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SUMS OF FOUR SQUARES WITH A CERTAIN RESTRICTION

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Abstract

Z.-W. Sun ['Refining Lagrange's four-square theorem', *J. Number Theory* **175** (2017), 169–190] conjectured that every positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N} = \{0, 1, \ldots\})$ with $x + 3y$ a square and also as $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 3y \in \{4^k : k \in \mathbb{N}\}$. In this paper, we confirm these conjectures via the arithmetic theory of ternary quadratic forms.

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1. Introduction

The Lagrange four-square theorem states that every positive integer can be written as the sum of four integral squares. In 1917, Ramanujan [8] claimed that there are at most 55 positive-definite integral diagonal quaternary quadratic forms that can represent all positive integers. Later, Dickson [2] confirmed that Ramanujan's claim is true for 54 forms in Ramanujan's list and pointed out that the quaternary form $x^2 + 2y^2 + 5z^2 + 5w^2$ included in his list represents all positive integers except 15.

In 2017, Z.-W. Sun [11] studied some refinements of Lagrange's theorem. For instance, he showed that for any $k \in \{4, 5, 6\}$, each positive integer n can be written as $x^k + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N} = \{0, 1, \ldots\}$. Sun called the integer polynomial $P(X, Y, Z, W)$ a *suitable polynomial* if every positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $P(x, y, z, w)$ a square. In the same paper, Sun showed that the polynomials X, $2X$, $X - Y$, $2X - 2Y$ are all suitable. Also, he showed that every positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y$ a square and he conjectured that $X + 2Y$ is a suitable polynomial. This conjecture was later confirmed by Y.-C. Sun and Z.-W. Sun in [9]. See [6, 12, 14] for recent progress on this topic. Sun [11] also investigated the polynomial $X + 3Y$ and obtained the following result.

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THEOREM 1.1 [11, Theorem 1.3(ii)]. *Assuming the GRH (Generalised Riemann Hypothesis), every positive integer n can be written as* $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x + 3y$ *a square.*

Based on calculations, Sun posed the following conjecture.

CONJECTURE 1.2 [11, Conjecture 4.1]. $X + 3Y$ is a suitable polynomial, that is, each positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y$ a square.

REMARK 1.3. With the help of a computer, Sun [13] verified this conjecture for n up to 10^8. Later, the authors verified the conjecture for n up to 4×10^9. For example,

$$9996 = 58^2 + 14^2 + 6^2 + 80^2 \quad \text{and} \quad 58 + 3 \times 14 = 10^2,$$
$$99999999 = 139^2 + 19^2 + 6866^2 + 7269^2 \quad \text{and} \quad 139 + 3 \times 19 = 14^2,$$
$$3999999999 = 2347^2 + 18^2 + 12671^2 + 15295^2 \quad \text{and} \quad 2347 + 3 \times 18 = 49^2.$$

We confirm this conjecture via the arithmetic theory of ternary quadratic forms.

THEOREM 1.4. *Every positive integer n can be written as* $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ *with* $x + 3y$ *a square.*

If we omit the condition that $x, y, z, w \in \mathbb{N}$, then Sun [12, Theorem 1.2(iii)] proved that every positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 3y \in \{4^k : k \in \mathbb{N}\}$ provided that any positive integer $m \equiv 9 \pmod{20}$ can be written as $5x^2 + 5y^2 + z^2$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid z$. Motivated by this, Sun also posed the following conjecture.

CONJECTURE 1.5. For every positive integer n, there exist $x, y, z, w \in \mathbb{Z}$ such that $n = x^2 + y^2 + z^2 + w^2$ and $x + 3y \in \{4^k : k = 0, 1, \ldots\}$.

REMARK 1.6. Note that there are infinitely many positive integers which do not satisfy the above conjecture if we add the condition that $x, y, z, w \in \mathbb{N}$. In fact, it is known that each integer of the form $4^{2r+1} \times 2$ $(r \in \mathbb{N})$ has only a single partition into four squares (see [4, page 86]), that is,

$$2 \times 4^{2r+1} = (2 \times 4^r)^2 + (2 \times 4^r)^2 + 0^2 + 0^2.$$

Clearly Conjecture 1.5 does not hold for any integer of this type if we add the condition that $x, y, z, w \in \mathbb{N}$.

