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Relativization of complexity and sensitivity

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Abstract. First notions of relative complexity function and relative sensitivity are introduced. It turns out that for any open factor map $\pi : (X, T) \to (Y, S)$ between topological dynamical systems with minimal (Y, S), π is positively equicontinuous if and only if the relative complexity function is bounded for each open cover of X; and that any non-trivial weakly mixing extension is relatively sensitive. Moreover, a relative version of the notable result that any M-system is sensitive if it is not minimal is obtained. Then notions of relative scattering and relative Mycielski's chaos are introduced. A relative disjointness theorem involving relative scattering is given. A relative version of the well-known result that any non-trivial scattering topological dynamical system is Li–Yorke chaotic is proved.

1. *Introduction*

By a *topological dynamical system* (TDS) we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map from X onto X.

Recall that a TDS (X, T) is *positively equicontinuous* if the transformation acts positively equicontinuously on X, that is, for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}_+$ and $d(x_1, x_2) < \delta$. Positively equicontinuous TDSs can be characterized using a concept called a complexity function. The complexity function of a TDS has been studied by many researchers in the past few years: a survey for symbolic systems was given in [**Fe**] and it was considered in [**BHM**] for a general TDS. Let (S, σ) be a symbolic system and $p_S(n)$ its complexity function. Then it is either eventually constant, or greater than n for all n; the symbolic systems having a bounded complexity function are exactly the equicontinuous ones. For a general TDS (X, T), the complexity function of a cover $\mathcal{C}$ (admitting a finite sub-cover) was introduced in [**BHM**] as the minimal cardinality of all sub-covers of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{C}$. They proved that equicontinuous transformations are exactly those such that any open cover has a bounded complexity function. In the same paper, the notions of scattering and 2-scattering were introduced.

It was proved that in the minimal case, the weakly mixing, scattering and 2-scattering properties are equivalent. In general, scattering evidently implies 2-scattering and whether the converse holds appeared as an open problem in [**BHM**]. This question was answered affirmatively in [**HY2**] by showing that 2-scattering is equivalent to scattering. Moreover, the authors of [**HY2**] generalized the notion of the complexity function along a subsequence of natural numbers, and so using it they could characterize some topological properties of TDSs, such as mild mixing, strong scattering, scattering and so on.

Even though abundant results have been obtained, the question remains open as to what extent the result in [**BHM**] is valid in a more general setting. We investigate this question and discuss some related topics in the first part of our paper.

Let (X, T) and (Y, S) be two TDSs. A *factor map* $\pi : (X, T) \to (Y, S)$ means a continuous surjective map satisfying $S\pi = \pi T$. In this case, we say that (X, T) is an *extension* of (Y, S), (Y, S) is a *factor* of (X, T). If, in addition, it is not one-to-one, then we say that π is *non-trivial*. We say that the factor map π is *positively equicontinuous* or π is a *positively equicontinuous extension* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}_+$, $d(x_1, x_2) < \delta$ and $\pi(x_1) = \pi(x_2)$.

First we introduce the notion of a relative complexity function for any given factor map π. We prove that if π is an open factor map between minimal TDSs, then it is positively equicontinuous if and only if each open cover has a bounded relative complexity function (Theorem 2.7), and that in the general case the former implies the latter, which generalizes the result in [**BHM**]. Moreover, we show that the assumptions of openness and minimality are necessary. Then, based on the idea of a relative complexity function we introduce the notions of relative complexity tuples, relative n-scattering ($n \in \mathbb{N} : n \geq 2$) and relative scattering. We show that if π is an open factor map between minimal invertible TDSs, then:

(1) the maximal positively equicontinuous factor of π is induced by the set of relative complexity pairs;
(2) any pair not on the diagonal is a relative complexity pair if and only if it is contained in the relative regionally proximal relation;
(3) relative scattering implies weak mixing.

We also present a relative disjointness theorem involving relative scattering.

Another area of progress in TDS made in the past few decades is the study of the chaotic behaviour of a TDS. Since the introduction of the so-called Li–Yorke's chaos in 1975 by Li and Yorke [**LY**], people have paid much attention to it (see, for example, [**AAB, BBCDS, BGKM, GW, HY1, LY**]). In [**HY1**], a long open problem was solved of whether Devaney's chaos implies Li–Yorke's chaos. Another long open problem was solved in [**BGKM**] by proving that positive entropy implies Li–Yorke's chaos. Just recently, this result was generalized to a more general setting in [**Z**]; namely, the author proved that positive conditional topological entropy implies Li–Yorke's chaos on fibres. With the help of the well-known Mycielski's theorem (see [**M**]) it is not hard to show that any non-trivial scattering TDS is Li–Yorke chaotic. A very nice explanation of the role of Cantor and Mycielski sets in topological dynamics is contained in [**A**]. Moreover, any minimal TDS is either equicontinuous or sensitive, and any M-system must be sensitive if it is not minimal (see [**GW**]).

Then the other part of our paper focuses on the chaotic behaviour on fibres for a given factor map between TDSs. Precisely, opposite to positive equicontinuity, we introduce the concepts of relative sensitivity and relative Mycielski's chaos (which is a stronger version of Li–Yorke's chaos on fibres), and prove that for a factor map between minimal TDSs it is either relatively sensitive or positively equicontinuous, and that any non-trivial weakly mixing factor map is relatively sensitive. Related to this, we present a relative version of Glasner–Weiss's result (Theorem 5.10). Moreover, we show that with necessary assumptions, relative 2-scattering implies relative Mycielski's chaos in many cases, including any open relatively sensitive factor map between invertible TDSs, any open non-trivial factor map between minimal invertible TDSs and so on.

The paper is organized as follows. In §2, we first introduce the notion of a relative complexity function, then, by localizing the concepts of a relative complexity function and positively equicontinuous extension, we prove that, for an open factor map between minimal TDSs, it is positively equicontinuous if and only if each open cover has a bounded relative complexity function. In §3, based on the idea of a relative complexity function, the notion of relative complexity tuples is given and basic properties are discussed. For an open factor map between minimal invertible TDSs, the relationship between relative complexity pairs and the relative regionally proximal relation is interpreted. In §4, we introduce the notions of relative scattering and relative Mycielski's chaos, and prove that relative scattering implies weak mixing if we consider an open factor map between minimal invertible TDSs. A relative disjointness theorem involving relative scattering is given. Then, with necessary assumptions, some cases are studied when relative 2-scattering implies relative Mycielski's chaos. In §5, we introduce the concept of relative sensitivity and prove that if π is a factor map between minimal TDSs, then it is either relatively sensitive or positively equicontinuous. Moreover, we obtain a relative version of the known Glasner–Weiss's result.

2. *Relative complexity functions and their application to positively equicontinuous extension*

In this section, we first introduce the notion of a relative complexity function. For any factor map between TDSs we prove that positive equicontinuity implies that each open cover has a bounded relative complexity function. Then we localize the concepts of a relative complexity function and positively equicontinuous extension, and prove that, for an open factor map between minimal TDSs, the factor map is positively equicontinuous if and only if each open cover has a bounded relative complexity function (Theorem 2.7).

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $\mathcal{U}$ a cover of X which admits a finite sub-cover. For $E \subseteq X$, set $N(\mathcal{U}, E)$ to be the minimum among the cardinalities of subsets of $\mathcal{U}$ which cover E, and let $N(\mathcal{U}|\pi) = \sup_{y\in Y} N(\mathcal{U}, \pi^{-1}(y))$. Put

$$\mathcal{C}(n, \mathcal{U}|\pi) = N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|\pi\right) \quad (n \in \mathbb{N}) \quad \text{and} \quad \mathcal{C}(\mathcal{U}|\pi) = \lim_{n\to\infty} \mathcal{C}(n, \mathcal{U}|\pi).$$

We say that $\mathcal{C}(\bullet, \mathcal{U}|\pi)$ is the *relative complexity function of $\mathcal{U}$ with respect to π*. We also regard $\mathcal{C}(\mathcal{U}|\pi)$ (it may be infinity) as the relative complexity function. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two covers of X admitting finite sub-covers. We say that $\mathcal{U}_1$ is *finer* than $\mathcal{U}_2$ (denoted by $\mathcal{U}_1 \succeq \mathcal{U}_2$ or $\mathcal{U}_2 \preceq \mathcal{U}_1$) if each element of $\mathcal{U}_1$ is contained in some element of $\mathcal{U}_2$. We have: (1) $\mathcal{C}(\mathcal{U}_2|\pi) \leq \mathcal{C}(\mathcal{U}_1|\pi)$ if $\mathcal{U}_2 \preceq \mathcal{U}_1$; (2) $\mathcal{C}(\mathcal{U}_1 \vee \mathcal{U}_2|\pi) \leq \mathcal{C}(\mathcal{U}_1|\pi) + \mathcal{C}(\mathcal{U}_2|\pi)$.

It is not hard to obtain the following.

LEMMA 2.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. If π is positively equicontinuous, then for any open cover $\mathcal{U}$ of X, $\mathcal{C}(\mathcal{U}|\pi) < \infty$.*

Proof. Let $\mathcal{U}$ be any open cover of X with $\epsilon > 0$ a Lebesgue number. Then there exists $\delta > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}_+$, $d(x_1, x_2) < \delta$ and $\pi(x_1) = \pi(x_2)$. Let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be an open cover of X with $\mathrm{diam}(V_i) < \delta$, $i = 1, \ldots, k$. Then for each $y \in Y$, $m \in \mathbb{Z}_+$ and i, $V_i \cap \pi^{-1}(y) \subseteq T^{-m} U_{i,m,y}$ for some $U_{i,m,y} \in \mathcal{U}$. So

$$N\bigg(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, \pi^{-1}(y)\bigg) \leq k$$

for each $n \in \mathbb{N}$. That is, $\mathcal{C}(\mathcal{U}|\pi) \leq k$. This completes the proof. □

In order to study positively equicontinuous extension using a relative complexity function and obtain the converse of Lemma 2.1 with some necessary assumptions, we localize the concepts of positive equicontinuity and a relative complexity function.

Definition 2.2. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that:

(1) the factor map π is *locally positively equicontinuous* (LPE) if for each $y \in Y$ and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, y) > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}_+$, $d(x_1, x_2) < \delta$ and $x_1, x_2 \in \pi^{-1}(y)$;

(2) a point $x \in X$ is *locally relatively positively equicontinuous* (LRPE) (denoted by $x \in E_{\mathrm{lre}}(X, T|\pi)$) if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(T^n x, T^n x') < \epsilon$ if $n \in \mathbb{Z}_+$, $d(x, x') < \delta$ and $\pi(x) = \pi(x')$.

It is clear that if π is positively equicontinuous then it is LPE, whereas the latter implies $E_{\mathrm{lre}}(X, T|\pi) = X$. In fact, we have the following.

PROPOSITION 2.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. Then π is LPE if and only if $E_{\mathrm{lre}}(X, T|\pi) = X$.*

Proof. Assume $E_{\mathrm{lre}}(X, T|\pi) = X$. Let $y \in Y$ and $\epsilon > 0$. For each $x \in \pi^{-1}(y)$ select $\delta_x > 0$ such that $d(T^n x, T^n x') < \epsilon/2$ if $n \in \mathbb{Z}_+$, $d(x, x') < \delta_x$ and $x' \in \pi^{-1}(y)$. Let $x_1, x_2, \ldots, x_m \in \pi^{-1}(y)$ $(m \in \mathbb{N})$ with

$$\bigcup_{i=1}^{m} B\bigg(x_i, \frac{\delta_{x_i}}{2}\bigg) \supseteq \pi^{-1}(y), \quad \text{where } B\bigg(x_i, \frac{\delta_{x_i}}{2}\bigg) = \bigg\{x \in X : d(x_i, x) < \frac{\delta_{x_i}}{2}\bigg\}.$$

Put $2\delta = \min_{1 \leq i \leq m} \delta_{x_i} > 0$. If $x', x'' \in \pi^{-1}(y)$ satisfy $d(x', x'') < \delta$, then for some i_0 we have $x', x'' \in B(x_{i_0}, \delta_{x_{i_0}})$, so $d(T^n x', T^n x'') < \epsilon$ for all $n \in \mathbb{Z}_+$. That is, π is LPE. □

The concept of a relative complexity function can be localized as follows.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $\mathcal{U}$ a cover of X which admits a finite sub-cover. For each point $y \in Y$, the *relative complexity function of $\mathcal{U}$ with*

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respect to π *at point* y is defined by

$$\mathcal{C}(n,\mathcal{U},y|\pi)=N\bigg(\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U},\pi^{-1}(y)\bigg)\quad(n\in\mathbb{N})$$

$$\text{and}\quad \mathcal{C}(\mathcal{U},y|\pi)=\lim_{n\to\infty}\mathcal{C}(n,\mathcal{U},y|\pi).$$

Then, LPE can be characterized as follows.

