Canad. Math. Bull. Vol. 64 (2), 2021, pp. 474–490
http://dx.doi.org/10.4153/S0008439520000570
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Complete boundedness of multiple operator integrals

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Abstract. In this paper, we characterize the multiple operator integrals mappings that are bounded on the Haagerup tensor product of spaces of compact operators. We show that such maps are automatically completely bounded and prove that this is equivalent to a certain factorization property of the symbol associated with the operator integral mapping. This generalizes a result by Juschenko-Todorov-Turowska on the boundedness of measurable multilinear Schur multipliers.

1 Introduction

A family $m = (m_{ij})_{i,j\in\mathbb{N}}$ of complex numbers is called a Schur multiplier if for any matrix $[a_{ij}] \in \mathcal{B}(\ell_2)$, the Schur product $T_m(a) = [m_{ij}a_{ij}]$ is the matrix of an element of $\mathcal{B}(\ell_2)$. Schur multipliers are an important tool in analysis, and play for instance a fundamental role in Perturbation Theory. See below for more information and references.

There is a well-known characterization of Schur multipliers due to Grothendieck in terms of factorization of the symbol m, see [18, Theorem 5.1]. It turns out, using the theory of operator spaces, that bounded Schur multipliers are completely bounded and in that case, the norm of T_m is equal to its complete norm. It is not yet known whether this is true for Schur multipliers defined on the Schatten classes. We refer the reader to [12] for recent developments regarding this question.

In this paper, we are interested in Schur multipliers in the multilinear setting. Effros and Ruan [11] introduced a Schur product as a multilinear map $T: M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined on the product of n copies of $M_n(\mathbb{C})$ and characterized the mappings T that extend to a complete contraction on the Haagerup tensor product $M_n(\mathbb{C}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} M_n(\mathbb{C})$. This result was generalized by Juschenko, Todorov and Turowska in [13] where they considered measurable multilinear Schur multipliers. They are defined as follows: let $n \in \mathbb{N}$ and let $(\Omega_1, \mu_1), \dots, (\Omega_n, \mu_n)$ be σ-finite measure spaces. Let $\phi \in L^\infty(\Omega_1 \times \cdots \times \Omega_n)$. We will identify $L^2(\Omega_i \times \Omega_j)$ with the space $\mathcal{S}^2(L^2(\Omega_i), L^2(\Omega_j))$ of Hilbert-Schmidt operators from $L^2(\Omega_i)$ into $L^2(\Omega_j)$. If $K_i \in L^2(\Omega_i \times \Omega_{i+1})$, $1 \leq i \leq n-1$, we let $\Lambda(\phi)(K_1, \dots, K_{n-1})$ to be the

Received by the editors August 21, 2019; revised May 6, 2020.
Published online on Cambridge Core July 27, 2020.
The author is supported by NSFC(11801573).
AMS subject classification: **46L07, 47B49**.
Keywords: Complete boundedness, multiplier, operator integrals.

https://doi.org/10.4153/S0008439520000570 Published online by Cambridge University Press

Hilbert-Schmidt operator with kernel

$$\int \phi(t_1, \dots, t_n) K_1(t_1, t_2) \dots K_{n-1}(t_{n-1}, t_n) \, d\mu_2(t_2) \dots d\mu_{n-1}(t_{n-1}) \in L^2(\Omega_1 \times \Omega_n),$$

which defines a multilinear mapping

$$\Lambda(\phi) : \mathcal{S}^2(L^2(\Omega_{n-1}), L^2(\Omega_n)) \times \cdots \times \mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2)) \to S^2(L^2(\Omega_1), L^2(\Omega_n)).$$

Using the notion of multilinear module mappings, the authors proved that if $\Lambda(\phi)$ extends to a bounded map on the Haagerup tensor product $\mathcal{S}^\infty(L^2(\Omega_{n-1}), L^2(\Omega_n)) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(L^2(\Omega_1), L^2(\Omega_2))$ into $\mathcal{S}^\infty(L^2(\Omega_1), L^2(\Omega_n))$, the extension is completely bounded [13, Lemma 3.3]. Using this fact, they characterized the functions ϕ that give rise to a (completely) bounded $\Lambda(\phi)$ in terms of the extended Haagerup tensor product $L^\infty(\Omega_1) \otimes_{eh} \cdots \otimes_{eh} L^\infty(\Omega_n)$, see [13, Theorem 3.4] and the remark following the theorem. We also refer the reader to [21] for more results on the case $n = 2$.

Let $A_1, \dots, A_n$ be normal operators and let $\lambda_{A_1}, \dots, \lambda_{A_n}$ be scalar-valued spectral measures associated with these operators, that is, λ_{A_i} is a finite measure on the Borel subsets of $\sigma(A_i)$ such that λ_{A_i} and E^{A_i}, the spectral measure of A_i, have the same sets of measure 0. For $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$ and $X_1, \dots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$, we formally define a multiple operator integral by

$$\left[\Gamma^{A_1,\dots,A_n}(\phi)\right](X_1, \dots, X_{n-1}) = \int_{\sigma(A_1)\times\cdots\times\sigma(A_n)} \phi(s_1, \dots, s_n) \, dE^{A_1}(s_1) \, X_1 \, dE^{A_2}(s_2) \dots X_{n-1} \, dE^{A_n}(s_n).$$

The theory of double operator integral (case $n = 2$) was developed by Birman and Solomyak in a series of three papers [1, 2, 3] and was then generalized to the case of multiple operator integrals [15, 22]. They play a prominent role in operator theory, especially in perturbation theory where they are a fundamental tool in the study of differentiability of operator functions. See [6, 7, 14, 16] where Fréchet and Gâteaux-differentiability of the mapping $f \mapsto f(A)$ are studied in the Schatten norms.

The definition of multiple operator integrals we will use in this paper is the one given in [5] and which is based on the construction of Pavlov [15]. See [16, 19] for other constructions of multiple operator integrals. The advantage of this definition is the property of w^*-continuity of the mapping $\phi \mapsto \Gamma^{A_1,\dots,A_n}(\phi)$ which allows to prove certain identities by simply checking them for functions with separated variables, see [5, 6] and the proof of Theorem 4.1.

In this paper, we prove that a characterization similar to the one established for measurable Schur multipliers in [13] holds in the setting of multiple operator integrals. Namely, we prove that if a multiple operator integral $\Gamma^{A_1,\dots,A_n}$ extends to a bounded mapping on the Haagerup tensor product $\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H})$ then the extension is completely bounded and that we have such extension if and only if ϕ has the following factorization: there exist separable Hilbert spaces $H_1, \dots, H_{n-1}$, $a_1 \in L^\infty(\lambda_{A_1}; H_1)$, $a_n \in L^\infty(\lambda_{A_n}; H_{n-1})$ and $a_i \in L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1}))$, $2 \le i \le n-1$ such that

$$\phi(t_1, \dots, t_n) = \langle a_1(t_1), [a_2(t_2) \dots a_{n-1}(t_{n-1})](a_n(t_n)) \rangle .$$

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Our proof rests on several properties of the Haagerup tensor product (Section 2.1) and the connection between multiple operator integrals and measurable multilinear Schur multipliers that we will present in Section 3.

2 Preliminaries

2.1 Operator spaces and the Haagerup tensor product

We refer to [17] and [20] for the theory of operator spaces. If $E \subset \mathcal{B}(H)$ and $F \subset \mathcal{B}(K)$ are two operator spaces, we denote by $CB(E,F)$ the Banach space of completely bounded maps from E into F equipped with the c.b. norm. If $\mathcal{H}$ is a Hilbert space, we will denote by $\mathcal{H}_c = \mathcal{B}(\mathbb{C},\mathcal{H})$ its column structure.

In [17, Chapter 5], Pisier defines the Haagerup tensor product $E_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} E_N$ of N operator spaces $E_1,\dots,E_N$. We will recall a few properties of the Haagerup tensor product that we will use in Section 4. The first one is the factorization of multilinear maps. If E is an operator space, then by [20, Proposition 9.2.2], a multilinear mapping $v : E_1 \times \cdots \times E_n \to E$ is completely bounded (in the sense of [20, Section 9.1], see also [4]) if and only if v extends to a completely bounded map $v : E_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} E_N \to E$. The following important theorem describes those maps.

