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Weak inclusion systems

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We define *weak inclusion systems* as a natural extension of *inclusion systems*. We prove that several properties of *factorisation systems* and *inclusion systems* remain valid under this extension and we obtain new properties as algebraic tools in abstract model theory.

1. Introduction

It is well known that the categorical approaches to computer science, in spite of the clarity of the proofs they handle, raise a serious problem: that of modelling properly practical concepts. This is because category theory has appeared as a result of the efforts of many mathematicians to unify mathematical concepts, rather than computing ones. For this reason, the categorical approaches to computing fields come together with their proper technical tools, which in many cases are different in shape but identical in spirit.

In this paper we propose *weak inclusion systems* as a mechanism of penetrating inside the objects and the morphisms of a category. Our main goal is to give both definitions and properties in the most general form, allowing the computer scientist to apply them as a technical device in his (or her) fields of interests.

Weak inclusion systems represent an analogy with factorisation systems (*e.g.* see Herrlich and Strecker (1973)), which have been used in many places with computer science (*e.g.* see Goguen and Burstall (1990) and Tarlecki (1984; 1986), and the older papers Németi (1982) and Németi and Sain (1981)).

Inclusion systems were first introduced in Diaconescu *et al.* (1993) as a categorical tool for the study of modularisation. In some cases, including modularisation, they are more useful than factorisation systems.

The rest of this paper is structured in four sections. Section 2 gives definitions and describes basic properties of weak inclusion systems. The intuition for inclusion systems is that they give for each morphism a *unique* object of factorisation, rather than merely up to isomorphism as in the case of factorisation systems. In this paper we prefer to use a weaker framework than that in Diaconescu *et al.* (1993), and we prove some old results about factorisation systems and obtain some new results. The same idea appears in Hilbedrik (in preparation), where the research goes in another direction.

We should mention that the category of many-sorted algebras, which is very much used in theoretical computing (especially in semantics), does not admit an inclusion system in the style of Diaconescu *et al.* (1993) but does admit a weak inclusion system.

A weak inclusion system for a category $\mathscr{C}$ consists of two subcategories $\mathscr{I}$ and $\mathscr{E}$ having the same objects as $\mathscr{C}$, such that $\mathscr{I}$ is a partial order and every morphism $f \in \mathscr{C}$ can be factored uniquely as $e; i$, where $e \in \mathscr{E}$ and $i \in \mathscr{I}$. In Hilbedrik (in preparation) every morphism of $\mathscr{E}$ is supposed to be an epic. We do not need this hypothesis here, even if in some interesting cases this hypothesis is useful. A similar restrictive assumption, but for factorisation systems, has appeared within a series of interesting papers due to Németi *et al*. (*e.g*. see Németi (1982) and Németi and Sain (1981)).

It is easy to prove that the category of inclusions $\mathscr{I}$ unambiguously determines $\mathscr{E}$. We show something more: a morphism is in $\mathscr{E}$ if and only if it has a diagonal-fill property. From this we give an equivalent definition for weak inclusion systems (see Definition 10) based only on the subcategory of inclusions.

Section 3 refers to categorical subobjects in the classical style (see Mac Lane (1971)). The category of inclusions is an independent system of representatives for (categorical) subobjects, that is, each subobject contains at most one inclusion. On the other hand, the category of inclusions is a complete system of representatives, that is, each subobject contains one inclusion if and only if each monic of $\mathscr{E}$ is an isomorphism. Within a category that has several weak inclusion systems, it is better to look, if possible, for an inclusion system in which $\mathscr{E}$ does not contain monics that are not isomorphisms.

Section 4 is concerned with building a weak inclusion system for a category $\mathscr{C}$, starting with both a category $\mathscr{D}$ that admits a weak inclusion system and a faithful functor $\mathscr{U}: \mathscr{C} \to \mathscr{D}$. We give two conditions and prove that each is sufficient for building a weak inclusion system for $\mathscr{C}$. One of them generalizes a construction from Diaconescu *et al*. (1993) where, by definition, a morphism i of $\mathscr{C}$ is an inclusion if and only if $\mathscr{U}(i)$ is an inclusion of $\mathscr{D}$. The two conditions have been used in Căzănescu (1972) where the first author has proved that they are sufficient for $\mathscr{U}$ to reflect limits and colimits, respectively. Since in the practical cases $\mathscr{U}$ is a forgetful functor from complicated structures to simpler ones, one can find weak inclusion systems for the complicated structures.

We give two examples where this result may be applied. One of them tells us that within an institution (see Goguen and Burstall (1992)), if the category of signatures has a weak inclusion system, the category of institutional theories has a weak inclusion system also.

The last section of the paper investigates pushout and pullback properties of weak inclusion systems. They may be useful in many categorical approaches to computing, especially for modularisation. Diaconescu *et al*. (1993) contains a particular form of them, which are much used.

2. On the weak inclusion system definition

The reader is referred to Mac Lane (1971) and Herrlich and Strecker (1973) and to Mitchell (1965) for all the necessary category theory background. Also, the reader who is acquainted with model theory (Chang and Keisler 1973) and categorical approaches to it (*e.g*. see Andréka and Németi (1981), Goguen and Burstall (1992) and Tarlecki (1984)) will understand more easily the rôle of weak inclusion systems. We will denote by $|\mathscr{C}|$ the class of objects of a category $\mathscr{C}$, and by $\|M\|$ the cardinal of a set M. The composition of the morphisms $f: A \to B$ and $g: B \to C$ is denoted by $f; g: A \to C$. If $f: A \to B$ is a morphism, we sometimes write $dom(f)$ instead of A and write $cod(f)$ instead of B.

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In this section we first present the definition of a factorisation system and then we give two equivalent definitions of a weak inclusion system.

2.1. *Factorisation systems*

Factorisation systems have been used in many places in computer science. As an example, in Goguen and Burstall (1992) and Tarlecki (1984) they have been used within the theory of institutions and abstract algebraic institutions, respectively.