PROPOSITION 2.4. *Let* $\pi:(X,T)\to(Y,S)$ *be a factor map between TDSs. Then:*

(1) *if* $x\notin E_{\mathrm{lre}}(X,T|\pi)$, *then there exists an open cover* $\mathcal{U}$ *of* X *with* $\mathcal{C}(\overline{\mathcal{U}},\pi(x)|\pi)=\infty$, *where* $\overline{\mathcal{U}}$ *consists of the closure of elements of* $\mathcal{U}$;

(2) $\pi^{-1}(y)\subseteq E_{\mathrm{lre}}(X,T|\pi)$ *if and only if* $\mathcal{C}(\mathcal{U},y|\pi)<\infty$ *for each open cover* $\mathcal{U}$ *of* X;

(3) π *is LPE if and only if* $\mathcal{C}(\mathcal{U},y|\pi)<\infty$ *for each open cover* $\mathcal{U}$ *of* X *and* $y\in Y$.

Proof. (1) Since $x\notin E_{\mathrm{lre}}(X,T|\pi)$, there exist $\epsilon>0$, a sub-sequence $\{t_n\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}_+$ and $\{x_n\}_{n\in\mathbb{N}}\subseteq\pi^{-1}(\pi x)$ such that $d(T^{t_n}x_n,T^{t_n}x)\geq\epsilon$ and $d(x_n,x)<1/n$. Let $\mathcal{U}$ be any open cover of X with diameter at most $\epsilon/2$. We claim that $\mathcal{C}(\overline{\mathcal{U}},\pi(x)|\pi)=\infty$. Otherwise, $\pi^{-1}(\pi x)$ admits a closed cover $\alpha=\{X_1,\ldots,X_m\}$, where $m=\mathcal{C}(\overline{\mathcal{U}},\pi(x)|\pi)$, such that for each $i\in\{1,\ldots,m\}$,

$$X_i\subseteq\bigcap_{n\geq0}T^{-n}\overline{U_{i,n}}\quad\text{for some }\overline{U_{i,n}}\in\overline{\mathcal{U}}\quad(n\in\mathbb{Z}_+).$$

This implies that $d(T^nx',T^nx'')\leq\epsilon/2$ if $n\in\mathbb{Z}_+$ and x',x'' are both in the same X_i. By choosing a sub-sequence, we may assume that $\{x_n\}_{n\in\mathbb{N}}\subseteq X_{i_0}$ for some $i_0\in\{1,\ldots,m\}$, then we have $x\in X_{i_0}$, which contradicts the selection of x and ϵ.

(2) By part (1), '$\Leftarrow$' is obvious. Now let us turn to the proof of '$\Rightarrow$'.

Let $\epsilon>0$ be a Lebesgue number of the cover $\mathcal{U}$. Since $\pi^{-1}(y)\subseteq E_{\mathrm{lre}}(X,T|\pi)$, proceeding in the same way as in the proof of Proposition 2.3 there exists $\delta>0$ such that $d(T^nx_1,T^nx_2)<\epsilon$ if $n\in\mathbb{Z}_+$, $d(x_1,x_2)<\delta$ and $\pi(x_1)=\pi(x_2)=y$. Let $\mathcal{V}=\{V_1,\ldots,V_k\}$ be any open cover of X with $\mathrm{diam}(V_i)<\delta$, $i=1,\ldots,k$. Then for each i,

$$V_i\cap\pi^{-1}(y)\subseteq\bigcap_{n\geq0}T^{-n}U_{i,n}\quad\text{for some }U_{i,n}\in\mathcal{U}\ (n\in\mathbb{Z}_+),$$

which implies $\mathcal{C}(\mathcal{U},y|\pi)\leq k$. This finishes the proof of part (2).

(3) This follows directly from Proposition 2.3 and part (2). □

The variational relation between a relative complexity function and its localization is interpreted in Lemma 2.6. The easy part is

$$\sup_{y\in Y}\mathcal{C}(n,\mathcal{U},y|\pi)=\mathcal{C}(n,\mathcal{U}|\pi)\quad\text{and}\quad\sup_{y\in Y}\mathcal{C}(\mathcal{U},y|\pi)=\mathcal{C}(\mathcal{U}|\pi).$$

Before giving the more difficult part, we need to recall some definitions.

Let $f:Y\to\mathbb{R}$ be a mapping (function) from the topological space Y to the space $\mathbb{R}$ of a real line. We say that the function f is:

(1) *upper-semicontinuous* if $f^{-1}((-\infty,t))$ is open in Y for all $t\in\mathbb{R}$;

(2) *lower-semicontinuous* if $f^{-1}((t,\infty))$ is open in Y for all $t\in\mathbb{R}$.

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Then f is continuous if and only if f is not only upper-semicontinuous but also lower-semicontinuous. The following lemma was given long ago; see, for example, [**Fu**, Lemma 1.28].

LEMMA 2.5. *Let $f : Y \to \mathbb{R}$ be an upper-semicontinuous (respectively, lower-semicontinuous) function, where Y is a compact metric space. Then the points of discontinuity lie in the union of countably many closed nowhere dense subsets. In particular, the set of continuous points forms a dense G_δ subset of Y, and so it is not empty.*

Recall that TDS (X, T) is *transitive* if for each pair of non-empty open subsets U and V, the *set of return times* $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}$ is non-empty; it is *weakly mixing* if $(X \times X, T \times T)$ is transitive. $x \in X$ is a *transitive point* (denoted by $x \in \mathrm{Tran}(X, T)$) if the *orbit* $\mathrm{orb}(x, T) = \{x, Tx, T^2x, \dots\}$ is dense in the space X. It is well known that (X, T) is transitive if and only if $\mathrm{Tran}(X, T)$ forms a dense G_δ subset of X. If $\mathrm{Tran}(X, T) = X$ we say that (X, T) is *minimal*. Then (X, T) is minimal if and only if for each non-empty open subset U of X there exists $N \in \mathbb{N}$ such that $\bigcup_{n=0}^{N} T^{-n}U = X$. For any minimal sub-system (X_0, T) each point of X_0 is called a *minimal point* or *almost periodic point*, denote by $\mathrm{AP}(X, T)$ the set of minimal points.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that π is *open* (respectively *semi-open*) if $\pi(U)$ is open (respectively $\pi(U)$ has a non-empty interior in Y) for each non-empty open subset U of X. Then we have the following.

LEMMA 2.6. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between TDSs with (Y, S) minimal. Then π is positively equicontinuous if and only if π is LPE.*

Proof. It is sufficient to prove that π is positively equicontinuous if π is LPE. Let d_X and d_Y be the metrics on X and Y, respectively.

Let $\epsilon > 0$ be fixed. Define a function $f : Y \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, as follows:

$$f(y) = \sup\left\{\delta > 0 : \sup_{n \in \mathbb{Z}_+} d_X(T^n x_1, T^n x_2) \leq \frac{\epsilon}{2} \text{ if } d_X(x_1, x_2) < \delta \text{ and } x_1, x_2 \in \pi^{-1}(y)\right\}.$$

Since π is LPE, such an f is well defined and $f(y) > 0$ for each $y \in Y$. Moreover, $\sup_{n \in \mathbb{Z}_+} d_X(T^n x_1, T^n x_2) \leq \epsilon/2$ if $d_X(x_1, x_2) \leq f(y)$ and $x_1, x_2 \in \pi^{-1}(y)$. Let $\delta_0 > 0$ and $y_0 \in Y$ with $f(y_0) < \delta_0$. Then there exist $x_{1,0}, x_{2,0} \in \pi^{-1}(y_0)$ such that $d_X(x_{1,0}, x_{2,0}) < \delta_0$, whereas $d_X(T^{n'} x_{1,0}, T^{n'} x_{2,0}) > \epsilon/2$ for some $n' \in \mathbb{Z}_+$. Let U_1 (respectively U_2) be an open neighbourhood of $x_{1,0}$ (respectively $x_{2,0}$) with

$$U_1 \cap U_2 = \emptyset, \quad d_X(x_1^*, x_2^*) < \delta_0 \quad \text{and} \quad d_X(T^{n'} x_1^*, T^{n'} x_2^*) > \frac{\epsilon}{2} \quad \text{if } x_1^* \in U_1,\ x_2^* \in U_2.$$

As π is open, $V \doteq \pi(U_1) \cap \pi(U_2) \ni y_0$ is an open subset of Y. It is not hard to conclude that $f^{-1}((0, \delta_0)) \supset V$, and so the function $f : Y \to \mathbb{R}^+$ is upper-semicontinuous.

By Lemma 2.5, choose $y_0 \in Y$ a continuous point of the function f and let $0 < 2\gamma_0 \leq \min\{\epsilon, f(y_0)\}$ such that $f^{-1}((f(y_0)/2, \infty)) \supseteq B_Y(y_0, \gamma_0) = \{y \in Y : d_Y(y, y_0) < \gamma_0\}$. As (Y, S) is minimal, there exists $N \in \mathbb{N}$ such that $\bigcup_{i=0}^{N} S^{-i} B_Y(y_0, \gamma_0) = Y$. For such N select $l_0 > 0$ with $d_X(T^i x_1, T^i x_2) \leq \gamma_0$ if $i = 0, 1, \dots, N$ and $d_X(x_1, x_2) < l_0$. Now if $x_1, x_2 \in X$ satisfy $\pi(x_1) = \pi(x_2)$ and $d_X(x_1, x_2) < l_0$, then:

(1) $d_X(T^i x_1, T^i x_2) \leq \gamma_0 \leq \min\{\epsilon, f(y_0)\}/2$ if $i = 0, 1, \ldots, N$;
(2) $y^* = S^{i_0}\pi(x_1) \in B_Y(y_0, \gamma_0)$ (and so $f(y^*) > f(y_0)/2$) for some $i_0 \in \{0, 1, \ldots, N\}$.
Note that $T^{i_0}x_1, T^{i_0}x_2 \in \pi^{-1}(y^*)$, by (1), $d_X(T^{i_0}x_1, T^{i_0}x_2) \leq f(y_0)/2 < f(y^*)$ and so

$$\sup_{n\in\mathbb{Z}_+} d_X(T^n T^{i_0}x_1, T^n T^{i_0}x_2) \leq \frac{\epsilon}{2}, \quad \text{i.e. } \sup_{n\geq i_0} d_X(T^n x_1, T^n x_2) \leq \frac{\epsilon}{2}.$$

Combining with (1) again, we have

$$\sup_{n\geq 0} d_X(T^n x_1, T^n x_2) \leq \frac{\epsilon}{2} < \epsilon.$$

i.e. π is positively equicontinuous. This completes the proof. □

The main result of this section is stated as follows, which appears as a direct corollary of Lemma 2.1, Proposition 2.4 and Lemma 2.6 (see [**BHM**] for the absolute case).