Theorem 2.1 *Let $E_1,\dots,E_n$ be operator spaces and let H_0 and H_n be Hilbert spaces. A linear mapping $u\colon E_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} E_n \to \mathcal{B}(H_n,H_0)$ is completely bounded if and only if there exist Hilbert spaces $H_1,\dots,H_{n-1}$ and completely bounded mappings $\phi_i\colon E_i \to \mathcal{B}(H_i,H_{i-1}), 1 \le i \le n$, such that*

$$u(x_1 \otimes \cdots \otimes x_n) = \phi_1(x_1)\dots\phi_n(x_n).$$

In this case we can choose $\phi_i, 1 \le i \le n$, such that

$$\|u\|_{cb} = \|\phi_1\|_{cb}\cdots\|\phi_n\|_{cb}.$$

Remark 2.2 When $H_0 = H_n = \mathbb{C}$ we can reformulate as follows: a linear functional $u\colon E_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} E_n \to \mathbb{C}$ is bounded (and therefore completely bounded) if and only if there exist Hilbert spaces $H_1,\dots,H_{n-1}$, $\alpha_1\colon E_1 \to (H_c)^*$ linear, $\alpha_i\colon E_i \to \mathcal{B}(H_i,H_{i-1}), 2 \le i \le n-1$ and $\alpha_n\colon E_n \to (H_{n-1})_c$ antilinear such that the α_j are completely bounded and

$$u(x_1,\dots,x_n) = \langle \alpha_1(x_1), [\alpha_2(x_2)\dots\alpha_{n-1}(x_{n-1})]\alpha_n(x_n)\rangle.$$

Recall that a map $s\colon X \to Y$ between two Banach spaces is called a quotient map if the injective map $\hat{s}\colon X/\ker(s) \to Y$ induced by s is a surjective isometry. If $E_1 \subset E_2$ are operator spaces, we equip E_2/E_1 with the quotient operator space structure (see e.g. [17, Section 2.4]). When E and F are operator spaces, a quotient map $u\colon E \to F$ is said to be a *complete metric surjection* if the associated mapping $\hat{u}\colon E/\ker(u) \to F$ is a completely isometric isomorphism.

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Proposition 2.3 *Let E_1, E_2, F_1, F_2 be operator spaces.*

(i) *If $q_i\colon E_i \to F_i$ is completely bounded, then $q_1 \otimes q_2\colon E_1 \otimes E_2 \to F_1 \overset{h}{\otimes} F_2$ defined by $(q_1 \otimes q_2)(e_1 \otimes e_2) = q_1(e_1) \otimes q_2(e_2)$ extends to a completely bounded map*

$$q_1 \otimes q_2\colon E_1 \overset{h}{\otimes} E_2 \to F_1 \overset{h}{\otimes} F_2.$$

(ii) *If $E_i \subset F_i$ completely isometrically, then $E_1 \overset{h}{\otimes} E_2 \subset F_1 \overset{h}{\otimes} F_2$ completely isometrically.*

(iii) *If $q_i\colon E_i \to F_i$ is a complete metric surjection, then $q_1 \otimes q_2\colon E_1 \overset{h}{\otimes} E_2 \to F_1 \overset{h}{\otimes} F_2$.*

(iv) *If $E_i \subset F_i$ are subspaces, let $p_i\colon F_i \to F_i/E_i$ be the canonical mappings. Then the induced map $p_1 \otimes p_2\colon F_1 \overset{h}{\otimes} F_2 \to F_1/E_1 \overset{h}{\otimes} F_2/E_2$ satisfies*

$$\ker(p_1 \otimes p_2) = \overline{E_1 \otimes F_2 + F_1 \otimes E_2}.$$

The second property is called the injectivity and the third one the projectivity of the Haagerup tensor product.

Proof We refer to [20, Proposition 9.2.5] for the proof of (i) and to [17, Corollary 5.7] for the proof of (ii) and (iii).

Let us prove (iv). Write $N = \overline{E_1 \otimes F_2 + F_1 \otimes E_2}$. Note that the inclusion $N \subset \ker(p_1 \otimes p_2)$. is clear. Therefore, to show the result, it is enough to show that

$$N^\perp \subset \ker(p_1 \otimes p_2)^\perp.$$

Let $\sigma\colon F_1 \overset{h}{\otimes} F_2 \to \mathbb{C}$ be such that $\sigma_{|N} = 0$. By Remark 2.2, there exist a Hilbert space H, $\alpha\colon F_1 \to (H_c)^*$ linear and $\beta\colon F_2 \to H_c$ antilinear, α and β completely bounded such that

$$\sigma(x, y) = \langle \alpha(x), \beta(y) \rangle, x \in F_1, y \in F_2.$$

Let $K = \overline{\alpha(F_1)}$ and denote by P_K the orthogonal projection onto K. Then we have, for any x and y,

$$\sigma(x, y) = \langle P_K\alpha(x), \beta(y) \rangle = \langle P_K\alpha(x), P_K\beta(y) \rangle.$$

Thus, by changing α into $P_K\alpha$ and β into $P_K\beta$, we can assume that α has a dense range. Similarly, setting $L = \overline{\beta(F_2)}$ and considering P_L, we may assume that β has a dense range.

By assumption, for any $e \in E_2$ and any $x \in E_1$, we have

$$0 = \sigma(x, e) = \langle \alpha(x), \beta(e) \rangle.$$

This implies that $\beta_{|E_2} = 0$. Similarly, we show that $\alpha_{|E_1} = 0$. Thus, we can consider

$$\widehat{\alpha}\colon F_1/E_1 \to H \quad \text{and} \quad \widehat{\beta}\colon F_2/E_2 \to H$$

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such that $\alpha = \widehat{\alpha} \circ p_1$ and $\beta = \widehat{\beta} \circ p_2$ and where F_1/E_1 and F_2/E_2 are equipped with their quotient structure. Now, define $\widehat{\sigma}: F_1/E_1 \overset{h}{\otimes} F_2/E_2 \to \mathbb{C}$ by

$$\widehat{\sigma}(s,t) = \langle \widehat{\alpha}(s), \widehat{\beta}(t) \rangle .$$

Then $\sigma = \widehat{\sigma} \circ (p_1 \otimes p_2)$, so that $\sigma \in \ker(p_1 \otimes p_2)^{\perp}$. ■

Finally, we recall the following [20, Proposition 9.3.3] which will be important in the last section.

Proposition 2.4 *Let E be an operator space and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For any $T \in CB(E, \mathcal{B}(\mathcal{H}, \mathcal{K}))$ we define a mapping $\sigma_T : \mathcal{K}^* \otimes E \otimes \mathcal{H} \to \mathbb{C}$ by setting*

$$\sigma_T(k^* \otimes e \otimes h) = \langle T(e)h, k \rangle .$$

Then, the mapping $T \mapsto \sigma_T$ induces a complete isometry

$$CB(E, \mathcal{B}(\mathcal{H}, \mathcal{K})) = \left((\mathcal{K}_c)^* \overset{h}{\otimes} E \overset{h}{\otimes} \mathcal{H}_c \right)^* .$$

2.2 Schatten classes

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For any $1 \leq p < +\infty$, let $\mathcal{S}^p(\mathcal{H}, \mathcal{K})$ be the space of compact operators $T : \mathcal{H} \to \mathcal{K}$ such that

$$\|T\|_p := \operatorname{tr}(|T|^p)^{\frac{1}{p}} < \infty.$$

Then $\| \cdot \|_p$ is a norm on $\mathcal{S}^p(\mathcal{H}, \mathcal{K})$ and $(\mathcal{S}^p(\mathcal{H}, \mathcal{K}), \| \cdot \|_p)$ is called the Schatten class of order p. When $p = \infty$, the space $\mathcal{S}^\infty(\mathcal{H}, \mathcal{K})$ will denote the space of compact operators equipped with the operator norm.

