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NONNEGATIVITY OF COVARIANCES BETWEEN FUNCTIONS OF ORDERED RANDOM VARIABLES

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In this article we investigate conditions by a unified method under which the covariances of functions of two adjacent ordered random variables are nonnegative. The main structural results are applied to several kinds of ordered random variable, such as delayed record values, continuous and discrete $\ell_\infty^{\leqslant}$-spherical order statistics, epoch times of mixed Poisson processes, generalized order statistics, discrete weak record values, and epoch times of modified geometric processes. These applications extend the main results for ordinary order statistics in Qi [28] and for usual record values in Nagaraja and Nevzorov [25].

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with a common distribution function F, and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Qi [28] proved that

$$\mathrm{Cov}(\phi(X_{i:n}), \phi(X_{i+1:n})) \ge 0, \qquad i = 1, \ldots, n-1, \tag{1.1}$$

for all measurable real-valued functions ϕ such the covariance exists. The result for the case $n = 2$ is due to Ma [23, 24]. Qi [28] and Li [22] gave counterexamples to illustrate that (1.1) does not hold for nonadjacent order statistics; that is, for any $n > 2$ and each pair (i, j), $1 \le i, j \le n$ and $|i - j| > 1$, there exists a function ϕ such that $\mathrm{Cov}(\phi(X_{i:n}), \phi(X_{j:n})) < 0$.

Nagaraja and Nevzorov [25] established the analogous result of (1.1) for usual record values (the exact definition can be found in the sequel). More precisely, let

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$\{X_{L(n)},\ n \in \mathbb{N}_+\}$ denote the record values of a sequence $\{X_n,\ n \in \mathbb{N}_+\}$ of i.i.d. random variables with a common distribution function F; here and henceforth $\mathbb{N}_+ = \{1, 2, \ldots\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$; They proved that if F is continuous, then

$$\mathrm{Cov}(\phi(X_{L(n)}), \phi(X_{L(n+1)})) \geq 0, \qquad n \in \mathbb{N}_+, \tag{1.2}$$

for all measurable real-valued functions ϕ such that the covariance exists. Counterexamples were also given in Nagaraja and Nevzorov [25] to show that (1.2) does not hold when F is discrete and that, for any $i, j \in \mathbb{N}_+$, $|i - j| \geq 2$ and a continuous distribution function F, there exists a function ϕ such that $\mathrm{Cov}(\phi(X_{L(i)}), \phi(X_{L(j)})) < 0$.

The purpose of this article is to investigate conditions by a unified method under which the covariances between functions of two adjacent ordered random variables are nonnegative. In Section 2 we give some structural theorems concerning general ordered random variables $T_1 \leq T_2 \leq \cdots \leq T_n$. These structural results are then applied to continuous and discrete ordered random variables in Sections 3 and 4, respectively. In Section 3 we consider delayed record values, continuous $\ell_\infty^{\leqslant}$-spherical order statistics, epoch times of mixed Poisson processes, and generalized order statistics. In Section 4 we consider discrete weak record values, discrete $\ell_\infty^{\leqslant}$-spherical order statistics, and epoch times of modified geometric processes. These applications extend (1.1) and (1.2) to the more general ordered random variables. Some counterexamples are presented in Section 5.

Throughout, "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively. When an expectation or a probability is conditioned on an event such as $\mathbf{X} = \mathbf{x}$, we assume that $\mathbf{x}$ is in the support of $\mathbf{X}$. Also, we denote by $[\mathbf{X}|A]$ any random vector (variable) whose distribution is the conditional distribution of $\mathbf{X}$ given event A.

2. MAIN RESULTS

Motivated by the idea used in the proof of Theorem 3.1 in Qi [28], we have the following two structural theorems, which give sufficient conditions to ensure nonnegativity of covariances between functions of two ordered random variables.

THEOREM 2.1: *Let $T_1 \leq T_2 \leq T_3 \leq T_4$ be four random variables such that, for any $s \leq t$,*

$$[(T_2, T_3)|T_1 = s, T_4 = t] \stackrel{d}{=} (V_{1:2}(s,t), V_{2:2}(s,t)), \tag{2.1}$$

where $\stackrel{d}{=}$ means equality in distribution and $V_{1:2}(s,t) \leq V_{2:2}(s,t)$ are the order statistics of i.i.d. random variables $V_1(s,t)$ and $V_2(s,t)$. Then

$$\mathrm{Cov}(\phi(T_2), \phi(T_3)) \geq 0 \tag{2.2}$$

for all measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: We use an idea from the proof of Theorem 3.1 in Qi [28]. First, assume that the family $\{V_i(s,t) : i = 1, 2, \forall, s \leq t\}$ is independent of $\{T_1, T_4\}$. For simplicity of notation, let $W_i = V_i(T_1, T_4)$ and $W_{i:2} = V_{i:2}(T_1, T_4)$ for $i = 1, 2$; that is, $W_{1:2}$ and $W_{2:2}$ are the order

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statistics of W_1 and W_2. By conditioning on T_1 and T_4, it follows from (2.1) that

$$\begin{aligned}\mathsf{E}[\phi(T_2)\phi(T_3)] &= \mathsf{E}\{\mathsf{E}[\phi(T_2)\,\phi(T_3)|T_1,T_4]\}\\ &= \mathsf{E}\{\mathsf{E}[\phi(W_{1:2})\,\phi(W_{2:2})|T_1,T_4]\}\\ &= \mathsf{E}\{\mathsf{E}[\phi(W_1)\,\phi(W_2)|T_1,T_4]\}\\ &= \mathsf{E}\{\mathsf{E}[\phi(W_1)|T_1,T_4]\}^2, \end{aligned} \tag{2.3}$$

where the last equality follows from the fact that, given (T_1, T_4), W_1 and W_2 are conditionally i.i.d. Similarly,

$$\mathsf{E}[\phi(T_2)] = \mathsf{E}\{\mathsf{E}[\phi(T_2)|T_1,T_4]\} = \mathsf{E}\{\mathsf{E}[\phi(W_{1:2})|T_1,T_4]\} = \mathsf{E}[\phi(W_{1:2})] \tag{2.4}$$

and

$$\mathsf{E}[\phi(T_3)] = \mathsf{E}[\phi(W_{2:2})]. \tag{2.5}$$

It is obvious that

$$\mathsf{E}[\phi(W_{1:2})] + \mathsf{E}[\phi(W_{2:2})] = 2\mathsf{E}[\phi(W_1)].$$

Therefore, from (2.3)–(2.5), we get that

$$\begin{aligned}\mathrm{Cov}(\phi(T_2),\phi(T_3)) &= \mathsf{E}\{\mathsf{E}[\phi(W_1)|T_1,T_4]\}^2 - \mathsf{E}[\phi(W_{1:2})]\,\mathsf{E}[\phi(W_{2:2})]\\ &= \mathrm{Var}(\mathsf{E}[\phi(W_1)|T_1,T_4]) + (\mathsf{E}[\phi(W_1)])^2 - \mathsf{E}[\phi(W_{1:2})]\,\mathsf{E}[\phi(W_{2:2})]\\ &= \mathrm{Var}(\mathsf{E}[\phi(W_1)|T_1,T_4]) + \frac{1}{4}\{\mathsf{E}[\phi(W_{1:2})] - \mathsf{E}[\phi(W_{2:2})]\}^2\\ &\geq 0.\end{aligned}$$