THEOREM 2.7. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between TDSs with (Y, S) minimal. Then the following statements are equivalent:*
(1) *π is positively equicontinuous;*
(2) *$\mathcal{C}(\mathcal{U}|\pi) < \infty$ for each open cover $\mathcal{U}$ of X;*
(3) *$\mathcal{C}(\mathcal{U}, y|\pi) < \infty$ for each open cover $\mathcal{U}$ of X and $y \in Y$;*
(4) *π is LPE;*
(5) *$E_{\mathrm{lre}}(X, T|\pi) = X$.*

It is not hard to show that the assumption of '(Y, S) is minimal' is necessary in the above theorem, even if the systems considered are both invertible. For example, put $X = \{(x, 0) : 0 \leq x \leq 1\} \cup \{(x, x) : 0 \leq x \leq 1\}$ and $Y = \{(x, 0) : 0 \leq x \leq 1\}$, inheriting the metrics from the real plane $\mathbb{R}^2$. The transformations $T : X \to X$ and $S : Y \to Y$ are given by $T(x, 0) = (x^{1/2}, 0)$, $T(x, x) = (x^{1/2}, x^{1/2})$ and $S(x, 0) = (x^{1/2}, 0)$. The factor map $\pi : (X, T) \to (Y, S)$ is defined as $\pi(x, 0) = (x, 0)$ and $\pi(x, x) = (x, 0)$. Clearly π is open. Meanwhile, $\pi^{-1}(y)$ contains at most two points of X for any $y \in Y$, which implies that for each given open cover of X the relative complexity function is at most 2, so all items of the theorem hold for π except item (1). However, obviously, the extension π is not positively equicontinuous. Note that the system (Y, S) in the example is not transitive, so we have the following.

Question 2.8. Is there an open factor map $\pi : (X, T) \to (Y, S)$ between TDSs with (Y, S) transitive such that Theorem 2.7(2)–(5) hold for π, whereas, π is not positively equicontinuous?

In fact, the assumption of 'π is open' is also essential, and we can find a factor map between minimal invertible TDSs (hence, it is semi-open, see [**Au**]) which is not positively equicontinuous. Let us recall the simple example that appeared as Example 6.5 in [**Go**]. Let $\alpha \in [0, 1]$ be irrational. Set $A_0 = \{e^{2\pi i\theta} : 0 \leq \theta \leq \frac{1}{2}\}$ and $A_1 = \{e^{2\pi i\theta} : \frac{1}{2} \leq \theta \leq 1\}$. Define $X = \big\{x = (x_r)_{r\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} : \bigcap_{r\in\mathbb{Z}} e^{2\pi i r\alpha} A_{x_r} \neq \emptyset\big\}$. Then $\emptyset \neq X \subseteq \{0, 1\}^{\mathbb{Z}}$ is closed and (X, T) forms a minimal invertible TDS where T is given by $(Tx)_n = x_{n+1}$ for each $n \in \mathbb{Z}$. Now let the minimal invertible TDS (Y, S) be the irrational rotation on the unit circle in the complex plan with $Sy = e^{2\pi i\alpha} y$. The factor map $\pi : (X, T) \to (Y, S)$

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is defined as $\pi(x) = \bigcap_{r\in\mathbb{Z}} e^{2\pi i r\alpha} A_{x_r}$ (it is well defined). It is not hard to check that there exists a countable subset $Y_0 \subseteq Y$ such that $\pi^{-1}(y)$ is a singleton if $y \in Y \setminus Y_0$ and $\pi^{-1}(y)$ contains exactly two points if $y \in Y_0$, whereas π is not positively equicontinuous.

3. *Relative complexity tuples*

In this section, based on the idea of a relative complexity function we introduce the notion of relative complexity tuples and prove that for an open factor map π between minimal TDSs, the maximal relative positively equicontinuous factor of π is induced by the set of relative complexity pairs. Moreover, after recalling the nth relative regionally proximal relation ($n \in \mathbb{N} : n \geq 2$) we show that, for any open factor map between minimal invertible TDSs, the nth relative regionally proximal relation is identical with the union of the nth diagonal and the set of relative complexity n-tuples.

Let us first start with the definition of relative complexity tuples. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $n \in \mathbb{N} : n \geq 2$. Set $R_\pi^{(n)} = \{(x_i)_1^n \in X^{(n)} : \pi(x_1) = \cdots = \pi(x_n)\}$ and $\Delta_n(X) = \{(x_i)_1^n \in X^{(n)} : x_1 = \cdots = x_n\}$, the nth *diagonal*.

Definition 3.1. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that $(x_i)_1^n \notin \Delta_n(X)$ is a *complexity n-tuple with respect to* π (denoted by $(x_i)_1^n \in \mathrm{Com}_n(X, T|\pi)$) if $\mathcal{C}(\{U_1^c, \ldots, U_n^c\}|\pi) = \infty$ when U_i is a closed neighbourhood of x_i with $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$ and $U_i = U_j$ if $x_i = x_j$, $1 \leq i, j \leq n$.

It is not hard to obtain $\mathrm{Com}_n(X, T|\pi) \subseteq R_\pi^{(n)} \setminus \Delta_n(X)$. Thus, we have (following [**B2**]) the following proposition.

PROPOSITION 3.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs.*

(1) *If $\mathcal{V}$ is an open cover of X with $\mathcal{C}(\mathcal{V}|\pi) = \infty$, then there exists an open cover $\mathcal{U} = \{U_1, U_2\}$ of X such that $\mathcal{C}(\mathcal{U}|\pi) = \infty$.*
(2) *If there is an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X with $\mathcal{C}(\mathcal{U}|\pi) = \infty$, then there exists $x_i \in U_i^c$ ($i = 1, \ldots, n$) such that $(x_i)_1^n \in \mathrm{Com}_n(X, T|\pi)$.*
(3) *$\mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X)$ is a closed, positively $T^{(n)}$-invariant subset of $X^{(n)}$, i.e. $T^{(n)}(\mathrm{Com}_n(X, T|\pi)) \subseteq \mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X)$.*
(4) *Let $\pi' : (Y, S) \to (Z, R)$ be a factor map between TDSs. Then:*
    (i) *$\mathrm{Com}_n(Y, S|\pi') \subseteq \pi \times \cdots \times \pi(\mathrm{Com}_n(X, T|\pi'\pi)) \subseteq \mathrm{Com}_n(Y, S|\pi') \cup \Delta_n(Y)$;*
    (ii) *$\mathrm{Com}_n(X, T|\pi) \subseteq \mathrm{Com}_n(X, T|\pi'\pi)$.*

Let $\pi : (X, T) \to (Y, S)$, $\pi_1 : (X, T) \to (Z, R)$ and $\pi_2 : (Z, R) \to (Y, S)$ be factor maps between TDSs with $\pi = \pi_2\pi_1$. In this case, we say that (Z, R) is a *factor of* π. If π_2 is non-trivial, then we say that the factor (Z, R) is *non-trivial*. If π_2 is positively equicontinuous, then we say that (Z, R) is a *positively equicontinuous factor of* π. Note that if π is open then π_2 is also open, as $\pi_2(W) = \pi(\pi_1^{-1}W)$ is an open subset of Y for each open subset W of Z. Then by Lemma 2.1, Theorem 2.7 and Proposition 3.2 we have the following.

COROLLARY 3.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs.*

(1) *If π is positively equicontinuous, then* $\mathrm{Com}_2(X, T|\pi) = \emptyset$.
(2) *Suppose that π is open and (Y, S) is minimal. Then:*

(i) *π is positively equicontinuous if and only if* $\mathrm{Com}_2(X, T|\pi) = \emptyset$*;*
(ii) *the maximal positively equicontinuous factor of π is induced by the closed positively invariant equivalence relation generated by* $\mathrm{Com}_2(X, T|\pi)$.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between invertible TDSs. Recall that π is *equicontinuous* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}$, $d(x_1, x_2) < \delta$ and $\pi(x_1) = \pi(x_2)$. Let $\pi_{-1} : (X, T^{-1}) \to (Y, S^{-1})$ be the factor map induced by π. One has $\mathrm{Com}_n(X, T|\pi) = \mathrm{Com}_n(X, T^{-1}|\pi_{-1})$, as

$$N\bigg(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}|\pi\bigg) = \sup_{y\in Y} N\bigg(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}, \pi^{-1}(S^{1-m}y)\bigg) = N\bigg(\bigvee_{i=0}^{m-1} T^{i}\mathcal{U}|\pi_{-1}\bigg)$$

for each $m \in \mathbb{N}$ and any cover $\mathcal{U}$ of X admitting a finite sub-cover. So by Proposition 3.2 the subset $\mathrm{Com}_n(X, T|\pi)$ is $T^{(n)}$-invariant, i.e. $T^{(n)}(\mathrm{Com}_n(X, T|\pi)) = \mathrm{Com}_n(X, T|\pi)$. Letting $n = 2$ we have the following easy fact.

COROLLARY 3.4. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between invertible TDSs with (Y, S) minimal. Then π is equicontinuous if and only if π is positively equicontinuous.*

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. Recall that π is *weakly mixing of order n* ($n \in \mathbb{N} : n \geq 2$) if the TDS $(R_\pi^{(n)}, T^{(n)})$ is transitive; when $n = 2$ we say that π is *weakly mixing*; and that π is *weakly mixing of all orders* if $(R_\pi^{(n)}, T^{(n)})$ is transitive for each $n \geq 2$. Then we have (see **[BHM]** for the absolute case) the following.

PROPOSITION 3.5. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. If π is weakly mixing of all orders, then* $\mathrm{Com}_n(X, T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$ *for each $n \in \mathbb{N} : n \geq 2$.*

Proof. For the proof it remains to show that $(x_i)_1^n \in \mathrm{Com}_n(X, T|\pi)$ for each $(x_i)_1^n \in R_\pi^{(n)}$ ($n \in \mathbb{N} : n \geq 2$) with $x_i \neq x_j$ if $i \neq j$. Fix such a tuple $(x_i)_1^n$. To complete the proof, we show that if $U_1, \ldots, U_n$ are n pairwise disjoint open neighbourhoods of $x_1, \ldots, x_n$ then $\mathcal{C}(\mathcal{U}|\pi) = \infty$ by constructing inductively $\{t_1 < t_2 < \cdots\} \subseteq \mathbb{Z}_+$ and $\{y_m : m \in \mathbb{N}\} \subseteq Y$ with $\mathcal{C}(t_m + 1, \mathcal{U}, y_m|\pi) \geq (n/(n-1))^m$, where $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. In fact, we claim the following.

CLAIM. *There exist $\{t_1 < t_2 < \cdots\} \subseteq \mathbb{Z}_+$ and $\{y_m : m \in \mathbb{N}\} \subseteq Y$ such that if $m \in \mathbb{N}$ and $s \doteq (s(i))_1^m \in \{1, \ldots, n\}^m$, then $\pi^{-1}(y_m) \cap \bigcap_{i=1}^m T^{-t_i} U_{s(i)} \neq \emptyset$.*

Proof. Set $t_1 = 0$ and $y_1 = \pi(x_1)$. Clearly $x_i \in \pi^{-1}(y_1) \cap U_i \neq \emptyset$, $1 \leq i \leq n$. Now assume that $t_1 < \cdots < t_k$ and y_k satisfying the claim are constructed. Then

$$W_1 \doteq \prod_{s\in\{1,\ldots,n\}^k} \bigg(\prod_{j=1}^n \bigcap_{i=1}^k T^{-t_i} U_{s(i)}\bigg) \cap R_\pi^{(n^{n^k})} \quad \text{and} \quad W_2 \doteq \prod_{s\in\{1,\ldots,n\}^k} \bigg(\prod_{j=1}^n U_j\bigg) \cap R_\pi^{(n^{n^k})}$$

are both non-empty open subsets of $R_\pi^{(n^{n^k})}$. Since π is weakly mixing of all orders, the TDS $(R_\pi^{(n^{n^k})}, T^{(n^{n^k})})$ is transitive. Say $(z_p)_1^{n^{n^k}} \in W_1 \cap \mathrm{Tran}(R_\pi^{(n^{n^k})}, T^{(n^{n^k})})$, then there exists

