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HAUSDORFF DIMENSION OF SETS OF ESCAPING POINTS AND ESCAPING PARAMETERS FOR ELLIPTIC FUNCTIONS

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Abstract Let $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ be a non-constant elliptic function. We prove that the Hausdorff dimension of the escaping set of f equals $2q/(q+1)$, where q is the maximal multiplicity of poles of f. We also consider the *escaping parameters* in the family $f_\beta = \beta f$, i.e. the parameters β for which the orbit of one critical value of f_β escapes to infinity. Under additional assumptions on f we prove that the Hausdorff dimension of the set of escaping parameters $\mathcal{E}$ in the family f_β is greater than or equal to the Hausdorff dimension of the escaping set in the dynamical space. This demonstrates an analogy between the dynamical plane and the parameter plane in the class of transcendental meromorphic functions.

Keywords: meromorphic functions; Julia set; critical values; escaping set; escaping parameters; Hausdorff dimension

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1. Introduction and main results

The understanding of the dynamics and geometry of elliptic functions has developed rapidly since the publication of the papers [**5**, **9**, **10**]. Although these functions are relatively 'regular', they manifest such unexpected features as the fact that the Hausdorff dimension of their Julia set is always larger than 1 (see [**9**]) or, in the non-recurrent case, that the corresponding Hausdorff measure always vanishes, whereas the packing measure, in the absence of parabolic points, is finite and positive (see [**10**]). A systematic exposition of the geometric measure theory and the ergodic theory of regular pseudo-non-recurrent elliptic functions is given in [**11**]. In spite of possible associations stemming from the name, this is not a narrow class of functions.

Let $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ be a transcendental meromorphic function, where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. For $n \in \mathbb{N}$, denote by f^n the nth iterate of f. The *Fatou set* $F(f)$ of f is the set of points $z \in \mathbb{C}$ such that all iterates $f^n(z)$ are well defined and $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of z. The complement $J(f)$ of $F(f)$ in $\bar{\mathbb{C}}$ is called the *Julia set* of f. Domínguez proved [**4**] that for transcendental meromorphic functions with poles, *the escaping set*

$$I(f) = \Big\{ z \in \mathbb{C} \colon \lim_{n\to\infty} f^n(z) = \infty \Big\}$$

is non-empty and $J(f) = \partial I(f)$. Several authors (see, for example, [**1**–**3**, **7**, **14**, **15**]) have studied properties of the escaping set of entire and meromorphic functions. It was shown by Kotus [**8**, Example 3] that if f is an elliptic function such that the closure of the post-critical set is disjoint from the set of poles, then the Hausdorff dimension $\dim_H(J(f)) \geqslant 2q/(1+q)$, where q is the maximal multiplicity of poles of f. The argument actually shows that

$$\dim_H(I(f)) \geqslant \frac{2q}{q+1}. \tag{1.1}$$

Moreover, Bergweiler and Kotus [**3**] and Kotus and Urbański [**9**] proved that if f is any elliptic function, then the upper bound on $\dim_H(I(f))$ is equal to the lower bound, i.e.

$$\dim_H(I(f)) \leqslant \frac{2q}{q+1}. \tag{1.2}$$

Our paper is divided into three parts. The first part of the paper (§§ 2 and 3) focuses on the generalization of (1.1) to the whole class of elliptic functions. Together with the estimate (1.2) it gives the following theorem.

Theorem 1.1. *Let f be a non-constant elliptic function. Then*

$$\dim_H(I(f)) = \frac{2q}{q+1},$$

where q is the maximal multiplicity of poles of f.

The second part of the paper (§ 4) is devoted to estimating the Hausdorff dimension of the escaping set in the parameter space for some families of elliptic functions. Let $f \colon \mathbb{C} \to \bar{\mathbb{C}}$ be an elliptic function such that one of its critical values, denoted by $f(c_1) \neq 0$, is a pole of the maximal multiplicity q. We define a one-parameter family of functions $f_\beta = \beta f$, $\beta \in \mathbb{C} \setminus \{0\}$. As a counterpart of the escaping set $I(f_\beta)$, we consider the set of escaping parameters in the family f_β, i.e.

$$\mathcal{E} = \Big\{ \beta \in \mathbb{C} \setminus \{0\} \colon \lim_{n\to\infty} f_\beta^n(c_1) = \infty \Big\}.$$

Remark 1.2. If $f(c_1) = 0$ is a pole, then $f_\beta(c_1) = 0$ and $f_\beta^2(c_1) = \infty$ for every $\beta \in \mathbb{C} \setminus \{0\}$. In this case $\mathcal{E} = \emptyset$.

The main result of the second part of the paper is the following theorem.

Theorem 1.3. *Let $f_\beta = \beta f$, $\beta \in \mathbb{C} \setminus \{0\}$, be a one-parameter family of elliptic functions such that one of the critical values of f, denoted by $f(c_1) \neq 0$, is a pole of the maximal multiplicity q. Then,*

$$\dim_H(\mathcal{E}) \geqslant \frac{2q}{q+1}.$$

Corollary 1.4. *For $q \nearrow \infty$ we have* $\dim_H(\mathcal{E}) \geqslant \dim_H(I(f)) \nearrow 2$.

A similar analogy between the dynamical and the parameter planes is known for the exponential family $f_\lambda(z) = \lambda e^z$ (see [**12**, **13**]). It follows from McMullen's result that the Hausdorff dimension of the escaping set $I(f_\lambda)$ equals 2. Qiu proved that the set of maps f_λ with the singular (asymptotic) value 0 approaching infinity also has Hausdorff dimension equal to 2. Theorem 1.3 is a generalization of Qiu's result, although it is valid under weaker assumptions on f. It is sufficient to assume that c_1 is any prepole of order 1, i.e. $f(c_1) \in f^{-1}(\infty)$, and the proof does not change.

In the third part of the paper (§ 5) we give some examples.

2. Notation and preliminary estimates

We fix a non-constant elliptic function $f\colon \mathbb{C} \to \overline{\mathbb{C}}$. Then f is periodic with respect to some lattice $\mathit{\Lambda} \subset \mathbb{C}$ defined by $\mathit{\Lambda} = [\lambda_1, \lambda_2] = \{l\lambda_1 + m\lambda_2 \colon l, m \in \mathbb{Z}\}$, where $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Im}(\lambda_1/\lambda_2) \neq 0$. Let

$$\mathcal{R} = \{t_1\lambda_1 + t_2\lambda_2 \colon 0 \leqslant t_1, t_2 \leqslant 1\} \tag{2.1}$$

be the basic fundamental parallelogram of f. Let q_b denote the multiplicity of $b \in f^{-1}(\infty)$ and let $q = \sup\{q_b \colon b \in f^{-1}(\infty)\}$. Since f has finitely many poles in $\mathcal{R}$, it follows by the periodicity of f that

$$q = \max\{q_b \colon b \in f^{-1}(\infty)\}. \tag{2.2}$$

Since f is periodic and has finitely many critical points in $\mathcal{R}$, there exists $\rho > 1$ such that all critical values of f are contained in $B(0, \rho - 1)$. In §§ 2 and 3, we consider only poles from the set

$$F = \{b \in f^{-1}(\infty) \colon |b| > \rho \text{ and } q_b = q\}. \tag{2.3}$$

We use the notation

$$U(a, r) = \left\{z \colon -\frac{3\pi}{4q} \leqslant \arg(z - a) \leqslant \frac{3\pi}{4q},\ |z - a| \leqslant r\right\},$$

for $a \in \mathbb{C}$ and $r > 0$.

Lemma 2.1. *Let $\varepsilon_0 \leqslant \frac{1}{3}$ be such that $\overline{B(b_i, \varepsilon_0)} \cap \overline{B(b_j, \varepsilon_0)} = \emptyset$, $b_i \neq b_j$, $b_i, b_j \in F$. There exist holomorphic functions G, H and constants $C_1, C_2 > 1$, $\varepsilon \in (0, \varepsilon_0)$, $\phi \in \mathbb{R}$ such that:*

(i) $f(z) = G(z)(z - b)^{-q}$, $f'(z) = H(z)(z - b)^{-q-1}$, $G(b), H(b) \neq 0$, $z \in B(b, \varepsilon)$;

(ii) $\dfrac{C_1^{-1}}{|z - b|^q} \leqslant |f(z)| \leqslant \dfrac{C_1}{|z - b|^q}$, $\dfrac{C_2^{-1}}{|z - b|^{q+1}} \leqslant |f'(z)| \leqslant \dfrac{C_2}{|z - b|^{q+1}}$, $z \in B(b, \varepsilon)$;

(iii) *f is one to one in each of the segments $U(b, \varepsilon)$;*

(iv) $\left\{z \in \bar{\mathbb{C}} \colon |z| \geqslant \dfrac{C_1}{\varepsilon^q},\ \phi - \dfrac{\pi}{8} \leqslant \arg(z) \leqslant \phi + \dfrac{9\pi}{8}\right\} \subset f(U(b, \varepsilon))$

for any $b \in F$.