We confirm the conjecture and obtain the following result.

THEOREM 1.7. *Every positive integer n can be written as* $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x + 3y \in \{4^k : k = 0, 1, \ldots\}$. *Moreover, if* $4 \nmid n$, *then there are* $x, y, z, w \in \mathbb{Z}$ *such that* $n = x^2 + y^2 + z^2 + w^2$ *and* $x + 3y \in \{1, 4\}$.

REMARK 1.8. For example,

$$99997 = (-98)^2 + 34^2 + 119^2 + 274^2 \quad \text{and} \quad -98 + 3 \times 34 = 4,$$
$$99999 = (-29)^2 + 10^2 + 33^2 + 313^2 \quad \text{and} \quad -29 + 3 \times 10 = 1.$$

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In Section 2 we will introduce some notation and prove some lemmas which are key elements in our proofs of the theorems. The proofs of our main results will be given in Section 3.

2. Notation and preparations

Throughout, for any prime p, we let $\mathbb{Z}_p$ denote the ring of p-adic integers and $\mathbb{Z}_p^\times$ the group of invertible elements in $\mathbb{Z}_p$. In addition, we set $\mathbb{Z}_p^{\times 2} = \{x^2 : x \in \mathbb{Z}_p^\times\}$ and let $M_3(\mathbb{Z}_p)$ denote the ring of 3×3 matrices with entries contained in $\mathbb{Z}_p$. We also adopt the standard notation of quadratic forms (see [1, 7]). Let

$$f(X, Y, Z) = aX^2 + bY^2 + cZ^2 + rYZ + sZX + tXY$$

be an integral positive-definite ternary quadratic form. Its associated matrix is

$$M_f := \begin{pmatrix} a & t/2 & s/2 \\ t/2 & b & r/2 \\ s/2 & r/2 & c \end{pmatrix}.$$

Let p be an arbitrary prime. We introduce the notation

$$\mathrm{q}(f) := \{f(x, y, z) : \ (x, y, z) \in \mathbb{Z}^3\},$$
$$\mathrm{q}_p(f) := \{f(x, y, z) : \ (x, y, z) \in \mathbb{Z}_p^3\}.$$

For $p > 2$, we say that f is unimodular over $\mathbb{Z}_p$ if its associated matrix $M_f \in M_3(\mathbb{Z}_p)$ and is invertible. We also let $\mathrm{gen}(f)$ denote the set of quadratic forms which are in the genus of f.

For any positive integer n, we say that n can be represented by $\mathrm{gen}(f)$ if there exists a form $f^* \in \mathrm{gen}(f)$ such that $n \in \mathrm{q}(f^*)$. When this occurs, we write $n \in \mathrm{q}(\mathrm{gen}(f))$. By [1, Theorem 1.3, page 129],

$$n \in \mathrm{q}(\mathrm{gen}(f)) \Leftrightarrow n \in \mathrm{q}_p(f) \quad \text{for all primes } p. \tag{2.1}$$

The first lemma involves local representations over $\mathbb{Z}_2$.

LEMMA 2.1 [5, pages 186–187]. *Let f be an integral positive-definite ternary quadratic form and let n be a positive integer. Then $n \in \mathrm{q}_2(f)$ if and only if*

$$f(X, Y, Z) \equiv n \pmod{2^{r+1}}$$

is solvable, where r is the 2-adic order of $4n$.

The next lemma concerns the global representations by the form $x^2 + 10y^2 + 10z^2$.