This completes the proof of the theorem. ■

THEOREM 2.2: *Let $T_1 \leq T_2 \leq T_3$ [resp. $T_2 \leq T_3 \leq T_4$] be three random variables such that, for any s,*

$$[(T_2,T_3)|T_1 = s] \stackrel{d}{=} (V_{1:2}(s), V_{2:2}(s))$$
$$[resp.\ [(T_2,T_3)|T_4 = s] \stackrel{d}{=} (V_{1:2}(s), V_{2:2}(s))], \tag{2.6}$$

where $V_{1:2}(s) \leq V_{2:2}(s)$ are the order statistics of i.i.d. random variables $V_1(s)$ and $V_2(s)$. Then (2.2) holds for all measurable functions $\phi: \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: The proof is the same as that of Theorem 2.1 with minor modification and, hence, is omitted. ■

Recall from Shaked [30] and Rinott and Pollak [29] that two random variables X_1 and X_2 are said to *positive function dependent* (PFD) if

$$\mathrm{Cov}(\phi(X_1), \phi(X_2)) \geq 0$$

for all real-valued function ϕ such that the covariance exists. It is noted that a number of interchangeable bivariate distributions (i.e., their joint distribution function is symmetric) are PFD. For example, if (X, Y) is conditionally i.i.d., then (X, Y) is PFD. Shaked [30] proved that the class of PFD distributions is closed under convolution, mixture, and convergence in distribution and also showed that not all PFD distributions are conditionally i.i.d.

Remark 2.3: From the proof of Theorem 2.1, it is seen that the independence property of $V_1(s, t)$ and $V_2(s, t)$ is used in (2.3). If, instead, $V_1(s, t)$ and $V_2(s, t))$ are PFD and interchangeable, then (2.3) is replaced by

$$\mathsf{E}[\phi(T_2)\phi(T_3)] = \mathsf{E}\{\mathsf{E}[\phi(W_1)\phi(W_2)|T_1, T_4]\} \geq \mathsf{E}\{\mathsf{E}[\phi(W_1)|T_1, T_4]\}^2,$$

and thus the conclusions of Theorems 2.1 and 2.2 are also valid.

We now give two special applications of Theorems 2.1 and 2.2. The corresponding results are stated as the following two theorems (Theorems 2.4 and 2.6), which will be used in Sections 3 and 4, respectively. Further applications of Theorems 2.1 and 2.2 will be given in Sections 3 and 4.

THEOREM 2.4: *Let W, Z_1, and Z_2 be independent random variables such that Z_i has an exponential distribution with failure rate λ_i for $i = 1,2$. If*

$$2\lambda_2 \geq \lambda_1, \tag{2.7}$$

then

$$\mathrm{Cov}\big(\phi(W + Z_1), \phi(W + Z_1 + Z_2)\big) \geq 0 \tag{2.8}$$

for all measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: First, assume that $2\lambda_2 > \lambda_1$. Let Z_3 be another exponential random variable, independent of everything else, with failure rate $\lambda_3 = 2\lambda_2 - \lambda_1 > 0$, and set $T_1 = W$ and $T_j = W + \sum_{i=1}^{j-1} Z_i$ for $j = 2, 3, 4$. Without loss of generality, assume that W is absolutely continuous with density function f_W. Then the joint density function of $(T_1, \ldots, T_4)$ is given by

$$f_{T_1,\ldots,T_4}(t_1, t_2, t_3, t_4) = f_W(t_1)\lambda_1\lambda_2\lambda_3 e^{-\lambda_3 t_4 + \lambda_1 t_1} e^{\delta(t_2+t_3)}, \qquad t_1 < t_2 < t_3 < t_4,$$

where $\delta = \lambda_2 - \lambda_1$. Hence, the conditional density function of $[(T_2,T_3)|T_1 = s, T_4 = t]$, $s < t$, is given by

$$\begin{aligned} g(t_2, t_3|s, t) &= \frac{e^{\delta(t_2+t_3)}}{\int\int_{s<x_2<x_3<t} e^{\delta(x_2+x_3)}\, dx_2\, dx_3} \\ &= 2!g_{s,t}(t_2)g_{s,t}(t_3), \qquad s < t_2 < t_3 < t, \end{aligned}$$

where

$$g_{s,t}(x) = \begin{cases} \dfrac{\delta e^{\delta x}}{e^{\delta t} - e^{\delta s}}, & s < x < t \\ 0, & \text{otherwise} \end{cases}$$

for $\delta \neq 0$, and

$$g_{s,t}(x) = \begin{cases} \dfrac{1}{t-s}, & s < x < t \\ 0, & \text{otherwise} \end{cases}$$

for $\delta = 0$; that is, condition (2.1) is satisfied where $V_1(s, t)$ and $V_2(s, t)$ are i.i.d. with density function $g_{s,t}$. Therefore, the desired result for this case now follows from Theorem 2.1.

Next, assume that $2\lambda_2 = \lambda_1$. Let δ and T_1, T_2, T_3 be as defined earlier. Then the conditional density function of $[(T_2, T_3)|T_1 = s]$ for any s is given by

$$\begin{aligned} g(t_2, t_3|s) &= \frac{e^{\delta(t_2+t_3)}}{\displaystyle\iint_{s<x_2<x_3} e^{\delta(x_2+x_3)}\, dx_2\, dx_3} \\ &= 2! g_s(t_2) g_s(t_3), \qquad s < t_2 < t_3, \end{aligned}$$

where

$$g_s(x) = \begin{cases} \lambda_2 e^{-\lambda_2(x-s)}, & x > s \\ 0, & \text{otherwise;} \end{cases}$$

that is, condition (2.6) is satisfied where $V_1(s)$ and $V_2(s)$ are i.i.d. with density function g_s. Therefore, the desired result for this case now follows from Theorem 2.2. This completes the proof of the theorem. ■

It is shown by Counterexample 5.1 that (2.8) does not hold if condition (2.7) is violated. To state and prove the next theorem, we need the following lemma.

LEMMA 2.5: *Let V_1 and V_2 be two discrete random variables with support $\mathcal{S}$ and with joint mass function given by*

$$\mathsf{P}(V_1 = x, V_2 = y) = \begin{cases} c\eta_x\eta_y, & x = y, x \in \mathcal{S} \\ \frac{1}{2}c\eta_x\eta_y, & x \neq y, (x, y) \in \mathcal{S}^2 \\ 0, & (x, y) \in \mathcal{S}^2, \end{cases}$$

where $\{\eta_x, x \in \mathcal{S}\}$ is a sequence of positive real numbers and c is the normalizing constant given by

$$c = \left[\sum_{x,y \in \mathcal{S}, x \leq y} \eta_x \eta_y\right]^{-1} < +\infty.$$

Then V_1 and V_2 are conditionally i.i.d. and, hence, PFD.

