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DEFINABLE HENSELIAN VALUATIONS

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Abstract. In this note we investigate the question when a henselian valued field carries a nontrivial $\emptyset$-definable henselian valuation (in the language of rings). This is clearly not possible when the field is either separably or real closed, and, by the work of Prestel and Ziegler, there are further examples of henselian valued fields which do not admit a $\emptyset$-definable nontrivial henselian valuation. We give conditions on the residue field which ensure the existence of a parameter-free definition. In particular, we show that a henselian valued field admits a nontrivial henselian $\emptyset$-definable valuation when the residue field is separably closed or sufficiently nonhenselian, or when the absolute Galois group of the (residue) field is nonuniversal.

§1. Introduction.

In a henselian valued field (K, v), many arithmetic or algebraic questions can be reduced, via the henselian valuation v, to simpler questions about the value group vK and the residue field Kv. By the celebrated Ax-Kochen/Ershov Principle, in fact, if the residue characteristic is 0, 'everything' can be so reduced: the 1st-order theory of (K, v) (as valued field) is fully determined by the 1st-order theory of vK (as ordered abelian group) and of Kv (as pure field). In that sense the valuation (with its two accompanying structures vK and Kv) 'knows' everything about K, especially the full 1st-order theory of K as pure field, or, as one may call it, the *arithmetic* of K.

Conversely, in all natural examples, and, as we will see, in most others as well, a henselian valuation v is so intrinsic to K that it is itself encoded in the arithmetic of K, or, to make this notion precise, that its valuation ring $\mathcal{O}_v$ is 1st-order definable in K. Well known examples are the classical fields $\mathbb{Q}_p$ and $\mathbb{C}((t))$ with their valuation rings

$$\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \mid \exists y\ 1 + px^2 = y^2\right\} \text{ (for } p \neq 2\text{)},$$
$$\mathbb{C}[[t]] = \left\{x \in \mathbb{C}((t)) \mid \exists y\ 1 + tx^2 = y^2\right\}.$$

Note that the second example uses the parameter t. This is not necessary though: one can also find a parameter-free definition of $\mathbb{C}[[t]]$ in $\mathbb{C}((t))$; however, as observed

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in [4], it can no longer be an existential definition: otherwise the definition would go up the tower of isomorphic fields

$$\mathbb{C}((t)) \subseteq \mathbb{C}((t^{1/2!})) \subseteq \mathbb{C}((t^{1/3!})) \subseteq \cdots .$$

thus leading to a 1st-order definition of a nontrivial valuation subring of the algebraically closed field $\mathbb{C}((t^{1/\infty})) = \bigcup_n \mathbb{C}((t^{1/n!}))$, contradicting quantifier elimination (every definable subset is finite or cofinite).

That $\mathbb{C}[[t]]$ *is* $\emptyset$-definable in $\mathbb{C}((t))$ follows from the more general fact that every henselian valuation with nondivisible archimedean value group is $\emptyset$-definable ([14]). This has recently been generalized to nondivisible regular value groups (those elementarily equivalent to archimedean ordered groups, see [9]). Note that there are also several recent preprints which discuss $\emptyset$-definability of a range of henselian valuations using only formulae of 'simple' quantifier type (i.e., definitions involving $\forall$-,$\exists$-,$\forall\exists$, or $\exists\forall$-formulae). To learn more about these exciting developments, we refer the reader to [4], [2], [7], and [17].

In this paper we will develop two new, fairly general criteria, one on the residue field and one on the absolute Galois group G_K of K to guarantee $\emptyset$-definability of (in the first case a given, in the second case, at least some) henselian valuation on K. It is well-known that separably and real closed fields admit no definable henselian valuations. Furthermore, by the work of Prestel and Ziegler ([18], Section 7) there are henselian valued fields which are neither separably nor real closed and which do not admit any $\emptyset$-definable henselian valuation. It is thus a natural question to ask which conditions on a henselian valued field (K, v) ensure that v is $\emptyset$-definable or that K admits at least some $\emptyset$-definable henselian valuation. In the present work, we focus on parameter-free definitions as a definition of a henselian valuation with parameters need not ensure the existence of a definable henselian valuation in elementarily equivalent fields. Note that there are also examples of henselian valuations which are not even definable with parameters (see [5], Theorem 4.4). The only known examples of henselian fields which admit no parameter-definable henselian valuations at all are separably and real closed fields.

The paper is organized as follows. In the next section, we discuss the main tools which we require. We recall the definition of p-henselian valuations and the canonical (p)-henselian valuation. Building on work of the second author (see [12]), the authors have shown that the canonical p-henselian valuation v_K^p is typically definable (Theorem 3.1 in [10]). We show that it is furthermore henselian iff it is coarser than the canonical henselian valuation.

The third section contains the main results of this paper. We begin by giving conditions on the residue field to make a henselian valuation definable. The first criterion says that the henselian valuation v on K is $\emptyset$-definable if, for some prime p, Kv allows a separable extension L with $L \neq L(p)$ that does not allow a p-henselian valuation (Theorem 3.6, cf. Section 2 for the definition of $L(p)$ and p-henselian). We deduce from this that any henselian valuation with finitely generated, hilbertian, PAC or simple but not separably closed residue field is $\emptyset$-definable. We use a similar method to show that a henselian valued field (K, v) where Kv is separably or real closed, but K isn't, admits some $\emptyset$-definable henselian valuation.