LEMMA 2.2. (i) *Let $n \equiv 1, 2 \pmod 4$ be a positive integer and λ be an odd integer with $0 < \lambda \le \sqrt{10n}$ and $5 \nmid \lambda$. Then there are $x, y, z \in \mathbb{Z}$ such that*

$$10n - \lambda^2 = x^2 + 10y^2 + 10z^2.$$

(ii) *Let* $n \equiv 3 \pmod 4$ *be a positive integer and* δ *be an integer with* $0 < \delta \le \sqrt{10n}$, $4 \mid \delta$ *and* $5 \nmid \delta$. *Then there are* $x, y, z \in \mathbb{Z}$ *such that*

$$10n - \delta^2 = x^2 + 10y^2 + 10z^2.$$

(iii) *Let* n *be a positive odd integer and* μ *be an integer with* $0 < \mu \le \sqrt{10n}$, $\mu \equiv 2 \pmod 4$ *and* $5 \nmid \mu$. *Then there are* $x, y, z \in \mathbb{Z}$ *such that*

$$10n - \mu^2 = x^2 + 10y^2 + 10z^2.$$

PROOF. Part (i). Let $f(X, Y, Z) = X^2 + 10Y^2 + 10Z^2$. For any prime $p \neq 2, 5$, it is clear that $f(X, Y, Z)$ is unimodular over $\mathbb{Z}_p$ and hence $\mathrm{q}_p(f) = \mathbb{Z}_p$. As $5 \nmid \lambda$, we have $10n - \lambda^2 \in \mathrm{q}_5(f)$ by the local square theorem (see [7, Section 63:1]). When $p = 2$, since $n \equiv 1, 2 \pmod 4$, we have $10n - \lambda^2 \equiv 1, 3 \pmod 8$. By the local square theorem again, $10n - \lambda^2 \in \mathbb{Z}_2^{\times 2}$ if $n \equiv 1 \pmod 4$ and $10n - \lambda^2 \in 3\mathbb{Z}_2^{\times 2}$ if $n \equiv 2 \pmod 4$. This implies that $10n - \lambda^2 \in \mathrm{q}_2(f)$ and hence $10n - \lambda^2 \in \mathrm{q}(\mathrm{gen}(f))$.

There are two classes in $\mathrm{gen}(f)$ and the one not containing f has a representative $g(X, Y, Z) = 4X^2 + 5Y^2 + 6Z^2 + 4ZX$. By (2.1), either $10n - \lambda^2 \in \mathrm{q}(f)$ or $10n - \lambda^2 \in \mathrm{q}(g)$. If $10n - \lambda^2 \in \mathrm{q}(f)$, then we are done. Suppose now that $10n - \lambda^2 \in \mathrm{q}(g)$, that is, there are $x, y, z \in \mathbb{Z}$ such that

$$10n - \lambda^2 = g(x, y, z) = 4x^2 + 5y^2 + 6z^2 + 4zx.$$

Clearly $2 \nmid y$. As $10n - \lambda^2 \equiv 2n - 1 \equiv 5 + 2z^2 \pmod 4$, we obtain $z \equiv n + 1 \pmod 2$. If $n \equiv 1 \pmod 4$, then $10n - \lambda^2 \equiv 1 \equiv 4x^2 + 5 \pmod 8$ and hence $x \equiv 1 \pmod 2$. This gives $x - y - z \equiv 0 \pmod 2$ in the case $n \equiv 1 \pmod 4$. In the case $n \equiv 2 \pmod 4$, as $z \equiv 1 \pmod 2$ and

$$g(X, Y, Z) = g(X + Z,\ Y,\ -Z), \tag{2.2}$$

there must exist $x', y', z' \in \mathbb{Z}$ with $2 \mid x' - y' - z'$ such that $10n - \lambda^2 = g(x', y', z')$. One can also easily verify that

$$f(X, Y, Z) = g\Big(\frac{X - Y - Z}{2},\ -Y + Z,\ Y + Z\Big). \tag{2.3}$$

By this equality and the above discussion, it is easy to see that $10n - \lambda^2 \in \mathrm{q}(f)$.