Recall that $\left(\mathcal{S}^1(\mathcal{H}, \mathcal{K})\right)^* = \mathcal{B}(\mathcal{K}, \mathcal{H})$ and that for $1 < p \leq +\infty$, $(\mathcal{S}^p(\mathcal{H}, \mathcal{K}))^* = \mathcal{S}^{p'}(\mathcal{K}, \mathcal{H})$ where p' is the conjugate exponent of p, for the duality pairing

$$\langle S, T \rangle = \operatorname{tr}(ST), \quad S \in \mathcal{S}^p(\mathcal{H}, \mathcal{K}), T \in \mathcal{S}^{p'}(\mathcal{K}, \mathcal{H}).$$

Using the Haagerup tensor product introduced in Subsection 2.1, we have, by [20, Proposition 9.3.4], a complete isometry

$$(\mathcal{H}_c)^* \overset{h}{\otimes} \mathcal{K}_c = \mathcal{S}^1(\mathcal{H}, \mathcal{K}). \tag{2.1}$$

where $\mathcal{S}^1(\mathcal{H}, \mathcal{K})$ is equipped with its operator space structure as the predual of $\mathcal{B}(\mathcal{K}, \mathcal{H})$.
Similarly, we have a complete isometry

$$\mathcal{K}_c \overset{h}{\otimes} (\mathcal{H}_c)^* = \mathcal{S}^\infty(\mathcal{H}, \mathcal{K}). \tag{2.2}$$

Finally, if (Ω_1, μ_1) and (Ω_2, μ_2) are two σ-finite measure spaces, we will identify $L^2(\Omega_1 \times \Omega_2)$ with the space $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ of Hilbert-Schmidt operators as

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follows. If $K \in L^2(\Omega_1 \times \Omega_2)$, the operator

$$(2.3)\qquad \begin{array}{rcl} X_K\colon L^2(\Omega_1) & \longrightarrow & L^2(\Omega_2) \\ f & \longmapsto & \displaystyle\int_{\Omega_1} K(t,\cdot) f(t) \mathrm{d}\mu_1(t) \end{array}$$

is a Hilbert-Schmidt operator and $\|X_K\|_2 = \|K\|_{L^2}$. Moreover, any element of $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ has this form. Hence, the space $L^2(\Omega_1 \times \Omega_2)$ is isometrically isomorphic to $\mathcal{S}^2(L^2(\Omega_1), L^2(\Omega_2))$ through the mapping $K \mapsto X_K$.

2.3 L^p_σ-spaces and Duality

Let (Ω, μ) be a σ-finite measure space and let F be a Banach space. For any $1 \le p \le +\infty$, we let $L^p(\Omega; F)$ denote the classical Bochner space of measurable functions $f\colon \Omega \to F$.

Assume that E is a separable Banach space. A function $f\colon \Omega \to E^*$ is said to be w^*-measurable if for all $e \in E$, the function $t \in \Omega \mapsto \langle \phi(t), e \rangle$ is measurable. We denote by $L^p_\sigma(\Omega; E^*)$ the space of all w^*-measurable $f\colon \Omega \to E^*$ such that $\|f(\cdot)\| \in L^p(\Omega)$, after taking quotient by the functions which are equal to 0 almost everywhere. Equipped with the norm

$$\|f\|_p = \| \|f(.)\| \|_{L^p(\Omega)},$$

$(L^p_\sigma(\Omega; E^*), \|.\|_p)$ is a Banach space.

Let $1 \le p' < +\infty$ be the conjugate exponent of p. Then we have an isometric isomorphism

$$L^p(\Omega; E)^* = L^{p'}_\sigma(\Omega; E^*)$$

through the duality pairing

$$\langle f, g \rangle := \int_\Omega \langle f(t), g(t) \rangle \, \mathrm{d}\mu(t) \, .$$

See [5, Section 4] and the references therein for a proof of that result and more information about L^p_σ-spaces.

Note that by [9, Chapter IV], the equality $L^p_\sigma(\Omega; E^*) = L^p(\Omega; E^*)$ is equivalent to E^* having the Radon-Nikodym property. It is for instance the case for Hilbert spaces.

The important identification we will need in this paper is the following. For any $f \in L^\infty_\sigma(\Omega; E^*)$, define

$$(2.4)\qquad u_f\colon \psi \in L^1(\Omega) \mapsto \left[e \in E \mapsto \int_\Omega \langle f(t), e \rangle \, \psi(t) \, \mathrm{d}t \right] \in E^*.$$

Then $f \mapsto u_f$ yields an isometric identification (see [10, Theorem 2.1.6])

$$(2.5)\qquad L^\infty_\sigma(\Omega; E^*) = \mathcal{B}(L^1(\Omega), E^*).$$

In particular, for a Hilbert space $\mathcal{H}$ we have the equality

$$(2.6)\qquad L^\infty(\Omega; \mathcal{H}) = \mathcal{B}(L^1(\Omega), \mathcal{H}).$$

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3 Multiple Operator Integrals

3.1 Multiple Operator Integrals Associated with Operators

Let $\mathcal{H}$ be a separable Hilbert space and let A be a (possibly unbounded) normal operator on $\mathcal{H}$. We denote by $\sigma(A)$ the spectrum of A and by E^A its spectral measure. A scalar-valued spectral measure for A is a positive measure λ_A on the Borel subsets of $\sigma(A)$ such that λ_A and E^A have the same sets of measure zero. Let e be a separating vector of the von Neumann algebra $W^*(A)$ generated by A (see [8, Corollary 14.6]).

Then, by [8, Proposition 15.3], the measure λ_A defined by

$$\lambda_A = \|E^A(.)e\|^2$$

is a scalar-valued spectral measure for A. We refer the reader to [8, Section 15] and [5, Section 2.1] for more details.

For any bounded Borel function $f\colon \sigma(A) \to \mathbb{C}$, we define $f(A) \in \mathcal{B}(\mathcal{H})$ by

$$f(A) := \int_{\sigma(A)} f(t)\, \mathrm{d}E^A(t),$$

and this operator only depends on the class of f in $L^\infty(\lambda_A)$. According to [8, Theorem 15.10], we obtain a w^*-continuous $*$-representation

$$f \in L^\infty(\lambda_A) \mapsto f(A) \in \mathcal{B}(\mathcal{H}).$$

Moreover, the space $L^\infty(\lambda_A)$ does not depend on the choice of the scalar-valued spectral measure.

Let $n \in \mathbb{N}, n \geq 1$ and let $E_1, \dots, E_n, E$ be Banach spaces. We denote by $\mathcal{B}_n(E_1 \times \cdots \times E_n, E)$ the space of n-linear continuous mappings from $E_1 \times \cdots \times E_n$ into E equipped with the norm

$$\|T\|_{\mathcal{B}_n(E_1\times\cdots\times E_n, E)} := \sup_{\|e_i\|\leq 1, 1\leq i\leq n} \|T(e_1, \dots, e_n)\|.$$

When $E_1 = \cdots = E_n = E$, we will simply write $\mathcal{B}_n(E)$.

Let $n \in \mathbb{N}, n \geq 2$ and let $A_1, A_2, \dots, A_n$ be normal operators in $\mathcal{H}$ with scalar-valued spectral measures $\lambda_{A_1}, \dots, \lambda_{A_n}$. We let

$$\Gamma^{A_1,A_2,\dots,A_n}\colon L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_n}) \to \mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H}))$$

be the unique linear map such that for any $f_i \in L^\infty(\lambda_{A_i}), i = 1, \dots, n$ and for any $X_1, \dots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$,

$$\begin{aligned}\left[\Gamma^{A_1,A_2,\dots,A_n}(f_1 \otimes \cdots \otimes f_n)\right](X_1, \dots, X_{n-1}) \\ = f_1(A_1)X_1 f_2(A_2) \cdots f_{n-1}(A_{n-1})X_{n-1} f_n(A_n).\end{aligned}$$

We have a natural inclusion $L^\infty(\lambda_{A_1}) \otimes \cdots \otimes L^\infty(\lambda_{A_n}) \subset L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right)$ which is w^*-dense. The following shows that $\Gamma^{A_1,A_2,\dots,A_n}$ extends to $L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right)$. It was proved in [5, Theorem 4 and Proposition 5].