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Proof. Fix $b \in F$. Since b is a pole of the multiplicity q, the formulae for f and f' given in (i) are obvious. Taking sufficiently small $\varepsilon \in (0, \varepsilon_0)$, we may assume that $G(z) \neq 0$ and $H(z) \neq 0$ for $z \in B(b, \varepsilon)$. There exist constants $C_1, C_2 > 1$ such that

$$C_1^{-1} \leqslant |G(z)| \leqslant C_1, \quad C_2^{-1} \leqslant |H(z)| \leqslant C_2, \quad z \in B(b, \varepsilon).$$

Hence,

$$\frac{C_1^{-1}}{|z-b|^q} \leqslant |f(z)| = \left| \frac{G(z)}{(z-b)^q} \right| \leqslant \frac{C_1}{|z-b|^q} \tag{2.4}$$

and

$$\frac{C_2^{-1}}{|z-b|^{q+1}} \leqslant |f'(z)| = \left| \frac{H(z)}{(z-b)^{q+1}} \right| \leqslant \frac{C_2}{|z-b|^{q+1}} \tag{2.5}$$

for $z \in B(b, \varepsilon)$. Shrinking ε if necessary, we can choose constants μ_1, μ_2, $0 < \mu_2 - \mu_1 < \pi/4$, such that $\mu_1 < \arg(G(z)) < \mu_2$, $z \in B(b, \varepsilon)$ and f is one to one in the segment $U(b, \varepsilon)$. Observe that the function $f(z)/G(z) = (z-b)^{-q}$ maps $U(b, \varepsilon)$ onto the set $\{z \in \bar{\mathbb{C}} \colon |z| \geqslant \varepsilon^{-q},\ -3\pi/4 \leqslant \arg(z) \leqslant 3\pi/4\}$. Hence,

$$\left\{ z \in \bar{\mathbb{C}} \colon |z| \geqslant \frac{C_1}{\varepsilon^q},\ -\frac{3\pi}{4} + \mu_2 \leqslant \arg(z) \leqslant \frac{3\pi}{4} + \mu_1 \right\} \subset f(U(b, \varepsilon)).$$

Since $0 < \mu_2 - \mu_1 < \pi/4$, there exists $\phi \in \mathbb{R}$ such that

$$\left\{ z \in \bar{\mathbb{C}} \colon |z| \geqslant \frac{C_1}{\varepsilon^q},\ \phi - \frac{\pi}{8} \leqslant \arg(z) \leqslant \phi + \frac{9\pi}{8} \right\} \subset f(U(b, \varepsilon)).$$

Note that the domain of G, H can be extended to $\bigcup_{b \in F} B(b, \varepsilon)$ and the constants C_1, C_2, ε, ϕ are universal for all poles in F because of the periodicity of f. □

Up to the end of § 3 the constants C_1, C_2, ε, ϕ are as in Lemma 2.1.

For $0 < r_1 < r_2$ we write

$$P(r_1, r_2) = \{z \colon r_1 < |z| < r_2\}.$$

We fix a pole $b_0 \in F$ of the maximum multiplicity q. We take $R_0 > 1$ such that $U(b_0, \varepsilon) \subset P(R_0, 2R_0)$ and define a constant

$$a_0 = \max\left\{2, \frac{C_1}{\varepsilon^q R_0}, \frac{\rho}{R_0}, \frac{8\max\{|\lambda_1 + \lambda_2|, |\lambda_1 - \lambda_2|\}}{R_0}\right\}. \tag{2.6}$$

Fix

$$a > a_0$$

and consider a sequence of radii

$$R_n = a^n R_0, \quad n \geqslant 1. \tag{2.7}$$

Let

$$P^+(R_n) = \{z \in \mathbb{C} \colon R_n < |z| < 2R_n,\ \phi < \arg(z) < \phi + \pi\}, \quad n \geqslant 1. \tag{2.8}$$

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The condition $a > a_0 \geqslant 2$ guarantees that the annuli $P(R_n, 2R_n)$ are pairwise disjoint. We consider the iterates f^n, $n \in \mathbb{N}$, which are defined outside a countable set of points.

Definition 2.2. We define the family of sets $\{\mathcal{A}_n(a)\}_{n\geqslant 0}$ as follows:

$$\begin{aligned}\mathcal{A}_0(a) &= \{A_0 = U(b_0,\varepsilon)\},\\ \mathcal{A}_n(a) &= \{A_n \subset A_{n-1} \in \mathcal{A}_{n-1}(a) \mid \exists b_n \in F\colon U(b_n,\varepsilon) \subset P^+(R_n),\\ &\qquad\qquad A_n \text{ is a component of } f^{-n}(U(b_n,\varepsilon))\}, \quad n \geqslant 1.\end{aligned}$$

Let

$$\mathcal{U}_n(a) = \bigcup_{A_n \in \mathcal{A}_n(a)} A_n, \qquad A(a) = \bigcap_{n=0}^{\infty} \mathcal{U}_n(a).$$

From now on, $\mathrm{vol}(A)$ signifies the Lebesgue measure of a set A.

Proposition 2.3. *For each $n \geqslant 0$ the set $\mathcal{A}_n(a)$ defined above is non-empty. Moreover, for $n \geqslant 1$, the number N_n of its elements contained in each $A_{n-1} \in \mathcal{A}_{n-1}(a)$ is greater than or equal to $3\pi R_n^2/8\mathrm{vol}(\mathcal{R})$.*

Proof. Obviously, $\mathcal{A}_0(a) \neq \emptyset$. We fix $n \geqslant 1$ and suppose that $\mathcal{A}_{n-1}(a) \neq \emptyset$. We will show that $\mathcal{A}_n(a)$ is non-empty. It follows from Lemma 2.1 that for all $b \in F$ we have

$$\{z \in \bar{\mathbb{C}}\colon |z| > R_n,\ \phi \leqslant \arg(z) \leqslant \phi + \pi\} \subset f(U(b,\varepsilon)),$$

as $R_n \geqslant R_1 = aR_0 > C_1\varepsilon^{-q}$ in view of (2.6). Take $A_{n-1} \in \mathcal{A}_{n-1}(a)$. Since $f^{n-1}(A_{n-1}) = U(b_{n-1},\varepsilon)$ for some pole $b_{n-1} \in F \cap P(R_{n-1}, 2R_{n-1})$, we have $f^n(A_{n-1}) \supset P^+(R_n)$.

Since $a > a_0$, we have $R_n \geqslant aR_0 \geqslant 8\max\{|\lambda_1+\lambda_2|, |\lambda_1-\lambda_2|\}$. Taking the poles $b \in F$ such that $B(b,\varepsilon) \subset P^+(R_k)$, we have

$$\bigcup_b (\mathcal{R}+b) \supset \left\{z\colon \frac{5R_n}{4} \leqslant |z| \leqslant \frac{7R_n}{4},\ \phi + \frac{\pi}{4} \leqslant \arg(z) \leqslant \phi + \frac{3\pi}{4}\right\}$$

and, consequently,

$$N_n = \frac{N_n \mathrm{vol}(\mathcal{R})}{\mathrm{vol}(\mathcal{R})} \geqslant \frac{\pi((1.75)^2 - (1.25)^2)R_n^2}{4\mathrm{vol}(\mathcal{R})} = \frac{3\pi R_n^2}{8\mathrm{vol}(\mathcal{R})} > 0.$$

The result follows. □

Remark 2.4. The elements A_n of $\mathcal{A}_n(a)$, $n \geqslant 1$, are pairwise disjoint as, by Lemma 2.1, f is injective in each segment $U(b,\varepsilon)$, $b \in F$, and the segments are pairwise disjoint.

In order to prove the lower bound on $\dim_H(I(f))$, we use the following result proved by McMullen [**12**, Proposition 2.2]. Here and throughout the paper, $\mathrm{diam}(A)$ denotes the diameter of a set A.