Part (ii). Keep notation as above. For the same reasons as in (i), $10n - \delta^2 \in \mathrm{q}_p(f)$ for any prime $p \neq 2$. When $p = 2$, by the local square theorem,

$$\mathbb{Z}_2^{\times} \subseteq \{2x^2 + 5y^2 + 5z^2 : x, y, z \in \mathbb{Z}_2\}. \tag{2.4}$$

Hence, $5n - \delta^2/2 \equiv 2x^2 + 5y^2 + 5z^2 \pmod 8$ is solvable. This implies that the congruence $10n - \delta^2 \equiv f(x, y, z) \pmod{16}$ is solvable. By Lemma 2.1 and the above, we obtain $10n - \delta^2 \in \mathrm{q}(\mathrm{gen}(f))$.

By (2.1), we have either $10n - \delta^2 \in \mathrm{q}(f)$ or $10n - \delta^2 \in \mathrm{q}(g)$. If $10n - \delta^2 \in \mathrm{q}(f)$, then we are done. Suppose that $10n - \delta^2 \in \mathrm{q}(g)$, that is, there are x, y, z such that $10n - \delta^2 = g(x, y, z)$. Then clearly $2 \mid y$. As $10n - \delta^2 \equiv 2 \equiv 2z^2 \pmod 4$, we

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obtain $2 \nmid z$. Hence, by (2.2), there must exist $x', y', z' \in \mathbb{Z}$ with $x' - y' - z' \equiv 0 \pmod 2$ such that $10n - \delta^2 = g(x', y', z')$. By (2.3), we clearly have $10n - \delta^2 \in \mathrm{q}(f)$.

Part (iii). Keep notation as above. Clearly $10n - \mu^2 \in \mathrm{q}_p(f)$ for any prime $p \neq 2$. When $p = 2$, by (2.4), the equation $5n - \mu^2/2 \equiv 2x^2 + 5y^2 + 5z^2 \pmod 8$ is solvable. This implies that the equation $10n - \mu^2 \equiv x^2 + 10y^2 + 10z^2 \pmod{16}$ is solvable. By Lemma 2.1 and the above, we have $10n - \mu^2 \in \mathrm{q}(\mathrm{gen}(f))$.

By (2.1), either $10n - \mu^2 \in \mathrm{q}(f)$ or $10n - \mu^2 \in \mathrm{q}(g)$. If $10n - \mu^2 \in \mathrm{q}(f)$, then we are done. Suppose now that $10n - \mu^2 \in \mathrm{q}(g)$, that is, there are x, y, z such that $10n - \mu^2 = g(x, y, z)$. Then clearly $2 \mid y$. Since $10n - \mu^2 \equiv 2 \equiv 2z^2 \pmod 4$, we get $2 \nmid z$. Then, by (2.2), there are $x', y', z' \in \mathbb{Z}$ with $x' - y' - z' \equiv 0 \pmod 2$ such that $10n - \mu^2 = g(x', y', z')$. By (2.3), we clearly have $10n - \mu^2 \in \mathrm{q}(f)$. This completes the proof. □

REMARK 2.3. Note that

$$\{x^2 + 10y^2 + 10z^2 : x, y, z \in \mathbb{Z}\} = \{x^2 + 5y^2 + 5z^2 : x, y, z \in \mathbb{Z} \text{ and } 2 \mid y - z\}.$$

Sun has studied the latter set in [10].

Sun and Sun [9, Theorem 1.1(ii)] proved that every positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 2y$ a square. Motivated by this, we obtain the following stronger result.

LEMMA 2.4. *Every odd integer* $n \geq 8 \times 10^6$ *can be written as* $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ *with* $x \leq y$ *and* $x + 2y$ *a square.*

PROOF. We first note that

$$8 \times 10^6 > \left[\left(\frac{2}{5^{1/4} - (4.5)^{1/4}}\right)^4\right],$$

where $[\cdot]$ denotes the floor function. If $n \geq 8 \times 10^6$, then $(5n)^{1/4} - (4.5n)^{1/4} > 2$ and hence there is an integer m with $(4.5n)^{1/4} \leq m \leq (5n)^{1/4}$ such that $m \equiv \frac{1}{2}(n-1) \pmod 2$.