Definition 1. A *factorisation system* for a category $\mathscr{C}$ consists of a class $\mathscr{M}$ of monics and a class $\mathscr{E}$ of epics in $\mathscr{C}$ such that

—$\mathscr{M}$ and $\mathscr{E}$ are subcategories of $\mathscr{C}$,
—both $\mathscr{M}$ and $\mathscr{E}$ contain all the ismorphisms of $\mathscr{C}$,
—every morphism f in $\mathscr{C}$ can be factored 'uniquely up to isomorphism', namely there exist $e \in \mathscr{E}$ and $m \in \mathscr{M}$ such that $f = e; m$ and for any other $e' \in \mathscr{E}$ and $m' \in \mathscr{M}$ with $f = e'; m'$ there is a unique isomorphism, let us say j, such that $e; j = e'$ and $j; m' = m$.

2.2. *First definition and properties*

Our first definition of a weak inclusion system is a natural extension of that of an inclusion system Diaconescu *et al.* (1993).

Definition 2. $\langle \mathscr{I}, \mathscr{E} \rangle$ is a *weak inclusion system* for a category $\mathscr{C}$ if $\mathscr{I}$ and $\mathscr{E}$ are two subcategories of $\mathscr{C}$ with $|\mathscr{I}| = |\mathscr{E}| = |\mathscr{C}|$ such that
1. $\mathscr{I}$ is a partial order, *i.e.*,

—$\|\mathscr{I}(A, B)\| \leqslant 1$ for each $A, B \in |\mathscr{C}|$
—$\mathscr{I}(A, B) \neq \phi$ and $\mathscr{I}(B, A) \neq \phi$ imply $A = B$.

2. Every morphism f in $\mathscr{C}$ can be factored uniquely as $e; i$ with $e \in \mathscr{E}$ and $i \in \mathscr{I}$.

The morphisms of $\mathscr{I}$ are called *inclusions*. If we had required that $\mathscr{E}$ should contain only epimorphisms and $\mathscr{I}$ admit finite coproducts (denoted by '$+$'), we would have obtained inclusion systems Diaconescu (1993) or *abstract* inclusion systems Hilbedrik (in preparation.

Example 1. If $\mathscr{E} = \mathscr{C}$ and $\mathscr{I}$ consists of all the identities of $\mathscr{C}$, then $\langle \mathscr{I}, \mathscr{E} \rangle$ is a weak inclusion system but not an inclusion system.

This example gives an uninteresting weak inclusion system, but it shows us that the condition '$\mathscr{E}$ contains only epics' is independent. On the other hand, it also shows that the concept of weak inclusion system is too general to include just the interesting cases.

It is well known that the two classes of morphisms of a *factorisation system* unambiguously determine one another. In the case of weak inclusion systems a weaker condition holds.

Lemma 3. If $\langle \mathscr{I}, \mathscr{E} \rangle$ and $< \mathscr{I}', \mathscr{E}' >$ are two weak inclusion systems for $\mathscr{C}$ and if $\mathscr{I} \subseteq \mathscr{I}'$, then $\mathscr{E}' \subseteq \mathscr{E}$.

Proof. Let $e' \in \mathscr{E}'$. Factor e' as $e; i$ with $e \in \mathscr{E}$ and $i \in \mathscr{I}$. Factor e as $e_1; i_1$ with $e_1 \in \mathscr{E}'$ and $i_1 \in \mathscr{I}'$. We have $e' = e_1; i_1; i$ with $i_1; i \in \mathscr{I}'$. But the factorisation is unique, therefore $e' = e_1$ and $i_1; i = 1_{cod(e_1)}$. Hence i_1 and i are identities and $e' = e$, that is $e' \in \mathscr{E}$. □

Note that $\mathscr{I} \subseteq \mathscr{I}'$ does not imply $\mathscr{E}' = \mathscr{E}$. This fact is easily seen by the next counter-example, where we forget the identity morphisms.

Example 2. Let $\mathscr{C}$ be a three-object category with three nonidentity morphisms f_1, f_2, f_3 such that $f_1; f_2 = f_3$. We can take two weak inclusion systems as $\mathscr{I} = \{f_2\}$, $\mathscr{E} = \{f_1\}$ and $\mathscr{I}' = \{f_1, f_2, f_3\}$, $\mathscr{E}' = \phi$.

Corollary 4. $\mathscr{I}$ unambiguously determines $\mathscr{E}$.

This corollary leads us to the idea of removing $\mathscr{E}$ from the above definition. We will spend the rest of this section doing this.

A category with a weak inclusion system is said to be *weak inclusive*, and we will use e to denote the morphisms from $\mathscr{E}$, and i to denote the morphisms from $\mathscr{I}$. For each $f \in \mathscr{C}$ we fix the notations $f = e_f; i_f$, where $e_f \in \mathscr{E}$ and $i_f \in \mathscr{I}$.

Fact 5. If $\langle \mathscr{I}, \mathscr{E} \rangle$ is a weak inclusion system for $\mathscr{C}$,
1. $\mathscr{I}$ contains only monics.
2. Each morphism in $\mathscr{I} \cap \mathscr{E}$ is an identity.
3. If $f; i \in \mathscr{I}$, then $f \in \mathscr{I}$.
4. If $f; i \in \mathscr{E}$, then i is an identity and $f \in \mathscr{E}$.
5. If $f; g \in \mathscr{E}$, then $g \in \mathscr{E}$.
6. Any co-equalizer is in $\mathscr{E}$.
7. Any retract is in $\mathscr{E}$.
8. All isomorphisms in $\mathscr{C}$ are in $\mathscr{E}$.

Proof.