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$t_{k+1} > t_k$ such that $(T^{t_{k+1}} z_p)_1^{n^{n^k}} \in W_2$. Let $y_{k+1} = \pi(z_1)$. It is easy to check that

$$\pi^{-1}(y_{k+1}) \cap \bigcap_{i=1}^{k} T^{-t_i} U_{s(i)} \cap T^{-t_{k+1}} U_j \neq \emptyset$$

for each $s \in \{1, \ldots, n\}^k$ and $j \in \{1, \ldots, n\}$. □

Let $\{t_1 < t_2 < \cdots\} \subseteq \mathbb{Z}_+$ and $\{y_m : m \in \mathbb{N}\} \subseteq Y$ be constructed as in the claim. Fix $m \in \mathbb{N}$. For each $s \in \{1, \ldots, n\}^m$, say $x_s \in \pi^{-1}(y_m) \cap \bigcap_{i=1}^m T^{-t_i} U_{s(i)}$. Let $X_m = \{x_s : s \in \{1, \ldots, n\}^m\} \subseteq \pi^{-1}(y_m)$. Obviously, $\#(X_m) = n^m$, where $\#(X_m)$ is the cardinality of the set X_m. Note that, each element of $\bigvee_{i=1}^m T^{-t_i}\mathcal{U}$, say $\bigcap_{i=1}^m T^{-t_i} U_{r(i)}^c$, contains just $(n-1)^m$ points in X_m, $\{x_s : s \in \{1, \ldots, n\}^m$ satisfies $s(1) \neq r(1), \ldots, s(m) \neq r(m)\}$. This implies $N\big(\bigvee_{i=1}^m T^{-t_i}\mathcal{U}, \pi^{-1}(y_m)\big) \geq (n/(n-1))^m$. Then

$$\begin{aligned}\mathcal{C}(t_m + 1, \mathcal{U}, y_m|\pi) &= N\bigg(\bigvee_{i=0}^{t_m} T^{-i}\mathcal{U}, \pi^{-1}(y_m)\bigg) \geq N\bigg(\bigvee_{i=1}^{m} T^{-t_i}\mathcal{U}, \pi^{-1}(y_m)\bigg)\\ &\geq \left(\frac{n}{n-1}\right)^m. \end{aligned}$$ □

Let $\pi : (X, T) \to (Y, S)$ be an open factor map between TDSs with π weakly mixing. We do not know whether $\mathrm{Com}_2(X, T|\pi) = R_\pi^{(2)} \setminus \Delta_2(X)$; at least the proof of Proposition 3.5 does not work. However, if (X, T) and (Y, S) are both minimal invertible TDSs, we can answer it positively. Before proceeding, we need a general version of the well-known Kuratowski–Ulam theorem (an example of the usual version is given in [**O**, Theorem 15.1, p. 56]).

LEMMA 3.6. *Let $\pi : X \to Y$ be an open continuous map between topological spaces, where X has a countable basis.*

(1) *Suppose that $E \subseteq X$ is closed with $E^\circ = \emptyset$, where E° denotes the interior of E in X. Then there exists $V \subseteq Y$ such that:*
   (i) *V contains the intersection of a countable dense open subsets of Y;*
   (ii) *$E \cap \pi^{-1}(y)$ has an empty interior in $\pi^{-1}(y)$ for each $y \in V$.*

(2) *Suppose that $E \subseteq X$ is a set of first category and Y is a complete metric space. Then there exists a dense G_δ subset V of Y such that $E \cap \pi^{-1}(y)$ is a subset of first category in $\pi^{-1}(y)$ for each $y \in V$.*

Proof. For the proof it is sufficient to prove part (1).

Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis of X. Let $y \in Y$. Obviously, $E \cap \pi^{-1}(y)$ has a non-empty interior in $\pi^{-1}(y)$ if and only if there exists $n \in \mathbb{N}$ such that $E \supseteq V_n \cap \pi^{-1}(y) \neq \emptyset$. Set

$$W_n = (\pi(V_n))^c \cup \{y \in Y : V_n \cap \pi^{-1}(y) \setminus E \neq \emptyset\} = (\pi(V_n))^c \cup \pi(V_n \setminus E).$$

Denote by V the set of points $y \in Y$ such that $E \cap \pi^{-1}(y)$ has an empty interior in $\pi^{-1}(y)$. Then we have $V = \bigcap_{n\in\mathbb{N}} W_n$. To finish the proof, it suffices to show that W_n contains a dense open subset of Y for any fixed $n \in \mathbb{N}$.

Clearly, W_n is dense in Y. Otherwise, there exists a non-empty open subset $U \subseteq \pi(V_n)$ such that $V_n \cap \pi^{-1}(y) \setminus E = \emptyset$ for each $y \in U$, which implies $E \supseteq V_n \cap \pi^{-1}(U) \neq \emptyset$, a contradiction to the assumption of '$E^\circ = \emptyset$'. Since π is open, $(\pi(V_n))^c \subseteq Y$ is closed, $\pi(V_n \setminus E)$ and $\pi(V_n)$ are both open. Then it is not hard to check that W_n contains a dense open subset of Y. In fact, as W_n is dense in Y, $\overline{\pi(V_n \setminus E)} \supseteq \pi(V_n)$, so $\overline{\pi(V_n \setminus E)} = \overline{\pi(V_n)} = X \setminus [(\pi(V_n))^c]^\circ$, i.e. $[(\pi(V_n))^c]^\circ \cup \pi(V_n \setminus E) \subseteq W_n$ is a dense open subset. This completes the proof. □

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $n \in \mathbb{N} : n \geq 2$. Recall that the nth *relative regionally proximal relation* $\mathrm{RP}_n(X, T|\pi) \subseteq R_\pi^{(n)}$ is introduced as: $(x_i)_1^n \in \mathrm{RP}_n(X, T|\pi)$ if and only if, for all $\epsilon > 0$ and all $U_{x_i} \in \mathcal{V}_{x_i}$, $i = 1, \ldots, n$,

$$\exists (x_i')_1^n \in R_\pi^{(n)} \cap \prod_{i=1}^n U_{x_i} \quad \text{and} \quad m \in \mathbb{Z}_+ \quad \text{with} \max_{1 \leq i < j \leq n} d(T^m x_i', T^m x_j') \leq \epsilon,$$

where $\mathcal{V}_{x_i}$ denotes the set of neighbourhoods of x_i $(i = 1, \ldots, n)$.

PROPOSITION 3.7. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between minimal invertible TDSs and $n \in \mathbb{N} : n \geq 2$. Then:*
(1) $\mathrm{RP}_n(X, T|\pi) \subseteq \mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X)$*;*
(2) *if π is weakly mixing of order n, then* $\mathrm{Com}_n(X, T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$.

Proof. It is not hard to obtain that $\mathrm{RP}_n(X, T|\pi) = R_\pi^{(n)}$ if π is weakly mixing of order n by selecting a transitive point of $(R_\pi^{(n)}, T^{(n)})$ in each $\prod_{i=1}^n U_{x_i}$. Then part (2) follows from part (1). Now we aim to prove part (1).

Assume the contrary. Let $(x_i)_1^n \in \mathrm{RP}_n(X, T|\pi) \setminus (\mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X))$.

As $(x_i)_1^n \notin \mathrm{Com}_n(X, T|\pi) = \mathrm{Com}_n(X, T^{-1}|\pi_{-1})$, we can select open subsets $U_i \ni x_i$ $(i = 1, \ldots, n)$ such that $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$, $U_i = U_j$ if $x_i = x_j$ and $\mathcal{C}(\mathcal{U}|\pi_{-1})$ is finite, where $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$. Then it is not hard to see that for each $y \in Y$, $\pi^{-1}(y)$ admits a closed cover $\mathcal{W}(y) \subseteq \bigvee_{m \geq 0} T^m \mathcal{U}$ with cardinality at most $\mathcal{C}(\mathcal{U}|\pi_{-1})$. Say $W(y) \in \mathcal{W}(y) \cap \pi^{-1}(y)$ has a non-empty interior in $\pi^{-1}(y)$.

Denote by S_n the collection of all permutations for $\{1, \ldots, n\}$. Set

$$C_\pi = \bigcap_{m \geq 0} (T^{(n)})^m \left(\bigcup_{i=1}^n \prod_{j=1}^n U_i^c \right) \cap R_\pi^{(n)}, \quad \text{and}$$

$$P_\pi = \bigcup_{m \geq 0} (T^{(n)})^m \left(\bigcup_{(s(1), \ldots, s(n)) \in S_n} \prod_{i=1}^n U_{s(i)} \right) \cap R_\pi^{(n)}.$$

Then we have:
(1) $C_\pi \cap P_\pi = \emptyset$, by the definitions of C_π and P_π;
(2) $C_\pi \subseteq R_\pi^{(n)}$ is closed and $\bigcup_{y \in Y} \prod_{j=1}^n W(y) \subseteq C_\pi$;
(3) $\Delta_n(X)$ is contained in the closure of P_π in $R_\pi^{(n)}$, as $(x_i)_1^n \in \mathrm{RP}_n(X, T|\pi)$ and (X, T) is minimal.

Let $\pi_n : (R_\pi^{(n)}, T^{(n)}) \to (Y, S)$ be the factor map determined by π. Then π_n is open, as π is open. Let $\mathrm{int}(C_\pi)$ be the interior of C_π in $R_\pi^{(n)}$. Now applying Lemma 3.6 to $C_\pi \setminus \mathrm{int}(C_\pi)$ one has that there exists a dense G_δ subset Y_0 of Y such that $C_\pi \setminus \mathrm{int}(C_\pi) \cap \prod_{j=1}^n \pi^{-1}(y)$ has an empty interior in $\prod_{j=1}^n \pi^{-1}(y)$ for each $y \in Y_0$, as $\pi_n^{-1}(y) = \prod_{j=1}^n \pi^{-1}(y)$.

Let $y \in Y_0$. Since $\prod_{j=1}^n W(y)$ has a non-empty interior in $\prod_{j=1}^n \pi^{-1}(y)$, write it as $\mathrm{int}\big(\prod_{j=1}^n W(y)\big)$ $(\neq \emptyset)$. By the above discussion we have

$$\bigcup_{y\in Y_0} \mathrm{int}\bigg(\prod_{j=1}^n W(y)\bigg) \subseteq \mathrm{int}(C_\pi). \tag{3.1}$$

Obviously $\mathrm{int}\big(\prod_{j=1}^n W(y)\big) \cap \Delta_n(X) \neq \emptyset$, this implies $\mathrm{int}(C_\pi) \cap \Delta_n(X) \neq \emptyset$. Thus,

$$C_\pi \cap P_\pi \supseteq \mathrm{int}(C_\pi) \cap P_\pi \neq \emptyset \text{ (by (3))},$$

a contradiction to (1). This means $\mathrm{RP}_n(X, T|\pi) \subseteq \mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X)$. □

LEMMA 3.8. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between invertible TDSs and $n \in \mathbb{N} : n \geq 2$. Then* $\mathrm{Com}_n(X, T|\pi) \subseteq \mathrm{RP}_n(X, T^{-1}|\pi_{-1})$.

Proof. The proof follows the idea of [**BHM**, Proposition 5.8].

Let $(x_i)_1^n \in R_\pi^{(n)} \setminus \mathrm{RP}_n(X, T^{-1}|\pi_{-1})$. Then there exist a closed neighbourhood U_{x_i} of x_i $(i = 1, \ldots, n)$ with $U_{x_i} \cap U_{x_j} = \emptyset$ if $x_i \neq x_j$, $U_{x_i} = U_{x_j}$ if $x_i = x_j$, and $\epsilon > 0$ such that if $(x_i')_1^n \in \prod_{i=1}^n U_{x_i} \cap R_\pi^{(n)}$ and $m \in \mathbb{Z}_+$, then $\max_{1\leq i<j\leq n} d(T^{-m}x_i', T^{-m}x_j') > \epsilon$.

Let $\{B_1, \ldots, B_k\}$ be a closed cover of X with: (1) $\mathrm{diam}(B_i) \leq \epsilon/2$ and $B_i^\circ \neq \emptyset$ $(i = 1, \ldots, k)$; (2) $x_i \in B_i^\circ \subseteq B_i \subseteq U_{x_i}$, $i = 1, \ldots, n$. Let $y \in Y$ and $m \in \mathbb{Z}_+$. We claim that

$$\forall j \in \{1, \ldots, k\}, \quad \exists i_j \in \{1, \ldots, n\} \quad \text{such that } B_j \cap \pi^{-1}(y) \subseteq T^{-m}(B_{i_j}^c). \tag{3.2}$$

Otherwise, we can select $(x_i'')_1^n \in \prod_{i=1}^n (T^m(B_j) \cap B_i \cap \pi^{-1}(T^m y)) \subseteq \prod_{i=1}^n U_{x_i} \cap R_\pi^{(n)}$. So $\max_{1\leq i<l\leq n} d(T^{-m}x_i'', T^{-m}x_l'') > \epsilon$, a contradiction to $\mathrm{diam}(B_j) \leq \epsilon/2$.