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Theorem 3.1 *$\Gamma^{A_1,A_2,\dots,A_n}$ extends to a unique w^*-continuous isometry still denoted by*

$$\Gamma^{A_1,A_2,\dots,A_n}: L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right) \longrightarrow \mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H})).$$

Definition 3.1 For $\phi \in L^\infty(\prod_{i=1}^n \lambda_{A_i})$, the transformation $\Gamma^{A_1,A_2,\dots,A_n}(\phi)$ is called a multiple operator integral associated with $A_1, A_2, \dots, A_n$ and ϕ.

The w^*-continuity of $\Gamma^{A_1,A_2,\dots,A_n}$ means that if a net $(\phi_i)_{i\in I}$ in $L^\infty(\prod_{i=1}^n \lambda_{A_i})$ converges to $\phi \in L^\infty(\prod_{i=1}^n \lambda_{A_i})$ in the w^*-topology, then for any $X_1, \dots, X_{n-1} \in \mathcal{S}^2(\mathcal{H})$, the net

$$\left(\left[\Gamma^{A_1,A_2,\dots,A_n}(\phi_i)\right](X_1,\dots,X_{n-1})\right)_{i\in I}$$

converges to $\left[\Gamma^{A_1,A_2,\dots,A_n}(\phi)\right](X_1,\dots,X_{n-1})$ weakly in $\mathcal{S}^2(\mathcal{H})$. We refer the reader to [5, Section 3.1] for more details.

3.2 Measurable Multilinear Schur Multipliers

Let $n \in \mathbb{N}$. Let $(\Omega_1, \mu_1), \dots, (\Omega_n, \mu_n)$ be σ-finite measure spaces, and let $\phi \in L^\infty(\Omega_1 \times \cdots \times \Omega_n)$. Let $\Omega = \Omega_2 \times \cdots \times \Omega_{n-1}$. For any $K_i \in L^2(\Omega_i \times \Omega_{i+1})$, $1 \le i \le n-1$, we let $\Lambda(\phi)(K_1,\dots,K_{n-1})$ be the function

$$(t_1,t_n) \mapsto \int_\Omega \phi(t_1,\dots,t_n) K_1(t_1,t_2)\dots K_{n-1}(t_{n-1},t_n)\, \mathrm{d}\mu_2(t_2)\dots \mathrm{d}\mu_{n-1}(t_{n-1})$$

By Cauchy-Schwarz inequality, $\Lambda(\phi)(K_1,\dots,K_{n-1}) \in L^2(\Omega_1 \times \Omega_n)$ and

$$\|\Lambda(\phi)(K_1,\dots,K_{n-1})\|_2 \le \|\phi\|_\infty \|K_1\|_2 \dots \|K_{n_1}\|_2. \tag{3.1}$$

Thus, $\Lambda(\phi)$ defines a bounded $(n-1)$-linear map

$$\Lambda(\phi): L^2(\Omega_1 \times \Omega_2) \times L^2(\Omega_2 \times \Omega_3) \times \cdots \times L^2(\Omega_{n-1} \times \Omega_n) \longrightarrow L^2(\Omega_1 \times \Omega_n),$$

or, equivalently, by (2.3) and the obvious equality $L^2(\Omega_i \times \Omega_j) = L^2(\Omega_j \times \Omega_i), 1 \le i, j \le n$, a bounded $(n-1)$-linear map

$$\Lambda(\phi): \mathcal{S}^2(L^2(\Omega_2), L^2(\Omega_1)) \times \cdots \times \mathcal{S}^2(L^2(\Omega_n), L^2(\Omega_{n-1})) \to \mathcal{S}^2(L^2(\Omega_n), L^2(\Omega_1)).$$

For simplicity, write $E_i = L^2(\Omega_i), 1 \le i \le n$. Then, the map $\Lambda: \phi \mapsto \Lambda(\phi)$ is a linear isometry

$$\Lambda: L^\infty(\Omega_1 \times \cdots \times \Omega_n) \longrightarrow B_{n-1}(\mathcal{S}^2(E_2,E_1) \times \cdots \times \mathcal{S}^2(E_n,E_{n-1}), \mathcal{S}^2(E_n,E_1)).$$

This follow, for example, from similar computations as those in the proof of [5, Proposition 8] or from [13, Theorem 3.1].

Let $\mathcal{H}$ be a separable Hilbert space and let $A_1, \dots, A_n$ be normal operators on $\mathcal{H}$. For any $1 \le i \le n$, let $e_i \in \mathcal{H}$ be such that

$$\lambda_{A_i}(\cdot) = \|E^{A^i}(\cdot)e_i\|^2.$$

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By [5, Subsection 4.2], the linear mappings $\rho_i: L^2(\sigma(A_i), \lambda_{A_i}) \to \mathcal{H}$ defined for any measurable subset $F \subset \sigma(A_i)$ by

$$\rho_i(\chi_F) = E^{A_i}(F)e_i$$

extends uniquely to an isometry $\rho_i: L^2(\sigma(A_i), \lambda_{A_i}) \to \mathcal{H}$. Hence, denoting by $\mathcal{H}_i$ the range of ρ_i, we get that $\rho_i: L^2(\sigma(A_i), \lambda_{A_i}) \equiv \mathcal{H}_i$ is a unitary.

In the next result, we will consider the map Λ introduced before and associated with the measure spaces $(\Omega_i, \mu_i) = (\sigma(A_i), \lambda_{A_i})$. We see any operator $T \in \mathcal{S}^2(\mathcal{H}_i, \mathcal{H}_j)$ as an element of $\mathcal{S}^2(\mathcal{H})$ by identifying T with the matrix $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^2\left(\mathcal{H}_i \overset{2}{\oplus} \mathcal{H}_i^{\perp}, \mathcal{H}_j \overset{2}{\oplus} \mathcal{H}_j^{\perp}\right)$. The following makes the connection between the multiple operator integrals associated with operators and the map Λ defined above. In particular, when one restricts the Hilbert space $\mathcal{H}$ to the subspaces $\mathcal{H}_i$, then the associated multiple operator integral coincides with Λ. It is the analogue of [5, Proposition 9] for n operators. The proof is similar and we leave it to the reader.

Proposition 3.2 *For any* $1 \le i \le n-1$, $K_i \in \mathcal{S}^2(L^2(\lambda_{A_{i+1}}), L^2(\lambda_{A_i}))$ *and set*

$$\widetilde{K}_i = \rho_i \circ K_i \circ \rho_{i+1}^{-1} \in \mathcal{S}^2(\mathcal{H}_{i+1}, \mathcal{H}_i).$$

For any $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$, $\Gamma^{A_1,\dots,A_n}(\phi)(\widetilde{K}_1, \dots, \widetilde{K}_{n-1})$ *belongs to* $\mathcal{S}^2(\mathcal{H}_n, \mathcal{H}_1)$ *and*

$$\Lambda(\phi)(K_1, \dots, K_{n-1}) = \rho_1^{-1} \circ \Gamma^{A_1,\dots,A_n}(\phi)(\widetilde{K}_1, \dots, \widetilde{K}_{n-1}) \circ \rho_n. \tag{3.2}$$

4 Characterization of the Complete Boundedness of Multiple Operator Integrals

Let $A_1, \dots, A_n$ be n normal operators on a separable Hilbert space $\mathcal{H}$ associated with scalar-valued spectral measures $\lambda_{A_1}, \dots, \lambda_{A_n}$. For $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$, $\Gamma^{A_1,\dots,A_n}(\phi)$ belongs to $\mathcal{B}_{n-1}(\mathcal{S}^2(\mathcal{H}))$, which is equivalent, by [5, Section 3.1], to having a continuous mapping defined on the projective tensor product of $n-1$ copies $\mathcal{S}^2(\mathcal{H})$ and still denoted by

$$\Gamma^{A_1,\dots,A_n}(\phi): \mathcal{S}^2(\mathcal{H}) \overset{\wedge}{\otimes} \cdots \overset{\wedge}{\otimes} \mathcal{S}^2(\mathcal{H}) \to \mathcal{S}^2(\mathcal{H}).$$

We will make this identification for the rest of the paper.

