

Uniformly perfect finitely generated simple left orderable groups

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(Received 10 April 2019 and accepted in revised form 18 July 2019)

Abstract. We show that the finitely generated simple left orderable groups G_ρ constructed by the first two authors in Hyde and Lodha [Finitely generated infinite simple groups of homeomorphisms of the real line. *Invent. Math.* (2019), doi:10.1007/s00222-019-00880-7] are uniformly perfect—each element in the group can be expressed as a product of three commutators of elements in the group. This implies that the group does not admit any homogeneous quasimorphism. Moreover, any non-trivial action of the group on the circle, which lifts to an action on the real line, admits a global fixed point. It follows that any faithful action on the real line without a global fixed point is *globally contracting*. This answers Question 4 of the third author [A. Navas. Group actions on 1-manifolds: a list of very concrete open questions. *Proceedings of the International Congress of Mathematicians*, Vol. 2. Eds. B. Sirakov, P. Ney de Souza and M. Viana. World Scientific, Singapore, 2018, pp. 2029–2056], which asks whether such a group exists. This question has also been answered simultaneously and independently, using completely different methods, by Matte Bon and Triestino [Groups of piecewise linear homeomorphisms of flows. *Preprint*, 2018, arXiv:1811.12256]. To prove our results,

we provide a characterization of elements of the group G_ρ which is a useful new tool in the study of these examples.

Key words: groups of homeomorphisms, orderable groups, simple groups
 2010 Mathematics Subject Classification: 43A07 (Primary); 20F05 (Secondary)

1. *Introduction*

In 1980 Rhemtulla asked whether there exist finitely generated non-trivial simple left orderable groups (see [9] for a discussion around the history of the problem and references). This question was answered in the affirmative by the first two authors in [9]. The construction takes as an input a certain *quasiperiodic* labeling ρ of the set $\frac{1}{2}\mathbf{Z}$, which is a map

$$\rho : \frac{1}{2}\mathbf{Z} \rightarrow \{a, b, a^{-1}, b^{-1}\}$$

that satisfies a certain set of axioms. (See the preliminaries section for details.) Such labelings exist and are easy to construct explicitly. For each such labeling ρ , one constructs an explicit group action $G_\rho < \text{Homeo}^+(\mathbf{R})$ which is a finitely generated simple left orderable group.

Given a group G and an element $f \in [G, G]$, the integer $\text{cl}(f)$ is defined as the smallest k such that f can be expressed as a product of k commutators of elements in G . We state our main theorem.

THEOREM 1.1. *Let ρ be a quasiperiodic labeling. Then $\text{cl}(f) \leq 3$ for each element $f \in G_\rho$.*

Recall that a homogeneous quasimorphism is a quasimorphism $\phi : G \rightarrow \mathbf{R}$ with the property that the restriction of ϕ to any cyclic subgroup is a homomorphism. As a consequence of Theorem 1.1 we obtain that the stable commutator length vanishes, and hence the group does not admit any non-trivial homogeneous quasimorphism. (We refer the reader to [4] as a general reference for these concepts). Using the work of Ghys [7], this allows us to show the following corollary.

COROLLARY 1.2. *Let ρ be a quasiperiodic labeling. Then every faithful action of G_ρ on \mathbf{S}^1 , which lifts to an action on the real line, admits a global fixed point on \mathbf{S}^1 .*

Recall that, for every action of a finitely generated group G by orientation-preserving homeomorphisms of the real line without global fixed points, we have one of three possibilities:

- (i) there is a σ -finite measure μ that is invariant under the action;
- (ii) the action is semiconjugate to a minimal action for which every small enough interval is sent into a sequence of intervals that converge to a point under well-chosen group elements, but this property does not hold for every bounded interval (here, by a semiconjugacy we roughly mean a factor action for which the factor map is a continuous, non-decreasing, proper map of the real line);
- (iii) the action is globally contracting; more precisely, it is semiconjugate to a minimal one for which the contraction property above holds for all bounded intervals.

We obtain the following result as an immediate consequence of Corollary 1.2.

COROLLARY 1.3. *Let ρ be a quasiperiodic labeling. Then any faithful action of the group G_ρ on \mathbf{R} without global fixed points is of type (iii).*

This answers the following question of the third author.

Question 1.4. [13, Question 4] Does there exist an infinite, finitely generated group that acts on the real line all of whose actions by orientation-preserving homeomorphisms of the line without global fixed points are of type (iii)?

Remark 1.5. The above question has been answered simultaneously and independently by Matte Bon and Triestino in [12]. They provide a new family of finitely generated simple left orderable groups, which are overgroups of the groups G_ρ , and prove the analog of Corollary 1.2 for that family. Their methods are completely different from ours.

Corollaries 1.2 and 1.3 should be compared with similar theorems for lattices in higher-rank simple Lie groups. For these, it is known that every action on the circle has a finite orbit; therefore, up to a finite-index group, they admit a global fixed point [3, 8]. However, it is still unknown whether they admit non-trivial actions on the line or not, yet several definitive results are known [10, 11, 14]. If one of these lattices admits such an action, it is not hard to see that it would also provide an affirmative answer to Question 1.4 (see [6]).

The proof of Theorem 1.1 uses the following new description of the group which is the main technical result of this paper. Let ρ be a quasiperiodic labeling. (Recall this notion from [9], or see Definition 2.5.) Given an $x \in \mathbf{R}$ and $n \in \mathbf{N}$, we define a word $\mathcal{W}(x, n)$ of length $2n + 1$ in the alphabet $\{a, a^{-1}, b, b^{-1}\}$ as follows. Let $y \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$ be such that $x \in [y - \frac{1}{2}, y + \frac{1}{2})$. Then

$$\mathcal{W}(x, n) = \rho(y - \frac{1}{2}n)\rho(y - \frac{1}{2}(n - 1)) \dots \rho(y) \dots \rho(y + \frac{1}{2}(n - 1))\rho(y + \frac{1}{2}n).$$

For each integer $n \in \mathbf{Z}$, we denote by ι_n the unique orientation-reversing isometry $\iota_n : [n, n + 1) \rightarrow (n, n + 1]$. We define the map $\iota : \mathbf{R} \rightarrow \mathbf{R}$ as

$$x \cdot \iota = x \cdot \iota_n \quad \text{where } x \in [n, n + 1) \text{ for } n \in \mathbf{Z}.$$

In what follows, by a *countably singular* piecewise linear homeomorphism we mean a piecewise linear homeomorphism with a countable set of singularities (or breakpoints).

Definition 1.6. Let K_ρ be the set of homeomorphisms $f \in \text{Homeo}^+(\mathbf{R})$ satisfying the following conditions.

- (1) f is a countably singular piecewise linear homeomorphism of \mathbf{R} with a discrete set of singularities, all of which lie in $\mathbf{Z}[\frac{1}{2}]$.
- (2) $f'(x)$, wherever it exists, is an integer power of 2.
- (3) There is a $k_f \in \mathbf{N}$ such that:
 - (3.a) whenever $x, y \in \mathbf{R}$ satisfy

$$x - y \in \mathbf{Z}, \quad \mathcal{W}(x, k_f) = \mathcal{W}(y, k_f),$$

we have

$$x - x \cdot f = y - y \cdot f;$$

(3.b) whenever $x, y \in \mathbf{R}$ satisfy

$$x - y \in \mathbf{Z}, \quad \mathcal{W}(x, k_f) = \mathcal{W}^{-1}(y, k_f),$$

we have

$$x - x \cdot f = y' \cdot f - y' \quad \text{where } y' = y \cdot \iota.$$

It is not hard to check that K_ρ is a group. If this is not clear to the reader, it will be a consequence of Theorem 1.8.

Remark 1.7. Note that, given an element $f \in K_\rho$ and a number $k_f \in \mathbf{N}$ satisfying the conditions of Definition 1.6, any number $k'_f \in \mathbf{N}$ such that $k'_f > k_f$ also satisfies the conditions of the definition.