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PROOF: Let U_0, U_1, and U_2 be independent discrete random variables with support $\mathcal{S}$ and with probability mass functions given respectively by

$$h_0(x) = \frac{\eta_x^2}{\sum_{y\in\mathcal{S}} \eta_y^2}, \qquad x \in \mathcal{S},$$

and

$$h_i(x) = \frac{\eta_x}{\sum_{y\in\mathcal{S}} \eta_y}, \qquad x \in \mathcal{S},\ i = 1, 2.$$

Let I be a Bernoulli random variable, independent of $\{U_0, U_1, U_2\}$, with probability mass function given by

$$\mathsf{P}(I = 0) = \frac{c}{2}\left(\sum_{x\in\mathcal{S}} \eta_x\right)^2, \qquad \mathsf{P}(I = 1) = \frac{c}{2}\sum_{x\in\mathcal{S}} \eta_x^2.$$

Straightforward computations yield that

$$(V_1, V_2) \stackrel{d}{=} (IU_0 + (1 - I)U_1, IU_0 + (1 - I)U_2),$$

which implies that given (U_0, I), (V_1, V_2) is conditionally i.i.d. and, hence, PFD. This completes the proof of the lemma. ■

THEOREM 2.6: *Let W, B_1, and B_2 be independent random variables such that B_i has a geometric distribution with parameter p_i for $i = 1, 2$; that is, $\mathsf{P}(B_i = n) = p_i(1 - p_i)^n$ for $n \in \mathbb{N}$. If*

$$(1 - p_2)^2 \le 1 - p_1, \tag{2.9}$$

then

$$\mathrm{Cov}(\phi(W + B_1), \phi(W + B_1 + B_2)) \ge 0$$

holds for all measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: The proof is similar to that of Theorem 2.4. First, assume that $(1 - p_2)^2 < 1 - p_1$. Let B_3 be another geometric random variable, independent of everything else, with parameter

$$p_3 = 1 - \frac{(1 - p_2)^2}{1 - p_1} > 0.$$

Let $T_1 = W$ and $T_j = W + \sum_{i=1}^{j-1} B_i$ for $j = 2, 3, 4$. Without loss of generality, assume that W is discrete with probability mass function f_W. Then the joint mass function of

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$(T_1, \ldots, T_4)$ is given by

$$\begin{aligned} \mathsf{P}(T_i = t_i, i = 1, \ldots, 4) &= f_W(t_1) \prod_{i=1}^{3} p_i (1 - p_i)^{t_{i+1} - t_i} \\ &= f_W(t_1) p_1 p_2 p_3 (1 - p_1)^{t_1} (1 - p_3)^{t_4} \left(\frac{1 - p_1}{1 - p_2} \right)^{t_2} \left(\frac{1 - p_2}{1 - p_3} \right)^{t_3} \\ &= f_W(t_1) p_1 p_2 p_3 (1 - p_1)^{t_1} (1 - p_3)^{t_4} \delta^{t_2 + t_3} \end{aligned}$$

for $t_1 \leq t_2 \leq t_3 \leq t_4$, where

$$\delta \equiv \frac{1 - p_1}{1 - p_2}.$$

Hence, the conditional mass function of $[(T_2, T_3) | T_1 = s,\ T_4 = t]$, $s \leq t$, is given by

$$\begin{aligned} \mathsf{P}(T_2 = t_2, T_3 = t_3 | T_1 = s, T_4 = t) &= \frac{\delta^{t_2 + t_3}}{\sum\limits_{0 \leq i \leq j \leq t - s} \delta^{2s + i + j}} \\ &= \begin{cases} 2! g_{s,t}(t_2, t_3), & s \leq t_2 < t_3 \leq t \\ g_{s,t}(t_2, t_3), & s \leq t_2 = t_3 \leq t, \end{cases} \end{aligned} \qquad \textbf{(2.10)}$$

where

$$g_{s,t}(x, y) = \begin{cases} c_{s,t} \delta^{x + y - 2s}, & s \leq x = y \leq t \\ \frac{1}{2} c_{s,t} \delta^{x + y - 2s}, & s \leq x \neq y \leq t \\ 0, & \text{otherwise}, \end{cases} \qquad (x, y) \in \{s, s + 1, \ldots, t\}^2,$$

is the joint mass function of some interchangeable random variables $V_1(s,\ t)$ and $V_2(s,\ t)$. Here, $c_{s,t} = [\sum_{0 \leq i \leq j \leq t-s} \delta^{i+j}]^{-1}$. Clearly, (2.10) means that condition (2.1) is satisfied. By Lemma 2.5, $(V_1(s,\ t),\ V_2(s, t))$ is conditionally i.i.d. and, hence, PFD. Therefore, the desired result for this case now follows from Remark 2.3.

Next, assume that $(1 - p_2)^2 = 1 - p_1$. Let T_1, T_2, and T_3 be as defined earlier. Then the conditional mass function of $[(T_2,\ T_3) | T_1 = s]$ is given by

$$\mathsf{P}(T_2 = t_2, T_3 = t_3 | T_1 = s) = \frac{(1 - p_2)^{t_2 + t_3}}{\sum\limits_{0 \leq i \leq j} (1 - p_2)^{2s + i + j}}, \qquad s \leq t_2 \leq t_3.$$

A similar argument to the above paragraph yields that (2.6) is satisfied and $(V_1(s),\ V_2(s))$ is PFD. Therefore, the desired result for this case now follows from Remark 2.3. This completes the proof of the theorem. ■

It is worthwhile to mention that for discrete random variables T_i's, if $\mathsf{P}(T_2 < T_3 | T_1 = s,\ T_4 = t) = 1$ for $s < t$, then representation (2.1) does not hold. So, in

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Theorem 2.6, if the geometric distribution is replaced by the one truncated at zero, then the conclusion of the theorem is in general not true.

3. APPLICATIONS TO ORDERED CONTINUOUS RANDOM VARIABLES

3.1. Delayed Record Values

Let $\{X_n, n \in \mathbb{N}_+\}$ be a sequence of i.i.d. random variables with a continuous distribution F. Let Y be a random variable independent of $\{X_n, n \in \mathbb{N}_+\}$. The delayed record value sequence is $\{X^Y_{L(n)}, n \in \mathbb{N}\}$, where $L(0) = 0$, $X^Y_{L(0)} = Y$,

$$L(n) = \inf\left\{i > L(n-1) : X_i > X^Y_{L(n-1)}\right\}, \qquad n \in \mathbb{N}_+,$$

and $X^Y_{L(n)}$ is the first X_i in the sequence after $X^Y_{L(n-1)}$ to exceed $X^Y_{L(n-1)}$; see Wei and Hu [38]. The reason for the adjective "delayed" is that record values are not kept until after a value Y has been observed. The usual record value sequence $\{X_{L(n)}, n \in \mathbb{N}_+\}$ is obtained with $Y = F^{-1}(0)$, where

$$F^{-1}(u) = \sup\{x : F(x) \le u\}, \qquad u \in [0, 1).$$

In this case, the superscript Y is suppressed. The record values have been extensively studied in the literature. For an excellent review, we refer to Ahsanullah [1, 2] and Arnold, Balakrishnan, and Nagaraja [4].

The following lemma presents a stochastic representation of delayed record values by partial sums of i.i.d. exponential random variables.

LEMMA 3.1 (Wei and Hu [38]): *Let $\{Z_n, n \in \mathbb{N}_+\}$ be a sequence of i.i.d. unit rate exponential random variables, independent of Y. If F is continuous, then*

$$\left\{X^Y_{L(0)}, X^Y_{L(1)}, X^Y_{L(2)}, \ldots\right\} \overset{d}{=} \{H(U), H(U + Z_1), H(U + Z_1 + Z_2), \ldots\},$$

where $U = -\ln(1 - F(Y))$ and $H(x) = F^{-1}(1 - e^{-x})$ for $x \in \mathbb{R}_+$.