1. Let i be an inclusion, and $f, g \in \mathscr{C}$ such that $f; i = g; i$. Factor f as $e_f; i_f$ and g as $e_g; i_g$. It follows that $e_f; i_f; i = e_g; i_g; i$, that is, $e_f = e_g$ and $i_f; i = i_g; i$. Since $\mathscr{I}$ is a partial order, we get $i_f = i_g$. Hence $f = g$.
2. Take $f \in \mathscr{I} \cap \mathscr{E}$. Then $f; 1_{cod(f)} = 1_{dom(f)}; f$, that is, $f = 1_{dom(f)} = 1_{cod(f)}$, because the factorisation is unique.
3. Factor f as $e_f; i_f$. Then $e_f = 1_{dom(f)}$ and $i_f; i = 1_{cod(f;i)}$, since $f; i$ admits a unique factorisation. Hence $f = i_f \in \mathscr{I}$.
4. Factor f as $e_f; i_f$. Then $e_f; i_f; i \in \mathscr{E}$, and then $i_f; i = 1_{cod(e_f)}$. We deduce that i_f and i are identities and $f = e_f \in \mathscr{E}$.
5. Factor g as $e_g; i_g$. Then $f; e_g; i_g \in \mathscr{E}$ and, moreover, $i_g = 1_{cod(g)}$ (by Part 4 of this fact). Hence $g = e_g$.
6. Let $r: A \to B$ be a co-equalizer for the diagram containing the two morphisms $u, v: C \to A$. Factor r as $e_r; i_r$ with $e_r: A \to B'$ and $i_r: B' \hookrightarrow B$. Since i_r is a monic (Part 1 of this fact), e_r is a cocone, and thus there is a unique $h: B \to B'$ such that $r; h = e_r$. It follows that $r; (h; i_r) = r$ and, since r is an epic, we obtain $h; i_r = 1_B$ and, furthermore, $i_r = 1_{B'}$ (by Part 4 of this fact). Therefore $r = e_r$, that is, r belongs to $\mathscr{E}$.

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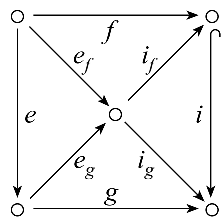
7. It is known that a retract r of f is a co-equalizer for the diagram containing the morphisms $1_{dom(r)}$ and $r;f$. Now, the conclusion follows from Part 6 of this fact.
8. This is obvious from Part 7, since each isomorphism is a retract. □

The following diagonal-fill lemma holds.

Lemma 6. (*Diagonal-fill*) For each morphism $f, g \in \mathscr{C}$ and for each $e \in \mathscr{E}$ and $i \in \mathscr{I}$, if $f;i = e;g$, there is a unique morphism $h \in \mathscr{C}$ such that $e;h = f$ and $h;i = g$.

Proof. Factor f as $e_f;i_f$ and g as $e_g;i_g$. Then $e_f;i_f;i = e;e_g;i_g$, and thus $e_f = e;e_g$ and $i_f;i = i_g$.



We take $h = e_g;i_f$. The uniqueness of h follows from $h;i = g$ and from Part 1 of Fact 5.

The following lemma represents the first step in removing $\mathscr{E}$ from the definition of a weak inclusion system.

Lemma 7. Let u be a morphism of $\mathscr{C}$. Then $u \in \mathscr{E}$ if and only if for each $f, g \in \mathscr{C}$ and $i \in \mathscr{I}$ such that $f;i = u;g$ there exists a morphism $h \in \mathscr{C}$ with $u;h = f$ and $h;i = g$.

Proof. The 'only if' part of this lemma is exactly the diagonal-fill lemma. Conversely, let u be a morphism from $\mathscr{C}$, and $e_u;i_u$ be its unique factorisation. If we take $f = e_u$ and $g = 1_{cod(u)}$, it follows that there is a morphism h that verifies $u;h = e_u$ and $h;i_u = 1_{cod(u)}$. By Part 4 of Fact 5 we deduce i_u is an identity. Hence $u = e_u$, that is, $u \in \mathscr{E}$. □

2.3. *Second definition*

In this section an equivalent definition of weak inclusion systems is presented. This new definition does not require $\mathscr{E}$ by hypothesis, because the class $\mathscr{E}$ is built knowing only the class $\mathscr{I}$ of inclusions.

Similar ideas may be found in some papers on factorisation systems (*e.g.* see Németi and Sain (1981)).

Definition 8. For each subcategory $\mathscr{I}$ (not necessarily of inclusions) of $\mathscr{C}$ we define the class of morphisms $\mathscr{E}_{\mathscr{I}}$ as follows: a morphism e of $\mathscr{C}$ belongs to $\mathscr{E}_{\mathscr{I}}$ if and only if it meets a weak form of the diagonal-fill lemma, namely for each morphism f, g of $\mathscr{C}$ and for each morphism i of $\mathscr{I}$ such that $f;i = e;g$ there exists a (not necessarily unique) morphism h that verifies $e;h = f$ and $h;i = g$.

Lemma 9. $\mathscr{E}_{\mathscr{I}}$ is a subcategory of $\mathscr{C}$ having the same objects as $\mathscr{C}$.

Proof. Obviously $1_A \in \mathscr{E}_{\mathscr{I}}$ for each object A in $\mathscr{C}$.

For $e_1, e_2 \in \mathscr{E}_{\mathscr{I}}$ we show that $e_1;e_2 \in \mathscr{E}_{\mathscr{I}}$. Let $f, g \in \mathscr{C}$ and $i \in \mathscr{I}$ such that $(e_1;e_2);g = f;i$. Since $e_1 \in \mathscr{E}_{\mathscr{I}}$, there exists $h_1 \in \mathscr{C}$ with $e_1;h_1 = f$ and $h_1;i = e_2;g$. As $e_2 \in \mathscr{E}_{\mathscr{I}}$, there is $h \in \mathscr{C}$ having the

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properties $e_2; h = h_1$ and $h; i = g$. Hence $(e_1; e_2); h = e_1; (e_2; h) = e_1; h_1 = f$ and $h; i = g$, that is, $e_1; e_2 \in \mathscr{E}_{\mathscr{I}}$. □

Lemma 7 proves that if $\langle \mathscr{I}, \mathscr{E} \rangle$ is a weak inclusion system, $\mathscr{E}_{\mathscr{I}} = \mathscr{E}$. Now we can define the notion of weak inclusion system as follows.