THEOREM 1.8. $K_\rho = G_\rho$.

This characterization provides a useful new definition of the groups G_ρ as groups of homeomorphisms of the real line satisfying a natural set of criterion. This also provides useful new structural results such as the following proposition. (In what follows, we denote by F' the commutator subgroup of Thompson’s group F .)

PROPOSITION 1.9. *Let ρ be a quasiperiodic labeling. Given any element $f \in G_\rho$, there are elements $g_1, g_2 \in G_\rho$ such that:*

- (1) $f = g_1 g_2$;
- (2) g_2 is a commutator in G_ρ ;
- (3) there is a subgroup $K < G_\rho$ such that K is isomorphic to a direct sum of finitely many copies of F' and $g_1 \in K$.

2. Preliminaries

We assume that $0 \in \mathbf{N}$. All actions will be right actions, unless otherwise specified. Given a group action $G < \text{Homeo}^+(\mathbf{R})$ and a $g \in G$, we denote by $\text{Supp}(g)$ the set

$$\text{Supp}(g) = \{x \in \mathbf{R} \mid x \cdot g \neq x\}.$$

Note that $\text{Supp}(g)$ is an open set, and that \mathbf{R} can be replaced by another 1-manifold. A homeomorphism $f : [0, 1] \rightarrow [0, 1]$ is said to be *compactly supported* in $(0, 1)$ if $\overline{\text{Supp}(f)} \subset (0, 1)$. Similarly, a homeomorphism $f : \mathbf{R} \rightarrow \mathbf{R}$ is said to be *compactly supported* in \mathbf{R} if $\overline{\text{Supp}(f)}$ is contained in a compact interval in \mathbf{R} . A point $x \in \mathbf{R}$ is said to be a *transition point* of f if

$$x \in \partial \text{Supp}(f) = \overline{\text{Supp}(f)} \setminus \text{Supp}(f).$$

Our construction uses in an essential way the structure and properties of Thompson’s group F . We shall only describe the features of F here that we need, and we direct the reader to [5] and [2] for more comprehensive surveys. Recall that the group $\text{PL}^+([0, 1])$ is the group of orientation-preserving piecewise linear homeomorphisms of $[0, 1]$. Recall that F is defined as the subgroup of $\text{PL}^+([0, 1])$ that satisfies the following conditions.

- (1) Each element has at most finitely many breakpoints. All breakpoints lie in the set of dyadic rationals, that is, $\mathbf{Z}[\frac{1}{2}]$.
- (2) For each element, the derivatives, wherever they exist, are powers of 2.

By *breakpoint* (or a singularity point) we mean a point where the derivative does not exist. For $r, s \in \mathbf{Z}[\frac{1}{2}] \cap [0, 1]$ such that $r < s$, we denote by $F_{[r,s]}$ the subgroup of elements whose support lies in $[r, s]$. We now present some well-known facts that we shall need. The group F satisfies the following properties:

- (1) F is 2-generated;
- (2) for each pair $r, s \in \mathbf{Z}[\frac{1}{2}] \cap [0, 1]$ such that $r < s$, the group $F_{[r,s]}$ is isomorphic to F and hence is also 2-generated;
- (3) F' is simple and consists of precisely the set of elements $g \in F$ such that $\overline{\text{Supp}(g)} \subset (0, 1)$.

An interval $I \subseteq [0, 1]$ is said to be a *standard dyadic interval* if it is of the form $[a/2^n, (a + 1)/2^n]$ such that $a, n \in \mathbf{N}, a < 2^n - 1$. The following lemmas give elementary facts about the action of F on the standard dyadic intervals.

LEMMA 2.1. *Let I, J be standard dyadic intervals in $(0, 1)$. Then there is an element $f \in F'$ such that:*

- (1) $I \cdot f = J$;
- (2) $f \upharpoonright I$ is linear.

LEMMA 2.2. *Let I_1, I_2 and J_1, J_2 be standard dyadic intervals in $(0, 1)$ such that*

$$\sup(I_1) < \inf(I_2), \quad \sup(J_1) < \inf(J_2).$$

Then there is an element $f \in F'$ such that:

- (1) $I_1 \cdot f = J_1$ and $I_2 \cdot f = J_2$;
- (2) $f \upharpoonright I_1$ and $f \upharpoonright I_2$ are linear.

Recall that we fix $\iota : (0, 1) \rightarrow (0, 1)$ as the unique orientation-reversing isometry. We say that an element $f \in F$ is *symmetric* if $f = \iota \circ f \circ \iota$. We say that a set $I \subset (0, 1)$ is *symmetric* if $I \cdot \iota = I$. Note that, given any symmetric set I with non-empty interior, we can find a non-trivial symmetric element $f \in F'$ such that $\text{Supp}(f) \subset \text{int}(I)$. We extend the map ι to \mathbf{R} as follows. For each integer $n \in \mathbf{Z}$, we denote the unique orientation-reversing isometry

$$\iota_n : [n, n + 1) \rightarrow (n, n + 1].$$

For $x \in \mathbf{R}$, we define the map $\iota : \mathbf{R} \rightarrow \mathbf{R}$ as

$$x \cdot \iota = x \cdot \iota_n \quad \text{where } x \in [n, n + 1) \text{ for } n \in \mathbf{Z}.$$

In this paper we shall also use the notation $\iota_{[x,y]} : [x, y) \rightarrow (x, y]$ or $\iota_I : I \rightarrow I$ to denote the unique orientation-reversing isometries between intervals of the form $[x, y)$ and $(x, y]$ (for $x, y \in \mathbf{R}$), or a compact subinterval I of \mathbf{R} . The usage of this notation will be made clear when it occurs. (Note that it differs from the ι defined above.)

Definition 2.3. We fix an element $c_0 \in F$ with the following properties:

- (1) the support of c_0 equals $(0, \frac{1}{4})$ and $x \cdot c_0 > x$ for each $x \in (0, \frac{1}{4})$;
- (2) $c_0 \upharpoonright (0, \frac{1}{16})$ equals the map $t \rightarrow 2t$.

Let

$$c_1 = \iota \circ c_0 \circ \iota, \quad v_1 = c_0 c_1.$$

Note that $v_1 \in F$ is a symmetric element. We define a subgroup H of F as

$$H = \langle F', v_1 \rangle.$$

Finally, we fix

$$v_2, v_3 : [0, 1] \rightarrow [0, 1]$$

as chosen homeomorphisms whose supports are contained in $(\frac{1}{16}, \frac{15}{16})$ and that generate the group $F_{[1/16, 15/16]}$.

The following lemma is [9, Lemma 2.4].

LEMMA 2.4. H is generated by v_1, v_2, v_3 . H' is simple and consists of precisely the set of elements of H (or F) that are compactly supported in $(0, 1)$. In particular, $H' = F'$.

Definition 2.5. We consider the additive group $\frac{1}{2}\mathbf{Z} = \{\frac{1}{2}k \mid k \in \mathbf{Z}\}$. A labeling is a map

$$\rho : \frac{1}{2}\mathbf{Z} \rightarrow \{a, b, a^{-1}, b^{-1}\}$$

which satisfies:

- (1) $\rho(k) \in \{a, a^{-1}\}$ for each $k \in \mathbf{Z}$;
- (2) $\rho(k) \in \{b, b^{-1}\}$ for each $k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$.

We regard $\rho(\frac{1}{2}\mathbf{Z})$ as a bi-infinite word with respect to the usual ordering of the integers. A subset $X \subseteq \frac{1}{2}\mathbf{Z}$ is said to be a *block* if it is of the form

$$\{k, k + \frac{1}{2}, \dots, k + \frac{1}{2}n\}$$

for some $k \in \frac{1}{2}\mathbf{Z}, n \in \mathbf{N}$. Note that each block is endowed with the usual ordering inherited from \mathbf{R} . The set of blocks of $\frac{1}{2}\mathbf{Z}$ is denoted as \mathbf{B} . To each block

$$X = \{k, k + \frac{1}{2}, \dots, k + \frac{1}{2}n\}$$

we assign a formal word

$$W_\rho(X) = \rho(k)\rho(k + \frac{1}{2}) \dots \rho(k + \frac{1}{2}n)$$

which is a word in the letters $\{a, b, a^{-1}, b^{-1}\}$. Such a formal word is called a *subword* of the labeling.