The next part discusses a second, Galois-theoretic criterion for the existence of a $\emptyset$-definable henselian valuation on a (nonseparably- and nonreal-closed) henselian valued field K (Theorem 3.15). It says that if K is henselian and G_K is *nonuniversal*, that is, that not every finite group is a subquotient of G_K, then K admits some $\emptyset$-definable nontrivial henselian valuation. In most cases, we will in fact define the canonical henselian valuation on K. This generalizes old results by Neukirch, Geyer, and Pop on henselian fields with prosolvable G_K. One class of examples is given by henselian NIP fields of positive characteristic.

These two criteria, one on the residue field of a given henselian valuation v on K and one on G_K in the presence of *some* henselian valuation on K, are fairly independent. One easily finds examples of the first kind where G_K *is* universal and examples where it isn't. Similarly, there are henselian fields K with nonuniversal G_K where every henselian valuation on K satisfies the criterion on the residue field and such where none of them does. What is common between the two criteria, however, is the method of proof which in either case depends on a careful analysis when, on a field K, the canonical p-henselian valuation v_K^p is already henselian. Although many fields have universal absolute Galois groups, the best known ones are hilbertian fields and PAC fields with nonabelian free absolute Galois group. Hence some of the main examples of henselian valued fields for which the second criterion fails are covered by the first one.

§2. Henselian and p-henselian valuations.

2.1. The canonical henselian valuation. We call a field K *henselian* if it admits some nontrivial henselian valuation. For any field K, there is a *canonical henselian valuation* on K. In this section, we recall the definition and discuss some of its properties. We use the following notation: For a valued field (K, v), we denote the valuation ring by $\mathcal{O}_v$, the residue field by Kv, the value group by vK, and the maximal ideal by $\mathfrak{m}_v$. For an element $a \in \mathcal{O}_v$, we write $\overline{a}$ to refer to its image in Kv.

THEOREM 2.1 (á la F.K. Schmidt). *If a field admits two independent nontrivial henselian valuations, then it is separably closed.*

PROOF. [6], Theorem 4.4.1. ⊣

One can deduce from this that the henselian valuations on a field form a tree: Divide the class of henselian valuations on K into two subclasses, namely

$$H_1(K) = \{\, v \text{ henselian on } K \mid Kv \neq Kv^{sep} \,\}$$

and

$$H_2(K) = \{\, v \text{ henselian on } K \mid Kv = Kv^{sep} \,\}.$$

A corollary of the above theorem is that any valuation $v_2 \in H_2(K)$ is *finer* than any $v_1 \in H_1(K)$, i.e., $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1(K)$ are comparable. Furthermore, if $H_2(K)$ is nonempty, then there exists a unique coarsest $v_K \in H_2(K)$; otherwise there exists a unique finest $v_K \in H_1(K)$. In either case, v_K is called the *canonical henselian valuation*. Note that if K is not separably closed and admits a nontrivial henselian valuation, then v_K is also nontrivial.

As we will usually define henselian valuations on finite Galois extensions later on, we often use the fact that coarsenings of v_K remain henselian when restricted to subfields of finite index:

Theorem 2.2 ([6], Theorem 4.4.4). *Let (L, w) be a valued field, and assume that L is not separably closed and that w is a (not necessarily proper) coarsening of v_L. If $K \subset L$ is a subfield such that L/K is finite, then $v = w|_K$ is a coarsening of v_K.*

2.2. p-henselianity. Throughout this section, let K be a field and p a prime.

Definition. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree. A valuation v on K is called *p-henselian* if v extends uniquely to $K(p)$. We call K *p-henselian* if K admits a nontrivial p-henselian valuation.

Clearly, this definition only imposes a condition on v if K admits Galois extensions of p-power degree.

Proposition 2.3 ([12], Propositions 1.2 and 1.3). *For a valued field (K, v), the following are equivalent*:

1. *v is p-henselian,*
2. *v extends uniquely to every Galois extension of K of p-power degree,*
3. *v extends uniquely to every Galois extension of K of degree p,*
4. *for every polynomial $f \in \mathcal{O}_v$ which splits in $K(p)$ and every $a \in \mathcal{O}_v$ with $\bar{f}(\overline{a}) = 0$ and $\bar{f}'(\overline{a}) \neq 0$, there exists $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $\overline{\alpha} = \overline{a}$.*

As for fields carrying a henselian valuation, there is again a canonical p-henselian valuation, due to the following analogue of Theorem 2.1:

Theorem 2.4 ([3], Corollary 1.5). *If K carries two independent nontrivial p-henselian valuations, then $K = K(p)$.*

We again divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{\, v \ p\text{-henselian on } K \mid Kv \neq Kv(p) \,\}$$

and

$$H_2^p(K) = \{\, v \ p\text{-henselian on } K \mid Kv = Kv(p) \,\}.$$

As before, one can deduce that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is nonempty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation*. Again, if K is p-henselian and $K \neq K(p)$ holds, then v_K^p is also nontrivial.