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Proposition 2.5. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a finite collection of disjoint compact subsets of $\mathbb{R}^d$, each of which has positive d-dimensional Lebesgue measure. Define $\mathcal{U}_n = \bigcup_{A_n \in \mathcal{A}_n} A_n$ and $A = \bigcap_{n=1}^{\infty} \mathcal{U}_n$. Suppose that for each $A_n \in \mathcal{A}_n$ there is $A_{n+1} \in \mathcal{A}_{n+1}$ and a unique $A_{n-1} \in \mathcal{A}_{n-1}$ such that $A_{n+1} \subset A_n \subset A_{n-1}$. If Δ_n, d_n are such that, for each $A_n \in \mathcal{A}_n$,*

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geqslant \Delta_n > 0, \quad \mathrm{diam}(A_n) \leqslant d_n \quad \text{and} \quad d_n \to 0 \text{ as } n \to \infty,$$

then $\dim_H(A) \geqslant d - \limsup_{n\to\infty} \sum_{j=1}^{n} |\log \Delta_j|/|\log d_n|$.

3. Proof of Theorem 1.1

Let $K = C_1^{(q+1)/q} C_2$. We prove the following lemma.

Lemma 3.1. *Let f be a non-constant elliptic function and let a_0 be the constant given in (2.6). Then, for every $a > a_0$, there is a subset $A(a)$ of $I(f)$ and for this subset*

$$\dim_H(A(a)) \geqslant \frac{2q}{q+1} - \frac{6\log 2 + 12q \log K/(q+1)}{\log a}.$$

Since $\dim_H(A(a)) \geqslant 2q/(q+1) - (6\log 2 + 12q\log K/(q+1))/\log a \nearrow 2q/(q+1)$ for $a \nearrow \infty$, we have that Theorem 1.1 follows from Lemma 3.1.

We fix $a > a_0$ and consider the sets $\mathcal{A}_n(a)$, $n \geqslant 0$, given in Definition 2.2. We drop the parameter a and keep the notation from the last section.

Lemma 3.2. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 1$. Then, for every $z \in A_n$,*

$$K^{-1} R_j^{(q+1)/q} \leqslant |f'(f^{j-1}(z))| \leqslant K(2R_j)^{(q+1)/q}, \quad j = 1, \dots, n.$$

Proof. Fix $z \in A_n$. We have $f^j(z) \in U(b_j, \varepsilon) \subset P(R_j, 2R_j)$, $j = 0, 1, \dots, n$. It follows from Lemma 2.1 that

$$\frac{C_1^{-1}}{|f^{j-1}(z) - b_{j-1}|^q} \leqslant |f^j(z)| \leqslant \frac{C_1}{|f^{j-1}(z) - b_{j-1}|^q}, \quad j = 1, \dots, n.$$

Thus,

$$\frac{C_1^{-1}}{2R_j} \leqslant \frac{C_1^{-1}}{|f^j(z)|} \leqslant |f^{j-1}(z) - b_{j-1}|^q \leqslant \frac{C_1}{|f^j(z)|} \leqslant \frac{C_1}{R_j}$$

and consequently

$$\left(\frac{C_1^{-1}}{2R_j}\right)^{(q+1)/q} \leqslant |f^{j-1}(z) - b_{j-1}|^{q+1} \leqslant \left(\frac{C_1}{R_j}\right)^{(q+1)/q}. \tag{3.1}$$

Again using Lemma 2.1, we obtain

$$\frac{C_2^{-1}}{|f^{j-1}(z) - b_{j-1}|^{q+1}} \leqslant |f'(f^{j-1}(z))| \leqslant \frac{C_2}{|f^{j-1}(z) - b_{j-1}|^{q+1}}, \quad j = 1, \dots, n,$$

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which, using (3.1), gives

$$\frac{R_j^{(q+1)/q}}{C_1^{(q+1)/q}C_2} \leqslant |f'(f^{j-1}(z))| \leqslant (2R_j)^{(q+1)/q}C_1^{(q+1)/q}C_2, \quad j=1,\dots,n.$$

□

The next lemma is devoted to estimating the derivatives $(f^n)'$, $n \geqslant 1$.

Lemma 3.3. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 1$. Then, for every $z \in A_n$,*

$$K^{-n}a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q} \leqslant |(f^n)'(z)| \leqslant (2^{(q+1)/q}K)^n a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q}.$$

Proof. Fix $z \in A_n$. We know that

$$(f^n)'(z) = \prod_{j=0}^{n-1} f'(f^j(z)).$$

Using Lemma 3.2, we obtain

$$\begin{aligned}\left|\prod_{j=0}^{n-1} f'(f^j(z))\right| &\leqslant 2^{(q+1)/q}KR_1^{(q+1)/q}\cdot\dots\cdot 2^{(q+1)/q}KR_n^{(q+1)/q}\\ &= (2^{(q+1)/q}K)^n(aR_0)^{(q+1)/q}\cdot\dots\cdot(a^nR_0)^{(q+1)/q}\\ &= (2^{(q+1)/q}K)^n a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q}.\end{aligned}$$

Analogously, we get the estimate from below

$$\left|\prod_{j=0}^{n-1} f'(f^j(z))\right| \geqslant K^{-n}a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q}.$$

Finally,

$$K^{-n}a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q} \leqslant |(f^n)'(z)| \leqslant (2^{(q+1)/q}K)^n a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q}.$$

□

From now on, $L(f, A)$ denotes distortion of a map f on a set A, i.e.

$$L(f, A) = \frac{\sup_{z\in A}|f'(z)|}{\inf_{z\in A}|f'(z)|}.$$

The next lemma follows immediately from Lemma 3.3.

Lemma 3.4. *Let $A_n \in \mathcal{A}_n$, $n \geqslant 1$. Then*

$$L(f^n, A_n) \leqslant 2^{(q+1)n/q}K^{2n} \quad \text{and} \quad \operatorname{diam}(A_n) \leqslant \frac{2\varepsilon K^n}{a^{(q+1)n(n+1)/2q}R_0^{(q+1)n/q}},$$

where ε is as in Lemma 2.1.

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Remark 3.5. Observe that $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$, since $a > a_0 \geqslant 2$.

By Lemma 3.4, the numbers d_n defined in Proposition 2.5 are equal to

$$d_n = 2\varepsilon K^n a^{-(q+1)n(n+1)/2q} R_0^{-(q+1)n/q}, \quad n \geqslant 1. \tag{3.2}$$

Lemma 3.6. *There exists $\gamma > 0$ such that*

$$\frac{\operatorname{vol}(\mathcal{U}_{n+1} \cap A_n)}{\operatorname{vol}(A_n)} \geqslant \frac{\gamma}{2^{6(q+1)n/q} K^{12n} a^{2(n+1)/q} R_0^{2/q}}$$

for each $A_n \in \mathcal{A}_n$, $n \geqslant 1$.

Proof. Recall that in Definition 2.2 we considered the segments

$$U(b, \varepsilon) = \left\{ z \in \mathbb{C}\colon -\frac{3\pi}{4q} \leqslant \arg(z - b) \leqslant \frac{3\pi}{4q},\ |z - b| \leqslant \varepsilon \right\},$$

where $b \in F$ and $\varepsilon > 0$ as in Lemma 2.1. Hence, $\operatorname{vol}(U(b, \varepsilon)) = 3\pi\varepsilon^2/4q$.

Take some $n \geqslant 1$ and $A_n \in \mathcal{A}_n$. There exists $b_n \in F$ such that A_n is a component of $f^{-n}(U(b_n, \varepsilon))$, where $U(b_n, \varepsilon) \subset P^+(R_n)$. Moreover, for each $A_k \in \mathcal{A}_{n+1}$ there is $b_{n+1} \in F$ such that A_k is a component of $f^{-n-1}(U(b_{n+1}, \varepsilon))$, $U(b_{n+1}, \varepsilon) \subset P^+(R_{n+1})$. There are finitely many sets $A_k \in \mathcal{A}_{n+1}$ contained in A_n. We denote by b_k the pole corresponding to A_k. Let $z_n = f^{-n}(b_n) \in A_n$, $z_k = f^{-n-1}(b_k) \in A_k$. Observe that f^n is conformal on A_n as all critical values of f are contained in $B(0, \rho - 1)$, so all the branches of f^{-1} are well defined on $\bigcup_{b \in F}(U(b, \varepsilon))$. Thus, $L(f^n, A_n) = L(f^{-n}, f^n(A_n))$. Hence,

$$\begin{aligned}
\operatorname{vol}(A_n) &= \iint_{U(b_n, \varepsilon)} |(f^{-n})'(z)|^2 \, \mathrm{d}z \\
&\leqslant \iint_{U(b_n, \varepsilon)} \Big(\sup_{z \in U(b_n, \varepsilon)} |(f^{-n})'(z)| \Big)^2 \, \mathrm{d}z \\
&= \operatorname{vol}(U(b_n, \varepsilon)) \Big(L(f^{-n}, U(b_n, \varepsilon)) \inf_{z \in U(b_n, \varepsilon)} |(f^{-n})'(z)| \Big)^2 \\
&\leqslant \frac{3\pi\varepsilon^2}{4q} (L(f^n, A_n) |(f^{-n})'(b_n)|)^2 \\
&= \frac{3\pi\varepsilon^2}{4q} \left(\frac{L(f^n, A_n)}{|(f^n)'(z_n)|} \right)^2.
\end{aligned} \tag{3.3}$$