Recall that, given a word $w_1 \dots w_n$ in the letters $\{a, b, a^{-1}, b^{-1}\}$, the formal inverse of the word is $w_n^{-1} \dots w_1^{-1}$. The formal inverse of $W_\rho(X)$ is denoted by $W_\rho^{-1}(X)$.

A labeling ρ is said to be *quasiperiodic* if the following conditions hold:

- (1) for each block $X \in \mathbf{B}$, there is an $n \in \mathbf{N}$ such that whenever $Y \in \mathbf{B}$ is a block of size at least n , then $W_\rho(X)$ is a subword of $W_\rho(Y)$;
- (2) for each block $X \in \mathbf{B}$, there is a block $Y \in \mathbf{B}$ such that $W_\rho(Y) = W_\rho^{-1}(X)$.

Note that by *subword* in the above we mean a string of consecutive letters in the word.

A non-empty finite word $w_1 \dots w_n$ for $w_i \in \{a, b, a^{-1}, b^{-1}\}$ is said to be a *permissible word* if n is odd and the following condition holds. For odd $i \leq n$ one has $w_i \in \{a, a^{-1}\}$, and for even $i \leq n$ one has $w_i \in \{b, b^{-1}\}$.

The following lemma is [9, Lemma 3.1].

LEMMA 2.6. Given any permissible word $w_1 \dots w_m$, there is a quasiperiodic labeling ρ of $\frac{1}{2}\mathbf{Z}$ and a block $X \in \mathbf{B}$ satisfying that $W_\rho(X) = w_1 \dots w_m$.

Following [9], we recall that to each labeling ρ we associate a group $G_\rho < \text{Homeo}^+(\mathbf{R})$ as follows.

Definition 2.7. Let $H < \text{Homeo}^+([0, 1])$ be the group defined in Definition 2.3. Recall from Lemma 2.4 that the group H is generated by the three elements v_1, v_2, v_3 defined in Definition 2.3. In what appears below, by \cong_T we mean that the restrictions are topologically conjugate via the unique orientation-preserving isometry that maps $[0, 1]$ to the respective interval. We define the homeomorphisms

$$\zeta_1, \zeta_2, \zeta_3, \chi_1, \chi_2, \chi_3 : \mathbf{R} \rightarrow \mathbf{R}$$

as follows: for each $i \in \{1, 2, 3\}$ and $n \in \mathbf{Z}$,

$$\begin{aligned} \zeta_i \upharpoonright [n, n + 1] &\cong_T v_i && \text{if } \rho(n + \frac{1}{2}) = b, \\ \zeta_i \upharpoonright [n, n + 1] &\cong_T (\iota \circ v_i \circ \iota) && \text{if } \rho(n + \frac{1}{2}) = b^{-1}, \\ \chi_i \upharpoonright [n - \frac{1}{2}, n + \frac{1}{2}] &\cong_T v_i && \text{if } \rho(n) = a, \\ \chi_i \upharpoonright [n - \frac{1}{2}, n + \frac{1}{2}] &\cong_T (\iota \circ v_i \circ \iota) && \text{if } \rho(n) = a^{-1}. \end{aligned}$$

The group G_ρ is defined as

$$G_\rho := \langle \zeta_1^{\pm 1}, \zeta_2^{\pm 1}, \zeta_3^{\pm 1}, \chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1} \rangle < \text{Homeo}^+(\mathbf{R}).$$

We denote the above generating set of G_ρ by

$$\mathbf{S}_\rho := \{ \zeta_1^{\pm 1}, \zeta_2^{\pm 1}, \zeta_3^{\pm 1}, \chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1} \}.$$

We also define subgroups

$$\mathcal{K} := \langle \zeta_1^{\pm 1}, \zeta_2^{\pm 1}, \zeta_3^{\pm 1} \rangle, \quad \mathcal{L} := \langle \chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1} \rangle$$

of G_ρ that are both isomorphic to H , and

$$\mathcal{K}' \cong \mathcal{L}' \cong F'.$$

Note that the definition of \mathcal{K}, \mathcal{L} requires us to fix a labeling ρ but we denote them as such for simplicity of notation.

Recall that in [9] we fixed notation for the natural isomorphisms

$$\lambda : H \rightarrow \mathcal{K}, \quad \pi : H \rightarrow \mathcal{L}$$

as follows: for each $f \in H, n \in \mathbf{Z}$,

$$\begin{aligned} \lambda(f) \upharpoonright [n, n + 1] &\cong_T f && \text{if } \rho(n + \frac{1}{2}) = b, \\ \lambda(f) \upharpoonright [n, n + 1] &\cong_T (\iota \circ f \circ \iota) && \text{if } \rho(n + \frac{1}{2}) = b^{-1}, \\ \pi(f) \upharpoonright [n - \frac{1}{2}, n + \frac{1}{2}] &\cong_T f && \text{if } \rho(n) = a, \\ \pi(f) \upharpoonright [n - \frac{1}{2}, n + \frac{1}{2}] &\cong_T (\iota \circ f \circ \iota) && \text{if } \rho(n) = a^{-1}. \end{aligned}$$

We also denote the naturally defined inverse isomorphisms by

$$\lambda^{-1} : \mathcal{K} \rightarrow H, \quad \pi^{-1} : \mathcal{L} \rightarrow H.$$

Note that the group G_ρ is defined for every labeling ρ . The following theorem is proved in [9].

THEOREM 2.8. *Let ρ be a quasiperiodic labeling. Then the group G_ρ is simple.*

For simplicity of notation, in what follows we will not explicitly mention the labeling ρ in what we now define. Recall that, given an $x \in \mathbf{R}$ and $n \in \mathbf{N}$, we define a word $\mathcal{W}(x, n)$ as follows. Let $y \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$ such that $x \in [y - \frac{1}{2}, y + \frac{1}{2})$. Then we define

$$\mathcal{W}(x, n) = \rho(y - \frac{1}{2}n)\rho(y - \frac{1}{2}(n - 1)) \dots \rho(y) \dots \rho(y + \frac{1}{2}(n - 1))\rho(y + \frac{1}{2}n).$$

Given a compact integer interval (i.e. with integer endpoints) $J \subset \mathbf{R}$ and $n_1, n_2 \in \mathbf{N}$, we define a word $\mathcal{W}(J, n_1, n_2)$ as follows. Let

$$y_1 = \inf(J) + \frac{1}{2}, \quad y_2 = (J) - \frac{1}{2}.$$

Then we define

$$\begin{aligned} \mathcal{W}(J, n_1, n_2) = & \rho(y_1 - \frac{1}{2}n_1)\rho(y_1 - \frac{1}{2}(n_1 - 1)) \dots \rho(y_1) \dots \rho(y_2) \dots \\ & \rho(y_2 + \frac{1}{2}(n_2 - 1))\rho(y_2 + \frac{1}{2}n_2). \end{aligned}$$

If $n_1 = n_2 = n$, we denote $\mathcal{W}(J, n_1, n_2)$ simply as $\mathcal{W}(J, n)$.

We denote by $\mathcal{W}^{-1}(x, n)$ and $\mathcal{W}^{-1}(J, n)$ the formal inverses of the words $\mathcal{W}(x, n)$ and $\mathcal{W}(J, n)$, respectively. We now state a few structural results about the groups G_ρ that were proved in [9]. For what follows, we assume that ρ is a quasiperiodic labeling. Note that the statement of the first lemma is slightly modified to suit the needs of this paper. However, the modification is entirely straightforward.