Note that unlike henselianity, being p-henselian does not go up arbitrary algebraic extensions, as a superfield might have far more extensions of p-power degree. Nevertheless, similar to Theorem 2.2, sometimes p-henselianity goes down:

Proposition 2.5. *Let K be a field, $K \neq K(p)$. Assume that L is a normal algebraic extension of K, where L is p-henselian and $L \neq L(p)$. If*

1. *$K \subseteq L \subsetneq K(p)$ or*
2. *L/K is finite*

then K is p-henselian.

Proof.

1. See [13], Proposition 2.10.
2. Assume K is not p-henselian, and let v be a valuation on K. By the first part of the proposition, v has infinitely many extensions to $K(p)$: If there were

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only n extensions of v to $K(p)$, then there would be some $L' \supset K$ finite, $L' \subsetneq K(p)$, such that v had n extensions to L'. The normal hull of L' and thus K would be p-henselian.

Now assume $L = K(a_1, \dots, a_m)$ finite and normal, then $K(p)(a_1, \dots, a_m) \subseteq L(p)$. If w is a valuation on L, then $v = w|_K$ has infinitely many prolongations to $K(p)$. As v has only finitely many prolongations to L, and all these are conjugate, w must have infinitely many prolongations to $K(p)(a_1, \dots, a_m)$ and hence to $L(p)$. ⊣

For any valued field, p-extensions of the residue field lift to p-extensions of the field.

Proposition 2.6 ([6], Theorem 4.2.6). *Let (K, v) be a valued field and p a prime. If $Kv \neq Kv(p)$, then $K \neq K(p)$.*

2.3. Defining p-henselian valuations. In this section, we recall a Corollary of the Main Theorem in [10] which is used in all of our proofs in later sections.

When it comes to henselian valued fields, real closed fields always play a special role. By o-minimality, no real closed field admits a definable henselian valuation, and there are real closed fields which admit no henselian valuations (like $\mathbb{R}$) whereas others do (like $\mathbb{R}((t^{\mathbb{Q}}))$). These difficulties are reflected by 2-henselian valuations on Euclidean fields. A field K is called *Euclidean* if $[K(2) : K] = 2$. Any Euclidean field is uniquely ordered, the positive elements being exactly the squares. If a Euclidean field has no odd-degree extensions, then it is real closed. In particular, there is an $\mathcal{L}_{ring}$-sentence ρ such that any field K with $K \neq K(2)$ models ρ iff it is nonEuclidean. Note that Euclidean fields are the only fields for which $K(p)$ can be a finite proper extension of K.

Theorem 2.7 (Corollary 3.3 in [10]). *Let p be a prime and consider the class of fields*

$$\mathcal{K} = \{K \mid K \ p\text{-henselian, with } \zeta_p \in K \text{ in case } \operatorname{char}(K) \neq p\}$$

There is a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ such that

1. *if $p \neq 2$ or Kv_2 is not Euclidean, then $\phi_p(x)$ defines the valuation ring of the canonical p-henselian valuation v_K^p, and*
2. *if $p = 2$ and Kv_2 is Euclidean, then $\phi_p(x)$ defines the valuation ring of the coarsest 2-henselian valuation v_K^{2*} such that Kv_K^{2*} is Euclidean.*

The existence of such a uniform definition of the canonical p-henselian makes sure that the different cases split into elementary classes:

Corollary 2.8. *The classes of fields*

$$\mathcal{K}_1 = \left\{ K \;\middle|\; K \ p\text{-henselian, with } \zeta_p \in K \text{ in case } \operatorname{char}(K) \neq p \text{ and } v_K^p \in H_1^p(K) \right\}$$

and

$$\mathcal{K}_2 = \left\{ K \;\middle|\; K \ p\text{-henselian, with } \zeta_p \in K \text{ in case } \operatorname{char}(K) \neq p \text{ and } v_K^p \in H_2^p(K) \right\}$$

are elementary classes in $\mathcal{L}_{ring}$.

Proof. The class

$$\{ K \mid K \ p\text{-henselian, with } \zeta_p \in K \text{ in case } \operatorname{char}(K) \neq p \}$$

is an elementary class in $\mathcal{L}_{ring}$ by Corollary 2.2 in [12]. The sentence dividing the class into the two elementary subclasses is the statement whether the residue field of the valuation defined by $\phi_p(x)$ as in Theorem 2.7 admits a Galois extension of degree p. Note that if $p = 2$ and Kv_2 is Euclidean, both v_K^2 and v_K^{2*} are elements of $H_1^p(K)$. ⊣

Remark. When one is only interested in defining henselian valuations, one can usually avoid to consider the special case of a Euclidean residue field: If (K, v) is a henselian valued field, K not real closed and Kv Euclidean, then—similarly to Proposition 2.6—K is also real, so $i \notin K$. Now $K(i)$ is a $\emptyset$-interpretable extension of K, and the unique prolongation w of v to $K(i)$ has a nonEuclidean residue field, namely $Kv(i)$. Thus, in order to get a parameter-free definition of v, it suffices to define w without parameters on $K(i)$.

However, the same argument does not work for p-henselian valuations, as there is no strong enough analogue of Theorem 2.2. Thus, for completeness' sake, we give Theorem 2.7 in its full generality.