Set $P_{n+1} = P^+(R_{n+1})$. Then

$$\begin{aligned}
\operatorname{vol}(\mathcal{U}_{n+1} \cap A_n) = \sum_{A_k \subset A_n} \operatorname{vol}(A_k) &= \sum_{b_k \in P_{n+1}} \operatorname{vol}(f^{-n-1}(U(b_k, \varepsilon))) \\
&= \sum_{b_k \in P_{n+1}} \iint_{U(b_k, \varepsilon)} |(f^{-n-1})'(z)|^2 \, \mathrm{d}z \\
&\geqslant \sum_{b_k \in P_{n+1}} \iint_{U(b_k, \varepsilon)} \Big(\inf_{z \in U(b_k, \varepsilon)} |(f^{-n-1})'(z)| \Big)^2 \, \mathrm{d}z
\end{aligned}$$

$$= \frac{3\pi\varepsilon^2}{4q}\sum_{b_k\in P_{n+1}}\left(\frac{\sup_{z\in U(b_k,\varepsilon)}|(f^{-n-1})'(z)|}{L(f^{-n-1},U(b_k,\varepsilon))}\right)^2$$
$$\geqslant \frac{3\pi\varepsilon^2}{4q}\sum_{b_k\in P_{n+1}}\left(\frac{|(f^{-n-1})'(b_k)|}{L(f^{-n-1},U(b_k,\varepsilon))}\right)^2$$
$$= \frac{3\pi\varepsilon^2}{4q}\sum_{z_k\in A_k\subset A_n}(L(f^{n+1},A_k)|(f^{n+1})'(z_k)|)^{-2}. \quad (3.4)$$

Now, combining (3.3) and (3.4), we estimate the density of the sets $\mathcal{U}_{n+1}\cap A_n$ in A_n:

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1}\cap A_n)}{\mathrm{vol}(A_n)} \geqslant \frac{\sum_{z_k\in A_k\subset A_n}(L(f^{n+1},A_k)|(f^{n+1})'(z_k)|)^{-2}}{(L(f^n,A_n)/|(f^n)'(z_n)|)^2}$$
$$= \frac{|(f^n)'(z_n)|^2}{(L(f^n,A_n))^2}\sum_{z_k\in A_k\subset A_n}(L(f^{n+1},A_k)|(f^{n+1})'(z_k)|)^{-2}. \quad (3.5)$$

Using Lemma 3.4, we obtain

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1}\cap A_n)}{\mathrm{vol}(A_n)}$$
$$\geqslant \frac{|\prod_{j=0}^{n-1}f'(f^j(z_n))|^2}{2^{2(q+1)n/q}K^{4n}}\sum_{z_k\in A_k\subset A_n}\frac{1}{2^{2(q+1)(n+1)/q}K^{4(n+1)}|\prod_{j=0}^{n}f'(f^j(z_k))|^2}$$
$$= \frac{1}{2^{2(q+1)(2n+1)/q}K^{4(2n+1)}}\sum_{z_k\in A_k\subset A_n}\frac{\prod_{j=0}^{n-1}|f'(f^j(z_n))|^2}{\prod_{j=0}^{n-1}|f'(f^j(z_k))|^2}\frac{1}{|f'(f^n(z_k))|^2}$$
$$= \frac{1}{2^{2(q+1)(2n+1)/q}K^{4(2n+1)}}\sum_{z_k\in A_k\subset A_n}\left(\prod_{j=0}^{n-1}\frac{|f'(f^j(z_n))|}{|f'(f^j(z_k))|}\right)^2\frac{1}{|f'(f^n(z_k))|^2}. \quad (3.6)$$

It follows from Lemma 3.2 that

$$|f'(f^j(z_n))|\geqslant K^{-1}R_{j+1}^{(q+1)/q} \quad\text{and}\quad |f'(f^j(z_k))|\leqslant (2R_{j+1})^{(q+1)/q}K, \quad j=0,1,\dots,n-1.$$

This implies that

$$\frac{|f'(f^j(z_n))|}{|f'(f^j(z_k))|}\geqslant\frac{1}{2^{(q+1)/q}K^2}, \quad j=0,1,\dots,n-1. \quad (3.7)$$

Using Lemma 3.2 repeatedly, we obtain

$$|f'(f^n(z_k))|\leqslant (2R_{n+1})^{(q+1)/q}K. \quad (3.8)$$

Putting (3.7) and (3.8) into (3.6) and by Proposition 2.3, we obtain

$$\frac{\mathrm{vol}(\mathcal{U}_{n+1}\cap A_n)}{\mathrm{vol}(A_n)}$$
$$\geqslant \frac{1}{2^{2(q+1)(2n+1)/q}K^{4(2n+1)}}\left(\frac{1}{2^{(q+1)/q}K^2}\right)^{2n}\frac{1}{2^{2(q+1)/q}R_{n+1}^{2(q+1)/q}K^2}\sum_{z_k\in A_k\subset A_n}1$$
$$= 2^{-2(q+1)(3n+2)/q}K^{-6(2n+1)}R_{n+1}^{-2(q+1)/q}N_{n+1}$$

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$$\geqslant 2^{-2(q+1)(3n+2)/q} K^{-6(2n+1)} R_{n+1}^{-2(q+1)/q} 3\pi R_{n+1}^2 (8\mathrm{vol}(\mathcal{R}))^{-1}$$
$$= 2^{-6(q+1)n/q} K^{-12n} R_{n+1}^{-2/q} \gamma$$
$$= 2^{-6(q+1)n/q} K^{-12n} a^{-2(n+1)/q} R_0^{-2/q} \gamma,$$

where $\gamma = (3\pi/8)2^{-4(q+1)/q} K^{-6}(\mathrm{vol}(\mathcal{R}))^{-1}$. □

By Lemma 3.6, the numbers Δ_n from Proposition 2.5 are given by

$$\Delta_n = \frac{\gamma}{2^{6(q+1)n/q} K^{12n} a^{2(n+1)/q} R_0^{2/q}}, \quad n \geqslant 1.$$

Assembling the preceding lemmas, we may now prove Lemma 3.1.

Proof of Lemma 3.1. Lemma 3.6 implies that

$$\sum_{j=1}^{n} |\log \Delta_j| \sim \frac{3(q+1)\log 2 + 6q \log K + \log a}{q} n^2 \quad \text{as } n \to \infty.$$

In view of Lemma 3.4, we have

$$|\log d_n| \sim \frac{q+1}{2q} n^2 \log a \quad \text{as } n \to \infty.$$

The result follows. □

4. Proof of Theorem 1.3

Let $\{c_i \in \mathcal{R} \colon f'(c_i) = 0,\ i = 1, \dots, k\}$ be the set of critical points of f in the basic fundamental parallelogram $\mathcal{R}$ defined in (2.1). Now, we assume that the critical value $f(c_1) \neq 0$ is a pole of the multiplicity q defined in (2.2).

We consider the one-parameter family of functions

$$f_\beta(z) = \beta f(z), \quad \beta \in B(1, r) \text{ for } 0 < r < \frac{1}{4} - \frac{1}{2\alpha + 4} \approx 0.04, \tag{4.1}$$

where $\alpha = \sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2$. The functions f_β are periodic and their critical points are the same as for the elliptic function f. We modify a definition of F given in (2.3). Now

$$F = \{f(c_1) + l\lambda_1 + m\lambda_2 \colon l, m \in \mathbb{Z}\}.$$

Lemma 4.1. *Let $\varepsilon_1 = \min\{\varepsilon_0, r|f(c_1)|\}$. There exists $\varepsilon \in (0, \varepsilon_1)$ such that:*

(i) $\dfrac{(2C_1)^{-1}}{|z-b|^q} \leqslant |f_\beta(z)| \leqslant \dfrac{2C_1}{|z-b|^q},\ \dfrac{(2C_2)^{-1}}{|z-b|^{q+1}} \leqslant |f'_\beta(z)| \leqslant \dfrac{2C_2}{|z-b|^{q+1}},\ z \in B(b, \varepsilon)$;

(ii) $\left\{z \in \bar{\mathbb{C}} \colon |z| \geqslant \dfrac{2C_1}{\varepsilon^q},\ \phi - \dfrac{\pi}{8} \leqslant \arg(z) \leqslant \phi + \dfrac{9\pi}{8}\right\} \subset f_\beta(U(b, \varepsilon))$;

(iii) *f is one to one in each of the segments $U(b, \varepsilon)$*

for any $b \in F$ and all $\beta \in B(1, r)$, where C_1, C_2, ϕ are from Lemma 2.1.