The purpose of this section is to characterize the functions $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$ such that $\Gamma^{A_1,\dots,A_n}(\phi)$ extends to a (completely) bounded map

$$\Gamma^{A_1,\dots,A_n}(\phi): \underbrace{\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H})}_{n-1 \text{ times}} \longrightarrow \mathcal{S}^\infty(\mathcal{H}).$$

We will also consider the measurable multilinear Schur multipliers $\Lambda(\phi)$. In [13], the authors studied and characterized the boundedness of measurable multilinear

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Schur multipliers

$$\mathcal{S}^\infty(L^2(\lambda_{A_{n-1}}), L^2(\lambda_{A_n})) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(L^2(\lambda_{A_1}), L^2(\lambda_{A_2})) \to \mathcal{S}^\infty(L^2(\lambda_{A_1}), L^2(\lambda_{A_n})).$$

They proved that we have such extension if and only if ϕ has a certain factorization that will be given in the theorem below. They also proved that the boundedness for the Haagerup norm in this setting implies the complete boundedness.

The proof of Theorem 4.1 below provides another proof of [13, Theorem 3.4]. We show that for multiple operator integrals, boundedness and complete boundedness are also equivalent and that the same characterization holds.

Theorem 4.1 *Let $n \in \mathbb{N}, n \geq 2$, let $A_1, \dots, A_n$ be normal operators on a separable Hilbert space $\mathcal{H}$ and let $\phi \in L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$. For any $1 \leq i \leq n$, let $E_i = L^2(\lambda_{A_i})$. The following are equivalent:*

(i) $\Gamma^{A_1,\dots,A_n}(\phi)$ *extends to a bounded mapping*

$$\Gamma^{A_1,\dots,A_n}(\phi)\colon \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H}).$$

(ii) $\Gamma^{A_1,\dots,A_n}(\phi)$ *extends to a completely bounded mapping*

$$\Gamma^{A_1,\dots,A_n}(\phi)\colon \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H}).$$

(iii) $\Lambda(\phi)$ *extends to a completely bounded mapping*

$$\Lambda(\phi)\colon \mathcal{S}^\infty(E_2, E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(E_n, E_{n-1}) \to \mathcal{S}^\infty(E_n, E_1).$$

(iv) *There exist separable Hilbert spaces $H_1, \dots, H_{n-1}$, $a_1 \in L^\infty(\lambda_{A_1}; H_1), a_n \in L^\infty(\lambda_{A_n}; H_{n-1})$ and $a_i \in L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1})), 2 \leq i \leq n-1$, such that*

$$\phi(t_1, \dots, t_n) = \langle a_1(t_1), [a_2(t_2) \dots a_{n-1}(t_{n-1})](a_n(t_n))\rangle \tag{4.1}$$

for a.-e. $(t_1, \dots, t_n) \in \sigma(A_1) \times \cdots \times \sigma(A_n)$.

In this case,

$$\begin{aligned} \left\|\Gamma^{A_1,\dots,A_n}(\phi)\right\| &= \left\|\Gamma^{A_1,\dots,A_n}(\phi)\right\|_{cb} \\ &= \|\Lambda(\phi)\|_{cb} = \inf\left\{\|a_1\|_\infty \cdots \|a_n\|_\infty \mid \phi \text{ as in } (4.1)\right\}. \end{aligned} \tag{4.2}$$

Remark 4.2 Using the normal Haagerup tensor product $\otimes_{\sigma h}$ of operator spaces, for which we refer the reader to [4], one can prove, by simply considering the bi-adjoint of $\Gamma^{A_1,\dots,A_n}(\phi)$, that the above four statements are also equivalent to:

(v) $\Gamma^{A_1,\dots,A_n}(\phi)$ *extends to a w^*-continuous and completely bounded mapping*

$$\Gamma^{A_1,\dots,A_n}(\phi)\colon \mathcal{B}^\infty(\mathcal{H}) \otimes_{\sigma h} \cdots \otimes_{\sigma h} \mathcal{B}^\infty(\mathcal{H}) \to \mathcal{B}^\infty(\mathcal{H}).$$

Proof Proof of (i) $\Leftrightarrow$ (ii)

Clearly (ii) $\Rightarrow$ (i) so we only prove (i) $\Rightarrow$ (ii). We keep the notation $\Gamma^{A_1,\dots,A_n}(\phi)$ for the associated multilinear map defined on $\mathcal{S}^\infty(\mathcal{H}) \times \cdots \times \mathcal{S}^\infty(\mathcal{H})$. Let $\mathcal{D} = W^*(A_1)'$ and $\mathcal{C} = W^*(A_n)'$ be the commutant of $W^*(A_1)$ and $W^*(A_n)$, respectively, where

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the von Neumann algebra $W^*(A)$ was defined in Section 3.1. Then $\Gamma^{A_1,\dots,A_n}(\phi)$ is a multilinear $(\mathcal{D},\mathcal{C})$-module map, that is, for any $d\in\mathcal{D}$, $c\in\mathcal{C}$, and any $X_1,\dots,X_{n-1}\in\mathcal{S}^\infty(\mathcal{H})$,

$$\left[\Gamma^{A_1,\dots,A_n}(\phi)\right](dX_1,\dots,X_{n-1}c)=d\left[\Gamma^{A_1,\dots,A_n}(\phi)\right](X_1,\dots,X_{n-1})c. \tag{4.3}$$

By density, it is sufficient to check this equality when $X_i\in\mathcal{S}^2(\mathcal{H})$. But in this case, by linearity and w^*-continuity of $\Gamma^{A_1,\dots,A_n}$, we can further assume that ϕ is an elementary tensor $\phi=f_1\otimes\cdots\otimes f_n$, where $f_i\in L^\infty(\lambda_{A_i})$. Then, since $f_1(A_1)\in W^*(A_1)$ and $f_n(A_n)\in W^*(A_n)$ we have

$$\begin{aligned}&\left[\Gamma^{A_1,\dots,A_n}(\phi)\right](dX_1,\dots,X_{n-1}c)\\&\quad=f_1(A_1)dX_1f_2(A_2)\dots f_{n-1}(A_{n-1})X_{n-1}cf_n(A_n)\\&\quad=df_1(A_1)X_1f_2(A_2)\dots f_{n-1}(A_{n-1})X_{n-1}f_n(A_n)c\\&\quad=d\left[\Gamma^{A_1,\dots,A_n}(\phi)\right](X_1,\dots,X_{n-1})c.\end{aligned}$$

Note that $W^*(A_1)$ has a separating vector and hence, by [8, Proposition 14.3], this vector is cyclic for $\mathcal{D}$. Similarly, $\mathcal{C}$ has a cyclic vector. It remains to apply [13, Lemma 3.3] to obtain the complete boundedness of $\Gamma^{A_1,\dots,A_n}(\phi)$ and the equality of the norms.