Now let $h(x, y, z) = x^2 + 5y^2 + 5z^2$. By [3, pages 112–113],

$$\mathrm{q}(h) = \{x \in \mathbb{N} : x \not\equiv \pm 2 \pmod 5 \text{ and } x \neq 4^k(8l + 7) \text{ for any } k, l \in \mathbb{N}\}. \tag{2.5}$$

As $5n - m^4 \geq 0$, $5n - m^4 \equiv 1, 2 \pmod 4$ and $5n - m^4 \not\equiv \pm 2 \pmod 5$, we have $5n - m^4 \in \mathrm{q}(h)$ by (2.5). Hence, there exist $s \in \mathbb{Z}$ and $z, w \in \mathbb{N}$ such that $5n - m^4 = s^2 + 5z^2 + 5w^2$. Clearly $-\sqrt{5n - m^4} \leq s \leq \sqrt{5n - m^4}$. Replacing s by $-s$ if necessary, we may assume that $s \in \mathbb{Z}$ and $s \equiv -2m^2 \pmod 5$. By the inequality

$$s + 2m^2 \geq -\sqrt{5n - m^4} + 2m^2 = \frac{5m^4 - 5n}{\sqrt{5n - m^4} + 2m^2} > 0,$$

we may write $s + 2m^2 = 5y$ for some $y \in \mathbb{N}$. This gives

$$5n - m^4 = (5y - 2m^2)^2 + 5z^2 + 5w^2$$

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and hence

$$n = (m^2 - 2y)^2 + y^2 + z^2 + w^2. \tag{2.6}$$

Let $x := m^2 - 2y$. Then

$$5x = 5m^2 - 10y = m^2 - 2s \geq m^2 - 2\sqrt{5n - m^4} = \frac{5(m^4 - 4n)}{m^2 + 2\sqrt{5n - m^4}} > 0.$$

This gives $x > 0$. Moreover,

$$5(y - x) = 3s + m^2 \geq -3\sqrt{5n - m^4} + m^2 = \frac{10(m^4 - 4.5n)}{m^2 + 3\sqrt{5n - m^4}} \geq 0.$$

This gives $x \leq y$. In view of the above, we can write $n = x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$, $x \leq y$ and $x + 2y = m^2$. This completes the proof. □

LEMMA 2.5. *For every integer* $n \not\equiv 0 \pmod 4$ *such that* $n \geq 4 \times 10^8$, *we can write* $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ *with* $x + 3y = 2m^2$ *for some* $m \in \mathbb{N}$.

PROOF. We divide our proof into two cases.

Case 1. $n \equiv 2 \pmod 4$. In this case, we write $n = 2n'$ for some odd integer $n' \geq 4 \times 10^7$. By Lemma 2.4, we can write $n' = x'^2 + y'^2 + z'^2 + w'^2$ $(x', y', z', w' \in \mathbb{N})$ with $x' \leq y'$, $z' \leq w'$ and $x' + 2y' = m_0^2$ for some $m_0 \in \mathbb{N}$. Then

$$n = 2n' = (y' - x')^2 + (y' + x')^2 + (w' - z')^2 + (z' + w')^2.$$

Letting $x := y' - x'$, $y := y' + x'$, $z := w' - z'$ and $w := z' + w'$,

$$n = x^2 + y^2 + z^2 + w^2$$

with $x + 3y = (y' - x') + 3(y' + x') = 2(x' + 2y') = 2m_0^2$.

Case 2. n is odd. We first note that

$$4 \times 10^8 > \left[\left(\frac{4\sqrt{2}}{10^{1/4} - 9^{1/4}}\right)^4\right].$$

If $n \geq 4 \times 10^8$, then $(10n)^{1/4}/\sqrt{2} - (9n)^{1/4}/\sqrt{2} > 4$ and hence there is an integer m with $(9n)^{1/4}/\sqrt{2} \leq m \leq (10n)^{1/4}/\sqrt{2}$ such that $2 \nmid m$ and $5 \nmid m$. By Lemma 2.2(iii), there exist $t \in \mathbb{Z}$ and $z, w \in \mathbb{N}$ such that

$$10n - 4m^4 = t^2 + 10z^2 + 10w^2.$$

Clearly $-\sqrt{10n - 4m^4} \leq t \leq \sqrt{10n - 4m^4}$. Replacing t by $-t$ if necessary, we may assume that $t \equiv -6m^2 \pmod{10}$. By the inequality