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Proof. Since $\beta \in B(1, r)$, where r is defined in (4.1), we have $1/2 < 1 - r < |\beta| < 1 + r < 2$, and then (i) is an easy consequence of Lemma 2.1 (ii). Moreover, $|\arg(\beta)| < \arcsin(1/4 - 1/(2\alpha + 4)) \approx 0.04$, so for sufficiently small $\varepsilon \in (0, \varepsilon_1)$ we have $\mu_1 \leqslant \arg(\beta G(z)) \leqslant \mu_2$, where $0 < \mu_2 - \mu_1 < \pi/4$ for all $\beta \in B(1, r)$. Arguing analogously to the proof of Lemma 2.1, we show (ii) and (iii). □

From now on, the constants C_1,C_2, ε, ϕ are as in Lemma 4.1.

In this section, we modify the definition of the sequence of radii given in § 2 (see (2.7)). Let $R_1 > 0$ be such that

$$B(f(c_1), \varepsilon) \subset P(R_1, 2R_1). \tag{4.2}$$

Next, we define a constant

$$\hat{a}_0 = \max\left\{2, \frac{2C_1}{(1-\alpha)\varepsilon^q R_1}, \frac{3M}{R_1}, \frac{(6M)^{2q}}{R_1^{2q+1}}, \frac{8\max\{|\lambda_1 + \lambda_2|, |\lambda_1 - \lambda_2|\}}{R_1}\right\}, \tag{4.3}$$

where $M = 2(2C_1)^{(q+1)/q} C_2$. For $a > \hat{a}_0$ we consider a sequence of radii $\{R_n\}_{n \geqslant 1}$ given by the formula $R_n = a^{n-1} R_1$. We consider auxiliary functions $g_n(\beta) = f_\beta^n(c_1)$, $n \in \mathbb{N}$, which are defined outside a countable set of parameters.

Definition 4.2. We define the following family of sets:

$$\begin{aligned}
&\mathcal{D}_0(a) = \{D_0 = B(1, r)\},\\
&\mathcal{D}_1(a) = \{D_1 = g_1^{-1}(U(f(c_1), \varepsilon)) \subset D_0\},\\
&\mathcal{D}_n(a) = \{D_n \subset D_{n-1} \in \mathcal{D}_{n-1}(a) \mid \exists b_n \in F \colon U(b_n, \varepsilon) \subset P^+(R_n),\\
&\qquad\qquad D_n \text{ is a component of } g_n^{-1}(U(b_n, \varepsilon))\}, \quad n \geqslant 2.
\end{aligned}$$

Let

$$\mathcal{V}_n(a) = \bigcup_{D_n \in \mathcal{D}_n(a)} D_n, \qquad D(a) = \bigcap_{n=0}^{\infty} \mathcal{V}_n(a).$$

Figure 1 illustrates the sets defined above for $q = 4$.

Proposition 4.3. *For each $n \geqslant 0$ the set $\mathcal{D}_n(a)$ defined above is non-empty. Moreover, for $n \geqslant 2$, the number N_n of its elements contained in each $D_{n-1} \in \mathcal{D}_{n-1}(a)$ is estimated from below by $3\pi R_n^2/8\mathrm{vol}(\mathcal{R})$.*

Proof. Obviously, $\mathcal{D}_0(a), \mathcal{D}_1(a) \neq \emptyset$. Assume that $\mathcal{D}_{n-1}(a) \neq \emptyset$ for some $n \geqslant 2$. Using Lemma 4.1, we have

$$\{z \in \bar{\mathbb{C}} \colon |z| > R_n,\ \phi \leqslant \arg(z) \leqslant \phi + \pi\} \subset f_\beta(U(b, \varepsilon))$$

for all poles $b \in F$ and $\beta \in B(1, r)$, as $R_n \geqslant R_2 = aR_1 > 2C_1(1-\alpha)^{-1}\varepsilon^{-q} > 2C_1\varepsilon^{-q}$ in view of (4.3). Since $g_{n-1}(D_{n-1}) = U(b_{n-1}, \varepsilon)$ for some pole $b_{n-1} \in F \cap P(R_{n-1}, 2R_{n-1})$, we have $g_n(D_{n-1}) = \{f_\beta(g_{n-1}(\beta)) \mid \beta \in D_{n-1}\} \supset P^+(R_n)$. As $R_n \geqslant 8\max\{|\lambda_1 + \lambda_2|, |\lambda_1 - \lambda_2|\}$, similarly to Proposition 2.3, we show the lower bound on N_n. □

In the next part of this section, we prove the following lemma.

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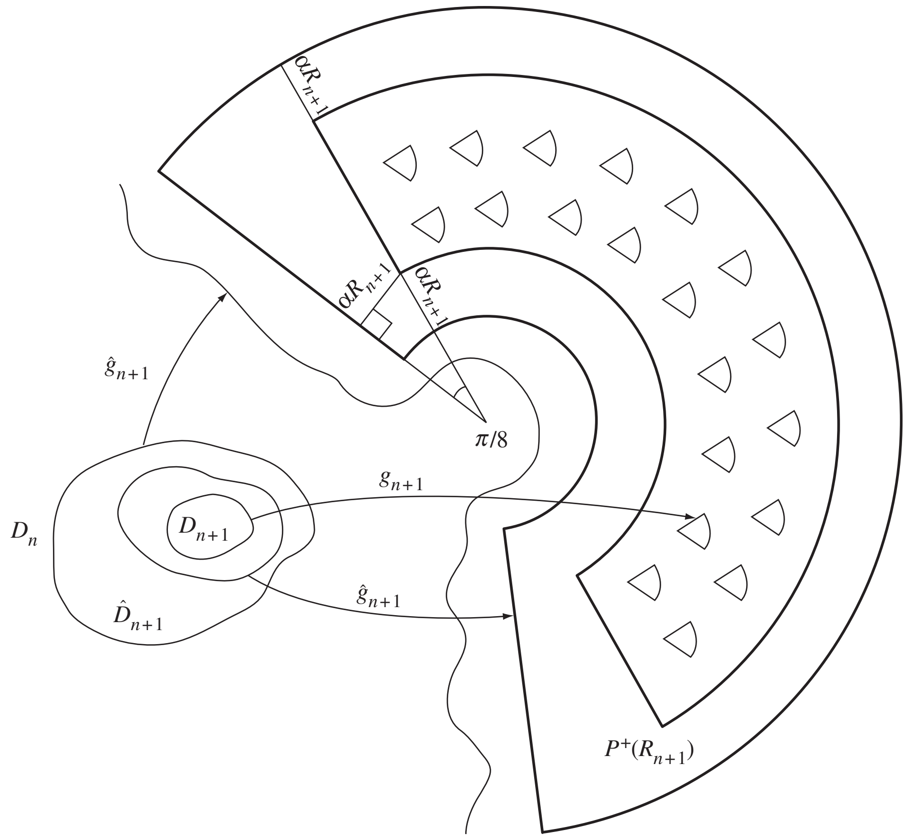

Figure 1. Sets from Definition 4.2 and the proof of Lemma 4.8.

Lemma 4.4. *Let f_β be the family of maps defined in (4.1) and let $\hat{a}_0$ be the constant given in (4.3). Then, for every $a > \hat{a}_0$ there is a subset $D(a)$ of $\mathcal{E}$, and for this subset*

$$\dim_H(D(a)) \geqslant \frac{2q}{q+1} - \frac{6\log 2 + 12q\log M/(q+1)}{\log a}.$$

Theorem 1.3 easily follows from Lemma 4.4.

In order to prove Lemma 4.4 we again use Proposition 2.5. We fix $a > \hat{a}_0$ and consider the sets $\mathcal{D}_n(a)$, $n \geqslant 0$, given in Definition 4.2. As before, for simplicity we suppress the explicit dependence on a in the notation.

Using Lemma 4.1, it is easy to prove the following estimate.

Lemma 4.5. *Let $D_n \in \mathcal{D}_n$, $n \geqslant 2$. Then, for all $\beta \in D_n$,*

$$M^{-1}R_{j+1}^{(q+1)/q} \leqslant |f'_\beta(f^j_\beta(c_1))| \leqslant M(2R_{j+1})^{(q+1)/q}, \quad j = 1,\ldots,n-1.$$

The next lemma is technical, but it is key to the proof in the following part of this section. We would like to note that the 'prime' in g'_n signifies differentiation with respect to β, whereas the 'prime' in f'_β signifies differentiation with respect to z.