THEOREM 3.2: *Let $\{X^Y_{L(n)}, n \in \mathbb{N}\}$ be a sequence of delayed record values of i.i.d. random variables $\{X_n, n \in \mathbb{N}_+\}$ with a continuous distribution function F. Then*

$$\mathrm{Cov}(\phi(X^Y_{L(n)}), \phi(X^Y_{L(n+1)})) \ge 0, \qquad n \in N_+,$$

for all measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: Let ϕ be any function such that the covariance exists. and define $\psi(x) = \phi(H(x))$ for $x \in \mathbb{R}_+$. Denote $W = U + \sum_{i=1}^{n-1} Z_i$, where U and the Z_i's are defined in Lemma 3.1. Then by Lemma 3.1 and Theorem 2.4, we have

$$\mathrm{Cov}(\phi(X^Y_{L(n)}), \phi(X^Y_{L(n+1)})) = \mathrm{Cov}(\psi(W + Z_n), \psi(W + Z_n + Z_{n+1})) \ge 0.$$

This completes the proof. ■

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An immediate consequence of Theorem 3.1 is (1.2) (Theorem 1 in Nagaraja and Nevzorov [25]). They proved it by using the properties of Laguerre polynomials and expanding the function $\phi(x)$ into a series in Laguerre polynomials.

3.2. Continuous $\ell_\infty^{\leqslant}$-Spherical Order Statistics

$\ell_\infty^{\leqslant}$-Spherical order statistics arise naturally in the Bayesian statistical theory of reliability; see Spizzichino [34, Sects. 1.4 and 4.3]. Nonnegative random variables $T_1 \le T_2 \le \cdots \le T_n$ are said to be *$\ell_\infty^{\leqslant}$-spherical order statistics* if their joint density function is of the form

$$f_{T_1,\ldots,T_n}(t_1,\ldots,t_n) = \begin{cases} \varphi(t_n), & 0 \le t_1 \le t_2 \le \cdots \le t_n \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

for some nonnegative function φ. The T_i's can be regarded as the order statistics of interchangeable random variables $X_1, X_2, \ldots, X_n$ with density function given by

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \frac{1}{n!}\varphi\left(\max_{i=1}^{n} x_i\right), \qquad (x_1,x_2,\ldots,x_n) \in \mathbb{R}_+^n$$

which is called *spherical in ℓ_∞-norm.* Define

$$Z_1 = T_1, \qquad Z_2 = T_2 - T_1, \qquad \ldots, \qquad Z_n = T_n - T_{n-1}.$$

Then $T_1, \ldots, T_n$ are $\ell_\infty^{\leqslant}$-spherical order statistics if and only if the density function of $(Z_1, \ldots, Z_n)$ is of the form

$$f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n) = \varphi\left(\sum_{i=1}^{n} z_i\right), \qquad (z_1,z_2,\ldots,z_n) \in \mathbb{R}_+^n,$$

which is called *Schur constant* (see Spizzichino [34]).

Shaked, Spizzichino, and Suter [32, 33] characterized, among other things, $\ell_\infty^{\leqslant}$-spherical distributions by means of epoch times of nonhomogeneous pure birth processes and by means of the uniform and general order statistics property.

THEOREM 3.3: *Let $T_1 \le T_2 \le \cdots \le T_n$ be $\ell_\infty^{\leqslant}$-spherical order statistics with density function of the form (3.1), and let ϕ be any measurable real-valued function such that the covariances below exist. Then*

$$\mathrm{Cov}(\phi(T_r), \phi(T_{r+1})) \ge 0 \tag{3.2}$$

for $r = 1, \ldots, n-2$. Moreover, if φ is differentiable and decreasing, then (3.2) holds for $r = n-1$.

PROOF: Fix $r \in \{1, 2, \ldots, n-2\}$. By Proposition 2.4 in Shaked et al. [33] we have

$$[(T_1, T_2, \ldots, T_{r+1}) | T_{r+2} = t] \stackrel{d}{=} (V_{1:r+1}, V_{2:r+1}, \ldots, V_{r+1:r+1}),$$

where $V_{1:r+1} \le V_{2:r+1} \le \cdots \le X_{r+1:r+1}$ are order statistics of i.i.d. uniform (0, t)

random variables $V_1, V_2, \ldots, V_{r+1}$. It can be checked that

$$[(T_r, T_{r+1})|T_{r-1} = s, T_{r+2} = t] \stackrel{d}{=} [(V_{r:r+1}, V_{r+1:r+1})|V_{r-1:r+1} = s] \\ \stackrel{d}{=} (V_{1:2}(s,t), V_{2:2}(s,t)),$$

where $V_{1:2}(s, t) \leq V_{2:2}(s, t)$ are the order statistics of i.i.d. uniform (s_0, t) random variables $V_1(s, t)$ and $V_2(s, t)$, and the second equality follows from Arnold, Balakrishnan, and Nagaraja [3, pp. 25–26]. Now, by Theorem 2.1, (3.2) follows.

If φ is differentiable and decreasing, Shaked et al. [33] proved that there exists a random variable T_{n+1} such that $T_1 \leq \cdots \leq T_n \leq T_{n+1}$ have an $\ell_\infty^{\leqslant}$-spherical density of the form

$$f_{T_1,\ldots,T_{n+1}}(t_1, \ldots, t_{n+1}) = \begin{cases} \widetilde{\varphi}(t_{n+1}), & 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1} \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\widetilde{\varphi}(x) = -\frac{d}{dx}\varphi(x), \qquad x \in \mathbb{R}_+.$$

Thus, (3.2) holds for $r = n - 1$ by the same reasoning as in the above paragraph. This completes the proof. ■

Counterexample 5.5 illustrates that (3.2) cannot be true for $r = n - 1$ if φ is differentiable but not decreasing.

Remark 3.4: By applying increasing transformation, the conclusion of Theorem 3.3 also holds for ordered random variables $T_1 \leq T_2 \leq \cdots \leq T_n$ with joint density of the form

$$f_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = \begin{cases} \varphi(a(t_n)) \prod_{i=1}^{n} a'(t_i), & 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \\ 0, & \text{otherwise}, \end{cases}$$

for some nonnegative function φ and some strictly increasing and differentiable function $a : \mathbb{R}_+ \to \mathbb{R}_+$.

We now consider epoch times of mixed Poisson processes. A counting process $\{N(t),\ t \in \mathbb{R}_+\}$ is said to be a mixed Poisson process if there exist a nonnegative random variable Λ and a unit rate homogeneous Poisson process $\{\tilde{N}(t),\ t \in \mathbb{R}_+\}$, independent of each other, such that

$$\{N(t), t \in \mathbb{R}_+\} \stackrel{d}{=} \{\widetilde{N}(\Lambda t), t \in \mathbb{R}_+\}. \tag{3.3}$$

Equivalently, $\{N(t),\ t \in \mathbb{R}_+\}$ is a mixed Poisson process if and only if the interepoch intervals $\{Z_i,\ i \in \mathbb{N}_+\}$ of $\{N(t),\ t \in \mathbb{R}_+\}$ are a mixture of i.i.d. exponential random variables; that is, for any $n \in \mathbb{N}_+$, the joint density of $(Z_1, Z_2, \ldots, Z_n)$ is of the form

$$f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n) = \int_{0-}^{\infty} \lambda^n e^{-\lambda(z_1+\cdots+z_n)} \mathrm{d}\, G(\lambda), \qquad (z_1,\ldots,z_n) \in \mathbb{R}_+^n, \tag{3.4}$$

where G is the distribution function of some nonnegative random variable Λ. Mixed Poisson processes play an important role in many branches of applied probability (for instance, in actuarial mathematics and physics). Grandell [16] provided a detailed coverage of the theory and applications of mixed Poisson processes.