Proof of (ii) $\Rightarrow$ (iii)

We use the same notation as in Subsection 3.2 where we introduced the subspaces $\mathcal{H}_i$ of $\mathcal{H}$, $1\le i\le n$, with $\mathcal{H}_i\equiv L^2(\sigma(A_i),\lambda_{A_i})$. For any $1\le i\le n-1$, $\mathcal{S}^\infty(\mathcal{H}_{i+1},\mathcal{H}_i)$ is a closed subspace of $\mathcal{S}^\infty(\mathcal{H})$ and by injectivity of the Haagerup tensor product (see Proposition 2.3), we have a closed subspace

$$\mathcal{S}^\infty(\mathcal{H}_2,\mathcal{H}_1)\overset{h}{\otimes}\cdots\overset{h}{\otimes}\mathcal{S}^\infty(\mathcal{H}_n,\mathcal{H}_{n-1})\subset\mathcal{S}^\infty(\mathcal{H})\overset{h}{\otimes}\cdots\overset{h}{\otimes}\mathcal{S}^\infty(\mathcal{H}).$$

By Proposition 3.2, the restriction of $\Gamma^{A_1,\dots,A_n}(\phi)$ to $\mathcal{S}^\infty(\mathcal{H}_2,\mathcal{H}_1)\overset{h}{\otimes}\cdots\overset{h}{\otimes}\mathcal{S}^\infty(\mathcal{H}_n,\mathcal{H}_{n-1})$ is valued in $\mathcal{S}^\infty(\mathcal{H}_n,\mathcal{H}_1)$. Moreover, this restriction is completely bounded and by the same proposition, we obtain the inequality

$$\|\Lambda(\phi)\|_{\mathrm{cb}}\le\left\|\Gamma^{A_1,\dots,A_n}(\phi)\right\|_{\mathrm{cb}}.$$

Proof of (iii) $\Rightarrow$ (iv)

In this part, the L^1–spaces will be equipped with their maximal operator space structure (Max) for which we refer the reader to [17, Chapter 3]. If (Ω,μ) is a measure space, the mapping $(f,g)\in L^2(\Omega)^2\mapsto fg\in L^1(\Omega)$ induces a quotient map

$$f\otimes g\in L^2(\Omega)\overset{\wedge}{\otimes}L^2(\Omega)\mapsto fg\in L^1(\Omega).$$

We can identify $L^2(\Omega)$ with its conjugate space so that by (2.1) we get a quotient map

$$q:\mathcal{S}^1(L^2(\Omega))\to L^1(\Omega)$$

which turns out to be a complete metric surjection.

Let $q_i:\mathcal{S}^1(L^2(\lambda_{A_i}))\to L^1(\lambda_{A_i})$, $i=1,\dots,n$ be defined as above. Recall the notation $E_i=L^2(\lambda_{A_i})$. Using Proposition 2.3 together with the associativity of the

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Haagerup tensor product, we get a complete metric surjection

$$Q = q_1 \otimes \cdots \otimes q_n : \mathcal{S}^1(E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^1(E_n) \to L^1(\lambda_{A_1}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} L^1(\lambda_{A_n}).$$

Let $N = \ker Q$ and let, for $1 \leq i \leq n$, $N_i = \ker q_i$. For any $1 \leq j \leq n$, let

$$F_j = \mathcal{S}^1(E_1) \otimes \cdots \otimes \mathcal{S}^1(E_{j-1}) \otimes N_j \otimes \mathcal{S}^1(E_j) \otimes \cdots \otimes \mathcal{S}^1(E_n).$$

By Proposition 2.3 (iv), we obtain that

$$N = \overline{F_1 + F_2 + \cdots + F_n}.$$

Assume that $\Lambda(\phi)$ extends to a completely bounded mapping

$$\Lambda(\phi) : \mathcal{S}^\infty(E_2, E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(E_n, E_{n-1}) \to \mathcal{S}^\infty(E_n, E_1).$$

Let $E = \mathcal{S}^\infty(E_2, E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(E_n, E_{n-1})$. By Proposition 2.4, we have a complete isometry

$$CB(E, \mathcal{B}(E_n, E_1)) = \left(((E_1)_c)^* \overset{h}{\otimes} E \overset{h}{\otimes} (E_n)_c \right)^*.$$

By (2.2) we have

$$E = (E_1)_c \overset{h}{\otimes} ((E_2)_c)^* \overset{h}{\otimes} (E_2)_c \overset{h}{\otimes} ((E_3)_c)^* \overset{h}{\otimes} \cdots \overset{h}{\otimes} (E_{n-1})_c \overset{h}{\otimes} ((E_n)_c)^*.$$

Thus, using (2.1) and the associativity of the Haagerup tensor product, we get that

$$CB(E, \mathcal{B}(E_n, E_1)) = \left(\mathcal{S}^1(E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^1(E_n) \right)^*.$$

Let $u : \mathcal{S}^1(E_1) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^1(E_n) \to \mathbb{C}$ induced by $\Lambda(\phi)$. For any $x_i \in \mathcal{S}^1(H_i), 1 \leq i \leq n$, we have

$$\begin{aligned} &u(x_1 \otimes \cdots \otimes x_n) \\ &\quad = \int_{\Omega_1 \times \cdots \times \Omega_n} \phi(t_1, \ldots, t_n)[q_1(x_1)](t_1) \ldots [q_n(x_n)](t_n) \, d\mu_1(t_1) \ldots d\mu_n(t_n). \end{aligned}$$

To see this, it is enough to check when the x_i are rank one operators and in that case, one can use the identifications above. In particular, the latter implies that u vanishes on $N = \ker Q$. Since Q is a complete metric surjection, we get a mapping

$$v : L^1(\lambda_{A_1}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} L^1(\lambda_{A_n}) \to \mathbb{C}$$

such that $u = v \circ Q$. An application of Theorem 2.1 with suitable restrictions using the separability of the spaces $L^1(\lambda_{A_i})$ gives the existence of separable Hilbert spaces $H_1, \ldots, H_{n-1}$ and completely bounded maps

$$\alpha_1 : L^1(\lambda_{A_1}) \to \mathcal{B}(H_1, \mathbb{C}) = (H_1)_c^*,$$

$$\alpha_i : L^1(\lambda_{A_i}) \to \mathcal{B}(H_i, H_{i-1}), 2 \leq i \leq n-1,$$

$$\alpha_n : L^1(\lambda_{A_n}) \to \mathcal{B}(\mathbb{C}, H_{n-1}) = (H_{n-1})_c$$

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such that for any $f_j \in L^1(\lambda_{A_j}), 1 \le j \le n$,

$$\nu(f_1 \otimes \cdots \otimes f_n) = \langle \alpha_1(f_1), [\alpha_2(f_2) \ldots \alpha_{n-1}(f_{n-1})](\alpha_n(f_n)) \rangle .$$

Since $L^1(\Omega_2)$ is equipped with the Max operator space structure, we have

$$CB(L^1(\lambda_{A_i}), \mathcal{B}(H_i, H_{i-1})) = \mathcal{B}(L^1(\lambda_{A_i}), \mathcal{B}(H_i, H_{i-1})).$$

Moreover, by (2.5), we have

$$\mathcal{B}(L^1(\lambda_{A_i}), \mathcal{B}(H_i, H_{i-1})) = L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1})).$$

Thus, for any $2 \le i \le n-1$, we associate with α_i an element $a_i \in L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1}))$. Similarly, we associate with α_1 an element $a_1 \in L^\infty(\lambda_{A_1}; H_1)$ and with α_n an element $a_n \in L^\infty(\lambda_{A_n}; H_{n-1})$. Using the identification (2.4), we obtain that

$$\phi(t_1, \ldots, t_n) = \langle a_1(t_1), [a_2(t_2) \ldots a_{n-1}(t_{n-1})](a_n(t_n)) \rangle$$

for a.-e. $(t_1, \ldots, t_n) \in \sigma(A_1) \times \cdots \times \sigma(A_n)$, and one can choose $a_1, \ldots, a_n$ such that we have the equality

$$\|\Lambda(\phi)\|_{\mathrm{cb}} = \|a_1\|_\infty \cdots \|a_n\|_\infty .$$

Proof of (iv) $\Rightarrow$ (ii)

Assume that there exist separable Hilbert space $H_1, \ldots, H_{n-1}$, $a_1 \in L^\infty(\lambda_{A_1}; H_1)$, $a_i \in L^\infty_\sigma(\lambda_{A_i}; \mathcal{B}(H_i, H_{i-1}))$, $2 \le i \le n-1$ and $a_n \in L^\infty(\lambda_{A_n}; H_{n-1})$ such that

$$\phi(t_1, \ldots, t_n) = \langle a_1(t_1), [a_2(t_2) \ldots a_{n-1}(t_{n-1})](a_n(t_n)) \rangle$$

for a.-e. $(t_1, \ldots, t_n) \in \sigma(A_1) \times \cdots \times \sigma(A_n)$. For any $1 \le i \le n-1$, let $(\varepsilon^i_k)_{k \ge 1}$ be a Hilbertian basis of H_i. for $k, l \ge 1$, define

$$a^1_k = \left\langle a_1, \varepsilon^1_k \right\rangle, a^i_{kl} = \left\langle \varepsilon^{i-1}_k, a_i \varepsilon^i_l \right\rangle \quad \text{and} \quad a^n_l = \left\langle \varepsilon^{n-1}_l, a_n \right\rangle .$$