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Lemma 4.6. *Let $D_n \in \mathcal{D}_n$, $n \geqslant 2$. Then, for every $\beta \in D_n$,*

$$g_n'(\beta) = \frac{1}{\beta} \prod_{k=1}^{n-1} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} \right].$$

Proof. Let $n = 2$. Note that $g_2(\beta) = f_\beta^2(c_1) = \beta f(\beta f(c_1))$. Thus,

$$\begin{aligned} g_2'(\beta) = f(\beta f(c_1)) + \beta f'(\beta f(c_1)) f(c_1) &= \frac{f_\beta^2(c_1)}{\beta} + \frac{f_\beta'(\beta f(c_1)) f_\beta(c_1)}{\beta} \\ &= \frac{1}{\beta} f_\beta'(\beta f(c_1)) \left[f_\beta(c_1) + \frac{f_\beta^2(c_1)}{f_\beta'(f_\beta(c_1))} \right] \\ &= \frac{1}{\beta} f_\beta'(f_\beta(c_1)) \left[f_\beta(c_1) + \frac{f_\beta^2(c_1)}{f_\beta'(f_\beta(c_1))} \right]. \end{aligned}$$

Suppose that the lemma is true for some $n \geqslant 2$. We show that it is true for $n+1$:

$$\begin{aligned} g_{n+1}(\beta) &= \beta f(g_n(\beta)), \\ g_{n+1}'(\beta) &= f(g_n(\beta)) + \beta f'(g_n(\beta)) g_n'(\beta) \\ &= \frac{f_\beta^{n+1}(c_1)}{\beta} + f_\beta'(f_\beta^n(c_1)) \frac{1}{\beta} \prod_{k=1}^{n-1} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} \right] \\ &= \frac{f_\beta^{n+1}(c_1)}{\beta} + \frac{1}{\beta} \prod_{k=1}^{n} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} \right] \\ &= \frac{1}{\beta} \prod_{k=1}^{n} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} + \frac{f_\beta^{n+1}(c_1)}{\prod_{i=1}^{n} f_\beta'(f_\beta^i(c_1))} \right] \\ &= \frac{1}{\beta} \prod_{k=1}^{n} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n+1} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} \right]. \end{aligned}$$

□

In the following lemma we estimate the derivative of g_n. We show that g_n' is comparable to the product of the derivatives of f_β over the trajectory of the critical value $f_\beta(c_1)$.

Lemma 4.7. *Let $D_n \in \mathcal{D}_n$, $n \geqslant 2$. Then, for every $\beta \in D_n$,*

$$\begin{aligned} &\frac{M^{1-n}}{2(1+r)} a^{(q+1)n(n-1)/2q} R_1^{(n(q+1)-1)/q} \\ &\qquad \leqslant |g_n'(\beta)| \\ &\qquad \leqslant \frac{5}{2(1-r)} (2^{(q+1)/q} M)^{n-1} a^{(q+1)n(n-1)/2q} R_1^{(n(q+1)-1)/q}. \end{aligned}$$

Proof. Fix $\beta \in D_n$. It follows from Lemma 4.5 that

$$\begin{aligned}\left|\prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1))\right| &\leqslant 2^{(q+1)/q} R_2^{(q+1)/q} M \times \cdots \times 2^{(q+1)/q} R_n^{(q+1)/q} M \\ &= (2^{(q+1)/q} M)^{n-1} (aR_1)^{(q+1)/q} \times \cdots \times (a^{n-1} R_1)^{(q+1)/q} \\ &= (2^{(q+1)/q} M)^{n-1} a^{(q+1)n(n-1)/2q} R_1^{(q+1)(n-1)/q}.\end{aligned}$$

Analogously, we get the estimate from below:

$$\left|\prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1))\right| \geqslant (M^{-1})^{n-1} a^{(q+1)n(n-1)/2q} R_1^{(q+1)(n-1)/q}.$$

Finally,

$$\begin{aligned}M^{1-n} a^{(q+1)n(n-1)/2q} R_1^{(q+1)(n-1)/q} &\leqslant \left|\prod_{k=1}^{n-1} f'_\beta(f^k_\beta(c_1))\right| \\ &\leqslant (2^{(q+1)/q} M)^{n-1} a^{(q+1)n(n-1)/2q} R_1^{(q+1)(n-1)/q}.\end{aligned} \tag{4.4}$$

Now, applying (4.4), we estimate the sum $\sum_{k=2}^{n} f^k_\beta(c_1)/\prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1))$:

$$\begin{aligned}\left|\sum_{k=2}^{n} \frac{f^k_\beta(c_1)}{\prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1))}\right| &\leqslant \sum_{k=2}^{n} \left|\frac{f^k_\beta(c_1)}{\prod_{i=1}^{k-1} f'_\beta(f^i_\beta(c_1))}\right| \\ &\leqslant \sum_{k=2}^{n} \frac{2R_k}{M^{1-k} a^{(q+1)k(k-1)/2q} R_1^{(q+1)(k-1)/q}} \\ &= \sum_{k=2}^{n} \frac{2a^{k-1} R_1 M^{k-1}}{a^{(q+1)k(k-1)/2q} R_1^{(q+1)(k-1)/q}} \\ &= \sum_{k=2}^{n} \frac{2M^{k-1}}{a^{((q+1)k-2q)(k-1)/2q} R_1^{(q+1)(k-1)/q-1}} \\ &= \frac{2M}{(aR_1)^{1/2q}} \sum_{k=2}^{n} \frac{M^{k-2}}{a^{(((q+1)k-2q)(k-1)-1)/2q} R_1^{(2(q+1)(k-1)-2q-1)/2q}}.\end{aligned}$$

Since $a > \hat{a}_0 \geqslant 2$ and $((q+1)k-2q)(k-1) \geqslant 2(q+1)(k-1)-2q$ for $q \in \mathbb{N}$, $k = 2, 3, \ldots,$ we have

$$\sum_{k=2}^{n} \frac{M^{k-2}}{a^{(((q+1)k-2q)(k-1)-1)/2q} R_1^{(2(q+1)(k-1)-2q-1)/2q}} \leqslant \sum_{k=2}^{n} \frac{M^{k-2}}{(aR_1)^{(2(q+1)(k-1)-2q-1)/2q}}.$$

Using the inequality $(2(q+1)(k-1)-2q-1)/2q \geqslant k-2$, $q \in \mathbb{N}$, $k = 2, 3, \ldots,$ and the fact that $a > \hat{a}_0 \geqslant \max\{R_1^{-1}, 3MR_1^{-1}\}$, we obtain

$$\sum_{k=2}^{n} \frac{M^{k-2}}{(aR_1)^{(2(q+1)(k-1)-2q-1)/2q}} \leqslant \sum_{k=2}^{n} \left(\frac{M}{aR_1}\right)^{k-2} \leqslant \sum_{k=2}^{\infty} \left(\frac{M}{aR_1}\right)^{k-2} = \frac{1}{1-M/aR_1} \leqslant \frac{3}{2}.$$

Hence,

$$\left|\sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f'_\beta(f_\beta^i(c_1))}\right| \leqslant \frac{2M}{(aR_1)^{1/2q}} \times \frac{3}{2} \leqslant \frac{R_1}{2},$$

because $a > \hat{a}_0 \geqslant (6M)^{2q} R_1^{-2q-1}$. Therefore,

$$\frac{R_1}{2} = R_1 - \frac{R_1}{2} \leqslant \left|f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f'_\beta(f_\beta^i(c_1))}\right| \leqslant 2R_1 + \frac{R_1}{2} = \frac{5R_1}{2}. \tag{4.5}$$

Plugging (4.4) and (4.5) into the formula for g'_n, we prove the lemma. □

In the next part of this section we estimate the diameters of D_n and the ratios $\mathrm{vol}(\mathcal{V}_{n+1} \cap D_n)/\mathrm{vol}(D_n)$, and in order to do that we prove that the functions g_n are conformal on $D_n \in \mathcal{D}_n$. Note that the maps g_n, $n \geqslant 2$, are holomorphic outside a countable set of points and have poles at $\beta_{n-1} \in \partial D_{n-1}$.