2.4. p-henselian valuations as henselian valuations. Let K be a henselian field and p a prime such that $K \neq K(p)$ holds. As any henselian valuation is in particular p-henselian, we have either $v_K^p \supseteq v_K$ or $v_K^p \subsetneq v_K$. In the first case, v_K^p is henselian. As we will make use of this fact several times later, we note here that this is in fact an equivalence:

Observation 2.9. *Let K be a henselian field with $K \neq K(p)$ for some prime p. Then v_K^p is henselian iff v_K^p coarsens v_K.*

Proof. Any coarsening of a henselian valuation—like v_K—is henselian. Conversely, assume that v_K^p is henselian and a proper refinement of v_K. Then, by the definition of v_K, we get $v_K^p \in H_2(K)$ and hence $v_K \in H_2(K)$. In this case, v_K^p has a proper coarsening with p-closed residue field, contradicting the definition of v_K^p. ⊣

§3. Main results.

3.1. Conditions on the residue field. We first want to show that we can use the canonical p-henselian valuation to define any henselian valuation which has not p-henselian residue field.

Proposition 3.1. *Let (K, v) be a nontrivially henselian valued field and p a prime. Assume that the residue field Kv is not p-henselian and that $Kv \neq Kv(p)$. If $p = 2$, assume further that Kv is not Euclidean. Then v is $\emptyset$-definable.*

Proof. Let p and (K, v) be as above. If $\mathrm{char}(K) \neq p$, we assume $\zeta_p \in K$ for now.

Note that $K \neq K(p)$ (Proposition 2.6). Thus, K is p-henselian. We claim that $v_K^p = v$. As v is henselian, it is in particular p-henselian and hence comparable to v_K^p. Since Kv is not p-henselian, v_K^p is a coarsening of v, as otherwise v_K^p would induce a p-henselian valuation on Kv ([6], Corollary 4.2.7). Assume v_K^p is a proper coarsening of v. Then we get $v \in H_2^p(K)$ and hence $Kv = Kv(p)$, contradicting our assumption on Kv. This proves the claim.

For $p = 2$, we get from our assumption that $Kv_K^2 = Kv$ is not Euclidean. Thus, v_K^p is henselian and $\emptyset$-definable by Theorem 2.7.

In case $\mathrm{char}(K) \neq p$ and K does not contain a primitive pth root of unity, we consider $K' = K(\zeta_p)$. As this is a $\emptyset$-definable extension of K, it suffices to define

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the—by henselianity unique—prolongation v' of v to K'. Since $K'v'$ is a finite normal extension of Kv of degree at most $p-1$, it still satisfies $K'v' \neq K'v'(p)$ and is furthermore not p-henselian by Proposition 2.5. Now v' is $\emptyset$-definable as above, and thus so is v. ⊣

Morally speaking, the proposition says that if we have a henselian valued field (K, v) such that the residue field is 'far away' from being henselian, then v is $\emptyset$-definable. Hence we will now consider well-known classes of examples of non-henselian fields and prove that any henselian valuation with such a residue field is $\emptyset$-definable.

Example. Let k be a finite field. Then $G_k \cong \hat{\mathbb{Z}}$, in particular $k \neq k(p)$ holds for all primes p. Note that k is not Euclidean since $\mathrm{char}(k) > 0$. As k admits no nontrivial valuations, k is also not p-henselian. Now by Proposition 3.1, if (K, v) is a nontrivially henselian valued field with $Kv = k$, then v is $\emptyset$-definable.

Probably the best known example of a nonhenselian field are the rationals. One way of showing that the rationals admit no nontrivial henselian valuation is via Hilbert's Irreducibility Theorem: No hilbertian field is henselian (see Lemma 15.5.4 in [8]). We will now show by a similar proof that furthermore any henselian valued field with hilbertian residue field satisfies the assumption of the above proposition. First, we recall the definition of hilbertianity.

Definition. Let K be a field and let T and X be variables. Then K is called *hilbertian* if for every polynomial $f \in K[T, X]$ which is separable, irreducible, and monic when considered as a polynomial in $K(T)[X]$ there is some $a \in K$ such that $f(a, X)$ is irreducible in $K[X]$.

Note that *Hilbert's Irreducibility Theorem* states that $\mathbb{Q}$ is hilbertian.

Examples of hilbertian fields include all infinite finitely generated fields, in particular number fields and function fields over finite fields.

Lemma 3.2. *If K is a hilbertian field then $K \neq K(p)$ for any prime p. Furthermore, K is neither Euclidean nor p-henselian.*

Proof. If K is hilbertian, then K is not Euclidean and $K \neq K(p)$ holds for any prime p by Corollary 16.3.6 in [8]. Let us first treat the case $\mathrm{char}(K) \neq p$. We may then assume that K contains a primitive pth root of unity as $K(\zeta_p)$ is again hilbertian, and if $K(\zeta_p)$ was p-henselian then so would be K by Proposition 2.5.

Let v be a nontrivial valuation on K. Choose $m \in \mathfrak{m}_v \setminus \{0\}$ and consider the irreducible polynomial $f(T, X) = X^p + mT - 1$ in $K(T)[X]$. If K is hilbertian, there exists an $a \in K^\times$ such that $f(a, X)$ is irreducible in $K[X]$. Furthermore, by exercise 13.4 in [8], a may be chosen in $\mathcal{O}_v$. But now $f(a, X)$ splits in $K(p)$, and has a simple zero in Kv. Hence by Proposition 2.3, v cannot be p-henselian.