LEMMA 2.9. [9, Lemma 5.1] *Let $f \in G_\rho$ be a non-identity element such that*

$$f = w_1 \dots w_k, \quad w_i \in \mathbf{S}_\rho \text{ for } 1 \leq i \leq k.$$

Then the following statements hold:

- (1) *the set of breakpoints of f is discrete and the set of transition points is also discrete;*
- (2) *there is an $m_f \in \mathbf{N}$ such that, for any compact interval J of length at least m_f , f fixes a point in J ;*
- (3) *for each $x \in \mathbf{R}$ and each $i \leq k$,*

$$x \cdot w_1 \dots w_i \in [x - (k + 1), x + (k + 1)].$$

LEMMA 2.10. [9, Lemma 5.3] *The action of G_ρ on \mathbf{R} is minimal.*

LEMMA 2.11. [9, Lemma 5.4] *For each pair of elements $m_1, m_2 \in \mathbf{Z}$ and a closed interval $I \subset (m_1, m_1 + 1)$, there is a word $w_1 \dots w_k$ in the generators \mathbf{S}_ρ such that*

$$I \cdot w_1 \dots w_k \subset (m_2, m_2 + 1)$$

and

$$I \cdot w_1 \dots w_i \subset [\inf\{m_1, m_2\}, \sup\{m_1 + 1, m_2 + 1\}]$$

for each $1 \leq i \leq k$.

We next state an elementary corollary of the third part of Lemma 2.9. The natural number m emerges as the word length of f in S_ρ .

COROLLARY 2.12. *Let $f \in G_\rho$. There is an $m \in \mathbf{N}$ such that, for any $x_1, x_2 \in \mathbf{R}$ so that $x_1 - x_2 \in \mathbf{Z}$, the following statements hold:*

(1) *if $\mathcal{W}(x_1, m) = \mathcal{W}(x_2, m)$ then*

$$x_1 - x_1 \cdot f = x_2 - x_2 \cdot f;$$

(2) *if $\mathcal{W}^{-1}(x_1, m) = \mathcal{W}(x_2, m)$ then*

$$x_1 - x_1 \cdot f = x_3 \cdot f - x_3 \quad \text{where } x_3 = x_2 \cdot t.$$

Finally, we shall also need the following folklore result (see the Appendix in [1] for a proof.)

THEOREM 2.13. *Every element in F' can be expressed as a product of at most two commutators of elements in F' .*

3. A characterization of elements of G_ρ

The goal of this section is to establish the characterization of elements in G_ρ as described in the introduction (Definition 1.6). In effect, this requires us to prove Theorem 1.8. Throughout this section we fix a quasiperiodic labeling ρ . Note that it follows from Corollary 2.12 that $G_\rho \subseteq K_\rho$. Therefore much of the rest of the paper will be devoted to proving that $K_\rho \subseteq G_\rho$. The proof of this requires us to establish some preliminary structural results about the group G_ρ .

The main structural result is Proposition 3.4. The proof of the main theorems and corollaries will follow from it. Proposition 3.4 will be proved in a subsequent section, and its proof involves the construction of a certain family of special elements in G_ρ .

Definition 3.1. A homeomorphism $f \in \text{Homeo}^+(\mathbf{R})$ is said to be *stable* if there exists an $n \in \mathbf{N}$ such that the following result holds. For any compact interval I of length at least n , there is an integer $m \in I$ such that f fixes a neighborhood of m pointwise. Given a stable homeomorphism $f \in \text{Homeo}^+(\mathbf{R})$ and an interval $[m_1, m_2]$, the restriction $f \upharpoonright [m_1, m_2]$ is said to be an *atom* of f , if:

- (1) $m_1, m_2 \in \mathbf{Z}$;
- (2) there is an $\epsilon > 0$ such that, for each $x \in (m_1 - \epsilon, m_1 + \epsilon) \cup (m_2 - \epsilon, m_2 + \epsilon)$, we have $x \cdot f = x$;
- (3) for any $m \in (m_1, m_2) \cap \mathbf{Z}$ and any $\epsilon > 0$, there is a point $x \in (m - \epsilon, m + \epsilon)$ such that $x \cdot f \neq x$.

In other words, an atom is the restriction of f to the closure of a maximal open interval J with the property that for each $m \in J \cap \mathbf{Z}$, f moves a point in any neighborhood of m .

Note that, given a stable homeomorphism f , there is a unique way to express \mathbf{R} as a union of integer intervals $\{I_\alpha\}_{\alpha \in P}$ such that $f \upharpoonright I_\alpha$ is an atom for each $\alpha \in P$ and different intervals intersect in at most one endpoint. For simplicity, we will refer just to the intervals I_α as the atoms of f .

Given an atom $f \upharpoonright I$, we refer to the intervals $[\inf(I), \inf(I) + 1]$ and $[\sup(I) - 1, \sup(I)]$ as the *head* and the *foot* of the atom, respectively. Note that it is possible that

an atom I_α has the same interval as head and foot, in which case $|I_\alpha| = 1$. Two atoms $f \upharpoonright [m_1, m_2]$ and $f \upharpoonright [m_3, m_4]$ are said to be *conjugate* if there is an integer translation $h(t) = t + z$ for $z \in \mathbf{Z}$ such that

$$f \upharpoonright [m_1, m_2] = h^{-1} \circ f \circ h \upharpoonright [m_3, m_4]$$

and *flip-conjugate* if there is an integer translation $h(t) = t + z$ for $z \in \mathbf{Z}$ such that

$$f \upharpoonright [m_1, m_2] = h^{-1} \circ (\iota_{[m_1, m_2]} \circ f \circ \iota_{[m_1, m_2]}) \circ h \upharpoonright [m_3, m_4]$$

where

$$\iota_{[m_1, m_2]} : [m_1, m_2] \rightarrow [m_1, m_2]$$

is the unique orientation-reversing isometry.

For a fixed $n \in \mathbf{N}$, we consider the set of *decorated atoms*:

$$\mathcal{T}_n(f) = \{(I_\alpha, n) \mid \alpha \in P\}.$$

We say that a pair of decorated atoms (I_α, n) and (I_β, n) are equivalent if either of the following statements holds:

- (1) I_α, I_β are conjugate and $\mathcal{W}(I_\alpha, n) = \mathcal{W}(I_\beta, n)$;
- (2) I_α, I_β are flip-conjugate and $\mathcal{W}(I_\alpha, n) = \mathcal{W}^{-1}(I_\beta, n)$.

The element f is said to be *uniformly stable* if it is stable and there are finitely many equivalence classes of decorated atoms for each $n \in \mathbf{N}$. Note that if there are finitely many equivalence classes of decorated atoms of f for some $n \in \mathbf{N}$, then this holds for any $n \in \mathbf{N}$. This is true since there are finitely many words of length n in $\{a, b, a^{-1}, b^{-1}\}$.

LEMMA 3.2. *Let $g \in K_\rho$. Then there exist $g_1, g_2 \in G_\rho$, where g_2 is a commutator of elements in G_ρ , such that $g_1^{-1}(gg_2^{-1})g_1 \in K_\rho$ is uniformly stable.*

Proof. Since $g \in K_\rho$, we know that there is a constant k_g that witnesses the conditions of Definition 1.6. Let $x \in \mathbf{R}$ be such that $x \cdot g > x$. Since ρ is quasiperiodic, there is a $y \in \mathbf{R}$ such that $x - y \in \mathbf{Z}$ and

$$\mathcal{W}^{-1}(y, k_g) = \mathcal{W}(x, k_g).$$

It follows from (3.b) in Definition 1.6 that $y' \cdot g < y'$ for $y' = y \cdot \iota$. Therefore, g admits a fixed point $p_0 \in \mathbf{R}$. A similar conclusion is achieved by starting with a point x for which $x \cdot g < x$.