Lemma 4.8. *For each $D_n \in \mathcal{D}_n$, $n \geqslant 1$, the map g_n is conformal on D_n.*

Proof. The map $g_1(\beta) = f_\beta(c_1) = \beta f(c_1)$ is linear, and so is one to one and holomorphic on D_1. By induction, we show that the maps g_n, $n \geqslant 2$, are conformal. Suppose that g_n, $n \geqslant 1$, is conformal on D_n; we prove that g_{n+1} is conformal on $D_{n+1} \subset D_n$. If $n = 1$, then we take the segment $U(b_1, \varepsilon) \subset P(R_1, 2R_1)$ with $b_1 = f(c_1)$, and if $n \geqslant 2$, we consider a segment $U(b_n, \varepsilon) \subset P^+(R_n)$. We know that D_n is a component of $g_n^{-1}(U(b_n, \varepsilon))$, $n \geqslant 1$. Let $\beta_n = g_n^{-1}(b_n) \in \partial D_n$. If $D_{n+1} \subset D_n \in \mathcal{D}_n$, then D_{n+1} is a component of $g_{n+1}^{-1}(U(b_{n+1}, \varepsilon))$ with $U(b_{n+1}, \varepsilon) \subset P^+(R_{n+1})$. We define a map $\hat{g}_{n+1}(\beta) = f_{\beta_n}(g_n(\beta)) = \beta_n f(g_n(\beta))$. It follows from Lemma 4.1 (ii) that

$$\hat{g}_{n+1}(D_n) \supset \left\{z \in \bar{\mathbb{C}} \colon |z| \geqslant \frac{2C_1}{\varepsilon^q},\ \phi - \frac{\pi}{8} \leqslant \arg(z) \leqslant \phi + \frac{9\pi}{8}\right\}.$$

We show that $\hat{g}_{n+1}$ is one to one in D_n. Take $\beta', \beta'' \in D_n$ such that $\hat{g}_{n+1}(\beta') = \hat{g}_{n+1}(\beta'')$. By definition of the map $\hat{g}_{n+1}$, we have $f(g_n(\beta')) = f(g_n(\beta''))$, where $g_n(\beta'), g_n(\beta'') \in g_n(D_n) = U(b_n, \varepsilon)$. By Lemma 4.1 (iii), f is one to one in $U(b_n, \varepsilon)$, so $g_n(\beta') = g_n(\beta'')$ and this implies that $\beta' = \beta''$. This follows from the injectivity of the map g_n.

There is a set $\hat{D}_{n+1} \subset D_n$ such that

$$\hat{g}_{n+1}(\hat{D}_{n+1}) = \left\{z \in \mathbb{C} \colon (1-\alpha)R_{n+1} < |z| < (2+\alpha)R_{n+1},\ \phi - \frac{\pi}{8} < \arg(z) < \phi + \frac{9\pi}{8}\right\} \tag{4.6}$$

for ϕ as in Lemma 4.1.

Now, we show that $D_{n+1} \subset \hat{D}_{n+1}$. Note that $\hat{g}_{n+1}(\beta) = (\beta_n/\beta) g_{n+1}(\beta)$. Since $g_{n+1}(D_{n+1}) = U(b_{n+1}, \varepsilon) \subset P^+(R_{n+1})$ and $0 < r < 1/4 - 1/(2\alpha + 4)$, for $\beta \in D_{n+1}$ we have

$$|\hat{g}_{n+1}(\beta)| > \frac{1-r}{1+r}R_{n+1} > \frac{3\alpha+8}{5\alpha+8}R_{n+1} \approx 0.92R_{n+1} > (1-\alpha)R_{n+1} \approx 0.62R_{n+1},$$

$$|\hat{g}_{n+1}(\beta)| < \frac{1+r}{1-r}2R_{n+1} < \frac{2(5\alpha+8)}{3\alpha+8}R_{n+1} \approx 2.17R_{n+1} < (2+\alpha)R_{n+1} \approx 2.38R_{n+1},$$

$$\begin{aligned}\arg(\hat{g}_{n+1}(\beta)) &< \phi + \pi + 2\max_{\beta\in B(1,r)}\arg(\beta) \\ &< \phi + \pi + 2\arcsin\left(\frac{1}{4} - \frac{1}{2\alpha+4}\right) \\ &\approx \phi + \pi + 0.08 < \phi + \frac{9\pi}{8},\end{aligned}$$

$$\arg(\hat{g}_{n+1}(\beta)) > \phi - 2\max_{\beta\in B(1,r)}\arg(\beta) > \phi - 2\arcsin\left(\frac{1}{4} - \frac{1}{2\alpha+4}\right) \approx \phi - 0.08 > \phi - \frac{\pi}{8}.$$

Thus, $\hat{g}_{n+1}(D_{n+1}) \subset \hat{g}_{n+1}(\hat{D}_{n+1})$. Since the map $\hat{g}_{n+1}$ is one to one in D_n, we have $D_{n+1} \subset \hat{D}_{n+1}$.

It follows from (4.6) that

$$\begin{aligned}U(b_{n+1},\varepsilon) &= g_{n+1}(D_{n+1}) \\ &\subset P^{+}(R_{n+1}) \\ &\subset \bigg\{z \in \mathbb{C}\colon (1-\alpha)R_{n+1} < |z| < (2+\alpha)R_{n+1}, \\ &\qquad\qquad \phi - \frac{\pi}{8} < \arg(z) < \phi + \frac{9\pi}{8}\bigg\} = \hat{g}_{n+1}(\hat{D}_{n+1}).\end{aligned}$$

Since $0 < r < 1/4 - 1/(2\alpha+4)$, taking $\zeta = g_n(\beta)$ for $\beta \in \partial\hat{D}_{n+1}$ we have

$$\begin{aligned}2r|f(\zeta)| < \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)|f(\zeta)| &= \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|\frac{\hat{g}_{n+1}(\beta)}{\beta_n}\right| \\ &\leqslant \left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\frac{(2+\alpha)R_{n+1}}{|\beta_n|} \\ &= \frac{\alpha R_{n+1}}{2|\beta_n|} \\ &< \alpha R_{n+1},\end{aligned}$$

as $|\beta_n| > 1 - r > 1/2$. Hence (see Figure 1),

$$\operatorname{dist}(\hat{g}_{n+1}(\beta), g_{n+1}(D_{n+1})) \geqslant \alpha R_{n+1} > 2r|f(\zeta)|.$$

Thus, for $\beta \in \partial\hat{D}_{n+1}$ and $w \in g_{n+1}(D_{n+1})$ we have

$$|\hat{g}_{n+1}(\beta) - w| \geqslant \operatorname{dist}(\hat{g}_{n+1}(\beta), g_{n+1}(D_{n+1})) > 2r|f(\zeta)|$$

and

$$|g_{n+1}(\beta) - \hat{g}_{n+1}(\beta)| = |\beta f(\zeta) - \beta_n f(\zeta)| = |\beta - \beta_n|\,|f(\zeta)| < 2r|f(\zeta)|.$$

Hence, $|\hat{g}_{n+1}(\beta) - w| > |g_{n+1}(\beta) - \hat{g}_{n+1}(\beta)|$ in the set $\partial\hat{D}_{n+1}$. Since the map g_{n+1} is holomorphic on $\operatorname{int} D_n$, the assumptions of the Rouché theorem are satisfied. It implies that the equations $\hat{g}_{n+1}(\beta) = w$ and $g_{n+1}(\beta) = w$ have the same number of roots in $\hat{D}_{n+1}$. Since the map $\hat{g}_{n+1}$ is one to one in $\hat{D}_{n+1}$, the former equation has a unique root for a given w. So does the latter. Since $D_{n+1} \subset \hat{D}_{n+1}$, we have that g_{n+1} is one to one in D_{n+1}. The map g_{n+1} is holomorphic on $\operatorname{int} D_n$, and so is conformal on D_{n+1}. □

Remark 4.9. In Lemma 4.8 we showed in fact that the segments $U(b_n, \varepsilon) \subset P^+(R_n)$, $n \geqslant 2$, are in one-to-one correspondence with the sets $D_n \subset \mathcal{V}_n \cap D_{n-1}$ for each $D_{n-1} \in \mathcal{D}_{n-1}$. Hence, each $\mathcal{D}_n$, $n \geqslant 1$, is a finite collection of the sets D_n. Moreover, the sets D_n are pairwise disjoint.

The next lemma follows immediately from Lemma 4.7.

Lemma 4.10. *Let $D_n \in \mathcal{D}_n$, $n \geqslant 2$. Then the distortion $L(g_n, D_n)$ satisfies*

$$L(g_n, D_n) \leqslant \frac{5(1+r)}{1-r} 2^{(q+1)(n-1)/q} M^{2(n-1)}$$

and

$$\operatorname{diam}(D_n) \leqslant \frac{4\varepsilon(1+r)M^{n-1}}{a^{(q+1)n(n-1)/2q} R_1^{(n(q+1)-1)/q}},$$

where ε is as in Lemma 4.1.