Puri [27] and Hayakawa [17] characterized mixed Poisson processes by using uniform order statistics property (see also Feigin [15] and Huang and Shoung [18]). It is seen from Shaked et al. [33] that a counting process $\{N(t),\, t \in \mathbb{R}_+\}$ is a mixed Poisson process if and only if, for all $n \in \mathbb{N}_+$, the first n epoch times of the process have an $\ell_\infty^{\leqslant}$-spherical distribution and that not all $\ell_\infty^{\leqslant}$-spherical order statistics are the epoch times of some mixed Poisson process.

An immediate consequence of Theorem 3.3 is the following corollary.

Corollary 3.5: *For epoch times $\{T_n, n \in \mathbb{N}_+\}$ of a mixed Poisson process, we have*

$$\mathrm{Cov}(\phi(T_n), \phi(T_{n+1})) \geq 0 \tag{3.5}$$

for $n \in \mathbb{N}_+$ and all measurable functions ϕ such that the covariance exists.

Recall that a counting process $\{N(t),\, t \in \mathbb{R}_+\}$ is said to be a nonhomogeneous pure birth process with intensity functions $\kappa_n \geq 0$ if the following hold:

1. $\{N(t),\, t \in \Re_+\}$ has the Markov property,
2. $\mathsf{P}(N(t+\Delta t) = n+1 | N(t) = n) = \kappa_n(t)\Delta t + \mathrm{O}(\Delta t)$ for $n \in \mathbb{N}$,
3. $\mathsf{P}(N(t+\Delta t) > n+1 | N(t) = n) = \mathrm{O}(\Delta t)$ for $n \in \mathbb{N}$,

where each κ_n is assumed to satisfy

$$\int_t^{\infty} \kappa_n(u)\, du = +\infty, \qquad t \in \mathbb{R}_+;$$

this ensures that, with probability 1, the process has a jump after any time point t. There is a close relationship between mixed Poisson processes and nonhomogeneous pure birth processes; see, for example, Grandell [16, Sect. 6.1] or Pfeifer and Heller [26].

Example 3.6 (Pólya process): Let $\{N(t),\, t \in \mathbb{R}_+\}$ be a nonhomogeneous pure birth process with intensity functions κ_n given by

$$\kappa_n(t) = \frac{\gamma + n}{\beta + t}, \qquad t \in \mathbb{R}_+, n \in \mathbb{N},$$

where $\gamma \geq 0$ and $\beta > 0$ are constants: It is known from Grandell [16, pp. 67] or Shaked et al. [32] that such a process is also a mixed Poisson process with G in

(3.4) having $\Gamma(\beta, \gamma)$ distribution, whose density function is given by

$$g(\lambda) = \frac{\beta^\gamma \lambda^{\gamma-1}}{\Gamma(\gamma)} e^{-\beta\lambda}, \qquad \lambda \in \mathbb{R}_+.$$

Therefore, Corollary 3.5 can be applied to the epoch times of such a process.

3.3. Generalized Order Statistics

The concept of generalized order statistics was introduced by Kamps [19, 20] as a unified approach to a variety of models of ordered random variables.

Definition 3.7: *Let $n \in \mathbb{N}_+$, $k > 0$, and $(m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$ be parameters such that*

$$\gamma_{r,n} = k + \sum_{j=r}^{n-1} (m_j + 1) > 0, \qquad r = 1, \ldots, n, \tag{3.6}$$

and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m}$ arbitrary if $n = 1$). If the random variables $U_{(r,n,\tilde{m},k)}$, $r = 1, \ldots, n$, possess a joint density of the form

$$f_{U_{(1,n,\tilde{m},k)},\ldots,U_{(n,n,\tilde{m},k)}}(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_{j,n}\right)\left(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ of $\mathbb{R}^n$, then they are called uniform generalized order statistics (GOSs, for short). Now, let F be an arbitrary distribution function. The random variables

$$X_{(r,n,\tilde{m},k)} = F^{-1}(U_{(r,n,\tilde{m},k)}), \qquad r = 1, \ldots, n,$$

are called the GOSs based on F.

In the particular case $m_1 = \cdots = m_{n-1} = m$, the above random variables are denoted by $U_{(r,n,m,k)}$ and $X_{(r,n,m,k)}$, $r = 1, \ldots, n$, respectively.

In the past 10 years, there is a vast amount of literature on studying various properties of GOSs. Khaledi and Kochar [21] and Cramer [9] investigated the dependence structure of GOSs. The structure of GOSs can be characterized by sums of independent exponential random variables as stated in Lemma 3.8 (see Cramer and Kamps [11]).

Lemma 3.8: *Let $X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}$ be GOSs based on a continuous distribution function F, and let $Z_1, \ldots, Z_n$ be independent exponential random variables with failure rates $\gamma_{1,n}, \ldots, \gamma_{n,n}$, respectively, where $\gamma_{n,n} = k$. Then*

$$\big(X_{(1,n,\tilde{m},k)}, X_{(2,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}\big) \stackrel{d}{=} \left(H(Z_1), H(Z_1+Z_2), \ldots, H\left(\sum_{i=1}^{n} Z_i\right)\right),$$

where $H(x) = F^{-1}(1 - e^{-x})$ for $x \in \mathbb{R}_+$.

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THEOREM 3.9: *Let $X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}$ be GOSs based on a continuous distribution function F. If $2\gamma_{r,n} \geq \gamma_{r-1,n}$ for some r, $2 \leq r \leq n$; then*

$$\mathrm{Cov}(\phi(X_{(r-1,n,\tilde{m},k)}), \phi(X_{(r,n,\tilde{m},k)})) \geq 0$$

for all measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the covariance exists.

PROOF: The proof is similar to that of Theorem 3.2 by using Lemma 3.8 and Theorem 2.4. ■

From (3.6), it follows that

$$2\gamma_{r,n} - \gamma_{r-1,n} = \gamma_{r+1,n} + (m_r - m_{r-1}) \quad \text{for } r = 2, \ldots, n-1$$

and

$$2\gamma_{n,n} - \gamma_{n-1,n} = k - (m_{n-1} + 1).$$

Thus, a sufficient condition for $2\gamma_{r,n} \geq \gamma_{r-1,n}$ is that $m_r \geq m_{r-1}$ for $r = 2, \ldots, n-1$ and $k \geq m_{n-1} + 1$ for $r = n$. In virtue of this observation, an immediate consequence of Theorem 3.9 is the following corollary.

COROLLARY 3.10: *Let $X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)}$ be GOSs based on a continuous distribution function F, and let ϕ be any measurable function such that the covariances below exist. Then*

$$\mathrm{Cov}(\phi(X_{(r-1,n,m,k)}), \phi(X_{(r,n,m,k)})) \geq 0 \quad \textit{for } r = 2, \ldots, n-1.$$

Moreover, if $k \geq m + 1$, then

$$\mathrm{Cov}(\phi(X_{(n-1,n,m,k)}), \phi(X_{(n,n,m,k)})) \geq 0.$$

It is worthwhile to mention that Theorem 3.9 and Corollary 3.10 do not hold in general for the case of nonadjacent GOSs, as shown by Counterexamples 5.2 and 5.3. Furthermore, Counterexample 5.4 shows that Theorem 3.9 might even not be true for the case of adjacent GOSs if $2\gamma_{r,n} < \gamma_{r-1,n}$.