Assume that $p_0 \in \mathbf{R} \setminus \mathbf{Z}$. The case where $p_0 \in \mathbf{R} \setminus (\frac{1}{2}\mathbf{Z} \setminus \mathbf{Z})$ is dealt with similarly. We find an element $l_2 \in F'$ such that l_2 is a commutator in F' and $g_2 = \lambda(l_2)$ coincides with g on a neighborhood of p_0 . Note that this is possible since p_0 is a fixed point of g and g satisfies the first two conditions of Definition 1.6. It follows that gg_2^{-1} fixes pointwise a subinterval I of non-empty interior.

Since the action of G_ρ on \mathbf{R} is minimal (see Lemma 2.11), we can find $g_1 \in G_\rho$ such that $0 \cdot g_1^{-1} \in I$. It follows that $g_1^{-1}(gg_2^{-1})g_1$ fixes a neighborhood of 0. From an application of quasiperiodicity and Definition 1.6, it follows that this element is uniformly stable. \square

The core of the proof of Theorem 1.8 reduces to the following proposition. To state it we first give the following definition.

Definition 3.3. Let $f \in \text{Homeo}^+(\mathbf{R})$ be uniformly stable. Let ζ be an equivalence class of elements in $\mathcal{T}_n(f)$. We define the homeomorphism f_ζ as

$$\begin{aligned} f_\zeta \upharpoonright I_\alpha &= f \upharpoonright I_\alpha && \text{if } (I_\alpha, n) \in \zeta, \\ f_\zeta \upharpoonright I_\alpha &= \text{id} \upharpoonright I_\alpha && \text{if } (I_\alpha, n) \notin \zeta. \end{aligned}$$

If ζ_1, \dots, ζ_m are the equivalence classes of elements in $\mathcal{T}_n(f)$, then the list of homeomorphisms $f_{\zeta_1}, \dots, f_{\zeta_m}$ is called the *cellular decomposition* of f .

PROPOSITION 3.4. *Given a uniformly stable element $f \in K_\rho$, there is an $n \in \mathbf{N}$ such that $f_\zeta \in G_\rho$ for each $\zeta \in \mathcal{T}_n(f)$. In particular, it follows that $f \in G_\rho$.*

4. *Special elements in G_ρ*

The proof of Proposition 3.4 requires the construction of a certain family of *special elements* in G_ρ . We define and construct them in this section. The construction of such elements is also a useful tool to study the groups G_ρ . Throughout the section we assume that ρ is a quasiperiodic labeling.

Recall the definitions of the subgroups $\mathcal{K}, \mathcal{L} \leq G_\rho$ from the preliminaries. Also recall the isomorphisms

$$\lambda : H \rightarrow \mathcal{K}, \quad \pi : H \rightarrow \mathcal{L}.$$

We consider the set of triples

$$\Omega = \{(W, k_1, k_2) \mid W \in \{a, b, a^{-1}, b^{-1}\}^{<\mathbf{N}}, k_1, k_2 \in \mathbf{N} \text{ such that } |W| = k_1 + k_2 + 1\}.$$

Definition 4.1. Given an element $f \in F'$ and $\omega \in \Omega$, we define the *special element* $\lambda_\omega(f) \in \text{Homeo}^+(\mathbf{R})$ as follows: for each $n \in \mathbf{Z}$, we let

$$\lambda_\omega(f) \upharpoonright [n, n + 1] = \lambda(f) \upharpoonright [n, n + 1] \quad \text{if } \begin{cases} \mathcal{W}([n, n + 1], k_1, k_2) = W, \\ \text{or} \\ \mathcal{W}([n, n + 1], k_2, k_1) = W^{-1}, \end{cases}$$

$$\lambda_\omega(f) \upharpoonright [n, n + 1] = \text{id} \upharpoonright [n, n + 1] \quad \text{otherwise.}$$

Similarly, we define the *special elements* $\pi_\omega(f) \in \text{Homeo}^+(\mathbf{R})$ as follows: for each $n \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$, we let

$$\pi_\omega(f) \upharpoonright [n, n + 1] = \pi(f) \upharpoonright [n, n + 1] \quad \text{if } \begin{cases} \mathcal{W}([n, n + 1], k_1, k_2) = W, \\ \text{or} \\ \mathcal{W}([n, n + 1], k_2, k_1) = W^{-1}, \end{cases}$$

$$\pi_\omega(f) \upharpoonright [n, n + 1] = \text{id} \upharpoonright [n, n + 1] \quad \text{otherwise.}$$

Given $\omega = (W, k_1, k_2)$ where $W = w_{-k_1} \dots w_0 \dots w_{k_2}$, we call w_0 the *central letter* of the word W .

Remark 4.2. Note the order of appearance of k_1, k_2 in $\mathcal{W}([n, n + 1], \cdot, \cdot)$ in the above definition.

The following lemma is a direct consequence of the definitions.

LEMMA 4.3. Consider $\omega_1 = (W_1, k_1, k_2)$ and $\omega_2 = (W_2, k_2, k_1)$ such that $W_1 = W_2^{-1}$. Then it follows that for each $f \in F'$,

$$\lambda_{\omega_1}(f) = \lambda_{\omega_2}(\iota \circ f \circ \iota), \quad \pi_{\omega_1}(f) = \pi_{\omega_2}(\iota \circ f \circ \iota).$$

In particular, by symmetry it follows that:

- (1) $\lambda_{\omega_1}(f) \in G_\rho$ for each $f \in F'$ if and only if $\lambda_{\omega_2}(f) \in G_\rho$ for each $f \in F'$;
- (2) $\pi_{\omega_1}(f) \in G_\rho$ for each $f \in F'$ if and only if $\pi_{\omega_2}(f) \in G_\rho$ for each $f \in F'$.

Remark 4.4. Note that $\lambda_\omega(f), \pi_\omega(f)$ for $\omega = (W, k_1, k_2)$ will be equal to the identity homeomorphism, or the trivial element of G_ρ , if W does not occur as a subword of the labeling ρ . Finally, note that $\lambda_\omega(f)$ is trivial if $w_0 \in \{a, a^{-1}\}$ and $\pi_\omega(f)$ is trivial if $w_0 \in \{b, b^{-1}\}$.

The key technical step in the proof of the main theorem is the following localization result.

PROPOSITION 4.5. Let $\omega \in \Omega$ and $f \in F'$. Then $\lambda_\omega(f), \pi_\omega(f) \in G_\rho$.

Proof. We show this for λ_ω ; the proof for π_ω is similar. Thanks to Lemma 4.3, we can assume without loss of generality that $\omega = (W, k_1, k_2)$ satisfies that the central letter of W equals b . As an appetizer, we first demonstrate the above proposition for $k_1, k_2 \in \{0, 1\}$. The statement in its full generality will then follow using an induction on n which is essentially similar to the base case.

The case $k_1 = k_2 = 0$. If $W = b$ then $\lambda_\omega(f) = \lambda(f)$.

The case $k_1 = 0, k_2 = 1$ or $k_1 = 1, k_2 = 0$. Consider the case $W = ba$. Given an $f \in F'$, we wish to show that $\lambda_\omega(f) \in G_\rho$. Since F' is generated by commutators, it suffices to show this in the case where f is a commutator. Since $f \in F'$, there is an $f_1 \in F'$ such that

$$\text{Supp}(f_1 f f_1^{-1}) \subset (\frac{1}{2}, 1).$$

Let $f_2 = f_1 f f_1^{-1}$. By self-similarity of F' , we note that f_2 is a commutator in $F'_{[1/2, 1]}$.

