epsilon, \bullet)$.

Let us observe the two technical factors. First, $\chi_\eta(\epsilon, w + \bar{\gamma}/\epsilon)$ and its derivatives vanish outside $\{w \mid \epsilon \boldsymbol{w} \in U_\eta\}$ and $\psi\big(\eta'^{-2}\big|(\tilde{y}_1^\epsilon - a')/\epsilon\big|^2\big)$ and its derivatives vanish outside $\{|r_{\epsilon,1}^1/\epsilon| \leqslant \eta'\}$. Second, as $\epsilon \searrow 0$,

$$\chi_\eta\left(\epsilon, w + \frac{\bar{\gamma}}{\epsilon}\right) = 1 + O(\epsilon^M), \qquad \psi\left(\frac{1}{\eta'^2}\left|\frac{\tilde{y}_1^\epsilon - a'}{\epsilon}\right|^2\right) = 1 + O(\epsilon^M) \tag{6.4}$$

in $\mathbf{D}_\infty$-topology for any (large) $M > 0$. Therefore, these two Wiener functionals have no influence in the coefficient of the expansion of $F(\epsilon, \bullet)$.

The expansion of $\langle \bar{\nu}, r_{\epsilon,1}^2\rangle/\epsilon^2 = \langle \bar{\nu}, \phi_1^2 + r_{\epsilon,1}^{\kappa_3}/\epsilon^2\rangle$ is indexed by Λ_2'. Hence, if the integrability issue were left aside, we would easily have an expansion of the following type:

$$\exp\left(\frac{\langle \bar{\nu}, r_{\epsilon,1}^2\rangle}{\epsilon^2}\right) \sim e^{\langle \bar{\nu}, \phi_1^2\rangle}(1 + \epsilon^{\rho_1}\Xi_{\rho_1} + \epsilon^{\rho_2}\Xi_{\rho_2} + \cdots), \tag{6.5}$$

where $0 = \rho_0 < \rho_1 < \rho_2 < \cdots$ are all the elements of Λ_3' in increasing order. Due to (6.4), it also should hold that

$$F(\epsilon, \bullet) \sim e^{\langle \bar{\nu}, \phi_1^2\rangle}(1 + \epsilon^{\rho_1}\Xi_{\rho_1} + \epsilon^{\rho_2}\Xi_{\rho_2} + \cdots). \tag{6.6}$$

Once (6.6) is actually shown, then we prove our main theorem (Theorem 2.3). Moreover, $\alpha_0 := \boldsymbol{E}[e^{\langle \bar{\nu}, \phi_1^2\rangle}\delta_0(\phi_1^1)] > 0$ is finite due to Assumption (A4).

Roughly speaking, the proof of (6.6) goes in the following way. First, Assumption (A4) implies $\boldsymbol{E}[e^{\langle \bar{\nu}, \phi_1^2\rangle}\delta_0(\phi_1^1)] < \infty$ since $\langle \bar{\nu}, \phi_1^2\rangle$ belongs to the inhomogeneous Wiener chaos of order 2 whose main term corresponds to the Hessian in (A4). Though it is not at all obvious, the Hessian is actually Hilbert–Schmidt on $\mathfrak{H} \times \mathfrak{H}$.

However, what we want to estimate is something like

$$\exp\left(\langle \bar{\nu}, \phi_1^2\rangle + \frac{\langle \bar{\nu}, r_{\epsilon,1}^{\kappa_3}\rangle}{\epsilon^2}\right) \cdot \delta_0\left(\phi_1^1 + \frac{r_{\epsilon,1}^2}{\epsilon}\right).$$

Here, thanks to the factor $\chi_\eta(\epsilon, w + \bar{\gamma}/\epsilon)$, we may assume $\epsilon \boldsymbol{w}$ stay in the bounded set U_η in the geometric rough path space. Hence, we can use the deterministic version of Taylor expansion of the Lyons–Itô map (Proposition 4.1) to prove that $\langle \bar{\nu}, r_{\epsilon,1}^{\kappa_3}\rangle/\epsilon^2$ and $r_{\epsilon,1}^2/\epsilon$ are actually so "small" that adding them does not destroy the integrability we need. □

§7. Examples

In this section, we provide two examples to which our main theorem (Theorem 2.3) applies. First, we recall the case of near points under the ellipticity condition.