Choosing the parameters appropriately, several other models of ordered random variables are seen to be particular cases. Ordinary order statistics of a random sample from a distribution F are a particular case of GOSs when $k = 1$ and $m_r = 0$ for $r = 1, \ldots, n-1$. When $k = 1$ and $m_r = -1$ for $r = 1, \ldots, n-1$, we get the first n record values from a sequence of i.i.d. random variables with distribution F. Some other models are as follows.

- *kth record values:* Fix $k \in \mathbb{N}_+$. Let $\{X_n, n \in \mathbb{N}_+\}$ be a sequence of i.i.d. random variables with a continuous distribution F. The random variables $L^{(k)}(1) = 1$ and

$$L^{(k)}(n+1) = \min\{j > L^{(k)}(n) : X_{j:j+k-1} > X_{L^{(k)}(n):L^{(k)}(n)+k-1}\}, \qquad n \in \mathbb{N},$$

are called *k-record times*, and

$$X_{L^{(k)}(n)} = X_{L^{(k)}(n):L^{(k)}(n)+k-1}$$

is called *nth k-record values* (see Kamps [20, p. 34] and Arnold et al. [4]). For $k = 1$, $X_{L^{(1)}(n)}$ reduces to $X_{L(n)}$. The first n k-records $(X_{L^{(k)}(1)}, \ldots, X_{L^{(k)}(n)})$ are the GOSs $(X_{(1,n,-1,k)}, \ldots, X_{(n,n,-1,k)})$ based on F. By Corollary 3.10, we have

$$\mathrm{Cov}\big(\phi(X_{L^{(k)}(n)}), \phi(X_{L^{(k)}(n+1)})\big) \geq 0, \qquad n \in \mathbb{N}_+,$$

for all measurable real-valued functions ϕ such that the covariance exists.

- *Progressive type II censored order statistics*: Progressive type II censoring has been suggested in the field of life-testing experiments. Suppose that N units are placed on a lifetime test. The failure times are described by i.i.d. random variables with a common distribution F. A number n $(n \leq N)$ of units are observed to fail. A predetermined number R_i of surviving units at the time of the ith failure are randomly selected and removed from further testing. Thus, $\sum_{i=1}^{n} R_i$ units are progressively censored; hence, $N = n + \sum_{i=1}^{n} R_i$. The n observed failure times are called progressive type II censored order statistics based on F, denoted by $T_1 \leq T_2 \leq \cdots \leq T_n$, which correspond to the GOSs based on F with parameters $k = R_n + 1$, $m_r = R_r$ and $\gamma_{r,n} = N - r + 1 - \sum_{i=1}^{r-1} R_i$ for $r = 1, \ldots, n-1$. For details on the model of progressive type II censoring, we refer to Balakrishnan and Aggarwala [6] and Cramer and Kamps [10]. If R_i is decreasing in i and F is continuous, then, by Theorem 3.9 and the comments after Theorem 3.9, we have

$$\mathrm{Cov}(\phi(T_r), \phi(T_{r+1})) \geq 0 \tag{3.7}$$

For $r = 1, \ldots, n-1$ and for all measurable real-valued functions ϕ such that the covariance exists.

- *Order statistics under multivariate imperfect repair Policy*$(p_1, \ldots, p_n)$: Suppose that n items with i.i.d. random lifetimes, with distribution function F, start to function at the same time 0. Upon failure, an item undergoes a repair and the repair is instantaneous. If i items have already been scrapped, then with probability p_{i+1}, the repair is unsuccessful and the item is scrapped, and with probability $1 - p_{i+1}$, the repair is successful and minimal (i.e., the item is restored to a working condition just prior to the failure). When an item fails and is successfully minimally repaired, the other functioning items "do not know" about the failure and repair. The ordered failure times $T_1 \leq T_2 \leq \cdots \leq T_n$ are the special case of GOSs based on F with parameters $k = p_n$, $m_r = (n-r+1)p_r - (n-r)p_{r+1} - 1$ and $\gamma_{r,n} = (n-r+1)p_r$ for $r = 1, \ldots, n-1$. For more details, see Shaked and Shanthikumar [31] and Belzunce, Mercader, and Ruiz [7]. Applying Theorem 3.9 yields that if

$$2(n-r)p_{r+1} \geq (n-r+1)p_r \tag{3.8}$$

for some r, $1 \leq r < n$, then (3.7) holds for measurable real-valued functions ϕ

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such that the covariance exists. A sufficient condition for (3.8) is that p_i is increasing in i.

- *Yule process*: A Yule process $\{N(t), t \in \mathbb{R}_+\}$, with initial population size θ, is a special homogeneous pure birth process with intensity functions

$$\kappa_i(t) = i\lambda, \qquad i \in \{\theta, \theta+1, \ldots\},$$

where $\lambda > 0$ is a constant. Let $T_1 \le T_2 \le \cdots \le T_n$ be the first n epoch times of the process. Then the T_i's can be regarded as the GOSs $X_{(i,n,m,k)}$ based on unit rate exponential distribution, where $k = \lambda(\theta + n - 1)$ and $m = -\lambda - 1$. Therefore, by Corollary 3.10, (3.7) holds for $r \in \mathbb{N}_+$ and all measurable functions ϕ.

4. APPLICATIONS TO ORDERED DISCRETE RANDOM VARIABLES

4.1. Discrete Weak Record Values

In the context of record values, a repetition of a record value can be regarded as a new record, and this makes sense for discrete distributions. This leads to the notion of weak records introduced by Vervaat [37]. Recently, a considerable amount of work has been done on weak record statistics; see Stepanov, Balakrishnan, and Hofmann [36], Wesolowski and López-Blázquez [39], Dembińska and López-Blázquez [13], Bairamov and Stepanov [5], Belzunce, Ortega, and Ruiz [8], Dembińska and Stepanov [14], Danielak and Dembińska [12], and references therein.

Formally, let $\{X, X_n, n \in \mathbb{N}_+\}$ be a sequence of i.i.d. discrete random variables with support being a subset of $\mathbb{N}$. The sequence of *weak record times* $\{L_w(n), n \in \mathbb{N}_+\}$ is defined by

$$L_w(1) = 1,$$
$$L_w(n+1) = \min\{j > L_w(n) : X_j \ge \max\{X_1, X_2, \ldots, X_{j-1}\}\}, \qquad n \in \mathbb{N}_+,$$

and $\{X_{L_w}(n), n \in \mathbb{N}_+\}$ is the sequence of *weak records*. The discrete weak record values possess the Markov property (see Vervaat [37]); that is,

$$\mathsf{P}(X_{L_w(n+1)} = j | X_{L_w(n)} = i) = \frac{\mathsf{P}(X = j)}{\mathsf{P}(X \ge i)}, \qquad i \le j. \tag{4.1}$$

Thus, the joint mass function of the first n weak record values is given by

$$\mathsf{P}(X_{L_w(1)} = j_1, \ldots, X_{L_w(n)} = j_n) = \mathsf{P}(X = j_n) \prod_{r=1}^{n-1} \eta_{j_r}$$

For $j_1 \le j_2 \le \cdots \le j_n$, where

$$\eta_j = \frac{\mathsf{P}(X = j)}{\mathsf{P}(X \ge j)} \tag{4.2}$$

is the failure rate function of X.