Now set $\mathcal{V} = \{B_1^c, \ldots, B_n^c\}$ to be an open cover of X. By (3.2), $\mathcal{C}(m, \mathcal{V}, y|\pi) \leq k$ for each $y \in Y$ and $m \in \mathbb{Z}_+$, so $\mathcal{C}(\mathcal{V}|\pi) \leq k$, which implies $(x_i)_1^n \notin \mathrm{Com}_n(X, T|\pi)$. □

As a corollary of Proposition 3.7 and Lemma 3.8 we have (see [**BHM**] for the absolute case) the following.

THEOREM 3.9. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between minimal invertible TDSs. Then* $\mathrm{RP}_n(X, T|\pi) = \mathrm{RP}_n(X, T^{-1}|\pi_{-1}) = \mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X)$.

Proof. As $\mathrm{Com}_n(X, T|\pi) = \mathrm{Com}_n(X, T^{-1}|\pi_{-1})$, then we have

$$\begin{aligned}\mathrm{RP}_n(X, T|\pi) &\subseteq \mathrm{Com}_n(X, T|\pi) \cup \Delta_n(X) \quad \text{(by Proposition 3.7)}\\ &= \mathrm{Com}_n(X, T^{-1}|\pi_{-1}) \cup \Delta_n(X)\\ &\subseteq \mathrm{RP}_n(X, T|\pi) \quad \text{(by Lemma 3.8)}.\end{aligned}$$

Note that if (X, T) is minimal, then so is (X, T^{-1}). So a similar discussion works for $\mathrm{RP}_n(X, T^{-1}|\pi_{-1})$. This finishes the proof. □

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4. *Relative scattering*

In this section, first we introduce the notions of relative n-scattering ($n \in \mathbb{N} : n \geq 2$) and relative scattering. We show that any relative scattering open factor map is weakly mixing if we consider minimal invertible TDSs. A relative disjointness theorem involving relative scattering is given. Then after introducing relative Mycielski's chaos (a stronger version of Li–Yorke's chaos on fibres) we generalize the result that any non-trivial scattering TDS is Li–Yorke chaotic.

First, let us define relative n-scattering and relative scattering.

Definition 4.1. Let $\pi : (X, T) \to (Y, S)$ be a non-trivial factor map between TDSs.

(1) Let $n \in \mathbb{N} : n \geq 2$. We say that π has *relative n-scattering* or (X, T) has *relative n-scattering* with respect to (Y, S) if $\mathrm{Com}_n(X, T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$.

(2) We say that π has *relative scattering* or (X, T) has *relative scattering* with respect to (Y, S) if for each $n \in \mathbb{N} : n \geq 2$, $\mathrm{Com}_n(X, T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$.

As a direct corollary of Proposition 3.2, by definition we have the following.

PROPOSITION 4.2. *Let $\pi : (X, T) \to (Y, S)$ be a non-trivial factor map between TDSs and $n \in \mathbb{N} : n \geq 2$. Suppose that π has relative n-scattering. Then we have the following.*

(1) *If $k \in \mathbb{N}$ with $n \geq k \geq 2$ then π has relative k-scattering.*

(2) *Let (Z, R) be a non-trivial factor of π. Then:*

   (i) *(Z, R) has relative n-scattering with respect to (Y, S);*

   (ii) *if π has relative scattering, then (Z, R) has relative scattering with respect to (Y, S).*

Let $\pi : (X, T) \to (Y, S)$ be a factor map between invertible TDSs. Recall that we say that $(x_1, x_2) \in X^{(2)}$ is a:

(1) *proximal pair* (denoted by $(x_1, x_2) \in \mathrm{P}(X, T)$) if there exists $\{n_i\}_1^\infty \subseteq \mathbb{Z}$ such that $|n_i| \to \infty$ and $d(T^{n_i}x_1, T^{n_i}x_2) \to 0$ as $i \to \infty$;

(2) *proximal pair for T* (denoted by $(x_1, x_2) \in \mathrm{P}_T(X, T)$) if there exists $\{n_i\}_1^\infty \subseteq \mathbb{Z}_+$ such that $n_i \to \infty$ and $d(T^{n_i}x_1, T^{n_i}x_2) \to 0$ as $i \to \infty$;

(3) *proximal pair for T^{-1}* (denoted by $(x_1, x_2) \in \mathrm{P}_{T^{-1}}(X, T)$) if there exists $\{n_i\}_1^\infty \subseteq \mathbb{Z}_+$ such that $n_i \to \infty$ and $d(T^{-n_i}x_1, T^{-n_i}x_2) \to 0$ as $i \to \infty$.

It is clear that $\mathrm{P}(X, T) = \mathrm{P}_T(X, T) \cup \mathrm{P}_{T^{-1}}(X, T)$. Then by Lemma 3.8 we have the following.

PROPOSITION 4.3. *Let $\pi : (X, T) \to (Y, S)$ be a non-trivial factor map between invertible TDSs. Suppose that π has relative 2-scattering. Then all of $\mathrm{P}(X, T) \cap R_\pi^{(2)}$, $\mathrm{P}_T(X, T) \cap R_\pi^{(2)}$ and $\mathrm{P}_{T^{-1}}(X, T) \cap R_\pi^{(2)}$ are dense G_δ subsets of $R_\pi^{(2)}$.*

Proof. As $\mathrm{Com}_2(X, T^{-1}|\pi_{-1}) = \mathrm{Com}_2(X, T|\pi) = R_\pi^{(2)} \setminus \Delta_2(X)$, using Lemma 3.8 we have $\mathrm{RP}_2(X, T|\pi) = \mathrm{RP}_2(X, T^{-1}|\pi_{-1}) = \overline{R_\pi^{(2)}}$. Note that $\mathrm{P}_T(X, T) \cap R_\pi^{(2)} = \bigcap_{m \in \mathbb{N}} R_\pi^{(2)}(m)$ and $\mathrm{RP}_2(X, T|\pi) = \bigcap_{m \in \mathbb{N}} R_\pi^{(2)}(m)$, where

$$R_\pi^{(2)}(m) = \bigcup_{n \in \mathbb{Z}_+} T^{-n} \times T^{-n} \left\{(x_1, x_2) \in R_\pi^{(2)} : d(x_1, x_2) < \frac{1}{m}\right\}.$$

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Then $R_\pi^{(2)}(m) \subseteq R_\pi^{(2)}$ forms a dense open subset (for all $m \in \mathbb{N}$), so $\mathrm{P}_T(X,T) \cap R_\pi^{(2)}$ is a dense G_δ subset of $R_\pi^{(2)}$. The same reasoning works for $\mathrm{P}(X,T) \cap R_\pi^{(2)}$ and $\mathrm{P}_{T^{-1}}(X,T) \cap R_\pi^{(2)}$. □

Using Theorem 3.9 we obtain (see [**BHM**] for the absolute case) the following.

THEOREM 4.4. *Let $\pi_X : (X,T) \to (Y,S)$ and $\pi_Z : (Z,R) \to (Y,S)$ be two factor maps between minimal invertible TDSs with π_X open. Suppose that π_X has relative scattering. Then TDS $(\pi_{X,Z}, T \times R)$ is transitive, where $\pi_{X,Z} = \{(x,z) \in X \times Z : \pi_X(x) = \pi_Z(z)\}$. In particular, π_X is weakly mixing.*

Proof. The proof follows the idea of [**G2**, Theorem 6.3].

For each $n \in \mathbb{N} : n \geq 2$, set

$$P_{\pi_X}^{(n)} = \bigcap \Big\{ \bigcup_{m \geq 0} (T^{(n)})^{-m} V : V \text{ is a neighbourhood of } \Delta_n(X) \text{ in } R_{\pi_X}^{(n)} \Big\}.$$

As π_X has relative scattering, $\mathrm{RP}_n(X, T|\pi_X) = R_{\pi_X}^{(n)}$ (by Theorem 3.9), using similar discussions as in the proof of Proposition 4.3 it is not hard to show that $P_{\pi_X}^{(n)}$ forms a dense G_δ subset of $R_{\pi_X}^{(n)}$. Since π_X is open, applying Lemma 3.6 to $R_{\pi_X}^{(n)} \setminus P_{\pi_X}^{(n)}$, there exists a dense G_δ subset Y_n of Y such that for each $y \in Y_n$, $P_{\pi_X}^{(n)} \cap \prod_{j=1}^n \pi^{-1}(y)$ is a dense G_δ subset of $\prod_{j=1}^n \pi^{-1}(y)$. Now put $Y^* = \bigcap_{n \geq 2} Y_n$. Then Y^* is a dense G_δ subset of Y.

Now let $W \subseteq \pi_{X,Z}$ be a positively $T \times R$-invariant subset with a non-empty interior in $\pi_{X,Z}$. To complete the proof it suffices to show that $W = \pi_{X,Z}$. Let U_X (respectively U_Z) be an open subset of X (respectively Y) such that $\emptyset \neq (U_X \times U_Z) \cap \pi_{X,Z} \subseteq W$. As π_X is open and (Z,R) is minimal (so π_Z is semi-open), without loss of generality we assume that $\pi_X(U_X) = \pi_Z(U_Z)$. Otherwise, replace U_X (respectively U_Z) by $U_X \cap \pi_X^{-1}(U_Y)$ (respectively $U_Z \cap \pi_Z^{-1}(U_Y)$), where U_Y is the interior of $\pi_Z(U_Z \cap \pi_Z^{-1}(\pi_X U_X))$ in Y (it is not hard to check that $\pi_X(U_X \cap \pi_X^{-1}(U_Y)) = \pi_Z(U_Z \cap \pi_Z^{-1}(U_Y)) \neq \emptyset$).

As π_X is open and $Y^* \subseteq Y$ is dense, there exists $y \in Y^*$ such that $y \in \pi_X(U_X)$. Since (Z,R) is minimal, there exist $1 \leq i_1 < \cdots < i_m$ such that

$$\bigcup_{j=1}^m R^{-i_j} U_Z \supseteq \pi_Z^{-1}(y) \quad \text{and} \quad \pi_Z^{-1}(y) \cap R^{-i_j} U_Z \neq \emptyset, \quad j = 1, \ldots, m.$$

Moreover, $W_m = \prod_{j=1}^m T^{-i_j} U_X \cap \prod_{j=1}^m \pi_X^{-1}(y) \neq \emptyset$, as $\pi_X(U_X) = \pi_Z(U_Z)$. Then by the construction of Y^*, there exists $(x_1, \ldots, x_m) \in W_m \cap P_{\pi_X}^{(m)}$, and so for some sequence $1 \leq p_1 < p_2 < \cdots$ and some point $x \in X$ we have $T^{p_k} x_j \to x$ as $k \to \infty$, $j = 1, \ldots, m$.