In case $\mathrm{char}(K) = p$, the same argument as above applies to the polynomial $f(T, X) = X^p + X + mT - 2$. ⊣

Combining Theorem 2.7 with Lemma 3.2, we also get:

Corollary 3.3. *Let (K, v) be a henselian valued field such that Kv is hilbertian. Then v is $\emptyset$-definable.*

Example. For any number field K and any ordered abelian group Γ, the power series valuation on $K((\Gamma))$ is $\emptyset$-definable.

Another well-known class of fields which are not henselian are nonseparably closed PAC fields. As in general—unlike hilbertian fields—PAC fields do not need to admit any Galois extensions of prime degree, we give a suitable generalization of Proposition 3.1. Any nonseparably closed PAC field has a finite Galois extension which is still PAC and which admits in turn Galois extensions of prime degree. This motivates the following

Definition. Let K be a field. We call K *virtually not p-henselian* if $p \mid \#G_K$ and there is some finite Galois extension L of K with $L \neq L(p)$ such that L is not p-henselian.

Note that if $K \neq K(p)$, then K is virtually not p-henselian iff it is not p-henselian by Proposition 2.5. We will now show a PAC field K is virtually not p-henselian for any prime p with $p \mid \#G_K$. First, we show that a PAC field K with $K \neq K(p)$ is not p-henselian using the same method as one uses to show that such a field is not henselian (see [8], Corollary 11.5.5).

Lemma 3.4 (Kaplansky–Krasner for p-henselian valuations). *Assume that (K, v) is a p-henselian valued field and take $f \in K[X]$ separable,* $\deg(f) > 1$, *such that f splits in $K(p)$. Suppose for each $\gamma \in vK$ there exists some $x \in K$ such that $v(f(x)) > \gamma$. Then f has a zero in K.*

Proof. Without loss of generality we may assume that f is monic and that $\deg(f) = n > 0$. Write

$$f(X) = \prod_{i=1}^{n}(X - x_i)$$

for $x_i \in K(p)$. Take $\gamma > n \cdot \max\{v(x_i - x_j) \mid 1 \leq i < j \leq n\}$ and choose $x \in K$ such that

$$v(f(x)) = \sum_{i=1}^{n} v(x - x_i) > \gamma.$$

Hence for some j with $1 \leq j \leq n$ we get $v(x - x_j) > \gamma/n$. If $x_j \notin K$, then there is some $\sigma \in \mathrm{Gal}(K(p)/K)$ such that $\sigma(x_j) \neq x_j$. Thus, we get

$$v(x - \sigma(x_j)) = v(\sigma(x - x_j)) = v(x - x_j) > \frac{\gamma}{n},$$

where the last equality holds as v is p-henselian. Therefore

$$v(x_j - \sigma(x_j)) \geq \min\{v(x_j - x), v(x - \sigma(x_j))\} > \frac{\gamma}{n}$$

which contradicts the choice of γ. Hence we conclude $x_j \in K$, so f has a zero in K. ⊣

Lemma 3.5. *Let K be a field and p a prime. If K is PAC and p-henselian, then we have $K = K(p)$.*

Proof. Assume that K is PAC and p-henselian. We show that $K = K(p)$ holds. Take $f \in K[X]$ a separable, irreducible polynomial with $\deg(f) > 1$ splitting in $K(p)$. It suffices to show that for all $c \in K^\times$ there exists an $x \in K$ such that $v(f(x)) \geq v(c)$, as then f has a zero in K.

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Consider the curve $g(X, Y) = f(X)f(Y) - c^2$. Consider $g(X, Y)$ as a polynomial over $K^{sep}[Y]$. Eisenstein's criterion ([8], Lemma 2.3.10(b)) applies over this ring to any linear factor of $f(Y)$, thus $g(X, Y)$ is absolutely irreducible. As K is PAC, there exist $x, y \in K$ such that $f(x)f(y) = c^2$. Thus, either $v(f(x)) \geq v(c)$ or $v(f(y)) \geq v(c)$ holds. ⊣

As being PAC passes up to algebraic extensions, any PAC field K is in particular not virtually p-henselian for all primes $p \mid \#G_K$. Furthermore, as real closed fields are not PAC, no PAC field is Euclidean.

We now give a stronger version of Proposition 3.1. The main difference is that the we drop the assumption on the residue field to admit a Galois extension of p-power degree for some prime p.

Theorem 3.6. *Let (K, v) be a nontrivially henselian valued field with $p \mid \#G_{Kv}$, and if $p = 2$ assume that Kv is not Euclidean. If Kv is virtually not p-henselian then v is $\emptyset$-definable on K.*

Proof. If Kv is virtually not p-henselian and $Kv \neq Kv(p)$, then v is $\emptyset$-definable by Proposition 3.1.

In case $Kv = Kv(p)$, by assumption there is a p-henselian finite Galois extension L of Kv with $L \neq L(p)$. As Kv is not Euclidean, L is also not Euclidean. By Proposition 2.5, we may assume that L contains a primitive pth root of unity in case $\mathrm{char}(Kv) \neq p$. Let $[L : Kv] = n$.

Consider any finite Galois extension M of K, with w the unique prolongation of v to M such that $Mw = L$ holds. As before, w is $\emptyset$-definable on M (since $w = v_M^p$ as in the proof of Proposition 3.1) and hence, by interpreting M in K using parameters, so is its restriction v to K.