EXAMPLE 7.1. Assume the ellipticity condition at the starting point a, that is, $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbf{R}^n$. Obviously, this implies (A1). If the end point a' is sufficiently close to the starting point a, then (A2), (A3) and (A4) are also satisfied. Therefore, our main theorem can be used.

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This was shown in [Ina16a] when $H \in (1/2, 1)$. The same proof works when $H \in (1/4, 1/2]$, too. The key of the proof is the implicit function theorem. Note that under the ellipticity assumption, the deterministic Malliavin covariance $Q(\gamma)$ is never degenerate and the implicit function theorem is available at every $\gamma \in \mathfrak{H}$.

Second, we provide an example, in which the coefficient vector fields satisfy the Hörmander condition, but not the ellipticity condition. This model was already studied in [Dri13].

EXAMPLE 7.2. (The "fractional diffusion" process on the Heisenberg group). Let $d = 2$, $n = 3$ and

$$V_0 = 0, \qquad V_1 = \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^3}, \qquad V_2 = \frac{\partial}{\partial x^2} - 2x^1 \frac{\partial}{\partial x^3}, \quad (x^1, x^2, x^3) \in \mathbf{R}^3.$$

Then, (A1) is satisfied at every point since $[V_1, V_2] = -4\partial/\partial x^3$. Fortunately, we can write down the solutions for (2.1) and (2.2) for any given initial condition $a = (a^1, a^2, a^3)$ as follows:

$$\begin{aligned} y_t &= \big(a^1 + \boldsymbol{w}^{1,1}_{0,t}, a^2 + \boldsymbol{w}^{1,2}_{0,t}, a^3 + 2(a^2 \boldsymbol{w}^{1,1}_{0,t} - a^1 \boldsymbol{w}^{1,2}_{0,t}) + 2(\boldsymbol{w}^{2,21}_{0,t} - \boldsymbol{w}^{2,12}_{0,t})\big), \\ \phi^0_t(\gamma) &= \big(a^1 + \gamma^1_t,\ a^2 + \gamma^2_t,\ a^3 + 2(a^2\gamma^1_t - a^1\gamma^2_t) + 2(\boldsymbol{\gamma}^{2,21}_{0,t} - \boldsymbol{\gamma}^{2,12}_{0,t})\big). \end{aligned}$$

Here, $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^1, \dots, \boldsymbol{\gamma}^{\lfloor 1/H \rfloor})$ is the natural lift of $\gamma = (\gamma^1_t, \gamma^2_t)_{0 \leqslant t \leqslant 1} \in \mathfrak{H}$ by means of Young integration and $\boldsymbol{\gamma}^{2,ij}$ stands for the (i, j)-component of $\boldsymbol{\gamma}^2$.

Let $a = (0, 0, 0)$, $a' = (\xi, \eta, 0) \in \mathbf{R}^3$ with $(\xi, \eta) \neq (0, 0)$. Then, (A2), (A3) and (A4) are also satisfied. (For a proof, see Lemma 7.4.)

Therefore, by Theorem 2.3 and Remark 2.4, we have the following asymptotics:

$$p_t((0, 0, 0), (\xi, \eta, 0)) \sim \exp\left(-\frac{\xi^2 + \eta^2}{2t^{2H}}\right) \frac{1}{t^{nH}} \{\alpha_0 + \alpha_2 t^{2H} + \alpha_4 t^{4H} + \cdots\}$$

as $t \searrow 0$ for some $\alpha_0 > 0$ and $\alpha_{2j} \in \mathbf{R}$ $(j = 1, 2, \dots)$.

REMARK 7.3. We make two remarks on Example 7.2.

- The definitions of V_1 and V_2 slightly differ from literature to literature.
- V_1 and V_2 are not of C^∞_b. So, precisely speaking, we cannot use Theorem 2.3 directly. However, there is no problem since the solution (y_t) has a very explicit expression. (The main purpose of imposing C^∞_b-condition is to ensure the existence of a global solution.)