Definition 10. $\mathscr{I}$ is a *weak inclusion system for* $\mathscr{C}$ if $\mathscr{I}$ is a subcategory of $\mathscr{C}$ such that

1. $|\mathscr{I}| = |\mathscr{C}|$ and $\mathscr{I}$ is a *partial order*,
2. $\mathscr{I}$ is $\mathscr{I}$-*right cancellable*, that is, if $i \in \mathscr{I}$ and $f; i \in \mathscr{I}$, then $f \in \mathscr{I}$,
3. for each $f \in \mathscr{C}$ there exist $e \in \mathscr{E}_{\mathscr{I}}$ and $i \in \mathscr{I}$ such that $f = e; i$.

The rôle of Condition 2 of this definition may seem a little obscure. We note that within many frameworks it is easier to verify a condition about inclusions like the above, than to prove the uniqueness of factorisation. For example, we use this definition fully in Section 4, where we build weak inclusion systems within abstract frameworks.

It is straightforward to show that the definition above is equivalent to the first definition of weak inclusion systems if and only if the uniqueness of factorisation holds.

Lemma 11. In the latter definition, for each $f \in \mathscr{C}$ there exist a *unique* $e \in \mathscr{E}_{\mathscr{I}}$ and a *unique* $i \in \mathscr{I}$ such that $f = e; i$.

Proof. Let $f = e'; i'$ be another factorisation of f. Since $e \in \mathscr{E}_{\mathscr{I}}$ and $i' \in \mathscr{I}$, it follows from the definition of $\mathscr{E}_{\mathscr{I}}$ that there is $h \in \mathscr{C}$ such that $e; h = e'$ and $h; i' = i$. There is also $h' \in \mathscr{C}$ such that $e'; h' = e$ and $h'; i = i'$. Since $\mathscr{I}$ is $\mathscr{I}$-right cancellable, it follows that h, $h' \in \mathscr{I}$. But $\mathscr{I}$ is a partial order, and therefore h and h' are identities. Thus the factorisation is unique. □

Since we have obtained an equivalent definition of weak inclusion systems involving only the category $\mathscr{I}$ of inclusions, we will sometimes denote an inclusion system by $\mathscr{I}$ rather than $\langle \mathscr{I}, \mathscr{E} \rangle$.

3. Subobjects

Let $\mathscr{C}$ be a category. For each object $A \in |\mathscr{C}|$, the equivalence relation $\sim_A$ is defined for each pair of monics $m: B \to A$ and $m': B' \to A$ by

$$m \sim_A m' \text{ if there is an isomorphism } j: B \to B' \text{ such that } m = j; m'.$$

A (*categorical*) *subobject* of A is a *coset* of $\sim_A$. We suppose that $\mathscr{C}$ has a weak inclusion system $\mathscr{I}$.

Proposition 12. Each subobject of A contains at most one inclusion.

Proof. Let $i_i \sim_A i_2$. Then there exists an isomorphism $j: B_1 \to B_2$ such that $i_1 = j; i_2$ and $j^{-1}; i_1 = i_2$. By the property of right cancellability it follows that $j, j^{-1} \in \mathscr{I}$, that is $B_1 = B_2$ and $i_1 = i_2$. □

In other words, the inclusions are an independent system of representatives for the (categorical) subobjects of $\mathscr{C}$.

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Definition 13. We say that $\mathscr{I}$ is a *complete system* for $\mathscr{C}$ if for each $A \in |\mathscr{C}|$, each subobject of A contains one inclusion.

Proposition 14. $\mathscr{I}$ is a complete system for $\mathscr{C}$ if and only if each monic in $\mathscr{E}$ is an isomorphism.

Proof. If $\mathscr{I}$ is a complete system and e is a monic from $\mathscr{E}$, there exist an inclusion i and an isomorphism j such that $e = j; i$. By Part 4 of Fact 5 it follows that $e = j$, that is, e is an isomorphism.

Conversely, let $m: B \to A$ be a monic in $\mathscr{C}$. Factor m as $e_m; i_m$ with $e_m \in \mathscr{E}(B, B')$. Since m is a monic, it follows that e_m is a monic, and, therefore, e_m is an isomorphism. Hence $m \sim_A i_m$. □

4. Constructions of weak inclusion systems

This section is concerned with building weak inclusion systems for a category. First, a common part is presented and then two distinct ways are described. Finally, we show how these results may be applied to build weak inclusion systems for two particular fields.

The theorems of the following two subsections may be used to construct weak inclusion systems for a category $\mathscr{C}$ using a faithful functor from $\mathscr{C}$ to a weak inclusive category.

We suppose in this section that $\mathscr{U}: \mathscr{C} \to \mathscr{D}$ is a faithful functor such that for each isomorphism $t: C \to C'$ if $\mathscr{U}(t)$ is an identity, $C = C'$.

A first consequence of this hypothesis is the following fact.

Fact 15. Let $t: A \to B$ and $s: B \to A$ be two morphisms in $\mathscr{C}$. If $\mathscr{U}(A) = \mathscr{U}(B)$ and $\mathscr{U}(t) = \mathscr{U}(s) = 1_{\mathscr{U}(A)}$, then $A = B$ and $t = s = 1_A$.