Thus, it remains to show that a definition can be found without parameters. The interpretation of Galois extensions of a fixed degree of K can be done uniformly with respect to the parameters (namely the coefficients of a minimal polynomial generating the extension). By Theorem 2.7, the definition of the p-henselian valuations on these can also be done uniformly. To make sure that the residue field of the canonical p-henselian valuation of a finite Galois extension of K corresponds to a field L as described above, we need to restrict to extensions M of K with $v_p^M \in H_1^p(M)$. By Corollary 2.8, this is a $\emptyset$-definable condition. Hence we get the desired definition by

$$\bigcap \Big(\mathcal{O}_{v_M^p} \cap K \,\Big|\, K \subseteq M \text{ Galois}, [M : K] = n,\ M \neq M(p),\ M \text{ not } p\text{-henselian}, \zeta_p \in M \text{ if } \mathrm{char}(M) \neq p,\ v_M^p \in H_1^p(M) \Big).$$ ⊣

As an immediate consequence, we have the following

Corollary 3.7. *Let p be a prime and let K be a field such that $p \mid \#G_K$ and that K is virtually not p-henselian. If $p = 2$, assume that K is not Euclidean. Then the power series valuation is $\emptyset$-definable on $K((\Gamma))$, for any ordered abelian group Γ.*

Combining Theorem 3.6 with Lemma 3.5, we get:

Corollary 3.8. *Let (K, v) be a henselian valued field such that Kv is PAC and not separably closed. Then v is $\emptyset$-definable.*

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Another application of Theorem 3.6 are henselian valued fields with simple residue fields. We call a field *simple* if $\text{Th}(K)$ is simple in the sense of Shelah (see [19] for some background on simplicity). In a simple theory, no orderings with infinite chains are interpretable. Thus, no simple field admits a definable valuation. Hence, by Theorem 2.7, simple fields cannot be p-henselian for any prime p. As all Galois extensions of a simple field are interpretable in K and thus again simple, any nonseparably closed simple field K is not virtually p-henselian for any p with $p \mid \#G_K$. Thus, we get the following

Corollary 3.9. *Let (K, v) be a henselian valued field such that Kv is simple and not separably closed. Then v is $\emptyset$-definable.*

Real closed and separably closed residue fields. In all our definitions of henselian valuations we showed so far that a given henselian valuation v on a field K coincided with both the canonical henselian valuation v_K and the canonical p-henselian valuation v_K^p for some prime p. However, it can happen that some v_K^p is henselian, but a proper coarsening of a given henselian valuation v. In this case v_K^p is again henselian and $\emptyset$-definable. An example for this are henselian valued fields with separably closed residue field:

Theorem 3.10. *Let K be a field which is not separably closed. Assume that K is henselian with respect to a valuation with separably closed residue field. Then K admits a nontrivial $\emptyset$-definable henselian valuation.*

Proof. We show first that G_K is pro-soluble. If K is henselian with respect to a valuation with separably closed residue field, then v_K has also separably closed residue field. Let w be the prolongation of v_K to K^{sep}. Recall that there is an exact sequence

$$I_w \longrightarrow G_K \longrightarrow G_{Kv_K},$$

where I_w denotes the inertia group of w over K (see [6], Theorem 5.2.7). Hence, as I_w is pro-soluble (see [6], Lemma 5.3.2), so is G_K.

Thus, there is some prime p with $K \neq K(p)$. But now v_K^p is indeed a (not necessarily proper) coarsening of v_K: Otherwise, the definition of v_K^p would imply $Kv_K \neq Kv_K(p)$. If K contains a primitive pth root of unity or $\text{char}(K) = p$, then v_K^p is $\emptyset$-definable and henselian. Else, we consider the $\emptyset$-definable extension $K(\zeta_p)$. Then the canonical henselian valuation on $K(\zeta_p)$ still has separably closed residue field, therefore $v^p_{K(\zeta_p)}|_K$ gives a $\emptyset$-definable henselian valuation on K. ⊣

Corollary 3.11. *Let K be a field and assume that K is not real closed. If K is henselian with respect to a valuation with real closed residue field, then K admits a nontrivial $\emptyset$-definable henselian valuation.*

Proof. If (K, v) is henselian and Kv is real closed, consider the unique prolongation w of v to $L = K(i)$. The residue field Lw is separably closed, so L admits a $\emptyset$-definable henselian valuation by Theorem 3.10. As v is the restriction of w to K, v is also $\emptyset$-definable on K. ⊣

3.2. Henselian fields with non-universal absolute Galois groups. In this section, we will give a Galois-theoretic condition to ensure the existence of a nontrivial $\emptyset$-definable henselian valuation on a henselian field.

The following group-theoretic definition is taken from [16].

Definition. Let G be a profinite group. We say that G is *universal* if every finite group occurs as a continuous subquotient of G.