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THEOREM 4.1: *For the sequence* $\{X_{L_w}(n), n \in \mathbb{N}_+\}$ *of weak record values, we have*

$$\mathrm{Cov}(\phi(X_{L_w(n)}), \phi(X_{L_w(n+1)})) \geq 0, \qquad n \in \mathbb{N}_+,$$

for all real-valued functions ϕ *such that the expectation exists.*

PROOF: From (4.1), it follows that the joint mass function of $\{X_{L_w}(r), r = n-1, \ldots, n+2\}$ is given by

$$\mathsf{P}(X_{L_w(r)} = j_r, r = n-1, \ldots, n+2) = \frac{\mathsf{P}(X_{L_w(n-1)} = j_{n-1})}{\mathsf{P}(X \geq j_{n-1})} \mathsf{P}(X = j_{n+2}) \eta_{j_n} \eta_{j_{n+1}}$$

for $j_{n-1} \leq j_n \leq j_{n+1} \leq j_{n+2}$ and $n \in \mathbb{N}_+$, where $X_{L_w(0)} = 0$. Thus,

$$\begin{aligned} &\mathsf{P}(X_{L_w(n)} = x, X_{L_w(n+1)} = y | X_{L_w(n-1)} = s, X_{L_w(n+2)} = t) \\ &= \frac{\eta_x \eta_y}{\sum_{s \leq i \leq j \leq t} \eta_i \eta_j} \\ &= \begin{cases} 2! g_{s,t}(x,y), & s \leq x < y \leq t \\ g_{s,t}(x,y), & s \leq x = y \leq t, \end{cases} \end{aligned} \tag{4.3}$$

where

$$g_{s,t}(x,y) = \begin{cases} c_{s,t}\eta_x^2, & s \leq x = y \leq t \\ \frac{1}{2} c_{s,t} \eta_x \eta_y, & s \leq x \neq y \leq t \\ 0, & \text{otherwise}, \end{cases} \qquad (x,y) \in \{s, s+1, \ldots, t\}^2,$$

is the joint mass function of some interchangeable random variable $V_1(s,t)$ and $V_2(s,t)$. Here, $c_{s,t} = \left[\sum_{s \leq i \leq j \leq t} \eta_x \eta_j\right]^{-1}$. Clearly, (4.3) means that condition (2.1) is satisfied. The desired result now follows from Lemma 2.5 and Remark 2.3. ∎

Now, let B_i be the number of weak record values that are equal to i for $i \in \mathbb{N}$. Then

$$M_r = \sum_{i=0}^{r} B_i$$

is the number of weak record values that are less than or equal to r, $r \in \mathbb{N}$. Denote by $P_i = \mathsf{P}(X \geq i)$ and by u_X the right end point of the support of X. Stepanov [35] proved that B_i, $i \in \mathbb{N}$, are independent,

$$\mathsf{P}(B_i = n) = \frac{P_{i+1}}{P_i}\left(1 - \frac{P_{i+1}}{P_i}\right)^n, \qquad n \in \mathbb{N}, \tag{4.4}$$

for $i = 0, 1, \ldots, u_X - 1$, and $\mathsf{P}(B_{u_X} = +\infty) = 1$ if $u_X < \infty$.

By Theorem 2.6, we obtain the next result, whose proof is trivial and, hence, is omitted.

Theorem 4.2: *Let η_i be the discrete failure rate of X, defined by (4.2). If $\eta_{r+1}^2 \le \eta_r$ for some $r \in \{0, 1, \ldots, u_X - 2\}$, then*

$$\mathrm{Cov}(\phi(M_r), \phi(M_{r+1})) \ge 0 \quad \textbf{(4.5)}$$

for all real-valued functions ϕ such that the expectation exists.

4.2. Discrete $\ell_\infty^{\leqslant}$-Spherical Order Statistics

Let $T_1 \le T_2 \le \cdots \le T_n$ be $\mathbb{N}$-valued random variables. $T_1, \ldots, T_n$ are said to be (discrete) $\ell_\infty^{\leqslant}$-spherical order statistics if their joint mass function of the form

$$p_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = \begin{cases} \varphi(t_n), & 0 \le t_1 \le t_2 \le \cdots \le t_n \\ 0, & \text{otherwise}, \end{cases} \quad \textbf{(4.6)}$$

for some nonnegative function φ (see Shaked et al. [33]).

The following result is a discrete analogue of Theorem 3.3. Its proof is a straightforward modification of the proof of Theorem 4.1 and, hence, is omitted.

Theorem 4.3: *Let $T_1 \le T_2 \le \cdots \le T_n$ be $\ell_\infty^{\leqslant}$-spherical order statistics on $\mathbb{N}^n$. Then*

$$\mathrm{Cov}(\phi(T_r), \phi(T_{r+1})) \ge 0 \quad \textbf{(4.7)}$$

for $r = 1, \ldots, n - 2$ and for all real-valued functions ϕ such the covariance exists.

We now modify the definition of a mixed geometric process introduced by Huang and Shoung [18].

Definition 4.4: *A discrete-time discrete-state process $\{M_t, t \in \mathbb{N}\}$ is called a modified mixed geometric process if there exists a random variable Θ that takes on values in (0,1) such that, given $\Theta = \theta$, the interepoch intervals Z_i, $i \in \mathbb{N}_+$, of the process are i.i.d. with $\mathsf{P}(Z_1 = z) = \theta(1 - \theta)^z$ for $z \in \mathbb{N}$.*

A modified mixed geometric process can have jumps larger than unity at the jump epochs. Denote by J_t the number of jumps occurring at time t; that is, $J_0 = M_0$ and $J_t = M_t - M_{t-1}$ for $t \in \mathbb{N}_+$. For a modified mixed geometric process, it is seen from Theorem 4.7 in Shaked et al. [33] that the first n epoch times $T_1, \ldots, T_n$ are $\ell_\infty^{\leqslant}$-spherical and that the jump amounts $J_0, J_1, \ldots, J_n$ have a Schur-constant mass function on $\mathbb{N}^n$, which implies that $M_0, M_1, \ldots, M_n$ are also $\ell_\infty^{\leqslant}$-spherical. By Theorem 4.3, we have the following corollary.

Corollary 4.5: *Let $\{T_n, n \in \mathbb{N}_+\}$ be the sequence of epoch times of a modified mixed geometric process $\{M_t, t \in \mathbb{N}\}$. Then (4.5) and (4.7) hold for all $r \in \mathbb{N}_+$ and all real-valued functions ϕ such that the covariances exist.*

5. COUNTEREXAMPLES

In this section several counterexamples are presented to illustrate that the conditions of the theorems and corollaries in the previous sections cannot be dropped off and that

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the nonnegativity property of the covariances do not hold for nonadjacent ordered random variables.