LEMMA 7.4. *Let the notation be as in Example 7.2. Then, (A2), (A3) and (A4) are satisfied for $a = (0, 0, 0)$, $a' = (\xi, \eta, 0) \in \mathbf{R}^3$ with $(\xi, \eta) \neq (0, 0)$. Moreover, $\bar{\gamma}_t = (\xi, \eta)R(1, t)$ and $\|\bar{\gamma}\|^2_{\mathfrak{H}} = \xi^2 + \eta^2$.*

Proof. First, note that we may assume without loss of generality that $a' = (\xi, 0, 0)$ with $\xi > 0$. The reason is as follows. If (w_t) is a two-dimensional fBm and $A \in \mathrm{SO}(2)$, then $\tilde{w} := Aw$ is again a two-dimensional fBm and $\boldsymbol{w}^{2,21}_{0,t} - \boldsymbol{w}^{2,12}_{0,t} = \tilde{\boldsymbol{w}}^{2,21}_{0,t} - \tilde{\boldsymbol{w}}^{2,12}_{0,t}$. So, if we choose suitable A, the problem reduces to the case $a' = (\xi, 0, 0)$ with $\xi > 0$.

Next, we fix some notation. We write $G(\gamma) = \phi^0_1(\gamma)$ for any $\gamma \in \mathfrak{H}$ or any two-dimensional continuous path γ such that the Young ODE (2.2) makes sense. Explicitly, $G(\gamma) = (\gamma^1_1, \gamma^2_1, 2\int_0^1 (\gamma^2_s\, d\gamma^1_s - \gamma^1_s\, d\gamma^2_s))$. We denote by $\mathfrak{H}_H(\mathbf{R}^m)$ the Cameron–Martin space of an

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m-dimensional fBm (so $\mathfrak{H} = \mathfrak{H}_H(\mathbf{R}^2)$). We write $I(\gamma) = \|\gamma\|^2_{\mathfrak{H}_H(\mathbf{R}^m)}/2$ for $\gamma \in \mathfrak{H}_H(\mathbf{R}^m)$, but suppress the dimension m. Recall that there exists a real-valued kernel $K(t, s) = K_H(t, s)$, $0 < s < t$, with the following properties:

(1) If (b_t) is an m-dimensional Brownian motion, then (w_t), defined by $w_t = \int_0^t K(t, s)\, db_s$, is an m-dimensional fBm with Hurst parameter H.
(2) Set $(\mathcal{K}f)_t = \int_0^t K(t, s) f_s\, ds$. Then, $\mathcal{K} = \mathcal{K}_H$ is a unitary isometry from $L^2([0, 1]; \mathbf{R}^m)$ to $\mathfrak{H}_H(\mathbf{R}^m)$.
(3) $R(t, t') = \int_0^{t\wedge t'} K(t, s)K(t', s)\, ds$.

For proofs, see [Nua06, Section 5.1] and [BHØZ08, Section 1.2]. There, an explicit expression of K is also given, but it is not used here.

Now we consider a one-dimensional problem before showing (A2). Let $m = 1$. We now prove that $\xi R(1, \bullet)$ is a unique minimizer of I among $\gamma \in \mathfrak{H}_H(\mathbf{R}^1)$ that connect 0 and $\xi > 0$ at time 1, that is,

$$\operatorname{argmin}\{I(\gamma) \mid \gamma \in \mathfrak{H}_H(\mathbf{R}^1), \gamma_1 = \xi\} = \{\xi R(1, \bullet)\}. \tag{7.1}$$

From (2) and (3) above we see that $(\mathcal{K}K(1, *))_\bullet = R(1, \bullet) \in \mathfrak{H}_H(\mathbf{R}^1)$ and

$$\|R(1, \bullet)\|^2_{\mathfrak{H}_H(\mathbf{R}^1)} = \|K(1, \bullet)\|^2_{L^2([0,1];\mathbf{R}^1)} = R(1, 1) = 1.$$