Note that for a field K, G_K is nonuniversal iff there is some $n \in \mathbb{N}$ such that the symmetric group S_n does not occur as a Galois group over any finite Galois extension of K (and then no S_m with $m \geq n$ will occur). The connection between nonuniversal absolute Galois groups and henselianity is given by the following statement:

Theorem 3.12 ([15], Theorem 3.1). *Let K be a field and let L and M be algebraic extensions of K which both carry nontrivial henselian valuations. Assume further that G_L is nontrivial pro-p and G_M nontrivial pro-q for primes $p < q$. Let v and w be (not necessarily proper) coarsenings of the canonical henselian valuations on L and M respectively, and, if $p = 2$ and Lv is real closed, assume v to be the coarsest henselian valuation on L with real closed residue field. Then either G_K is universal or $v|_K$ and $w|_K$ are comparable and the coarser valuation is henselian on K.*

Example. All of the following profinite groups are nonuniversal:

1. pro-abelian groups,
2. pro-nilpotent groups,
3. pro-soluble groups,
4. any group G such that $p \nmid \#G$ for some prime p.

Nonabelian free profinite groups are of course universal, and so are absolute Galois groups of hilbertian fields.

Now we can use Theorem 3.12 to deduce henselianity from p- and q-henselianity:

Proposition 3.13. *Suppose G_K is nonuniversal, and $K(p) \neq K \neq K(q)$ for two primes $p < q$. In case $p = 2$, assume further that K is not Euclidean. If K is p- and q-henselian, then K is henselian.*

Proof. Consider the henselization L' (respectively M') of K with respect to the canonical p-henselian valuation v_K^p (the canonical q-henselian valuation v_K^q) on K. Then define L to be the fixed field of a p-Sylow subgroup of $G_{L'}$, and M accordingly.

Claim: L is not separably closed.

Proof of Claim: We need to show that L' is not p-closed. But if $\alpha \in K(p)$ has degree p^n over K, then—as v_K^p is p-henselian—α is moved by some element of $D(K(p)/K)$. As decomposition groups behave well in towers, we get $\alpha \notin L$.

In case $p = 2$, the same argument shows that L is also not real closed. Since L is p-henselian and G_L is pro-p, L is also henselian, and likewise is M. Now we consider the canonical henselian valuations v_L on L and the canonical henselian valuation v_M on M. If $p = 2$ and Lv_L is real closed, we replace v_L by the coarsest henselian valuation on L with real closed residue field. As L is not real closed, this is again a nontrivial henselian valuation.

By Theorem 3.12, the restrictions $v_L|_K$ and $v_M|_K$ are comparable and the coarser one is henselian. As L and M are algebraic extensions of K, none of the restrictions is trivial. Hence K is henselian. ⊣

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Proposition 3.14. *Let G_K be nonuniversal. Assume that there are two primes $p < q$ with $p, q \mid \#G_K$ and such that $K(p) \neq K \neq K(q)$ holds. If K is henselian, then K is henselian with respect to a nontrivial $\emptyset$-definable valuation.*

Proof. As long as we define a coarsening of v_K without parameters, we may assume that $\zeta_p, \zeta_q \in K$ if $\mathrm{char}(K) \neq p$ or q respectively: The only special case is when $p = 2$ and K is Euclidean and $G_{K(i)}$ is pro-q. Then $K(i)$ already contains ζ_q and thus $v_{K(i)}^q = v_{K(i)}$ is a nontrivial $\emptyset$-definable henselian valuation on $K(i)$. In this case, its restriction to K is nontrivial $\emptyset$-definable henselian valuation on K.

So now assume $\zeta_p, \zeta_q \in K$. In particular, in case $p = 2$, K is not formally real and so Kv_K^2 cannot be Euclidean. All these extensions still have nonuniversal absolute Galois groups.

As K is henselian, it is in particular p- and q-henselian. We consider the canonical p-henselian (q-henselian) valuation v_K^p (v_K^q respectively) on K. If v_K^p or v_K^q is henselian, then we have found a $\emptyset$-definable henselian valuation.

But this must always be the case: Assume that neither v_K^p nor v_K^q is henselian. Then v_K is a proper coarsening of v_K^p, and thus Kv_K is p-henselian and satisfies $Kv_K \neq Kv_K(p)$. Similarly, Kv_K is q-henselian and $Kv_K \neq Kv_K(q)$ holds. Therefore, by Proposition 3.13, Kv_K is henselian. This contradicts the definition of v_K. ⊣

We can now prove our main result on henselian fields with nonuniversal absolute Galois group.

Theorem 3.15. *Let K be henselian, and assume that G_K is nonuniversal. If K is neither separably nor real closed, then K admits a $\emptyset$-definable henselian valuation. If Kv_K is neither separably nor real closed, then v_K is $\emptyset$-definable.*

Proof. By assumption, K is neither separably nor real closed. If K is henselian and Kv_K is separably closed (respectively real closed), then K admits a $\emptyset$-definable henselian valuation by Theorem 3.10 (respectively Corollary 3.11). Thus, we may assume from now on that Kv_K is neither separably nor real closed.

In this case, v_K is the finest henselian valuation on K and thus Kv_K is not henselian. Furthermore, there is some prime p with $p \mid \#G_{Kv_K}$. Assume first that G_{Kv_K} is pro-p, then it follows that $Kv_K \neq Kv_K(p)$ and thus $K \neq K(p)$ (Proposition 2.6). In particular, v_K must be a coarsening of v_K^p. But if v_K was a proper coarsening of v_K^p, then Kv_K would be p-henselian and hence—as G_{Kv_K} is pro-p—henselian or real closed. Since we have assumed that Kv_K is neither real closed nor henselian, we get $v_K = v_K^p$. As in previous proofs (see for example the proof of 3.14), we may assume $\zeta_p \in K$ if $\mathrm{char}(K) \neq p$, so v_K is $\emptyset$-definable.