Then $a^1_k \in L^\infty(\lambda_{A_1})$, $a^i_{kl} \in L^\infty(\lambda_{A_i})$, $2 \le i \le n-1$, and $a^n_l \in L^\infty(\lambda_{A_n})$. To see this, simply note that for $2 \le i \le n-1$,

$$a^i_{kl} = \mathrm{tr}(a_i(\cdot) \circ (\varepsilon^{i-1}_k \otimes \varepsilon^i_l)).$$

For $N \ge 1$ and $1 \le i \le n-1$, let P^i_N be the orthogonal projection onto $\mathrm{Span}(\varepsilon^i_1, \ldots, \varepsilon^i_N)$. Then define

$$\phi_N = \left\langle P^1_N(a(t_1))), [a_2(t_2) P^2_N a_3(t_3) P^3_N \ldots a_{n-1}(t_{n-1}) P^{n-1}_N](a_n(t_n)) \right\rangle .$$

It is clear that $(\phi_N)_{N \ge 1}$ is bounded in $L^\infty(\lambda_{A_1} \times \cdots \times \lambda_{A_n})$ and that $\phi_N \to \phi$ pointwise when $N \to \infty$. Therefore, by the Dominated Convergence Theorem, we have that $\phi_N \to \phi$ for the w^*–topology. This implies, by w^*– continuity of $\Gamma^{A_1, \ldots, A_n}$, that for any X_j in $\mathcal{S}^2(\mathcal{H})$, $1 \le j \le n-1$,

$$\left[\Gamma^{A_1, \ldots, A_n}(\phi_N)\right](X_1 \otimes \cdots \otimes X_{n-1}) \to \left[\Gamma^{A_1, \ldots, A_n}(\phi)\right](X_1 \otimes \cdots \otimes X_{n-1})$$

weakly in $\mathcal{S}^2(\mathcal{H})$.

Assume that $(\Gamma^{A_1, \ldots, A_n}(\phi_N))_N$ is uniformly bounded in $CB(\mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}), \mathcal{S}^\infty(\mathcal{H}))$. Then, the above approximation property together with the

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density of $\mathcal{S}^2$ into $\mathcal{S}^\infty$ imply that $\Gamma^{A_1,\ldots,A_n}(\phi)$ is completely bounded as well with $\|\Gamma^{A_1,\ldots,A_n}(\phi)\|_{cb} \leq \sup_N \|\Gamma^{A_1,\ldots,A_n}(\phi_N)\|_{cb}$.

We will show now that for any $N \geq 1$, $\|\Gamma^{A_1,\ldots,A_n}(\phi_N)\|_{cb} \leq \|a_1\|_\infty \ldots \|a_n\|_\infty$. For any $N \geq 1$ and a.-e. $(t_1,\ldots,t_n) \in \sigma(A_1)\times\cdots\times\sigma(A_n)$, we have

$$\phi_N(t_1,\ldots,t_n) = \sum_{k_1,\ldots,k_{n-1}=1}^{N} a^1_{k_1}(t_1)a^2_{k_1k_2}(t_2)\ldots a^{n-1}_{k_{n-2}k_{n-1}}(t_{n-1})a^n_{k_n}(t_n),$$

so that for any $X_1,\ldots,X_{n-1} \in \mathcal{S}^2(\mathcal{H})$,

$$\begin{aligned}&\left[\Gamma^{A_1,\ldots,A_n}(\phi_N)\right](X_1\otimes\cdots\otimes X_{n-1})\\ &= \sum_{k_1,\ldots,k_{n-1}=1}^{N} a^1_{k_1}(A_1)X_1a^2_{k_1k_2}(A_2)X_2\ldots X_{n-2}a^{n-1}_{k_{n-2}k_{n-1}}(A_{n-1})X_{n-1}a^n_{k_n}(A_n).\end{aligned}$$

Note that the latter can be written as

$$\left[\Gamma^{A_1,\ldots,A_n}(\phi_N)\right](X_1\otimes\cdots\otimes X_{n-1}) = A^1_N(X_1\otimes I_N)A^2_N(X_2\otimes I_N)\cdots(X_{n-1}\otimes I_N)A^n_N,$$

where

$$A^1_N = [a^1_1(A_1)\ a^1_2(A_1)\ldots a^1_N(A_1)]\colon \ell^N_2(\mathcal{H}) \to \mathcal{H},$$

$$A^i_N = [a^i_{kl}(A_i)]_{\substack{1\leq k\leq N\\ 1\leq l\leq N}}\colon \ell^N_2(\mathcal{H}) \to \ell^N_2(\mathcal{H}),\ 2\leq i\leq n-1$$

and

$$A^n_N = [a^n_1(A_n)\ a^n_2(A_n)\ldots A^n_N(A_n)]^t\colon \mathcal{H} \to \ell^N_2(\mathcal{H}).$$

The notation $X\otimes I_N$ stands for the element of $\mathcal{B}(\ell^N_2(\mathcal{H}))$ whose matrix is the $N\times N$ diagonal matrix $\mathrm{diag}(X,\ldots,X)$.

For any $N\geq 1$ and any $1\leq i\leq n$, let π_N and π_i be the $*-$representations defined by

$$\begin{array}{rcl}\pi_N\colon \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{B}(\ell^N_2(\mathcal{H}))\\ X & \longmapsto & X\otimes I_N\end{array} \quad\text{and}\quad \begin{array}{rcl}\pi_{A_i}\colon L^\infty(\lambda_{A_i}) & \longrightarrow & \mathcal{B}(\mathcal{H})\\ f & \longmapsto & f(A_i)\end{array}.$$

By [17, Proposition 1.5], π_N and π_{A_i} are completely bounded with cb-norm less than 1. Note that the element $[a^i_{kl}]_{1\leq k,l\leq N} \in M_N(L^\infty(\lambda_B))$ has a norm less than $\|a_i\|_\infty$. Thus, the latter implies that $A^i_N = [\pi_{A_i}(a^i_{kl})]_{1\leq k,l\leq N}$ has an operator norm less than $\|a_i\|_\infty$. Similarly (using column and row matrices), we show that A^1_N and A^n_N have a norm less than $\|a_1\|_\infty$ and $\|a_n\|_\infty$, respectively. Finally, write

$$\left[\Gamma^{A_1,\ldots,A_n}(\phi_N)\right](X_1\otimes\cdots\otimes X_{n-1}) = \sigma^1_N(X_1)\sigma^2_N(X_2)\ldots\sigma^{n-1}_N(X_{n-1}),$$

where for any $1\leq i\leq n-2$, $\sigma^i_N(X_1) = A^i_N\pi_N(X_i)$ and $\sigma^N_{n-1}(X_{n-1}) = A^{n-1}_N\pi_N(X_{n-1})A^n_N$. By the easy part of Wittstock theorem (see e.g. [17, Theorem 1.6]), σ^i_N and σ^N_{n-1} are completely bounded with cb-norm less than $\|a_i\|_\infty$ and $\|a_{n-1}\|_\infty\|a_n\|_\infty$, respectively. Hence, by Theorem 2.1, we get that $\Gamma^{A_1,\ldots,A_n}(\phi_N)$ is completely bounded with cb-norm less than $\|a_1\|_\infty\ldots\|a_n\|_\infty$. ∎

https://doi.org/10.4153/S0008439520000570 Published online by Cambridge University Press

We now give an example of a class of functions for which the multiple operator integrals will be completely bounded (in the sense of the paper) for any normal operators. We will identify a bounded Borel function $\psi : \mathbb{C}^n \to \mathbb{C}$ with the class of the restriction $\tilde{\psi} = \psi_{|\sigma(A_1)\times\sigma(A_2)\times\cdots\times\sigma(A_n)}$ in $L^\infty\left(\prod_{i=1}^n \lambda_{A_i}\right)$. Then we will denote by $\Gamma^{A_1,A_2,\ldots,A_n}(\psi)$ the multiple operator integral $\Gamma^{A_1,A_2,\ldots,A_n}(\tilde{\psi})$.