Take $\gamma \in \mathfrak{H}_H(\mathbf{R}^1)$ with $\gamma_1 = \xi$ arbitrarily. Since $\mathcal{K}$ is an isometry, there exists $f \in L^2([0, 1]; \mathbf{R}^1)$ such that $\gamma = \mathcal{K}(f + \xi K(1, \bullet))$ uniquely. From $(\mathcal{K}K(1, *))_\bullet = R(1, \bullet)$ and $\gamma_1 = \xi$, we have $(\mathcal{K}f)_1 = 0$, which implies $K(1, \bullet)$ and f are orthogonal in $L^2([0, 1]; \mathbf{R}^1)$. Hence we can deduce $\|\gamma\|^2_{\mathfrak{H}} = \|f + \xi K(1, \bullet)\|^2_{L^2([0,1];\mathbf{R}^1)} = \|f\|^2_{L^2([0,1];\mathbf{R}^1)} + \xi^2$. The minimum of $I(\gamma)$ is achieved only when $f = 0$. Thus, we have shown (7.1). It is easy to see from (7.1) that (A2) holds with $\bar{\gamma} = (\xi, 0)R(1, \bullet) \in \mathfrak{H}$.

Next, we prove (A3). Recall that non-degeneracy of the deterministic Malliavin covariance matrix $Q(\bar{\gamma})$ of G at $\bar{\gamma}$ is equivalent to the surjectivity of the tangent map $DG(\bar{\gamma}) : \mathfrak{H} \to \mathbf{R}^3$. Note that $\bar{\gamma} \in \mathfrak{H}_{1/2}(\mathbf{R}^2)$ since $H > 1/4$. From the third assertion in [Bis84, Theorem 1.10 in Chapter 1], we see that $DG(\gamma)$ is surjective for every $\gamma \in \mathfrak{H}_{1/2}(\mathbf{R}^2)$ with $\gamma \neq 0$ (in the reference, it is assumed that the distribution linearly spanned by $\{V_1, V_2\}$ is fat). Since C^1-paths which start at 0 are dense in $\mathfrak{H}_{1/2}(\mathbf{R}^2)$, there are three C^1-paths h_1, h_2, h_3 which start at 0 such that $\{D_{h_i} G(\bar{\gamma})\}_{1\leqslant i\leqslant 3}$ spans $\mathbf{R}^3$. Recalling that C^1-paths which start at 0 belong to $\mathfrak{H}$ (see [FV06]), we conclude that the tangent map of $G : \mathfrak{H} \to \mathbf{R}^3$ at $\bar{\gamma}$ is also surjective, which is equivalent to (A3).

Finally, we prove (A4). Let $\bar{\nu} = \bar{\nu}(\bar{\gamma}) \in \mathbf{R}^3$ be the Lagrange multiplier at $\bar{\gamma}$ for I under the condition that $G = a'$. Let $f : (-\epsilon_0, \epsilon_0) \to K_a^{a'}$ be as in (A4). Then, in the same way as [Ina16b, Proposition 5.5], we have

$$\frac{d^2}{du^2}\bigg|_{u=0} \frac{\|f(u)\|^2_{\mathfrak{H}}}{2} = \|f'(0)\|^2_{\mathfrak{H}} - \langle \bar{\nu} D^2 G(\bar{\gamma})\langle f'(0), f'(0)\rangle\rangle_{\mathbf{R}^3},$$

where $D^2 G(\bar{\gamma})\langle k, k\rangle = D^2_k G(\bar{\gamma})$. It is well known that

$$\bar{\nu}^i = \sum_{j=1}^{3} [G(\bar{\gamma})^{-1}]_{ij} D_{\bar{\gamma}} G^j(\bar{\gamma}).$$

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By straightforward computation, we have for every $\gamma, h \in \mathfrak{H}$

$$D_h G(\gamma) = \left(h_1^1, h_1^2, 2 \int_0^1 (h_s^2 \, d\gamma_s^1 - h_s^1 \, d\gamma_s^2 + \gamma_s^2 \, dh_s^1 - \gamma_s^1 \, dh_s^2) \right).$$