Assume the contrary that $\pi_{X,Z} \setminus W \neq \emptyset$, then there exist open subsets $V_X \subseteq X$ and $V_Z \subseteq Z$ such that $\emptyset \neq (V_X \times V_Z) \cap \pi_{X,Z} \subseteq \pi_{X,Z} \setminus W$ and $\pi_X(V_X) = \pi_Z(V_Z)$. As (X,T) is minimal, it makes no difference to assume that $x \in V_X$ and so there exists $t \geq \max\{i_1, \ldots, i_m\}$ such that $T^t x_1, \ldots, T^t x_m \in V_X$. As $\pi_X(V_X) = \pi_Z(V_Z)$, $\pi_Z(z) = \pi_X(T^t x_1) = S^t y$ for some $z \in V_Z$, then $R^{-t} z \in \pi_Z^{-1}(y)$ and so $R^{-t} z \in R^{-i_{j_0}} U_Z$ for some $j_0 \in \{1, \ldots, m\}$. One has $(T^t x_{j_0}, z) \in (V_X \times V_Z) \cap \pi_{X,Z} \subseteq \pi_{X,Z} \setminus W$ and

$$\begin{aligned}(T^t x_{j_0}, z) &= (T^{t-i_{j_0}}(T^{i_{j_0}} x_{j_0}), R^{t-i_{j_0}}(R^{i_{j_0}-t} z)) \\ &\in (T \times R)^{t-i_{j_0}}(U_X \times U_Z) \cap \pi_{X,Z} \in (T \times R)^{t-i_{j_0}}(W) \subseteq W,\end{aligned}$$

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as W is positively $T \times R$-invariant and $t \geq \max\{i_1, \ldots, i_m\}$. A contradiction, which implies $\pi_{X,Z} = W$, that is, $(\pi_{X,Z}, T \times R)$ is a transitive TDS. □

Now we present a result involving relative disjointness.

Let $\pi_X : (X, T) \to (Y, S)$ and $\pi_Z : (Z, R) \to (Y, S)$ be two factor maps between TDSs, and $\pi_1 : X \times Z \to X$, $\pi_2 : X \times Z \to Z$ the projections. $J \subseteq X \times Z$ is called a *joining* of (X, T) and (Z, R) over (Y, S) if J is closed, positively $T \times R$-invariant with $\pi_1(J) = X$, $\pi_2(J) = Z$ and $\pi_X \times \pi_Z(J) = \Delta_2(Y)$. Clearly, $X \times_Y Z$ is a joining, where

$$X \times_Y Z = \bigcup_{y \in Y} \pi_X^{-1}(y) \times \pi_Z^{-1}(y).$$

Call (X, T) and (Z, R) *disjoint* over (Y, S) if $X \times_Y Z$ contains no proper sub-joining of (X, T) and (Z, R) over (Y, S).

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. Say that π is *minimal* if X is the only closed positively T-invariant subset with π-image Y. The proof of the following theorem is similar to that of [**B2**, Proposition 6] (see also [**HYZ**, Theorem 2.5]).

THEOREM 4.5. *Let $\pi_X : (X, T) \to (Y, S)$ and $\pi_Z : (Z, R) \to (Y, S)$ be two factor maps between TDSs, where π_X is open and π_Z is minimal. Suppose that π_X has relative scattering and π_Z is positively equicontinuous. If (Y, S) is invertible, then (X, T) and (Z, R) are disjoint over (Y, S).*

In the remainder of this section, we focus on the relation between relative 2-scattering and Li–Yorke's chaos on fibres (for the definition of Li–Yorke's chaos, see, for example, [**LY**]). It is known that any non-trivial scattering TDS is Li–Yorke chaotic (see [**HY1**]). What happens if we consider a factor map between TDSs? We include here a relative version of this result (Theorem 4.7).

For the reader's convenience, before proceeding we make some preparations. Let X be a complete metric space and $K \subseteq X$. We say that K is a *Mycielski subset* if it has the form $K = \bigcup_{j \in \mathbb{N}} C_j$, where each C_j is a non-empty Cantor subset of X. We restate here a version of Mycielski's theorem [**M**, Theorem 1] which we use.

MYCIELSKI'S THEOREM. *Let X be a complete metric space with no isolated points. Suppose that for each $n \in \mathbb{N}$, R_n is a subset of first category in $X^{(r_n)}$, and let G_j, $j = 1, 2, \ldots$, be a sequence of non-empty open subsets of X. Then there exist Cantor subsets $C_j \subseteq G_j$ such that for each $n \in \mathbb{N}$ the Mycielski set $K = \bigcup_{j \in \mathbb{N}} C_j$ has the property that for every $x_1, \ldots, x_{r_n}$ distinct elements of K, $(x_i)_1^{r_n} \notin R_n$.*

A direct application of Mycielski's theorem is the following.

LEMMA 4.6. *Let X be a compact metric space with no isolated points. If $R \subseteq X^{(2)}$ contains a dense G_δ subset of $X^{(2)}$, then there exists a dense Mycielski subset K in X such that $(K \times K) \setminus \Delta_2(X) \subseteq R$.*

Let (X, T) be a TDS. Recall that we say that the pair $(x_1, x_2) \in X^{(2)}$ is *asymptotic* (denoted by $(x_1, x_2) \in \mathrm{AR}(X, T)$) if $d(T^n x_1, T^n x_2) \to 0$ as $n \to \infty$. Let $k, m \in \mathbb{N}$. Put

$$\mathrm{AR}_{k,m}(X, T) = \bigcap_{j \geq k} T^{-j} \times T^{-j} \left\{ (x_1, x_2) \in X^{(2)} : d(x_1, x_2) \leq \frac{1}{m} \right\}.$$

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It is obvious that for each $k, m \in \mathbb{N}$, $\mathrm{AR}_{k,m}(X, T)$ is a closed subset of $X^{(2)}$ and $\mathrm{AR}(X, T) = \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \mathrm{AR}_{k,m}(X, T)$ is a $T^{(2)}$-invariant $F_{\sigma\delta}$ subset of $X^{(2)}$.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that π has *relative Mycielski's chaos* if there exist $y \in Y$ and a Mycielski subset $K \subseteq \pi^{-1}(y)$ such that $(K \times K) \setminus \Delta_2(X) \subseteq \mathrm{P}_T(X, T) \setminus \mathrm{AP}(X, T)$.

The main result of [**Z**] tells us that if the factor map π has positive conditional topological entropy, then π has relative Mycielski's chaos (for details, see, for example, [**Z**, Theorem 4.2], but instead of relative Mycielski's chaos there we say Li–Yorke's chaos on fibres). Meanwhile, in the following we show that in many cases relative 2-scattering implies relative Mycielski's chaos which is stated as Theorem 4.7. We delay the definitions of sensitivity (for a TDS) and relative sensitivity (for a factor map between TDSs) and the proof of the second part of Theorem 4.7 until next section.

THEOREM 4.7. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between invertible TDSs, where π has relative 2-scattering. Suppose that there exists a subset $Y_0 \subseteq Y$ of second category in Y such that for each $y \in Y$, $\pi^{-1}(y)$ satisfies property $(*)$:*

There exists a non-empty open subset L_y of the sub-space $\pi^{-1}(y)$ such that K_y, the closure of L_y in $\pi^{-1}(y)$, is complete with no isolated points. $(*)$

Then one has the following.

(1) *If $(\mathrm{AR}(X, T) \cap R_\pi^{(2)}) \setminus \Delta_2(X)$ is a subset of first category in $R_\pi^{(2)}$, then π has relative Mycielski's chaos.*

(2) *In particular, π has relative Mycielski's chaos if one of the following holds:*
   - (i) *π is relatively sensitive;*
   - (ii) *(X, T) is minimal;*
   - (iii) *(X, T) is a transitive TDS which is not sensitive.*

Proof. By assumptions, it is obvious that π is non-trivial; by Proposition 4.3 one has $\mathrm{P}_T(X, T) \cap R_\pi^{(2)}$ is a dense G_δ subset of $R_\pi^{(2)}$, as π has relative 2-scattering.

(1) As $(\mathrm{AR}(X, T) \cap R_\pi^{(2)}) \setminus \Delta_2(X)$ is a subset of first category in $R_\pi^{(2)}$,

$$M \doteq \Delta_2(X) \cup ((\mathrm{P}_T(X, T) \setminus \mathrm{AR}(X, T)) \cap R_\pi^{(2)})$$

contains a dense G_δ subset of $R_\pi^{(2)}$. Since π is open, applying Lemma 3.6 to $R_\pi^{(2)} \setminus M$ there exists a dense G_δ subset Y_0' of Y such that for each $y \in Y_0'$, $M \cap (\pi^{-1}(y) \times \pi^{-1}(y))$ contains a dense G_δ subset of $\pi^{-1}(y) \times \pi^{-1}(y)$.

Set $Y_M \doteq Y_0 \cap Y_0' \neq \emptyset$. For each $y \in Y_M$, by assumptions $M \cap (K_y \times K_y)$ contains a dense G_δ subset of $K_y \times K_y$, then $M_y \doteq (\mathrm{P}_T(X, T) \setminus \mathrm{AR}(X, T)) \cap (K_y \times K_y)$ contains a dense G_δ subset of $K_y \times K_y$, as K_y has no isolated points. Now applying Lemma 4.6 to K_y and M_y, we obtain a dense Mycielski subset $K \subseteq K_y$ such that $(K \times K) \setminus \Delta_2(X) \subseteq M_y \subseteq \mathrm{P}_T(X, T) \setminus \mathrm{AR}(X, T)$. This means that π has relative Mycielski's chaos.

(2) The proof of this part will be presented at the end of next section. □

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5. *Relative sensitivity*

In this section, first we introduce the concept of relative sensitivity, then we prove that for a factor map between minimal TDSs it is either relatively sensitive or positively equicontinuous, and that any non-trivial weakly mixing factor map is relatively sensitive. We also present a relative version of the well-known result in [**GW**] that if an M-system is not minimal, then it is sensitive; for details, see Theorem 5.10.

The definition of relative sensitivity is stated as follows.

Definition 5.1. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that:

(1) π is *relatively sensitive* if there exists $\epsilon > 0$ such that for each $\delta > 0$ and $x \in X$ there exists $(x_1, x_2) \in R_\pi^{(2)}$ with $d(x, x_i) < \delta$ $(i = 1, 2)$ and $d(T^n x_1, T^n x_2) > \epsilon$ for some $n \in \mathbb{Z}_+$;

(2) $x \in X$ is *relative positively equicontinuous* (RPE) (denoted by $x \in E_{\mathrm{re}}(X, T|\pi)$) if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(T^n x_1, T^n x_2) < \epsilon$ if $n \in \mathbb{Z}_+$ and $(x_1, x_2) \in R_\pi^{(2)}$ with $d(x, x_i) < \delta$ $(i = 1, 2)$.

Remark 5.2. Recall that we say that (X, T) is *sensitive* if there exists $\epsilon > 0$ such that for each $\delta > 0$ and $x \in X$ there exists $x' \in X$ such that $d(x, x') < \delta$ and for some $n \in \mathbb{Z}_+$ with $d(T^n x, T^n x') > \epsilon$. It is obvious that if π is relatively sensitive, then (X, T) is sensitive.

By definition and the results obtained in §2, we have:

(1) $E_{\mathrm{re}}(X, T|\pi) \subseteq E_{\mathrm{lre}}(X, T|\pi)$;

(2) $E_{\mathrm{re}}(X, T|\pi) = X$ if and only if π is positively equicontinuous;

(3) $E_{\mathrm{re}}(X, T|\pi) = \emptyset$ if π is relatively sensitive.

PROPOSITION 5.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive TDSs.*

(1) *Then $E_{\mathrm{re}}(X, T|\pi) = \emptyset$ if and only if π is relatively sensitive.*

(2) *Suppose $E_{\mathrm{re}}(X, T|\pi) \neq \emptyset$. Then* $\mathrm{Tran}(X, T) \subseteq E_{\mathrm{re}}(X, T|\pi)$. *Moreover, if π is minimal, then* $\pi^{-1}(\mathrm{Tran}(Y, S)) \subseteq E_{\mathrm{re}}(X, T|\pi)$.

Proof. As $\mathrm{Tran}(X, T) = \pi^{-1}(\mathrm{Tran}(Y, S))$ if π is minimal, our proof will be finished once we show that if there exists $x_0 \in \mathrm{Tran}(X, T) \setminus E_{\mathrm{re}}(X, T|\pi)$, then π is relatively sensitive.