Proof. Since $\mathscr{U}$ is faithful and $\mathscr{U}(t; s) = \mathscr{U}(1_A)$, we deduce that $t; s = 1_A$. Similarly, it follows that $s; t = 1_B$, that is, t is an isomorphism, so by the hypothesis above, we obtain $A = B$. □

Lemma 16. Let $\mathscr{U}: \mathscr{C} \to \mathscr{D}$ be a functor as above. If $\mathscr{I}$ is a subcategory of $\mathscr{D}$ with $|\mathscr{I}| = |\mathscr{D}|$ and $\mathscr{I}$ is a partial order, then the family $\mathscr{I}_{\mathscr{C}} = \{i \in \mathscr{C} \mid \mathscr{U}(i) \in \mathscr{I}\}$ has the same objects as $\mathscr{C}$ and it is a partial order also. Moreover, if $\mathscr{I}$ is $\mathscr{I}$-right cancellable, $\mathscr{I}_{\mathscr{C}}$ is $\mathscr{I}_{\mathscr{C}}$-right cancellable.

Proof. It is obvious that $\mathscr{I}_{\mathscr{C}}$ is a subcategory of $\mathscr{C}$ with $|\mathscr{I}_{\mathscr{C}}| = |\mathscr{C}|$.

Let $i_1, i_2 \in \mathscr{I}_{\mathscr{C}}(A, B)$. Then $\mathscr{U}(i_1), \mathscr{U}(i_2) \in \mathscr{I}(\mathscr{U}(A), \mathscr{U}(B))$, and, since $\mathscr{I}$ is a partial order, it follows that $\mathscr{U}(i_1) = \mathscr{U}(i_2)$, and thus $i_1 = i_2$, because $\mathscr{U}$ is faithful. Hence $\|\mathscr{I}_{\mathscr{C}}(A, B)\| \leqslant 1$ for each A, $B \in |\mathscr{C}|$.

Let $i_1 \in \mathscr{I}_{\mathscr{C}}(A, B)$ and $i_2 \in \mathscr{I}_{\mathscr{C}}(B, A)$. Then $\mathscr{U}(i_1) \in \mathscr{I}(\mathscr{U}(A), \mathscr{U}(B))$ and $\mathscr{U}(i_2) \in \mathscr{I}(\mathscr{U}(B), \mathscr{U}(A))$, and, because $\mathscr{I}$ is a partial order, it follows that $\mathscr{U}(A) = \mathscr{U}(B)$ and $\mathscr{U}(i_1), \mathscr{U}(i_2)$ are identities. The fact above yields $A = B$.

Let $f \in \mathscr{C}(A, B)$ and $i \in \mathscr{I}_{\mathscr{C}}(B, C)$ be two morphisms such that $f; i \in \mathscr{I}_{\mathscr{C}}$. Applying $\mathscr{U}$, we obtain $\mathscr{U}(f); \mathscr{U}(i) \in \mathscr{I}$ and $\mathscr{U}(i) \in \mathscr{I}$. Since $\mathscr{I}$ is $\mathscr{I}$-right cancellable, it follows that $\mathscr{U}(f) \in \mathscr{I}$, that is, $f \in \mathscr{I}_{\mathscr{C}}$. □

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4.1. *Final morphisms*

In the following we define the notion of $\mathcal{U}$-final morphism. This definition generalizes a concept within topological spaces and has been taken over from Căzănescu (1972).

Definition 17. A morphism $f: A \to B$ of $\mathcal{C}$ is *$\mathcal{U}$-final* (see Căzănescu (1972)) if for each morphism $h: \mathcal{U}(B) \to \mathcal{U}(C)$ of $\mathcal{D}$ such that $\mathcal{U}(f);h$ is the image by $\mathcal{U}$ of a morphism in $\mathcal{C}(A, C)$, there exists $h': B \to C$ in $\mathcal{C}$ with $\mathcal{U}(h') = h$.

Theorem 18. Let $\mathcal{U}: \mathcal{C} \to \mathcal{D}$ be a functor as above. Assume that for each morphism $f: \mathcal{U}(C) \to D$ there exists a $\mathcal{U}$-final morphism $g: C \to C'$ such that $\mathcal{U}(C') = D$ and $\mathcal{U}(g) = f$. If $\mathcal{I}$ is a weak inclusion system for $\mathcal{D}$, then $\mathcal{I}_{\mathcal{C}} = \{i \in \mathcal{C} \mid \mathcal{U}(i) \in \mathcal{I}\}$ is a weak inclusion system for $\mathcal{C}$. Moreover, $f \in \mathcal{E}_{\mathcal{I}_{\mathcal{C}}}$ if and only if f is $\mathcal{U}$-final and $\mathcal{U}(f) \in \mathcal{E}_{\mathcal{I}}$.

Proof. We prove this result using Definition 10. By the lemma above, $\mathcal{I}_{\mathcal{C}}$ is a subcategory of $\mathcal{C}$ with $|\mathcal{I}_{\mathcal{C}}| = |\mathcal{C}|$, $\mathcal{I}_{\mathcal{C}}$ is a partial order and $\mathcal{I}_{\mathcal{C}}$ is $\mathcal{I}_{\mathcal{C}}$-right cancellable.

We show that $e \in \mathcal{C}(A, B)$ is $\mathcal{U}$-final and $\mathcal{U}(e) \in \mathcal{E}_{\mathcal{I}}$ imply $e \in \mathcal{E}_{\mathcal{I}_{\mathcal{C}}}$. Let $f \in \mathcal{C}(A, C)$, $g \in \mathcal{C}(B, D)$ and $i \in \mathcal{I}_{\mathcal{C}}(C, D)$ such that $e;g = f;i$. Since $\mathcal{U}(e);\mathcal{U}(g) = \mathcal{U}(f);\mathcal{U}(i)$, $\mathcal{U}(e) \in \mathcal{E}_{\mathcal{I}}$ and $\mathcal{U}(i) \in \mathcal{I}$, there exists a morphism $h: \mathcal{U}(B) \to \mathcal{U}(C)$ such that $\mathcal{U}(e);h = \mathcal{U}(f)$ and $h;\mathcal{U}(i) = \mathcal{U}(g)$. But e is $\mathcal{U}$-final, therefore there is $h': B \to C$ with $\mathcal{U}(h') = h$. Consequently there exists $h' \in \mathcal{C}(B, C)$ such that $e;h' = f$ and $h';i = g$ (by the faithfulness of $\mathcal{U}$), that is, $e \in \mathcal{E}_{\mathcal{I}_{\mathcal{C}}}$.