Thanks to the explicit forms of $\bar{\gamma}$ and G, we can easily see that $D_{\bar{\gamma}} G^j(\bar{\gamma}) = (\xi, 0, 0)$. Since the second component of $\bar{\gamma}$ is zero, the second and the third components of $D_h G(\bar{\gamma})$ do not depend on h^1, the first component of h. This implies that

$$Q(\bar{\gamma}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & * \end{pmatrix}, \qquad Q(\bar{\gamma})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

and hence $\bar{\nu} = (\xi, 0, 0)$. Since G^1 is linear, $D^2 G^1 \equiv 0$. Thus, we have shown that $\langle \bar{\nu}, D^2 G(\bar{\gamma}) \langle f'(0), f'(0) \rangle \rangle_{\mathbf{R}^3} = 0$, which clearly implies (A4). ∎

Remark 7.5. When $H = 1/2$, Young ODE (2.2) controlled by $\gamma \in \mathfrak{H}$ is called skeleton ODE in probability theory and has been thoroughly studied in both probability theory and control theory. In that case, (A1)–(A4) are known to be a natural set of assumptions with many concrete examples.

On the other hand, when $H \neq 1/2$, not much seems to be known for this ODE. This is the main reason why we cannot systematically find examples for our main theorem in this section. Therefore, we believe that a "fractional version of the control theory of ODEs" needs to be investigated for future developments of this topic. (The fraction version could also be useful in relaxing the unique minimizer assumption (A2).)

§8. Technical lemma

The following lemma to be used in the proof of Lemma 5.6 is a slight modification of [Nua06, Lemma 2.3.1]. For readers' convenience, we prove it in this section.

Lemma 8.1. *Let $\{A^\epsilon\}_{0<\epsilon<1}$ be a family of real symmetric non-negative definite $n \times n$ random matrices. Let μ_1 and μ_2 be non-negative constants. We assume that for every $1 < p < \infty$, there exist positive constants $c_{p,1}$ and $c_{p,2}$ such that*

$$\boldsymbol{P}(|A^\epsilon| > 1/\xi) \leqslant c_{p,1} (\epsilon^{-\mu_1} \xi)^p, \qquad \sup_{|v|=1} \boldsymbol{P}(\langle v, A^\epsilon v \rangle_{\mathbf{R}^n} < \xi) \leqslant c_{p,2} (\epsilon^{-\mu_2} \xi)^p$$

for any $0 < \epsilon < 1$ and $0 < \xi < 1$. Then, for all $0 < \epsilon < 1$, $\mu > \mu_1 \vee \mu_2$ and $1 < r < \infty$, we have

$$\boldsymbol{E}[\lambda_{\min}(A^\epsilon)^{-r}]^{1/r} \leqslant c \epsilon^{-\mu}.$$

Here, c is a positive constant, which is independent of ϵ and μ but depends on r.

Proof. Fix $0 < \epsilon < 1$. Let $\mu > \mu_1 \vee \mu_2$ and set

$$r_0 = \frac{(1 + 2n)(\mu_1 \vee \mu_2)}{\mu - (\mu_1 \vee \mu_2)} \vee 1.$$

First of all, we show that, for every $r \geqslant r_0$, there exists a positive constant $\tilde{c}_r$ such that

$$\boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi) \leqslant \tilde{c}_r \epsilon^{-r\mu} \xi^{r+1} \tag{8.1}$$

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for all $0 < \xi < 1$. Here, we can choose $\tilde{c}_r$ as a constant which is independent of ϵ, ξ, and μ but depends on r. Note that the restriction $r \geqslant r_0$ and $\mu > \mu_1 \vee \mu_2$ imply $r\mu \geqslant (r+1+2n)\mu_i$ for $i = 1, 2$. Note

$$\boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi) \leqslant \boldsymbol{P}(|A^\epsilon| \geqslant 1/\xi) + \boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi, |A^\epsilon| \leqslant 1/\xi).$$