Now consider the case that there are (at least) two primes $p < q$ with $p, q \mid \#G_{Kv_K}$. Thus, also $p, q \mid \#G_K$ holds. If $Kv_K(p) \neq Kv_K \neq Kv_K(q)$, then—using Proposition 2.6 once more—we have $K(p) \neq K \neq K(q)$. By the proof of Proposition 3.14, one of v_K^p and v_K^q is henselian. Say v_K^p is henselian, then we get $v_K \subset v_K^p$ by Observation 2.9. But v_K is also a coarsening of v_K^p, as $Kv_K \neq Kv_K(p)$. Thus, we conclude $v_K = v_K^p$, and hence v_K is again $\emptyset$-definable.

Finally, if there are two primes $p, q \mid G_{Kv_K}$, but $Kv_K = Kv_K(p)$ or $Kv_K = Kv_K(q)$, we want to consider finite Galois extensions L of Kv_K with $L(p) \neq L \neq L(q)$. Let M be a finite Galois extension of K, and let w be the unique prolongation of v_K to M. Note that G_M is again nonuniversal and, as Mw is still neither separably nor real closed, $w = v_M$ holds. If $Mw(p) \neq Mw \neq Mw(q)$, then w is $\emptyset$-definable on M by v_M^p or v_M^q as above. Say $w = v_M^p$. As w is in particular q-henselian and $Mw \neq Mw(q)$, we get $w \supset v_M^q$. Thus, in any case the finest common coarsening of v_M^p and v_M^q is equal to the coarser one of the two and furthermore $\emptyset$-definable and henselian.

Now we fix an integer n such that there is a Galois extension M of K (containing ζ_p and ζ_q if necessary) such that $Mw(p) \neq Mw \neq Mw(q)$. Just like in the proof of 3.6, we get a parameter-free definition of v by

$$\bigcap \big((\mathcal{O}_{v_M^p} \cdot \mathcal{O}_{v_M^q}) \cap K \mid K \subseteq M \text{ Galois, } [M:K]=n,\ M(p) \neq M \neq M(q), \\ \zeta_p \in M \text{ if } \mathrm{char}(M) \neq p,\ \zeta_q \in M \text{ if } \mathrm{char}(M) \neq q,\ v_M^p \in H_1^p(M),\ v_M^q \in H_1^q(M) \big). \quad \dashv$$

Remark. In fact, it suffices to assume for the proof of the above theorem that K is t-henselian rather than henselian. This is a generalization of henselianity introduced in [18]. Like henselianity, t-henselianity goes up to finite extensions and implies p-henselianity for any prime p. These are the only properties of henselianity needed in the proof. In particular, we get that any field with a nonuniversal absolute Galois group which is elementarily equivalent to a henselian field is in fact henselian itself (since a $\emptyset$-definable henselian valuation gives rise to a nontrivial henselian valuation on any field with the same elementary theory). Thus, for any field with a nonuniversal absolute Galois group, henselianity is an elementary property in $\mathcal{L}_{ring}$.

Our Galois-theoretic condition is moreover also a condition on the residue field.

Observation 3.16. *Let (K, v) be a henselian valued field. Then G_K is nonuniversal iff G_{Kv} is nonuniversal.*

Proof. Recall the exact sequence

$$I_v \longrightarrow G_K \longrightarrow G_{Kv}.$$

If G_K is nonuniversal, then some finite group does not appear as a Galois group over any finite extension of K, and hence the same holds for Kv.
On the other hand, if G_{Kv} is nonuniversal, there is some $n_0 \in \mathbb{N}$ such that neither S_n nor A_n (for $n \geq n_0$) occur as a subquotients of G_{Kv}. As I_v is soluble, S_n (for $n \geq \max\{5, n_0\}$) is not a subquotient of G_K, either. $\dashv$

In particular, we can use the observation to define a range of power series valuations.

Corollary 3.17. *Let K be a field with G_K nonuniversal. Let Γ be a nontrivial ordered abelian group, and assume that Γ is nondivisible in case that K is separably or real closed. Then there is a $\emptyset$-definable nontrivial henselian valuation on $K((\Gamma))$. If K is not henselian and neither separably nor real closed, then the power series valuation is definable.*

Proof. The first statement is immediate from the previous Observation and Theorem 3.15. The second also follows from Theorem 3.15: If K is not henselian, then the power series valuation is exactly the canonical henselian valuation. $\dashv$

One example of fields with nonuniversal absolute Galois group are NIP fields of positive characteristic. We call a field NIP if $\mathrm{Th}(K)$ is NIP in the sense of Shelah (see [1] for some background on NIP theories). In [11] (Corollary 4.5), the authors

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show that if K is an infinite NIP field of characteristic $p > 0$, then $p \nmid \#G_K$. Thus, we get the following

COROLLARY 3.18. *Let* (K, v) *be a nontrivially henselian valued field,* K *not separably closed. If*

- K *is NIP and* $\text{char}(K) > 0$, *or*
- Kv *is NIP and* $\text{char}(Kv) > 0$,

then K *admits a nontrivial* $\emptyset$*-definable henselian valuation.*

PROOF. The first statement follows from Theorem 3.15. The second statement is now a consequence of Observation 3.16. ⊣

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