$$t + 6m^2 \geq -\sqrt{10n - 4m^4} + 6m^2 = \frac{10(4m^4 - n)}{6m^2 + \sqrt{10n - 4m^4}} > 0,$$

we can write $t + 6m^2 = 10y$ for some $y \in \mathbb{N}$. This implies that

$$10n - 4m^4 = (10y - 6m^2)^2 + 10z^2 + 10w^2$$

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and hence

$$n = (2m^2 - 3y)^2 + y^2 + z^2 + w^2.$$

Let $x := 2m^2 - 3y$. Then

$$10x = 2m^2 - 3t \geq 2m^2 - 3\sqrt{10n - 4m^4} = \frac{10(4m^4 - 9n)}{2m^2 + 3\sqrt{10n - 4m^4}} \geq 0.$$

This gives $x \geq 0$. In view of the above, we can write $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y = 2m^2$.

This completes the proof of Lemma 2.5. □

3. Proofs of the main results

Proof of Theorem 1.4. We prove our result by induction on n. When $n < 4 \times 10^9$, we can verify the desired result by computer. Assume now that $n \geq 4 \times 10^9$. We divide the remaining proof into two cases.

Case 1. $n \equiv 0 \pmod 4$. If $16 \mid n$, the desired result follows immediately from the induction hypothesis. We now assume that $16 \nmid n$. Then we can write $n = 4n'$ with $n' \not\equiv 0 \pmod 4$. By Lemma 2.5, there exist $x_1, y_1, z_1, w_1, m_1 \in \mathbb{N}$ such that $n' = x_1^2 + y_1^2 + z_1^2 + w_1^2$ and $x_1 + 3y_1 = 2m_1^2$. Clearly we can write $n = 4n' = x^2 + y^2 + z^2 + w^2$ with $x + 3y = (2m_1)^2$, where $x = 2x_1, y = 2y_1, z = 2z_1, w = 2w_1 \in \mathbb{N}$.

Case 2. $n \equiv 1, 2, 3 \pmod 4$. We first note that

$$4 \times 10^9 > \left[\left(\frac{8}{10^{1/4} - 9^{1/4}}\right)^4\right] > \left[\left(\frac{4}{10^{1/4} - 9^{1/4}}\right)^4\right].$$

If $n \geq 4 \times 10^9$, then we have $(10n)^{1/4} - (9n)^{1/4} > 8$. Hence, we can find an integer m with $(9n)^{1/4} \leq m \leq (10n)^{1/4}$ satisfying the following condition:

$$\begin{cases} 2 \nmid m \text{ and } 5 \nmid m & \text{if } n \equiv 1, 2 \pmod 4, \\ 4 \mid m \text{ and } 5 \nmid m & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

By Lemma 2.2, there exist $u \in \mathbb{Z}$ and $z, w \in \mathbb{N}$ such that

$$10n - m^4 = u^2 + 10z^2 + 10w^2.$$

Clearly $-\sqrt{10n - m^4} \leq u \leq \sqrt{10n - m^4}$. Replacing u by $-u$ if necessary, we may assume that $u \in \mathbb{Z}$ and $u \equiv -3m^2 \pmod{10}$. By the inequality

$$u + 3m^2 \geq -\sqrt{10n - m^4} + 3m^2 = \frac{10(m^4 - n)}{\sqrt{10n - m^4} + 3m^2} > 0,$$

we can write $s + 3m^2 = 10y$ for some $y \in \mathbb{N}$. This gives

$$10n - m^4 = (10y - 3m^2)^2 + 10z^2 + 10w^2$$

and hence

$$n = (m^2 - 3y)^2 + y^2 + z^2 + w^2.$$

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Let $x := m^2 - 3y$. Then

$$10x = m^2 - 3u \geq m^2 - 3\sqrt{10n - m^4} = \frac{10(m^4 - 9n)}{m^2 + 3\sqrt{10n - m^4}} \geq 0.$$

This gives $x \geq 0$ and hence $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y = m^2$.