The first term satisfies $\boldsymbol{P}(|A^\epsilon| \geqslant 1/\xi) \leqslant c_{r+1,1}(\epsilon^{-\mu_1}\xi)^{r+1} \leqslant c_{r+1,1}\epsilon^{-r\mu}\xi^{r+1}$ from the assumption and $r\mu \geqslant (r+1)\mu_1$. The second term is estimated as follows. We denote by S^{n-1} and $B(v, \rho)$ the unit sphere centered at the origin and the open ball with center $v \in \mathbf{R}^n$ and radius $\rho > 0$. Since S^{n-1} is compact, there exist vectors $v_1, \ldots, v_m \in S^{n-1}$ so that $\{B(v_i, \xi^2/2)\}_{i=1}^m$ covers S^{n-1}. In addition, $\lambda_{\min}(A^\epsilon) = \inf_{v \in S^{n-1}} \langle v, A^\epsilon v\rangle_{\mathbf{R}^n}$ holds. From these facts, on the event $\{\lambda_{\min}(A^\epsilon) < \xi, |A^\epsilon| < 1/\xi\}$, we can choose $v \in S^{n-1}$ such that $\langle v, A^\epsilon v\rangle_{\mathbf{R}^n} < \xi$ with $v \in B(v_i, \xi^2/2)$ for some $1 \leqslant i \leqslant m$. Hence,

$$\langle v_i, A^\epsilon v_i\rangle_{\mathbf{R}^n} \leqslant \langle v, A^\epsilon v\rangle_{\mathbf{R}^n} + |\langle v_i - v, A^\epsilon v\rangle_{\mathbf{R}^n}| + |\langle v_i, A^\epsilon (v_i - v)\rangle_{\mathbf{R}^n}| < 2\xi,$$

which implies

$$\{\lambda_{\min}(A^\epsilon) < \xi, |A^\epsilon| < 1/\xi\} \subset \bigcup_{i=1}^m \{\langle v_i, A^\epsilon v_i\rangle_{\mathbf{R}^n} < 2\xi\}.$$

Moreover, we see that the number m of the vectors may depend on ξ; however $m \leqslant c_n \xi^{-2n}$ holds, where c_n is a constant depending only on n. Therefore, the term $\boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi, |A^\epsilon| \leqslant 1/\xi)$ is estimated by

$$\sum_{i=1}^m \boldsymbol{P}(\langle v_i, A^\epsilon v_i\rangle_{\mathbf{R}^n} < 2\xi) \leqslant m c_{r+1+2n,2}(\epsilon^{-\mu_2} 2\xi)^{r+1+2n} \leqslant c_{r,n}\epsilon^{-r\mu}\xi^{r+1},$$

where $c_{r,n} = 2^{r+1+2n} c_n c_{r+1+2n,2}$. In the last estimate, we used $r\mu \geqslant (r+1+2n)\mu_2$. The estimates above imply

$$\boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi) \leqslant c_{r+1,1}\epsilon^{-r\mu}\xi^{r+1} + c_{r,n}\epsilon^{-r\mu}\xi^{r+1},$$

which implies (8.1) with $\tilde{c}_r = c_{r+1,1} + c_{r,n}$.

Next, we complete the proof. It follows from (8.1) that, for all $r \geqslant r_0$,

$$\begin{aligned}
\boldsymbol{E}[\lambda_{\min}(A^\epsilon)^{-r}] &= r\int_0^\infty \xi^{-r-1}\boldsymbol{P}(\lambda_{\min}(A^\epsilon) < \xi)\, d\xi \\
&\leqslant r\int_0^1 \xi^{-r-1}\tilde{c}_r\epsilon^{-r\mu}\xi^{r+1}\, d\xi + r\int_1^\infty \xi^{-r-1} d\xi \\
&= r\tilde{c}_r\epsilon^{-r\mu} + 1 \\
&\leqslant (1 + r\tilde{c}_r)\epsilon^{-r\mu}.
\end{aligned}$$

For $1 \leqslant r < r_0$, we have

$$\boldsymbol{E}[\lambda_{\min}(A^\epsilon)^{-r}]^{1/r} \leqslant \boldsymbol{E}[\lambda_{\min}(A^\epsilon)^{-r_0}]^{1/r_0} \leqslant (1 + r_0\tilde{c}_{r_0})^{1/r_0}\epsilon^{-\mu}.$$

These imply the assertion. ∎

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