Example 4.3 Let C_b be the space of bounded and continuous functions $f : \mathbb{C} \to \mathbb{C}$. Let $n \geq 1$ be an integer. We define the integral tensor product of C_b, denoted by $C_b \widehat{\otimes}_i \cdots \widehat{\otimes}_i C_b$, as the space of functions $\phi : \mathbb{C}^n \to \mathbb{C}$ such that there exist a σ-finite measure space (Σ, μ) and functions $h_i : \mathbb{C} \times \Sigma \to \mathbb{C}$, $1 \leq i \leq n$, such that for a.e. $w \in \Sigma$, $t \mapsto h_i(t, w) \in C_b$,

$$\int_\Sigma \|h_1(\cdot, w)\|_\infty \cdots \|h_n(\cdot, w)\|_\infty \, \mathrm{d}\mu(w) < +\infty \tag{4.4}$$

and for every $t_1, \ldots, t_n \in \mathbb{C}$,

$$\phi(t_1, \ldots, t_n) = \int_\Sigma h_1(t_1, w) \cdots h_n(t_n, w) \, \mathrm{d}\mu(w). \tag{4.5}$$

The integral projective norm $\|\phi\|_i$ of ϕ is the infimum of the quantities (4.4) over all representations of ϕ as above.

Let $A_1, \ldots, A_n$ be normal operators on $\mathcal{H}$ and let $\phi \in C_b \widehat{\otimes}_i \cdots \widehat{\otimes}_i C_b$. Then we have a completely bounded multiple operator integral $\Gamma^{A_1,\ldots,A_n}(\phi) : \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H})$ with $\left\|\Gamma^{A_1,\ldots,A_n}(\phi)\right\|_{\mathrm{cb}} \leq \|\phi\|_i$.

Proof To show this, we will find another factorization of ϕ as in (4.5) that satisfies of Theorem 4.1(iv). First, note that by changing Σ if necessary, we can assume that for almost every $w \in \Sigma$, $\|h_1(\cdot, w)\|_\infty \cdots \|h_n(\cdot, w)\|_\infty > 0$. We define $g_i : \mathbb{C} \times \Sigma \to \mathbb{C}$, $1 \leq i \leq n$, for almost every (t_i, σ) by

$$g_1(t_1, w) = \frac{h_1(t_1, w)}{\sqrt{\|h_1(\cdot, w)\|_\infty}} \sqrt{\|h_2(\cdot, w)\|_\infty \cdots \|h_n(\cdot, w)\|_\infty},$$

$$g_i(t_i, w) = \frac{\overline{h_i}(t_i, w)}{\|h_i(\cdot, w)\|_\infty} \quad \text{if } 2 \leq i \leq n-1,$$

and

$$g_n(t_n, w) = \frac{\overline{h_n}(t_n, w)}{\sqrt{\|h_n(\cdot, w)\|_\infty}} \sqrt{\|h_1(\cdot, w)\|_\infty \cdots \|h_{n-1}(\cdot, w)\|_\infty}.$$

It is straightforward to check that for every $t_1, t_n \in \mathbb{C}$, $g_1(t_1, \cdot)$ and $g_n(t_n, \cdot)$ belong to $L^2(\Sigma, \mu)$ and we have, setting $\alpha = \int_\Sigma \|h_1(\cdot, w)\|_\infty \cdots \|h_n(\cdot, w)\|_\infty \, \mathrm{d}\mu(w)$,

$$\|g_1(t_1, \cdot)\|_{L^2(\Sigma,\mu)} \leq \sqrt{\alpha} \quad \text{and} \quad \|g_n(t_n, \cdot)\|_{L^2(\Sigma,\mu)} \leq \sqrt{\alpha}.$$

Moreover, the proof of [6, Proposition 5.4] shows that the associated mappings (after taking their restriction to the spectrum of A_1 and A_n) $a_1 : t_1 \in \sigma(A_1) \mapsto g_1(t_1, \cdot) \in L^2(\Sigma, \mu)$ and $a_n : t_n \in \sigma(A_n) \mapsto g_n(t_n, \cdot) \in L^2(\Sigma, \mu)$ are continuous, hence measur-

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able, so that, after taking the classes of these functions in L^∞, $a_1 \in L^\infty(\lambda_{A_1}, L^2(\Sigma,\mu))$ and $a_n \in L^\infty(\lambda_{A_n}, L^2(\Sigma,\mu))$.

Now, notice that for all $2 \le i \le n-1$, g_i is bounded on $\sigma(A_i) \times \Sigma$ and for almost every $t_i \in \sigma(A_i)$, $a_i(t_i) \in \mathcal{B}(L^2(\Sigma,\mu))$ be the multiplication map by $g_i(t_i,\cdot)$. This defines a mapping $a_i : \sigma(A_i) \to \mathcal{B}(L^2(\Sigma,\mu))$ that is bounded by $\|g_i\|_\infty = 1$. To prove that this map is w^*-measurable, it is sufficient, by linearity and density, to show that for any rank-one operator $T \in \mathcal{S}^1(L^2(\Sigma,\mu))$, $t_i \in \sigma(A_i) \mapsto \operatorname{tr}(a_i(t_i)T)$ is measurable. Such an operator T can be written as $T = b_1 \otimes b_2$ with $b_1, b_2 \in L^2(\Sigma,\mu)$ and $T(f) = \langle f, b_1\rangle\, b_2$. Now, one easily checks that $a_i(t_i)T = b_1 \otimes a_i(t_i)b_2 = b_1 \otimes g_i(t_i,\cdot)b_2$. Hence,

$$\begin{aligned}\operatorname{tr}(a_i(t_i)T) &= \int_\Sigma \overline{b_1}(w) g_i(t_i,w) b_2(w)\mathrm{d}w \\ &= \int_\Sigma g_i(t_i,w) b(w)\mathrm{d}w = \langle g_i(t_i,\cdot), b\rangle_{L^\infty,L^1}, \end{aligned} \tag{4.6}$$

where $b = \overline{b_1} b_2 \in L^1(\Sigma,\mu)$. Since $g_i \in L^\infty(\lambda_{A_i} \times \mu)$, we get that $t_i \mapsto g(t_i,\cdot)$ is a w^*-measurable map from $\sigma(A_i)$ into $L^\infty(\mu)$. Together with equality (4.6), this implies that a_i is w^*-measurable and hence $a_i \in L^\infty_\sigma(\lambda_{A_i}, \mathcal{B}(L^2(\Sigma,\mu)))$.

Finally, we check that

$$\phi(t_1,\dots,t_n) = \langle a_1(t_1), [a_2(t_2)\dots a_{n-1}(t_{n-1})](a_n(t_n))\rangle$$

for a.-e. $(t_1,\dots,t_n) \in \sigma(A_1)\times\cdots\times\sigma(A_n)$ and that $\|a_1\|_\infty \cdots \|a_n\|_\infty \le \alpha$. Taking the infimum over all representations of ϕ gives $\left\|\Gamma^{A_1,\dots,A_n}(\phi)\right\|_{cb} \le \|\phi\|_i$. ∎

Example 4.4 For fixed normal operator $A_1,\dots,A_n$ on $\mathcal{H}$, one can define in a similar way the space $L^\infty(\lambda_{A_1})\widehat{\otimes}_i\cdots\widehat{\otimes}_i L^\infty(\lambda_{A_n})$. Then, any ϕ in this space induces a completely bounded multiple operator integral $\Gamma^{A_1,\dots,A_n}(\phi)\colon \mathcal{S}^\infty(\mathcal{H}) \overset{h}{\otimes} \cdots \overset{h}{\otimes} \mathcal{S}^\infty(\mathcal{H}) \to \mathcal{S}^\infty(\mathcal{H})$. This can be proved using the same ideas as in Example 4.3. We refer the reader to [13] for another proof.

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