These conclusions complete the proof. □

Proof of Theorem 1.7. When $n = 1, 2, \ldots, 10$, we can verify our result by computer. Now assume that $n > 10$. By Lemma 2.2(i)–(ii), for any positive integer n with $4 \nmid n$, there are integers a, z, w such that

$$10n - \varepsilon^2 = a^2 + 10z^2 + 10w^2,$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \equiv 1, 2 \pmod 4, \\ 4 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Without loss of generality, we assume that $a \equiv -3\varepsilon \pmod{10}$ (otherwise we can replace a by $-a$). Writing $a = 10y - 3\varepsilon$,

$$10n - \varepsilon^2 = (10y - 3\varepsilon)^2 + 10z^2 + 10w^2$$

and hence

$$n = (\varepsilon - 3y)^2 + y^2 + z^2 + w^2.$$

Letting $x := \varepsilon - 3y$,

$$n = x^2 + y^2 + z^2 + w^2 \quad \text{and} \quad x + 3y = \varepsilon. \tag{3.1}$$

In addition, according to Lemma 2.2(iii), for any positive odd integer n, there are $b, z', w' \in \mathbb{Z}$ such that

$$10n - 2^2 = b^2 + 10z'^2 + 10w'^2.$$

As above, we may write $b = 10y' - 6$ for some $y' \in \mathbb{Z}$. This gives

$$10n - 2^2 = (10y' - 6)^2 + 10z'^2 + 10w'^2$$

and hence

$$n = (2 - 3y')^2 + y'^2 + z'^2 + w'^2.$$

Letting $x' = 2 - 3y'$,

$$n = x'^2 + y'^2 + z'^2 + w'^2 \quad \text{and} \quad x' + 3y' = 2. \tag{3.2}$$

Now let $n = 2n' \equiv 2 \pmod 4$ be a positive integer. By (2.5), we see that there are integers a_1, z_1', w_1' such that

$$5n' - \eta^2 = a_1^2 + 5z_1'^2 + 5w_1'^2,$$

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where

$$\eta = \begin{cases} 4 & \text{if } n' \equiv 1 \pmod 4, \\ 1 & \text{if } n' \equiv 3 \pmod 4. \end{cases}$$

Without loss of generality, we can assume that $a_1 \equiv -2\eta \pmod 5$ and then write $a_1 = 5y_1' - 2\eta$. This gives

$$5n' - \eta^2 = (5y_1' - 2\eta)^2 + 5z_1'^2 + 5w_1'^2$$

and hence

$$n' = (\eta - 2y_1')^2 + y_1'^2 + z'^2 + w'^2.$$

Letting $x_1' := \eta - 2y_1'$, we obtain $n' = x_1'^2 + y_1'^2 + z_1'^2 + w_1'^2$ with $x_1' + 2y_1' = \eta$. As

$$n = 2n' = (y_1' - x_1')^2 + (y_1' + x_1')^2 + (z_1' - w_1')^2 + (z_1' + w_1')^2,$$

letting $x_1 := y_1' - x_1', y_1 := y_1' + x_1', z_1 = z_1' - w_1'$ and $w_1 := z_1' + w_1'$,

$$n = x_1^2 + y_1^2 + z_1^2 + w_1^2 \quad \text{and} \quad x_1 + 3y_1 = 2\eta. \tag{3.3}$$

By (3.2) and (3.3), for every positive integer n_0 with $4 \nmid n$ there are integers x_0, y_0, z_0, w_0 such that

$$n_0 = x_0^2 + y_0^2 + z_0^2 + w_0^2 \quad \text{and} \quad x_0 + 3y_0 \in \{2 \times 1, 2 \times 2^2\}. \tag{3.4}$$

Now we prove our result by induction on n. If $16 \mid n$, then the desired result follows from the induction hypothesis. If $4 \nmid n$, then (3.1) implies the desired result. If $n = 4n'$ for some $n' \not\equiv 0 \pmod 4$, then by (3.4) one can easily verify our result.

These conclusions complete the proof. □

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