Let

$$f_2 = [f_3, f_4] \quad \text{for } f_3, f_4 \in F'_{[1/2, 1]} \subset F'_{[0, 1]}.$$

Let $f'_4 \in F'_{[0, 1/2]} \subseteq F'_{[0, 1]}$ be such that $f'_4 = h f_4 h^{-1}$ where $h(t) = t + \frac{1}{2}$. We claim that

$$\lambda_\omega(f_2) = [\lambda(f_3), \pi(f'_4)].$$

Consider an interval $[n, n + 1]$ where $n \in \mathbf{Z}$. If either

$$\rho(n + \frac{1}{2})\rho(n + 1) = ba$$

or

$$\rho(n)\rho(n + \frac{1}{2}) = a^{-1}b^{-1}$$

then

$$[\lambda(f_3), \pi(f'_4)] \upharpoonright [n, n + 1] = \lambda_\omega(f_2) \upharpoonright [n, n + 1].$$

If $\rho(n)\rho(n + \frac{1}{2})\rho(n + 1) \in \{ab^{-1}a, ab^{-1}a^{-1}, a^{-1}ba^{-1}\}$ then

$$(\text{Supp}(\lambda(f_3)) \cap [n, n + 1]) \cap (\text{Supp}(\pi(f'_4)) \cap [n, n + 1]) = \emptyset$$

and hence

$$[\lambda(f_3), \pi(f'_4)] \upharpoonright [n, n + 1] = \text{id} \upharpoonright [n, n + 1] = \lambda_\omega(f_2) \upharpoonright [n, n + 1].$$

Since the central letter of W is b , we obtain

$$\lambda_\omega(f) = \lambda(f_1^{-1})\lambda_\omega(f_2)\lambda(f_1) \in G_\rho.$$

The cases $W \in \{a^{-1}b, ab, ba^{-1}\}$ are very similar and are left as a pleasant visual exercise for the reader.

The general case. We perform an induction on $\sup\{k_1, k_2\}$. Let the inductive hypothesis hold for $n \in \mathbb{N}$. Consider a word

$$W = w_{-k_1} \dots w_0 \dots w_{k_2}, \quad w_i \in \{a, a^{-1}, b, b^{-1}\}$$

such that $\sup\{k_1, k_2\} = n + 1$. There are three cases:

- (1) $k_2 > k_1$;
- (2) $k_1 > k_2$;
- (3) $k_1 = k_2$.

The first two cases are symmetric, and we deal with $k_2 > k_1$ and $k_1 = k_2$.

The case $k_2 > k_1$. Assume as above that $w_0 = b$. We wish to show that, given an $f \in F'$, $\lambda_\omega \in G_\rho$. Since F' is generated by commutators, as above it suffices to show this in the case where f is a commutator.

Since $f \in F'$, there is an $f_1 \in F'$ such that

$$\text{Supp}(f_1 f f_1^{-1}) \subset (\frac{1}{2}, 1).$$

Let

$$f_2 = f_1 f f_1^{-1}.$$

As before, by self-similarity of F' , we note that f_2 is a commutator in $F'_{[1/2, 1]}$.

Let

$$f_2 = [f_3, f_4], \quad f_3, f_4 \in F'_{[1/2, 1]} \subset F'_{[0, 1]}.$$

Let $f'_4 \in F'_{[0, 1/2]} \subseteq F'_{[0, 1]}$ be such that $f'_4 = h f_4 h^{-1}$ where $h(t) = t + \frac{1}{2}$.

We define

$$W_1 = w_{-k_1} \dots w_{-1} w_0, \quad W_2 = w_1 \dots w_{k_2},$$

$$l_1 = k_1, \quad l_2 = 0, \quad l_3 = 0, \quad l_4 = k_2 - 1, \quad \omega_1 = (W_1, l_1, l_2), \quad \omega_2 = (W_2, l_3, l_4).$$

Note that the central letter of W_1 is b and the central letter of W_2 is $w_1 \in \{a, a^{-1}\}$. From our inductive hypothesis, we know that $\lambda_{\omega_1}(h), \lambda_{\omega_2}(h) \in G_\rho$ for each $h \in F'$. One checks that:

- (1) if $w_1 = a$ then

$$\lambda_\omega(f_2) = [\lambda_{\omega_1}(f_3), \pi_{\omega_2}(f'_4)];$$

- (2) if $w_1 = a^{-1}$ then

$$\lambda_\omega(f_2) = [\lambda_{\omega_1}(f_3), \pi_{\omega_2}(f''_4)], \quad f''_4 = \iota \circ f'_4 \circ \iota.$$

Since $w_0 = b$, it follows that

$$\lambda_\omega(f) = \lambda(f_1^{-1})\lambda_\omega(f_2)\lambda(f_1) \in G_\rho.$$

The case $k_1 = k_2$. Assume as above that $w_0 = b$. We wish to show that, given an $f \in F'$, $\lambda_\omega \in G_\rho$. Once again, as above it suffices to show this in the case where f is a commutator.

Just as above, we fix an $f_1 \in F'$ such that

$$\text{Supp}(f_1 f f_1^{-1}) \subset (\frac{1}{2}, 1).$$

Let

$$f_2 = f_1 f f_1^{-1}.$$

As before, by self-similarity of F' , we note that f_2 is a commutator in $F'_{[1/2, 1]}$.

Let

$$f_2 = [f_3, f_4], \quad f_3, f_4 \in F'_{[1/2, 1]} \subset F'_{[0, 1]}.$$

Let $f'_4 \in F'_{[0, 1/2]} \subseteq F'_{[0, 1]}$ such that $f'_4 = h f_4 h^{-1}$ where $h(t) = t + \frac{1}{2}$. Let $f_5 \in F'$ be an element such that

$$f_5 \upharpoonright \text{Supp}(f_3) = h^{-1}(t) \upharpoonright \text{Supp}(f_3).$$

Let

$$W_1 = w_{-k_1} \dots w_0, \quad W_2 = w_1 \dots w_{k_2}$$

and

$$l_1 = k_1 - 1, \quad l_2 = 1, \quad l_3 = 0, \quad l_4 = k_2 - 1, \quad \omega_1 = (W_1, l_1, l_2), \quad \omega_2 = (W_2, l_3, l_4).$$

Note that w_{-1}, w_1 are the central letters of W_1, W_2 , respectively.

Let

$$f''_3 = \iota \circ f'_3 \circ \iota \quad \text{if } w_{-1} = a^{-1}$$

and

$$f''_3 = f'_3 \quad \text{if } w_{-1} = a.$$

Let

$$f''_4 = \iota \circ f'_4 \circ \iota \quad \text{if } w_1 = a^{-1}$$

and

$$f''_4 = f'_4 \quad \text{if } w_1 = a.$$

From our inductive hypothesis, we know that $\lambda_{\omega_1}(k), \lambda_{\omega_2}(k) \in G_\rho$ for each $k \in F'$. One checks that

$$\lambda_\omega(f_2) = [\lambda(f_5^{-1})\pi_{\omega_1}(f''_3)\lambda(f_5), \pi_{\omega_2}(f''_4)].$$

Since $w_0 = b$, it follows that

$$\lambda_\omega(f) = \lambda(f_1^{-1})\lambda_\omega(f_2)\lambda(f_1) \in G_\rho. \quad \square$$

5. Epilog

The goal of this section is to prove Proposition 3.4, and subsequently the results stated in the Introduction. We consider a uniformly stable element $f \in K_\rho$. Let $\{I_\alpha\}_{\alpha \in P}$ be the set of atoms of f . From Definition 1.6, we know that there is a $k_f \in \mathbf{N}$ such that parts (3.a), (3.b) of the definition hold.

LEMMA 5.1. *Let $f \in K_\rho$ and $\{I_\alpha\}_{\alpha \in P}$ be as above. There is a number $l_f > k_f$ such that the following statement holds. Consider $n, m \in \mathbf{Z}$, $\alpha \in P$ such that $[n, n + 1]$, $[m, m + 1]$ are respectively the head and the foot of I_α . Assume that $n \neq m$ (and hence I_α has a distinct head and foot). Then it follows that*

$$\mathcal{W}([n, n + 1], l_f) \neq \mathcal{W}([m, m + 1], l_f),$$

$$\mathcal{W}([n, n + 1], l_f) \neq \mathcal{W}^{-1}([m, m + 1], l_f).$$

Proof. First we claim that

$$\mathcal{W}([n, n + 1], k_f + 2) \neq \mathcal{W}([m, m + 1], k_f + 2).$$

From the definition of the atoms of f , there is an $\epsilon > 0$ such that f fixes each point in $[n - \epsilon, n + \epsilon]$. However, there is a point in $[m - \epsilon, m + \epsilon]$ that is moved by f . It follows from Definition 1.6 that either

$$\mathcal{W}(n - \frac{1}{2}, k_f) \neq \mathcal{W}(m - \frac{1}{2}, k_f)$$

or

$$\mathcal{W}(n + \frac{1}{2}, k_f) \neq \mathcal{W}(m + \frac{1}{2}, k_f).$$

Therefore, the claim follows.