Throughout this section, let Z_1, Z_2, and Z_3 be independent exponential random variables with failure rates λ_1, λ_2, and λ_3, and denote their means by μ_1, μ_2, and μ_3. Choose $\phi(x) = x^2 - x$. Since $\mathsf{E}[Z_i^2] = 2\mu_i^2$, $\mathsf{E}[Z_i^3] = 6\mu_i^3$ and $\mathsf{E}[Z_i^4] = 24\mu_i^4$ for each i, it is easy to see that

$$\mathrm{Var}(\phi(Z_1)) = \mathsf{E}[Z_1^4 - 2Z_1^3 + Z_1^2] - [\mathsf{E}Z_1^2 - \mathsf{E}Z_1]^2 = 20\mu_1^4 - 8\mu_1^3 + \mu_1^2$$

and

$$\mathrm{Cov}(\phi(Z_1), Z_1 Z_2) = \left\{\mathsf{E}[Z_1^3 - Z_1^2] - (\mathsf{E}Z_1^2 - \mathsf{E}Z_1)\mathsf{E}Z_1\right\}\mathsf{E}Z_2 = \mu_2[4\mu_1^3 - \mu_1^2],$$
$$\mathrm{Cov}(\phi(Z_1), Z_1 Z_3) = \mu_3[4\mu_1^3 - \mu_1^2].$$

Then we have

$$\begin{aligned}\mathrm{Cov}(\phi(Z_1), \phi(Z_1 + Z_2 + Z_3)) &= \mathrm{Cov}\left(\phi(Z_1), \sum_{i=1}^{3} \phi(Z_i) + 2Z_1Z_2 + 2Z_1Z_3 + 2Z_2Z_3\right) \\ &= \mathrm{Var}(\phi(Z_1)) + 2\,\mathrm{Cov}(\phi(Z_1), Z_1Z_2) \\ &\quad + 2\,\mathrm{Cov}(\phi(Z_1), Z_1Z_3) \\ &= 20\mu_1^4 - 8\mu_1^3 + \mu_1^2 + 2(\mu_2 + \mu_3)(4\mu_1^3 - \mu_1^2)\end{aligned} \tag{5.1}$$

and

$$\mathrm{Cov}(\phi(Z_1), \phi(Z_1 + Z_2)) = 20\mu_1^4 - 8\mu_1^3 + \mu_1^2 + 2\mu_2(4\mu_1^3 - \mu_1^2). \tag{5.2}$$

Counterexample 5.1: Choose $\phi(x) = x^2 - x$ and $\lambda_1 = 6$, $\lambda_2 = 2$ such that $2\lambda_2 - \lambda_1 < 0$. From (5.2), it follows that

$$\mathrm{Cov}(\phi(Z_1), \phi(Z_1 + Z_2)) = -\frac{1}{324} < 0.$$

This shows that Theorem 2.4 does not hold if condition (2.7) is violated.

Counterexample 5.2: Let $X_{(1,3,\tilde{m},4)} \le X_{(2,3,\tilde{m},4)} \le X_{(3,3,\tilde{m},4)}$ be GOSs based on the standard exponential distribution with $\tilde{m} = (2, 1)$, and choose $(\lambda_1, \lambda_2, \lambda_3) = (9, 6, 4)$ and $\phi(x) = x^2 - x$. By Lemma 3.8, we get

$$(X_{(1,3,\tilde{m},4)}, X_{(2,3,\tilde{m},4)}, X_{(3,3,\tilde{m},4)}) \stackrel{d}{=} (Z_1, Z_1 + Z_2, Z_1 + Z_2 + Z_3).$$

From (5.1), straightforward computations give

$$\mathrm{Cov}(\phi(X_{(1,3,\tilde{m},4)}), \phi(X_{(3,3,\tilde{m},4)})) = \mathrm{Cov}(\phi(Z_1), \phi(Z_1 + Z_2 + Z_3)) = -\frac{51}{6 \times 9^4} < 0.$$

Clearly, $2\gamma_{r,3} > \gamma_{r-1,3}$ for $r = 2, 3$, satisfying the assumption of Theorem 3.9. This shows that Theorem 3.9 does not hold for the case of nonadjacent GOSs.

Counterexample 5.3: Let $X_{(1,3,1,4)} \le X_{(2,3,1,4)} \le X_{(3,3,1,4)}$ be GOSs based on the standard exponential distribution, and choose $(\lambda_1, \lambda_2, \lambda_3) = (8, 6, 4)$ and $\phi(x) = x^2 - x$. By Lemma 3.8, we get

$$(X_{(1,3,1,4)}, X_{(2,3,1,4)}, X_{(3,3,1,4)}) \stackrel{d}{=} (Z_1, Z_1 + Z_2, Z_1 + Z_2 + Z_3).$$

Straightforward computations give

$$\mathrm{Cov}(\phi(X_{(1,3,1,4)}), \phi(X_{(3,3,1,4)})) = \mathrm{Cov}(\phi(Z_1), \phi(Z_1 + Z_2 + Z_3)) = -\frac{5}{3072} < 0.$$

This shows that Corollary 3.10 does not hold for the case of nonadjacent GOSs.

Counterexample 5.4: Let $X_{(1,3,\tilde{m},1)} \le X_{(2,3,\tilde{m},1)} \le X_{(3,3,\tilde{m},1)}$ be GOSs based on the standard exponential distribution with $\tilde{m} = (3, 4)$, and choose $(\lambda_1, \lambda_2, \lambda_3) = (10, 6, 1)$ and $\phi(x) = x^2 - x$. It is easy to see from Lemma 3.8 that

$$(X_{(1,3,\tilde{m},1)}, X_{(2,3,\tilde{m},1)}, X_{(3,3,\tilde{m},1)}) \stackrel{d}{=} (Z_1, Z_1 + Z_2, Z_1 + Z_2 + Z_3).$$

Straightforward computations give

$$\begin{aligned}
&\mathrm{Cov}(\phi(X_{(2,3,\tilde{m},1)}), \phi(X_{(3,3,\tilde{m},1)})) \\
&\quad = \mathrm{Cov}(\phi(Z_1 + Z_2), \phi(Z_1 + Z_2 + Z_3)) \\
&\quad = 20\mu_1^4 + 20\mu_2^4 - 8\mu_1^3 - 8\mu_2^3 + \mu_1^2 + \mu_2^2 + 2(2\mu_2 + \mu_3)(4\mu_1^3 - \mu_1^2) \\
&\qquad + 2(2\mu_1 + \mu_3)(4\mu_2^3 - \mu_2^2) + 12\mu_1^2\mu_2^2 + 4\mu_1\mu_2\mu_3(\mu_1 + \mu_2) \\
&\quad = -0.0069 \\
&\quad < 0.
\end{aligned}$$

This shows that Theorem 3.9 might even not be true for adjacent GOSs if $2\gamma_{3,3} < \gamma_{2,3}$.

Counterexample 5.5: Let (T_1, T_2) have an $\ell_\infty^{\lessgtr}$-spherical density of the form

$$f_{T_1,T_2}(t_1, t_2) = \begin{cases} 3t_2, & 0 \le t_1 \le t_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then the marginal densities of T_1 and T_2 are respectively given by

$$f_{T_1}(t_1) = \begin{cases} \frac{3}{2}(1 - t_1^2), & 0 \le t_1 \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{T_2}(t_2) = \begin{cases} 3t_2^2, & 0 \le t_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Choose $\phi(x) = x^2 - x$. Then $\mathsf{E}[\phi(T_1)] = -7/40$, $\mathsf{E}[\phi(T_2)] = -3/20$, and $\mathsf{E}[\phi(T_1)\phi(T_2)] = 11/420$. Therefore, $\mathrm{Cov}(\phi(T_1), \phi(T_2)) = -1/16800 < 0$. This shows, by a limiting argument, that (3.2) in Theorem 3.3 cannot be true for $r = n-1$ if φ is differentiable but not decreasing.

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