Now we show the existence of a factorisation. Let f be a morphism in $\mathcal{C}(A, B)$. Factor $\mathcal{U}(f)$ as $e_{\mathcal{U}(f)};i_{\mathcal{U}(f)}$ with $e_{\mathcal{U}(f)} \in \mathcal{E}_{\mathcal{I}}(\mathcal{U}(A), T)$ and $i_{\mathcal{U}(f)} \in \mathcal{I}(T, \mathcal{U}(B))$. By hypothesis there exists a $\mathcal{U}$-final morphism $e_f \in \mathcal{C}(A, T')$ such that $\mathcal{U}(e_f) = e_{\mathcal{U}(f)} \in \mathcal{E}_{\mathcal{I}}$, and hence $e_f \in \mathcal{E}_{\mathcal{I}_{\mathcal{C}}}$. Since e_f is $\mathcal{U}$-final and $\mathcal{U}(f) = \mathcal{U}(e_f);i_{\mathcal{U}(f)}$, there exists $i_f \in \mathcal{C}(T', B)$ with $\mathcal{U}(i_f) = i_{\mathcal{U}(f)}$. Hence $i_f \in \mathcal{I}_{\mathcal{C}}$ and $f = e_f;i_f$.

Therefore $\mathcal{I}_{\mathcal{C}}$ is a weak inclusion system. Note that if f above belongs to $\mathcal{E}_{\mathcal{I}_{\mathcal{C}}}$, we have $f = e_f$ by the uniqueness of factorisation, and, therefore, f is $\mathcal{U}$-final and $\mathcal{U}(f) \in \mathcal{E}_{\mathcal{I}}$. □

4.2. *Initial morphisms*

The concept of $\mathcal{U}$-initial morphism Căzănescu (1972) is the dual of the notion of $\mathcal{U}$-final morphism. We remind the reader of the definition.

Definition 19. A morphism $f: A \to B$ of $\mathcal{C}$ is *$\mathcal{U}$-initial* (see Căzănescu (1972)) if for each morphism $h: \mathcal{U}(C) \to \mathcal{U}(A)$ of $\mathcal{D}$ such that $h;\mathcal{U}(f)$ is the image by $\mathcal{U}$ of a morphism in $\mathcal{C}(C, B)$, there exists $h': C \to A$ in $\mathcal{C}$ with $\mathcal{U}(h') = h$.

Theorem 20. Let $\mathcal{U}: \mathcal{C} \to \mathcal{D}$ be a functor as above. Assume that for each morphism $f: D \to \mathcal{U}(C)$ there exists a $\mathcal{U}$-initial morphism $g: C' \to C$ such that $\mathcal{U}(C') = D$ and $\mathcal{U}(g) = f$. If $\mathcal{I}$ is a weak inclusion system for $\mathcal{D}$, then $\mathcal{J} = \{f \in \mathcal{C} \mid \mathcal{U}(f) \in \mathcal{I}$, and f is $\mathcal{U}$-initial$\}$ is a weak inclusion system for $\mathcal{C}$. Moreover, for each $g \in \mathcal{C}$, $g \in \mathcal{E}_{\mathcal{J}}$ if and only if $\mathcal{U}(g) \in \mathcal{E}_{\mathcal{I}}$.

Proof. Since the $\mathcal{U}$-initial morphisms of $\mathcal{C}$ form a subcategory with the same objects as $\mathcal{C}$, we deduce by Lemma 16 that $\mathcal{J}$ is a subcategory having the same objects as $\mathcal{C}$ and that

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$\mathscr{J}$ is a partial order. Since $f;g$ is $\mathscr{U}$-initial implies f is $\mathscr{U}$-initial, we deduce that $\mathscr{J}$ is $\mathscr{J}$-right cancellable.

For each morphism e of $\mathscr{C}$ we show $\mathscr{U}(e) \in \mathscr{E}_{\mathscr{I}}$ implies $e \in \mathscr{E}_{\mathscr{J}}$. Let $e;g = f;i$ where $i \in \mathscr{J}$. Since $\mathscr{U}(e);\mathscr{U}(g) = \mathscr{U}(f);\mathscr{U}(i)$, $\mathscr{U}(e) \in \mathscr{E}_{\mathscr{I}}$ and $\mathscr{U}(i) \in \mathscr{I}$, there exists h in $\mathscr{D}$ such that $\mathscr{U}(e);h = \mathscr{U}(f)$ and $h;\mathscr{U}(i) = \mathscr{U}(g)$. Since i is $\mathscr{U}$-initial, there exists h' in $\mathscr{C}$ with $\mathscr{U}(h') = h$. We deduce that $e;h' = f$ and $h';i = g$. Therefore $e \in \mathscr{E}_{\mathscr{J}}$.

Let $h: A \to C$ be a morphism from $\mathscr{C}$ and $\mathscr{U}(h) = e;i$ be the unique factorisation of $\mathscr{U}(h)$ in $\mathscr{D}$ with $e \in \mathscr{E}_{\mathscr{I}}(\mathscr{U}(A), D)$ and $i \in \mathscr{I}(D, \mathscr{U}(C))$. By hypothesis there exist $B \in |\mathscr{C}|$ and a $\mathscr{U}$-initial morphism $i_h \in \mathscr{C}(B, C)$ such that $\mathscr{U}(B) = D$ and $\mathscr{U}(i_h) = i$. Therefore $i_h \in \mathscr{J}$. Moreover, as i_h is $\mathscr{U}$-initial and $e;\mathscr{U}(i_h) = \mathscr{U}(h)$, there exists $e_h \in \mathscr{C}(A, B)$ such that $\mathscr{U}(e_h) = e$. Hence e_h is in $\mathscr{E}_{\mathscr{J}}$ and $h = e_h;i_h$ is a factorisation for h.