Let $l = \sup\{|I_\beta| \mid \beta \in P\}$. Note that from Definition 1.6 it follows that l is finite. For $l_f = k_f + l$ it follows that

$$\mathcal{W}([n, n + 1], l_f) \neq \mathcal{W}^{-1}([m, m + 1], l_f).$$

(To see this, assume by way of contradiction that the equality holds. This would imply that there is a number $t \in [n, m]$ such that $\rho(t) = \rho(t)^{-1}$, which is impossible.) It follows that both inequalities hold for $l_f = k_f + l$. □

Definition 5.2. Given any $f \in K_\rho$ that is uniformly stable, we define the number emerging from the proof of the above lemma as

$$l_f = k_f + l, \quad l = \sup\{|I_\beta| \mid \beta \in P\}.$$

Note that l_f satisfies both the conditions of Definition 1.6 and the conclusion of Lemma 5.1.

Since f is uniformly stable, we can consider the cellular decomposition of f as decorated atoms $\mathcal{T}_{l_f}(f)$. Let ζ_1, \dots, ζ_m be the equivalence classes of $\mathcal{T}_{l_f}(g)$. The homeomorphisms $f_{\zeta_1}, \dots, f_{\zeta_m}$ form the resulting cellular decomposition. To prove Proposition 3.4 we would like to show that $f_{\zeta_1}, \dots, f_{\zeta_m} \in G_\rho$.

LEMMA 5.3. Let $f \in K_\rho$, $\{I_\alpha\}_{\alpha \in P}$ and $\mathcal{T}_f(f)$ be as above. Consider $n, m \in \mathbf{Z}$, $\alpha \in P$ such that:

- (1) $[n, n + 1], [m, m + 1]$ are subintervals of I_α ;
- (2) $[n, n + 1]$ is either the head or the foot of I_α and $[m, m + 1]$ is neither the head nor the foot of I_α .

Then it follows that

$$\begin{aligned} \mathcal{W}([n, n + 1], l_f) &\neq \mathcal{W}([m, m + 1], l_f), \\ \mathcal{W}([n, n + 1], l_f) &\neq \mathcal{W}^{-1}([m, m + 1], l_f). \end{aligned}$$

Proof. Assume that $[n, n + 1]$ is the head of I_α . (The proof for the foot is similar.) Note that by definition, $f \upharpoonright (n - \epsilon, n + \epsilon) = \text{id}$ for some $\epsilon > 0$. However, $f \upharpoonright (m - \epsilon, m + \epsilon) \neq \text{id}$. So from part (3) of Definition 1.6 our conclusion follows. □

Definition 5.4. An element $g \in G_\rho$ is said to preserve the atoms of f if the following statements hold:

- (1) for each $\alpha \in P$, g pointwise fixes a neighborhood of $\text{inf}(I_\alpha), \text{sup}(I_\alpha)$;
- (2) if (I_α, l_f) and (I_β, l_f) are equivalent, then

$$\begin{aligned} g \upharpoonright I_\alpha &\cong_T g \upharpoonright I_\beta \quad \text{if } \mathcal{W}(I_\alpha, l_f) = \mathcal{W}(I_\beta, l_f), \\ g \upharpoonright I_\alpha &\cong_T \iota_\beta \circ g \circ \iota_\beta \upharpoonright I_\beta \quad \text{if } \mathcal{W}(I_\alpha, l_f) = \mathcal{W}^{-1}(I_\beta, l_f) \end{aligned}$$

where $\iota_\beta : I_\beta \rightarrow I_\beta$ is the unique orientation-reversing isometry.

Note that these properties are closed under composition of elements, and hence we define a subgroup of G_ρ ,

$$\mathcal{M}_f = \{g \in G_\rho \mid g \text{ is atom preserving for } f\}.$$

Special elements in G_ρ provide a natural source of atom-preserving elements, as is observed in the proof of the following lemma.

LEMMA 5.5. The restriction $\mathcal{M}_f \upharpoonright \text{int}(I_\alpha)$ for each $\alpha \in P$ does not admit a global fixed point.

Proof. Let $x \in \text{int}(I_\alpha)$. We would like to show the existence of an element $g \in \mathcal{M}_f$ such that $x \cdot g \neq x$. Let $n_1 = \text{inf}(I_\alpha), n_2 = \text{sup}(I_\alpha)$. There are two cases:

- (1) $x \in (\frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}) \cap \text{int}(I_\alpha)$;
- (2) $x \in (n - \frac{1}{2} + \epsilon, n + \frac{1}{2} - \epsilon)$ for $n \in \mathbf{Z} \cap \text{int}(I_\alpha)$ and $\epsilon > 0$.

Let $g \in F'$ be an element such that $(\epsilon, 1 - \epsilon) \subset \text{Supp}(g)$. In the latter case, from an application of Lemmas 5.1 and 5.3, it is easy to see that the special element $\pi_{\omega_1}(g)$ for

$$\omega_1 = (\mathcal{W}([n - \frac{1}{2}, n + \frac{1}{2}], l_f), l_f, l_f)$$

is atom preserving. Moreover, $x \cdot \pi_{\omega_1}(g) \neq x$, since $(\epsilon, 1 - \epsilon) \subset \text{Supp}(g)$.

In the former case, the special element is $\lambda_{\omega_2}(g)$ for

$$\omega_2 = (\mathcal{W}(x, l_f), l_f, l_f)$$

is atom preserving, and $x \cdot \lambda_{\omega_2}(g) \neq x$ since $\frac{1}{2} \in (\epsilon, 1 - \epsilon) \subset \text{Supp}(g)$. □

Proof of Proposition 3.4. Let f_{ζ_j} be an element in the cellular decomposition of f . We would like to show that $f_{\zeta_j} \in G_\rho$. For each $\alpha \in P$, let

$$J_\alpha = \text{Supp}(f_{\zeta_j}) \cap I_\alpha.$$

From an application of Lemma 5.5, we find an element $g \in \mathcal{M}_f < G_\rho$ such that for each $\alpha \in P$ such that $J_\alpha \neq \emptyset$ one of the following statements holds:

- (1) $J_\alpha \cdot g$ is a subset of the head of I_α ;
- (2) $J_\alpha \cdot g$ is a subset of the foot of I_α .

Indeed, if $\alpha, \beta \in P$ are such that $\mathcal{W}^{-1}(I_\alpha, l_f) = \mathcal{W}(I_\beta, l_f)$, then $J_\alpha \cdot g$ being a subset of the head of I_α implies that $J_\beta \cdot g$ is a subset of the foot of I_β .

It follows from an application of Lemmas 5.1 and 5.3 that

$$g^{-1} f_{\zeta_j} g = \lambda_\omega(h)$$

where $h \in F'$ and

$$\omega = (\mathcal{W}(I_\alpha, l_f), l_f, l_f)$$

for some (or any) I_α such that $J_\alpha \cdot g$ is a subset of the head of I_α . In particular,

$$f_{\zeta_j} = g \lambda_\omega(h) g^{-1}.$$

Since, by Proposition 4.5, $\lambda_\omega(h) \in G_\rho$, we conclude that $f_{\zeta_j} \in G_\rho$. □

We can now finish the proof of Theorem 1.8, Proposition 1.9, Theorem 1.1, and Corollaries 1.2 and 1.3.