Since $x_0 \notin E_{\mathrm{re}}(X, T|\pi)$, there exists $\epsilon_0 > 0$ such that for each $\delta' > 0$ there exist $(x_1, x_2) \in R_\pi^{(2)}$ and $n \in \mathbb{Z}_+$ such that $d(x_0, x_i) < \delta'$ $(i = 1, 2)$ and $d(T^n x_1, T^n x_2) > \epsilon_0$. Now let $x \in X$ and $\delta > 0$. Say $m \in \mathbb{Z}_+$ with $d(T^m x_0, x) < \delta$, as $x_0 \in \mathrm{Tran}(X, T)$. Let $\delta_x > 0$ such that if $d(x_0, x^*) < \delta_x$, then $d(T^m x^*, x) < \delta$. For this δ_x, there exist $(x_1, x_2) \in R_\pi^{(2)}$ and $n \in \mathbb{Z}_+$ such that $d(x_0, x_i) < \delta_x$ $(i = 1, 2)$ and $d(T^n x_1, T^n x_2) > \epsilon_0$ (it makes no difference to assume $n > m$ by selecting small enough $\delta_x > 0$). So

$$(T^m x_1, T^m x_2) \in R_\pi^{(2)}; \quad d(x, T^m x_i) < \delta \quad (i = 1, 2);$$
$$d(T^{n-m}(T^m x_1), T^{n-m}(T^m x_2)) > \epsilon_0.$$

That is, π is relatively sensitive. This completes the proof. □

Then we have (see [**AAB**] for the absolute case) the following.

COROLLARY 5.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between minimal TDSs. Then π is either relatively sensitive or positively equicontinuous.*

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THEOREM 5.5. *Let $\pi : (X, T) \to (Y, S)$ be a non-trivial factor map between TDSs. If π is weakly mixing, then it is relatively sensitive.*

Proof. As (X, T) is transitive, it suffices to prove that $E_{\mathrm{re}}(X, T|\pi) = \emptyset$ (by Proposition 5.3). Let $x_0 \in X$ and $\epsilon > 0$ be small enough. Since TDS $(R_\pi^{(2)}, T^{(2)})$ is transitive, for each neighbourhood U of x_0, $\sup_{n\in\mathbb{Z}_+} d(T^n x_1, T^n x_2) > \epsilon$ if $(x_1, x_2) \in \mathrm{Tran}(R_\pi^{(2)}, T^{(2)}) \cap (U \times U)$ (as $\epsilon > 0$ is small enough), i.e. $x_0 \notin E_{\mathrm{re}}(X, T|\pi)$. This finishes the proof. □

Let (X, T) be a TDS. Recall that we say that (X, T) is an *M-system* if (X, T) is transitive and $\mathrm{AP}(X, T)$ is dense in X. It is well known that if an M-system is not minimal, then it is sensitive (see [**GW**]). Now we aim to prove a relative version of this result.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. We say that $x_0 \in X$ is a *relative minimal point* of π (denoted by $x \in \mathrm{AP}(X, T|\pi)$) if $\pi(X_0) = Y$ and X_0 contains no proper closed positively T-invariant subset which projects onto Y, where $X_0 = \overline{\{T^i x_0 : i \in \mathbb{Z}_+\}}$. In this case, we say that X_0 is a *relative minimal sub-system* of π.

Before proceeding, we need some notation. Let (X, T) be a TDS, $x \in X$ and $B \subseteq X$ a non-empty subset of X. Set $N(x, B) = \{n \in \mathbb{Z}_+ : T^n(x) \in B\}$. Let $\delta > 0$. Write $B_X(x, \delta) = \{x' \in X : d(x, x') < \delta\}$. Let $P, Q \subseteq \mathbb{Z}_+$. Put $P - Q = \{p - q \geq 0 : p \in P, q \in Q\}$. Then we can characterize the minimal factor map between transitive TDSs by the set of return times as follows.

LEMMA 5.6. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive TDSs. Then the following statements are equivalent.*

(1) *π is minimal.*

(2) *For each $x_0 \in \mathrm{Tran}(X, T)$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\pi^{-1}(\pi x_0) \subseteq \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon). \tag{5.1}$$

(3) *For each $x_0 \in \mathrm{Tran}(X, T)$ and $\epsilon > 0$, there exist $N \in \mathbb{N}$ and $\delta > 0$ such that*

$$\pi^{-1} B_Y(\pi x_0, \delta) \cap \{T^i x_0 : i \in \mathbb{Z}_+\} \subseteq \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon). \tag{5.2}$$

(4) *For each $x_0 \in \mathrm{Tran}(X, T)$ and $\epsilon > 0$, there exist $N \in \mathbb{N}$ and $\delta > 0$ such that*

$$N(\pi x_0, B_Y(\pi x_0, \delta)) \subseteq N(x_0, B_X(x_0, \epsilon)) - \{0, 1, \ldots, N\}. \tag{5.3}$$

(5) *There exists $x_0 \in \mathrm{Tran}(X, T)$ such that if $\epsilon > 0$, then (5.1) holds for some $N \in \mathbb{N}$.*

(6) *There exists $x_0 \in \mathrm{Tran}(X, T)$ such that if $\epsilon > 0$, then there exist $N \in \mathbb{N}$ and $\delta > 0$ such that for which (5.2) holds.*

(7) *There exists $x_0 \in \mathrm{Tran}(X, T)$ such that if $\epsilon > 0$, then there exist $N \in \mathbb{N}$ and $\delta > 0$ such that for which (5.3) holds.*

Proof. The proof is divided into three steps.

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Step 1. (1) ⇔ (2) ⇔ (5). (2) ⇒ (5) is obvious.

((1) ⇒ (2)). Let $x_0 \in \mathrm{Tran}(X, T)$ and $\epsilon > 0$. Set $X_0 = \bigcup_{i\in\mathbb{Z}_+} T^{-i} B_X(x_0, \epsilon)$. Then $X \setminus X_0 \subseteq X$ is closed positively T-invariant, so $\pi(x_0) \notin \pi(X \setminus X_0)$, i.e. $\pi^{-1}(\pi x_0) \subseteq X_0$, as π is minimal and $\pi(x_0) \in \mathrm{Tran}(Y, S)$. Then there exists $N \in \mathbb{N}$ such that (5.1) holds.

((5) ⇒ (1)). If π is not minimal, then there exists a closed positively T-invariant subset X^* of X such that $\pi(X^*) = Y$ and $x_0 \notin X^*$. Let $\epsilon > 0$ with $X^* \cap B_X(x_0, \epsilon) = \emptyset$. So $X^* \cap \bigcup_{i\in\mathbb{Z}_+} T^{-i} B_X(x_0, \epsilon) = \emptyset$. By assumption, for such $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that (5.1) holds. This implies $X^* \cap \pi^{-1}(\pi x_0) = \emptyset$, a contradiction. Then π is minimal.

Step 2. (2) ⇔ (3) and (5) ⇔ (6).

((2) ⇒ (3), (5) ⇒ (6)). Fix $x_0 \in \mathrm{Tran}(X, T)$. Let $\epsilon > 0$. By assumption (5.1) holds for some $N \in \mathbb{N}$, then $\pi(x_0) \notin \pi\big(X \setminus \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon)\big)$. Let $\delta > 0$ with

$$B_Y(\pi x_0, \delta) \cap \pi\left(X \Big\backslash \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon)\right) = \emptyset \Rightarrow \pi^{-1} B_Y(\pi x_0, \delta) \subseteq \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon). \tag{5.4}$$

In particular, $\pi^{-1} B_Y(\pi x_0, \delta) \cap \{T^i x_0 : i \in \mathbb{Z}_+\} \subseteq \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon)$.

((3) ⇒ (2), (6) ⇒ (5)). Fix $x_0 \in \mathrm{Tran}(X, T)$. Let $\epsilon > 0$. By assumption (5.2) holds for some $N \in \mathbb{N}$ and $\delta > 0$. Now we claim that (5.1) holds for 2ϵ and N. In fact, let $x \in \pi^{-1}(\pi x_0)$. Since $x_0 \in \mathrm{Tran}(X, T)$, there exists a sequence $\{n_j\}_{j\in\mathbb{N}}$ of $\mathbb{Z}_+$ such that $\pi(T^{n_j} x_0) \in B_Y(\pi x_0, \delta)$ $(j \in \mathbb{N})$ and $T^{n_j} x_0 \to x$. By (5.2) one has

$$x \in \overline{\pi^{-1} B_Y(\pi x_0, \delta) \cap \{T^i x_0 : i \in \mathbb{Z}_+\}} \subseteq \bigcup_{i=0}^{N} \overline{T^{-i} B_X(x_0, \epsilon)} \subseteq \bigcup_{i=0}^{N} T^{-i} B_X(x_0, 2\epsilon).$$

Step 3. (3) ⇔ (4) and (6) ⇔ (7). It follows from the following easy facts: if $x_0 \in X$, $N \in \mathbb{N}, \epsilon, \delta > 0$ and $j \in \mathbb{Z}_+$, then:

(1) $T^j x_0 \in \pi^{-1} B_Y(\pi x_0, \delta) \Leftrightarrow j \in N(\pi x_0, B_Y(\pi x_0, \delta))$; and

(2) $T^j x_0 \in \bigcup_{i=0}^{N} T^{-i} B_X(x_0, \epsilon) \Leftrightarrow j \in N(x_0, B_X(x_0, \epsilon)) - \{0, 1, \ldots, N\}$. □

Then we have the following useful corollary.

COROLLARY 5.7. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $x_0 \in X$ with $\pi(x_0) \in \mathrm{Tran}(Y, S)$. Then $x_0 \in \mathrm{AP}(X, T|\pi)$ if and only if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $N(\pi x_0, B_Y(\pi x_0, \delta)) \subseteq N(x_0, B_X(x_0, \epsilon)) - \{0, 1, \ldots, N\}$.*

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $x_0 \in X$. Put

$$\Omega(T, x_0|\pi) = \{x \in X : \exists n_i \to \infty, \{x_i\} \subseteq \pi^{-1}(\pi x_0) \text{ such that } x_i \to x_0, T^{n_i} x_i \to x\},$$
$$\omega(T, x_0) = \{x \in X : \exists n_i \to \infty \text{ such that } T^{n_i} x_0 \to x\} \subseteq \Omega(T, x_0|\pi).$$

It is clear that $x_0 \in \mathrm{Tran}(X, T)$ if and only if $\omega(T, x_0) = X$.

Let (X, T) be a TDS and $x_0 \in X$. It is notable that $x_0 \in \mathrm{AP}(X, T)$ if and only if for each open neighbourhood U_0 of x_0 the set of return times $N(x_0, U_0) = \{0 \le a_1 < a_2 < \cdots\}$ is *syndetic*, i.e. there exists a constant $M < \infty$ which bounds all $a_{i+1} - a_i$ $(i \in \mathbb{N})$. Then we have the following.

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LEMMA 5.8. *Let* $\pi : (X, T) \to (Y, S)$ *be a factor map between TDSs. Suppose* $x_0 \in E_{\mathrm{lre}}(X, T|\pi)$. *Then we have the following.*

(1) $\omega(T, x_0) = \Omega(T, x_0|\pi)$.

(2) *Suppose that there exists a sequence* $\{x_i\}_{i\in\mathbb{N}} \subseteq \pi^{-1}(\pi x_0) \cap \mathrm{Tran}(X, T)$ *(respectively* $\pi^{-1}(\pi x_0) \cap \mathrm{AP}(X, T)$, $\pi^{-1}(\pi x_0) \cap \mathrm{AP}(X, T|\pi)$*) such that* $x_i \to x_0$. *Then* $x_0 \in \mathrm{Tran}(X, T)$ *(respectively* $x_0 \in \mathrm{AP}(X, T)$, $x_0 \in \mathrm{AP}(X, T|\pi)$*).*

Proof. Let $\epsilon > 0$ be fixed. Since $x_0 \in E_{\mathrm{lre}}(X, T|\pi)$, there exists $\delta > 0$ with $\delta \leq \epsilon$ such that if $j \in \mathbb{Z}_+$, $x' \in \pi^{-1}(\pi x_0)$ and $d(x_0, x') \leq \delta$, then $d(T^j x_0, T^j x') \leq \epsilon/2$.