Remark 4.11. Observe that $\operatorname{diam}(D_n) \to 0$ as $n \to \infty$, since $a > \hat{a}_0 \geqslant 2$.

By Lemma 4.10, the numbers d_n defined in Proposition 2.5 are equal to

$$d_n = 4\varepsilon(1+r)M^{n-1} a^{-(q+1)n(n-1)/2q} R_1^{-(n(q+1)-1)/q}, \quad n \geqslant 2, \tag{4.7}$$

and $d_1 = \operatorname{diam}(A_1) < 2r$.

In the next lemma we estimate from below the density of the sets $\mathcal{V}_{n+1} \cap D_n$ in the set $D_n \in \mathcal{D}_n$, $n \geqslant 2$. The proof is very similar to the proof of Lemma 3.6.

Lemma 4.12. *There exists $M' > 0$ such that*

$$\frac{\operatorname{vol}(\mathcal{V}_{n+1} \cap D_n)}{\operatorname{vol}(D_n)} \geqslant \frac{M'}{2^{6(q+1)n/q} M^{12n} a^{2n/q} R_1^{2/q}}$$

for each $D_n \in \mathcal{D}_n$, $n \geqslant 2$.

Proof. Take some $n \geqslant 2$ and $D_n \in \mathcal{D}_n$. There exists $b_n \in F$ such that D_n is a component of $g_n^{-1}(U(b_n, \varepsilon))$, where $U(b_n, \varepsilon) \subset P^+(R_n)$. Moreover, for each $D_k \in \mathcal{D}_{n+1}$ there is $b_{n+1} \in F$ such that D_k is a component of $g_{n+1}^{-1}(U(b_{n+1}, \varepsilon))$, $U(b_{n+1}, \varepsilon) \subset P^+(R_{n+1})$. There are finitely many sets $D_k \in \mathcal{D}_{n+1}$ contained in D_n. We denote by b_k the pole corresponding to D_k. Let $\beta_n = g_n^{-1}(b_n) \in D_n$, $\beta_k = g_{n+1}^{-1}(b_k) \in D_k$. It follows from Lemma 4.8 that g_n is conformal on D_n, so $L(g_n, D_n) = L(g_n^{-1}, g_n(D_n))$. Hence,

$$\operatorname{vol}(D_n) \leqslant \frac{3\pi\varepsilon^2}{4q} \left(\frac{L(g_n, D_n)}{|g_n'(\beta_n)|}\right)^2 \tag{4.8}$$

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and

$$\operatorname{vol}(\mathcal{V}_{n+1} \cap D_n) = \sum_{D_k \subset D_n} \operatorname{vol}(D_k) \geqslant \frac{3\pi\varepsilon^2}{4q} \sum_{\beta_k \in D_k \subset D_n} (L(g_{n+1}, D_k)|g'_{n+1}(\beta_k)|)^{-2}. \quad (4.9)$$

It follows from (4.8) and (4.9) that

$$\frac{\operatorname{vol}(\mathcal{V}_{n+1} \cap D_n)}{\operatorname{vol}(D_n)} \geqslant \frac{|g'_n(\beta_n)|^2}{(L(g_n, D_n))^2} \sum_{\beta_k \in D_k \subset D_n} (L(g_{n+1}, D_k)|g'_{n+1}(\beta_k)|)^{-2}.$$

Then, by Lemma 4.6, (4.5) and Lemma 4.10, we obtain

$$\frac{\operatorname{vol}(\mathcal{V}_{n+1} \cap D_n)}{\operatorname{vol}(D_n)} \geqslant \frac{((1-r)/(1+r))^6}{5^6 2^{2(q+1)(2n-1)/q} M^{4(2n-1)}} \times \sum_{\beta_k \in A_k \subset A_n} \left(\prod_{j=1}^{n-1} \frac{|f'_{\beta_n}(f^j_{\beta_n}(c_1))|}{|f'_{\beta_k}(f^j_{\beta_k}(c_1))|} \right)^2 \frac{1}{|f'_{\beta_k}(f^n_{\beta_k}(c_1))|^2}. \quad (4.10)$$

By Lemma 4.5, we have

$$|f'_{\beta_k}(f^n_{\beta_k}(c_1))| \leqslant (2R_{n+1})^{(q+1)/q} M \quad \text{and} \quad \frac{|f'_{\beta_n}(f^j_{\beta_n}(c_1))|}{|f'_{\beta_k}(f^j_{\beta_k}(c_1))|} \geqslant \frac{1}{2^{(q+1)/q} M^2} \quad (4.11)$$

for $j = 1, 2, \dots, n-1$. Putting (4.11) into (4.10) and using Proposition 4.3, we obtain

$$\begin{aligned} \frac{\operatorname{vol}(\mathcal{V}_{n+1} \cap D_n)}{\operatorname{vol}(D_n)} &\geqslant \left(\frac{1-r}{1+r}\right)^6 \frac{1}{5^6 2^{2(q+1)(2n-1)/q} M^{4(2n-1)}} \left(\frac{1}{2^{(q+1)/q} M^2}\right)^{2(n-1)} \frac{1}{2^{2(q+1)/q} M^2 R_{n+1}^{2(q+1)/q}} \\ &\quad \times \sum_{\beta_k \in D_k \subset D_n} 1 \\ &= \left(\frac{1-r}{1+r}\right)^6 \frac{2^{2(q+1)/q} N_{n+1}}{5^6 2^{6(q+1)n/q} M^{6(2n-1)} R_{n+1}^{2(q+1)/q}} \\ &\geqslant \frac{M'}{2^{6(q+1)n/q} M^{12n} a^{2n/q} R_1^{2/q}}, \end{aligned}$$

where

$$M' = \frac{3\pi(1-r)^6 2^{2(q+1)/q} M^6}{8 \times 5^6 (1+r)^6 \operatorname{vol}(\mathcal{R})}.$$

□

By Lemma 4.12, the numbers Δ_n from Proposition 2.5 are given by

$$\Delta_n = \frac{M'}{2^{6(q+1)n/q} M^{12n} a^{2n/q} R_1^{2/q}}, \quad n \geqslant 2.$$

Lemma 4.4 is a consequence of McMullen's result (Proposition 2.5) and Lemmas 4.10 and 4.12. We omit the proof since it is analogous to the proof of Lemma 3.1.

5. Examples

In this section we give two examples of functions satisfying the assumptions of Theorem 1.3. First, we recall a definition of the Weierstrass elliptic function.

Definition 5.1. For any lattice Λ, the Weierstrass elliptic function is defined by the formula

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right), \quad z \in \mathbb{C}.$$

The numbers $\alpha_1(\Lambda) = 60\sum_{w\in\Lambda\setminus\{0\}} w^{-4}$, $\alpha_2(\Lambda) = 140\sum_{w\in\Lambda\setminus\{0\}} w^{-6}$ are invariants of the lattice Λ in the following sense: for any lattice Λ', if $\alpha_1(\Lambda) = \alpha_1(\Lambda')$ and $\alpha_2(\Lambda) = \alpha_2(\Lambda')$, then $\Lambda = \Lambda'$.

The first example was given by Hawkins and Koss in [**5**].

Example 5.2. Let $\Gamma = [\gamma_1, \gamma_2]$ be the lattice with the invariants $\alpha_1(\Gamma) = 0$, $\alpha_2(\Gamma) = 4$. Set $\Lambda = [\lambda_1, \lambda_2]$ with $\lambda_1 = \sqrt[3]{\mathrm{e}^{4\pi\mathrm{i}/3}\gamma_1^2/m}$, where m is an odd negative number and $\lambda_2 = \lambda_1\gamma_2/\gamma_1$. Then all critical values of $\wp_\Lambda$ are poles.

The same authors showed the following example in [**6**].

Example 5.3. Let Λ be the lattice with the invariants $\alpha_1(\Lambda) \approx 26.5626$ and $\alpha_2(\Lambda) \approx -26.2672$. Then $\wp_\Lambda$ has an attracting fixed point $p \approx 1.5566$, i.e. $\wp_\Lambda(p) = p$ and $|\wp'_\Lambda(p)| < 1$ such that $\wp^n_\Lambda(e_1) \to p$, $\wp^n_\Lambda(e_2) \to p$, where $e_1 \approx 1.4206$ and $e_2 \approx 1.5539$ are two critical values of $\wp_\Lambda$. The third critical value $e_3 \approx -2.9746$ is a pole.

Weiestrass elliptic functions given in both examples satisfy the assumptions of Theorem 1.3 with $q = 2$. Thus, in both cases $\dim_H(\mathcal{E}) \geqslant 4/3$.

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