Proof of Theorem 1.8. We know that $G_\rho \leq K_\rho$. It remains to show that, given $g \in K_\rho$, one has $g \in G_\rho$. Using Lemma 3.2, we know that there exist $g_1, g_2 \in G_\rho$ such that $h = g_1^{-1}(g g_2^{-1})g_1 \in K_\rho$ is uniformly stable. Using Proposition 3.4, we conclude that $h \in G_\rho$. Therefore it follows that $g \in G_\rho$. □

Proof of Proposition 1.9. Let $h \in K_\rho = G_\rho$. Thanks to Lemma 3.2, we know that there are elements $f_1, f_2 \in G_\rho$ such that f_2 is a commutator of elements in G_ρ and the element $f = f_1^{-1}(h f_2^{-1})f_1$ is uniformly stable.

CLAIM. *There is a subgroup $K < G_\rho$ such that $K \cong F' \oplus \dots \oplus F'$ and $f \in K$.*

Note that the claim implies that

$$h f_2^{-1} \in f_1 K f_1^{-1} < G_\rho, \quad f_1 K f_1^{-1} \cong F' \oplus \dots \oplus F'.$$

So the conclusion of Proposition 1.9 for h follows from this claim.

Proof of Claim. We know that $f \in G_\rho$ is a uniformly stable element. Let $\{I_\alpha\}_{\alpha \in P}$ be the atoms of f . Let l_f be the constant from Definition 5.2. Let the cellular decomposition of f as decorated atoms $\mathcal{T}_{l_f}(f)$ be $f_{\zeta_1}, \dots, f_{\zeta_m}$. Here we represent the equivalence classes of decorated atoms in $\mathcal{T}_{l_f}(f)$ as ζ_1, \dots, ζ_m . For each $1 \leq i \leq m$, set $L_i = |I_\alpha|$ where $(I_\alpha, l_f) \in \zeta_i$. (Recall that $|I_\alpha| = |I_\beta|$ whenever $(I_\alpha, l_f), (I_\beta, l_f) \in \zeta_i$.) For each $1 \leq i \leq m$, define the canonical isomorphism

$$\phi_i : F' \rightarrow F'_{[0, L_i]}$$

where $F_{[0, L_i]}$ is the standard copy of F supported on the interval $[0, L_i]$.

For each $1 \leq i \leq m$, we have

$$\{\mathcal{W}(I_\alpha, l_f) \mid (I_\alpha, l_f) \in \zeta_i\} = \{W_i, W_i^{-1}\}$$

for words W_1, \dots, W_m . Define a map

$$\phi : \bigoplus_{1 \leq i \leq m} F' \rightarrow \text{Homeo}^+(\mathbf{R})$$

as follows. For $\alpha \in P$ and $1 \leq i \leq m$,

$$\phi(g_1, \dots, g_m) \upharpoonright I_\alpha \cong_T \phi_i(g_i) \quad \text{if } (I_\alpha, l_f) \in \zeta_i \text{ and } \mathcal{W}(I_\alpha, l_f) = W_i,$$

$$\phi(g_1, \dots, g_m) \upharpoonright I_\alpha \cong_T \iota_{L_i} \circ \phi_i(g_i) \circ \iota_{L_i} \quad \text{if } (I_\alpha, l_f) \in \zeta_i \text{ and } \mathcal{W}(I_\alpha, l_f) = W_i^{-1}$$

where $\iota_{L_i} : [0, L_i] \rightarrow [0, L_i]$ is the unique orientation-reversing isometry. It is easy to check that this is an injective group homomorphism. Moreover, the image of each element under ϕ satisfies the conditions of Definition 1.6. Therefore, the image of ϕ lies in $K_\rho = G_\rho$ and contains $f = \phi(\phi_1^{-1}(f_{\zeta_1}), \dots, \phi_m^{-1}(f_{\zeta_m}))$. \square

Proof of Theorem 1.1. Let $f \in G_\rho$. We know from Lemma 3.2 that there is a commutator $f_1 \in G_\rho$ and an $f_2 \in G_\rho$ such that $f_0 = f_2(ff_1^{-1})f_2^{-1}$ is uniformly stable. By Proposition 1.9, we know that there is a subgroup of G_ρ that contains f_0 and is isomorphic to a direct sum of copies of F' . Since by Theorem 2.13 every element in F' can be expressed as a product of at most two commutators of elements in F' , the same holds for a direct sum of copies of F' . It follows that f_0 can be expressed as a product of at most two commutators of elements in G_ρ . Therefore, f can be expressed as a product of at most three commutators of elements in G_ρ . \square

Proof of Corollary 1.2. This follows from a theorem of Ghys [7], according to which such an action by orientation-preserving homeomorphisms of the circle induces a homogeneous quasimorphism (the rotation number), which is non-trivial in case of absence of a global fixed point. Since by Theorem 1.1 the stable commutator length of G_ρ vanishes, this quasimorphism must be trivial. Therefore, every such action of G_ρ on \mathbf{S}^1 must admit a global fixed point. \square

Proof of Corollary 1.3. The group G_ρ for a quasiperiodic labeling ρ cannot admit a type (i) action since it is not locally indicable (recall that G_ρ is a simple group). For a type (ii) action of G_ρ it is easy to construct an element $h \in \text{Homeo}^+(\mathbf{R})$ such that h commutes with each element of G_ρ . Upon taking a quotient, this provides a faithful fixed-point-free action of G_ρ on the circle, which contradicts Corollary 1.2. \square

Acknowledgements. A. Navas was funded by the projects FONDECYT 1160541, FONDECYT 265667 and the PREI-DGAPA of UNAM. Y. Lodha was funded by a Swiss National Science Foundation grant ‘Ambizione’ (no. PZ00P2_174137). C. Rivas was funded by the FONDECYT project 1181548.

REFERENCES

- [1] T. Altinel and A. Muranov. Interpretation of the arithmetic in certain groups of piecewise affine permutations of an interval. *J. Inst. Math. Jussieu* **8**(4) (2009), 623–652.
- [2] J. Belk. Thompson’s group F . *Preprint*, 2004, arXiv:0708.3609.
- [3] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.* **12**(2) (2002), 219–280.
- [4] D. Calegari. *scl* (*MSJ Memoirs*, 20). Mathematical Society of Japan, Tokyo, 2009.
- [5] J. W. Cannon, W. J. Floyd and W. R. Parry. Introductory notes on Richard Thompson’s groups. *Enseign. Math.* **42**(3–4) (1996), 215–256.
- [6] B. Deroin, A. Navas and C. Rivas. Groups, orders, and dynamics. *Preprint*, arXiv:1408.5805 (version 2016).
- [7] É. Ghys. Groupes d’homéomorphismes du cercle et cohomologie bornée. *The Lefschetz Centennial Conference, Part III (Mexico City, 1984) (Contemp. Math., 58, III)*. American Mathematical Society, Providence, RI, 1987, pp. 81–106.
- [8] É. Ghys. Actions de réseaux sur le cercle. *Invent. Math.* **137**(1) (1999), 199–231.
- [9] J. Hyde and Y. Lodha. Finitely generated infinite simple groups of homeomorphisms of the real line. *Invent. Math.* (2019), doi:10.1007/s00222-019-00880-7.
- [10] L. Lifschitz and D. Morris Witte. Bounded generation and lattices that cannot act on the line. *Pure Appl. Math. Q.* **4**(1) (2008), 99–126.
- [11] L. Lifschitz and D. Morris Witte. Isotropic nonarchimedean S -arithmetic groups are not left orderable. *C. R. Math. Acad. Sci. Paris* **339**(6) (2004), 417–420.
- [12] N. Matte Bon and M. Triestino. Groups of piecewise linear homeomorphisms of flows. *Preprint*, 2018, arXiv:1811.12256.
- [13] A. Navas. Group actions on 1-manifolds: a list of very concrete open questions. *Proceedings of the International Congress of Mathematicians, Vol. 2*. Eds. B. Sirakov, P. Ney de Souza and M. Viana. World Scientific, Singapore, 2018, pp. 2029–2056.
- [14] D. Witte. Arithmetic groups of higher Q -rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.* **122**(2) (1994), 333–340.