Therefore $\mathscr{J}$ is a weak inclusion system. Note that if h above is in $\mathscr{E}_{\mathscr{J}}$, then $h = e_h$ by the uniqueness of factorisation, and, therefore, $\mathscr{U}(h) \in \mathscr{E}_{\mathscr{I}}$. □

4.3. *Examples*

In this section we show how to build weak inclusion systems for two fields of interest. Note that **S***et*, the category of sets and functions, is a weak inclusive category where the inclusions really are inclusions.

4.3.1. *Topological spaces*

Let $\mathscr{U}: \mathbf{T}op \to \mathbf{S}et$ be the forgetful functor from topological spaces to sets. An application $f: (T, \sigma) \to (T', \sigma')$ is a morphism in **T***op* if it is a continuous application, *i.e.* for any $D' \in \sigma'$ it follows that $f^{-1}(D') \in \sigma$.

The following proposition appears in Căzănescu (1972) and we omit its proof.

Proposition 21. The morphism $f: (T, \sigma) \to (T', \sigma')$ is a $\mathscr{U}$-final morphism if and only if for each $D' \subseteq T'$, $D' \in \sigma'$ iff $f^{-1}(D') \in \sigma$. The morphism f is $\mathscr{U}$-initial if and only if the topology σ is generated by $\{f^{-1}(D') \mid D' \in \sigma'\}$.

Now we show that the functor $\mathscr{U}$ satisfies the hypotheses in Section 4.

For the first hypothesis at the beginning of Section 4, note that if $1_T: (T, \sigma) \to (T, \sigma')$ is an isomorphism, $\sigma = \sigma'$.

For the hypothesis of Theorem 18, let $f: (T, \sigma) \to T'$ be a function. We take $\sigma' = \{D' \subseteq T' \mid f^{-1}(D') \in \sigma\}$. It follows that (T', σ') is a topological space and f is a morphism. By the previous proposition we deduce that f is $\mathscr{U}$-final.

By Theorem 18,

$$\mathscr{I} = \{f: (T, \sigma) \to (T'\sigma') \mid T \subseteq T' \text{ and } f \text{ is a continuous inclusion}\}$$

is a weak inclusion system for **T***op*. Note that (T, σ) is a topology denser than the induced topology, that is, the induced topology is included in σ.

Since f belongs to $\mathscr{E}_{\mathscr{I}}$ iff f is a $\mathscr{U}$-final continuous surjection, we deduce that the weak inclusion system $\mathscr{I}$ is complete.

For the hypothesis of Theorem 20, let $f: T \to (T', \sigma')$ be a function. If σ is the topology generated by $\{f^{-1}(D) \mid D \in \sigma'\}$, then $f: (T, \sigma) \to (T', \sigma')$ is $\mathscr{U}$-initial.

By Theorem 20,

$$\mathscr{I} = \{(A, \sigma) \hookrightarrow (B, \tau) \mid A \subseteq B \text{ and } \sigma \text{ is the induced topology}\}$$

is a weak inclusion system. Since f belongs to $\mathscr{E}_{\mathscr{I}}$ iff f is a continuous surjection, we deduce that the weak inclusion system $\mathscr{I}$ is not complete because $\mathscr{E}_{\mathscr{I}}$ contains monics that are not isomorphisms.

4.3.2. *Institutional theories*

The following example will be easily understood by anyone familiar with the theory of institutions (*e.g.* see Goguen and Burstall (1993) or Tarlecki (1984)).

An institution consists of two functors denoted $\mathbf{S}en: \mathbf{S}ign \to \mathbf{S}et$ and $\mathbf{M}od: \mathbf{S}ign \to \mathbf{C}at^{op}$ together with a *Satisfaction Relation* $\models$ as a family of relations $\{\models_\Sigma \mid \Sigma \in |\mathbf{S}ign|\}$ where $\models_\Sigma \subseteq |\mathbf{M}od(\Sigma)| \times \mathbf{S}en(\Sigma)$, which verifies the *Satisfaction Condition*, that is, for any signature morphism $\phi: \Sigma \to \Sigma'$

$$(\forall M \in |\mathbf{M}od(\Sigma')|)\,(\forall e \in \mathbf{S}en(\Sigma))\; M \models_{\Sigma'} \mathbf{S}en(\phi)\,(e) \Leftrightarrow \mathbf{M}od(\phi)\,(M) \models_\Sigma e.$$

If $E \subseteq \mathbf{S}en(\Sigma)$, we will use $E^\bullet$ to denote the set of sentences satisfied in any model that satisfies E.

The following *Closure Lemma* (Goguen and Burstall 1993) holds.

Lemma 22. For each signature morphism $\phi: \Sigma \to \Sigma'$ and each $E \subseteq \mathbf{S}en(\Sigma)$ we have $\mathbf{S}en(\phi)\,(E^\bullet) \subseteq \mathbf{S}en(\phi)\,(E)^\bullet$.

A theory is a pair (Σ, E) where $\Sigma \in |\mathbf{S}ign|$ and E is a closed set of Σ-sentences, that is, $E \subseteq \mathbf{S}en(\Sigma)$ with $E = E^\bullet$. A morphism of theories $\phi: (\Sigma, E) \to (\Sigma', E')$ is a signature morphism with $\mathbf{S}en(\phi)\,(E) \subseteq E'$. We will denote by $\mathscr{T}h(\mathscr{Y})$ the category of the theories of an institution $\mathscr{Y}$. We will show that if $\mathbf{S}ign$ is a weak inclusive category, $\mathscr{T}h(\mathscr{Y})$ is a weak inclusive category also.

Let $\mathscr{U}: \mathscr{T}h(\mathscr{Y}) \to \mathbf{S}ign$ be the forgetful functor that preserves only the signature of a theory.