(1) It suffices to prove $\omega(T, x_0) \supseteq \Omega(T, x_0|\pi)$. Let $x \in \Omega(T, x_0|\pi)$ with $n_i \to \infty$, $\{x_i\} \subseteq \pi^{-1}(\pi x_0)$ such that $x_i \to x_0$ and $T^{n_i} x_i \to x$. Select $i \in \mathbb{N}$ such that $d(x_0, x_i) \leq \delta$ and $d(T^{n_i} x_i, x) \leq \epsilon/2$. Then $d(T^{n_i} x_0, T^{n_i} x_i) \leq \epsilon/2$, $d(T^{n_i} x_0, x) \leq d(T^{n_i} x_0, T^{n_i} x_i) + d(T^{n_i} x_i, x) \leq \epsilon$. Letting $\epsilon \to 0+$, one has $x \in \omega(T, x_0)$, i.e. $\Omega(T, x_0|\pi) \subseteq \omega(T, x_0)$.

(2) The first part of the proof follows by the assumption $\Omega(T, x_0|\pi) = X$. Then by part (1) one has $\omega(T, x_0) = X$, which implies $x_0 \in \mathrm{Tran}(X, T)$.

Now let us turn to the second part. We aim to prove $x_0 \in \mathrm{AP}(X, T)$ by showing that $N(x_0, B_X(x_0, 3\epsilon))$ is syndetic, as $\epsilon > 0$ is arbitrary. Let $n_0 \in \mathbb{N}$ with $d(x_0, x_{n_0}) < \delta$. Thus, $N(x_0, B_X(x_0, 3\epsilon)) \supseteq N(x_{n_0}, B_X(x_{n_0}, \epsilon))$ is syndetic, as $x_{n_0} \in \mathrm{AP}(X, T)$ and if $m \in N(x_{n_0}, B_X(x_{n_0}, \epsilon))$, then

$$d(x_0, T^m x_0) \leq d(x_0, x_{n_0}) + d(x_{n_0}, T^m x_{n_0}) + d(T^m x_{n_0}, T^m x_0) < \delta + \epsilon + \epsilon \leq 3\epsilon.$$

It remains to prove the third part. Set $y_0 = \pi(x_0)$. Then $y_0 \in \mathrm{Tran}(Y, S)$, as $y_0 = \pi(x_i)$ and $x_i \in \mathrm{AP}(X, T|\pi)$. Let $i_0 \in \mathbb{N}$ with $d(x_0, x_{i_0}) < \delta$. Then $d(T^j x_0, T^j x_{i_0}) < \epsilon$ for each $j \in \mathbb{Z}_+$. As $x_{i_0} \in \mathrm{AP}(X, T|\pi)$ and $y_0 = \pi(x_{i_0})$, by Corollary 5.7

$$N(y_0, B_Y(y_0, \delta_1)) \subseteq N(x_{i_0}, B_X(x_{i_0}, \epsilon)) - \{0, 1, \ldots, N\} \tag{5.5}$$

for some $\delta_1 > 0$ and $N \in \mathbb{N}$. Note that $N(x_{i_0}, B_X(x_{i_0}, \epsilon)) \subseteq N(x_0, B_X(x_0, 3\epsilon))$. By (5.5)

$$N(y_0, B_Y(y_0, \delta_1)) \subseteq N(x_0, B_X(x_0, 3\epsilon)) - \{0, 1, \ldots, N\}. \tag{5.6}$$

As $\pi(x_0) = y_0 \in \mathrm{Tran}(Y, S)$, by Corollary 5.7, $x_0 \in \mathrm{AP}(X, T|\pi)$, as $\epsilon > 0$ is arbitrary. □

As an application of Lemma 5.8 we have the following.

COROLLARY 5.9. *Let* $\pi : (X, T) \to (Y, S)$ *be a factor map between transitive TDSs. Suppose that* π *is not relatively sensitive.*

(1) *Assume that* $y \in \mathrm{Tran}(Y, S)$ *satisfies that* $\mathrm{Tran}(X, T) \cap \pi^{-1}(y)$ *is dense in* $\pi^{-1}(y)$. *Then* $E_{\mathrm{lre}}(X, T|\pi) \cap \pi^{-1}(y) = E_{\mathrm{re}}(X, T|\pi) \cap \pi^{-1}(y) = \mathrm{Tran}(X, T) \cap \pi^{-1}(y)$.

(2) *Assume that* π *is open. Then there exists a dense* G_δ *subset* Y_0 *of* Y *such that for each* $y \in Y_0$, $E_{\mathrm{lre}}(X, T|\pi) \cap \pi^{-1}(y) = E_{\mathrm{re}}(X, T|\pi) \cap \pi^{-1}(y) = \mathrm{Tran}(X, T) \cap \pi^{-1}(y)$.

Proof. (1) By Proposition 5.3, $E_{\mathrm{lre}}(X, T|\pi) \cap \pi^{-1}(y) \supseteq E_{\mathrm{re}}(X, T|\pi) \cap \pi^{-1}(y) \supseteq \mathrm{Tran}(X, T) \cap \pi^{-1}(y)$. Then the conclusion follows from Lemma 5.8 directly.

(2) Note that $\mathrm{Tran}(X, T)$ is a dense G_δ subset of X. As π is open, applying Lemma 3.6 to $X \setminus \mathrm{Tran}(X, T)$ we obtain a dense G_δ subset Y_1 of Y such that $\mathrm{Tran}(X, T) \cap \pi^{-1}(y)$ is a dense G_δ subset of $\pi^{-1}(y)$ for each $y \in Y_1$. Set $Y_0 = Y_1 \cap \mathrm{Tran}(Y, S)$. Then Y_0 is a dense G_δ subset of Y. Now applying part (1) to each $y \in Y_0$, we deduce the conclusion. □

Then we obtain a relative version of the well-known Glasner–Weiss's result that if an M-system is not minimal then it is sensitive.

THEOREM 5.10. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive TDSs. Then we have the following.*

(1) $\mathrm{AP}(X, T|\pi)$ *is dense in X if and only if the union $\cup X_0$ is dense in X, where $X_0 \subseteq X$ varies over all relative minimal sub-systems of π.*

(2) *Suppose that* $\mathrm{AP}(X, T|\pi)$ *is dense in X and π is not relatively sensitive. Then the following statements are equivalent:*
   (i) *π is minimal;*
   (ii) *if $x_0 \in \mathrm{Tran}(X, T)$, then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathrm{AP}(X, T|\pi) \cap \pi^{-1}(\pi x_0)$ such that $x_i \to x_0$;*
   (iii) *there exist $x_0 \in \mathrm{Tran}(X, T)$ and a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathrm{AP}(X, T|\pi) \cap \pi^{-1}(\pi x_0)$ such that $x_i \to x_0$.*

Proof. (1) If follows from some standard arguments.

(2) (i) $\Rightarrow$ (ii) As (X, T) and (Y, S) are both transitive and π is minimal, then the conclusion follows from the fact that $\mathrm{AP}(X, T|\pi) = \mathrm{Tran}(X, T) = \pi^{-1}(\mathrm{Tran}(Y, S))$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) By Proposition 5.3, $x_0 \in E_{\mathrm{re}}(X, T|\pi) \subseteq E_{\mathrm{lre}}(X, T|\pi)$, then $x_0 \in \mathrm{AP}(X, T|\pi)$ follows from Lemma 5.8 and the assumption. That is, π is minimal. □

Remark 5.11. We have some remarks about Theorem 5.10.

(1) Note that letting (Y, S) be trivial, the definition of relative sensitivity recovers the definition of sensitivity, then a direct corollary of Theorem 5.10 is that if an M-system is not minimal, then it is sensitive.

(2) Due to discussions with Professor E. Glasner [**G3**], the first item may be viewed as a possible definition of an *M-extension* for a factor map between transitive TDSs.

Before ending the proof of Theorem 4.7, we need (see [**HY1**] for the absolute case) the following.

LEMMA 5.12. *Let $\pi : (X, T) \to (Y, S)$ be an open factor map between TDSs. Suppose that π is relatively sensitive. Then* $\mathrm{AR}(X, T) \cap R_\pi^{(2)}$ *is a subset of first category in $R_\pi^{(2)}$.*

Proof. As π is relatively sensitive, there exists $\epsilon > 0$ such that for each open $\emptyset \neq U \subseteq X$ there exist $(x_1, x_2) \in R_\pi^{(2)} \cap (U \times U)$ and $n \in \mathbb{Z}_+$ with $d(T^n x_1, T^n x_2) > \epsilon$.

Now we claim that if $m > 2/\epsilon$, then the closed subset $\mathrm{AR}_{k,m}(X, T) \cap R_\pi^{(2)}$ has an empty interior in $R_\pi^{(2)}$, which implies that $\mathrm{AR}(X, T) \cap R_\pi^{(2)}$ is a subset of first category in $R_\pi^{(2)}$. Otherwise, there exist $\delta' > 0$ and $(x_1^*, x_2^*) \in R_\pi^{(2)}$ such that if $j \geq k$ and $(x_1', x_2') \in R_\pi^{(2)}$ with $d(x_i^*, x_i') < \delta'$ $(i = 1, 2)$, then $d(T^j x_1', T^j x_2') \leq 1/m$. For $i = 1, 2$, put $U_i = B_X(x_i^*, \delta')$ and set $V_1 = U_1 \cap \pi^{-1}(\pi(U_1) \cap \pi(U_2))$. Then V_1 is an open neighbourhood of x_1^*, as π is open. If $(x_1, x_2) \in R_\pi^{(2)} \cap (V_1 \times V_1)$, by the definition of V_1 there exists $x' \in U_2$ such that $\pi(x') = \pi(x_1)$ $(= \pi(x_2))$, then $(x', x_1), (x', x_2) \in R_\pi^{(2)} \cap (U_2 \times U_1)$, which implies

$$\sup_{j\geq k} d(T^j x', T^j x_i) \leq \frac{1}{m}, \quad i = 1, 2 \quad \Rightarrow \quad \sup_{j\geq k} d(T^j x_1, T^j x_2) \leq \frac{2}{m} < \epsilon.$$

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It makes no difference to assume $\sup_{j\in\mathbb{Z}_+} d(T^j x_1, T^j x_2) < \epsilon$ by selecting $\delta' > 0$ and shrinking V_1 appropriately. A contradiction to the selection of ϵ. This ends the proof. □

Remark 5.13. The assumption of 'π is open' is again essential. Let us recall the example constructed at the end of §2. As a factor map between minimal invertible TDSs, π is not positively equicontinuous, so π is relatively sensitive (using Theorem 5.5). Whereas, $\mathrm{AR}(X, T) \cap R_\pi^{(2)} = R_\pi^{(2)}$. In fact, for each $(x_1, x_2) \in R_\pi^{(2)}$, $d(T^n x_1, T^n x_2) \to 0$ as $|n| \to \infty$.

Now we are ready to finish proving Theorem 4.7.

Remainder of Proof of Theorem 4.7. For the proof of part (2), by the discussions in part (1) we only need prove that if only one of the assumptions holds, then $(\mathrm{AR}(X, T) \cap R_\pi^{(2)}) \setminus \Delta_2(X)$ is a subset of first category in $R_\pi^{(2)}$.

(i) This follows from Lemma 5.12, as π is relatively sensitive.

(ii) Assume that (X, T) is minimal. By Corollary 5.4, π is either relatively sensitive or positively equicontinuous. Then by (i) we only consider the case that π is positively equicontinuous. In this case, π is equicontinuous (by Corollary 3.4), which implies $\emptyset = (\mathrm{AR}(X, T) \cap R_\pi^{(2)}) \setminus \Delta_2(X)$.

(iii) Now assume that (X, T) is a transitive TDS which is not sensitive. It is well known that in this case $\mathrm{AR}(X, T) = \Delta_2(X)$ (for an alternative version see, for example, [**GW**, Lemma 1.2]). This finishes our proof. □

Remark 5.14. Let (X, T) be a non-trivial transitive invertible TDS. If (X, T) is scattering, then X is a complete metric space with no isolated points. Then by (i) and (iii) of Theorem 4.7(2) we obtain again the known fact that any non-trivial scattering TDS is Li–Yorke chaotic (note that scattering implies transitivity; see [**BHM**]).

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