We show that the faithful functor $\mathscr{U}$ satisfies the hypotheses in Section 4.

For the first hypothesis at the beginning of Section 4 note that if $1_\Sigma: (\Sigma, E) \to (\Sigma, E')$ is a theory isomorphism $E = E'$.

Fact 23. Let $\phi: (\Sigma, E) \to (\Sigma', E')$ be a theory morphism. Then
1. ϕ is $\mathscr{U}$-final iff $\mathbf{S}en(\phi)\,(E)^\bullet = E'$.
2. ϕ is $\mathscr{U}$-initial iff $E = \mathbf{S}en(\phi)^{-1}(E')$.

This fact shows that the hypotheses in Theorems 18 and 20 are fulfilled.

By Theorem 18,

$$\mathscr{I} = \{\phi: (\Sigma, E) \to (\Sigma', E') \mid \phi: \Sigma \hookrightarrow \Sigma' \text{ is an inclusion}\}$$

is a weak inclusion system for $\mathscr{T}h(\mathscr{Y})$.

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On the other hand, by Theorem 20,

$$\mathscr{J} = \{\phi : (\Sigma, E) \to (\Sigma', E') \mid \phi : \Sigma \hookrightarrow \Sigma' \text{ and } E = \mathbf{S}en(\phi)^{-1}(E')\}$$

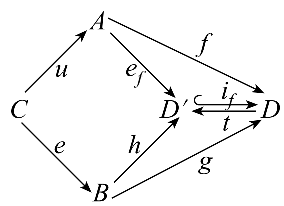
is another weak inclusion system for $\mathscr{T}h(\mathscr{Y})$.

5. Pushout and pullback properties

This section is concerned with giving preservation properties of the morphisms from both $\mathscr{E}$ and $\mathscr{I}$ by pushouts and pullbacks, respectively. Similar results are known for a lot of particular cases of weak inclusion systems as well as for factorisation systems.

Proposition 24. Let $\mathscr{C}$ be a weak inclusive category with $\langle \mathscr{I}, \mathscr{E} \rangle$ its weak inclusion system. Let $u: C \to A$ be in $\mathscr{C}$ and $e: C \to B$ be in $\mathscr{E}$. If $\langle f: A \to D, g: B \to D \rangle$ is a pushout in $\mathscr{C}$ of these morphisms, $f \in \mathscr{E}$.

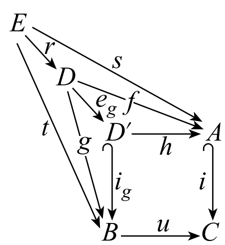
Proof. Factor f as $e_f; i_f$ with $e_f: A \to D'$.



Note that $e; g = (u; e_f); i_f$, and therefore, according to the diagonal-fill lemma, there is a unique morphism $h: B \to D'$ such that $e; h = u; e_f$ and $h; i_f = g$. Since $\langle e_f, h \rangle$ is a co-cone for $\langle u, e \rangle$, there is a unique $t: D \to D'$ such that $f; t = e_f$ and $g; t = h$. For the morphism $t; i_f$ we have $f; (t; i_f) = e_f; i_f = f$ and $g; (t; i_f) = h; i_f = g$. Since $\langle f, g \rangle$ is a pushout, it follows that $t; i_f = 1_D$, and because $1_D \in \mathscr{E}$, it yields $i_f = 1_D$ by Part 4 in Fact 5. Hence $f = e_f$.

Proposition 25. Let $\mathscr{C}$ be a weak inclusive category with pullbacks. Let $i: A \hookrightarrow C$ be an inclusion and $u: B \to C$ be a morphism. Then there is a unique pullback in $\mathscr{C}$ for these morphisms such that the opposite arrow of i is also in $\mathscr{I}$.

Proof. Existence
Let $\langle f: D \to A,\ g: D \to B \rangle$ be a pullback of $\langle i, u \rangle$. Factor g as $e_g; i_g$ with $e_g: D \to D'$.



Note that $f; i = e_g;\ (i_g; u)$, and therefore, according to the diagonal-fill lemma, there is a unique morphism $h: D' \to A$ such that $e_g; h = f$ and $h; i = i_g; u$. Therefore $\langle h, i_g \rangle$ is a cone for $\langle i, u \rangle$. Let $\langle s: E \to A,\ t: E \to B \rangle$ be another cone for $\langle i, u \rangle$. Since $\langle f, g \rangle$ is a pullback, it follows that there is a unique $r: E \to D$ such that $r; f = s$ and $r; g = t$. So the morphism r;

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e_g verifies $(r;e_g);h = s$ and $(r;e_g);i_g = t$. The fact that i_g is a monic implies the uniqueness of $r;e_g$ with these properties. Hence $\langle h, i_g \rangle$ is a pullback.
Uniqueness
If $\langle h_1 : D_1 \to A, i_1 : D_1 \hookrightarrow B\rangle$ and $\langle h_2 : D_2 \to A, i_2 : D_2 \hookrightarrow B\rangle$ are two pullbacks of $\langle i, u\rangle$, then according to Part 3 in Fact 5 it follows that $D_1 = D_2$. Now it is obvious that $h_1 = h_2$ (i is a monic). □

The following corollary appears in Diaconescu (1993) as a particular case of the above proposition for inclusion systems.

Corollary 26. Let $\mathscr{C}$ be a weak inclusive category that has pullbacks. Let $i_A : A \hookrightarrow C$ and $i_B : B \hookrightarrow C$ be two inclusions. Then there is a unique pullback in $\mathscr{C}$ such that all the arrows are inclusions.

Proof. By the previous proposition, let $\langle h : D \to A, i : D \hookrightarrow B\rangle$ be the unique pullback such that $i \in \mathscr{I}$. Since $h;i_A = i;i_B$ is in $\mathscr{I}$, one deduces $h \in \mathscr{I}$ by Part 3 in Fact 5. □

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