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The space of immersed surfaces in a manifold

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Abstract

We study the cohomology of the space of immersed genus g surfaces in a simply-connected manifold. We compute the rational cohomology of this space in a stable range which goes to infinity with g. In fact, in this stable range we are also able to obtain information about torsion in the cohomology of this space, as long as we localise away from $(g-1)$.

1. *Introduction and statement of results*

Let M be a smooth manifold, not necessarily compact and possibly with boundary, and write $\mathring{M}$ for its interior. Let Σ_g be a closed orientable surface of genus g, and $\mathrm{Imm}(\Sigma_g, \mathring{M})$ be the space of immersions of Σ_g into the interior of M, equipped with the Whitney C^∞-topology. The group $\mathrm{Diff}^+(\Sigma_g)$ of orientation preserving diffeomorphisms is also equipped with the Whitney C^∞-topology, and acts continuously on $\mathrm{Imm}(\Sigma_g, \mathring{M})$ by precomposition of functions. The *space of immersed surfaces of genus g in M* is defined to be the quotient space

$$\mathcal{I}_g(M) := \mathrm{Imm}(\Sigma_g, \mathring{M})/\mathrm{Diff}^+(\Sigma_g),$$

so that a point in $\mathcal{I}_g(M)$ is represented by an unparametrised immersed oriented surface of genus g in the interior of M.

We propose to study the cohomology of this space. The differential topology of such spaces of unparametrised immersions has been studied in detail by Cervera–Mascaró–Michor [**CMM91**] and Michor–Mumford [**MM05**], and the most elementary observation is that the action of $\mathrm{Diff}^+(\Sigma_g)$ on $\mathrm{Imm}(\Sigma_g, \mathring{M})$ is not free: for example, if an immersion is a covering space of its image, the group of covering transformations lies in the stabiliser. Thus the homotopy-type of $\mathcal{I}_g(M)$ is not as directly related to the homotopy-types of $\mathrm{Imm}(\Sigma_g, \mathring{M})$ and $\mathrm{Diff}^+(\Sigma_g)$ as one might like. To exert better homotopical control we may work with the homotopy quotient (i.e. Borel construction),

$$\mathcal{I}^h_g(M) := \mathrm{Imm}(\Sigma_g, \mathring{M})/\!\!/\mathrm{Diff}^+(\Sigma_g),$$

which enjoys better formal properties although it is perhaps not so geometrically meaningful. (In fact, $\mathrm{Imm}(\Sigma_g, \mathring{M})$ admits the structure on an infinite-dimensional smooth manifold, and the action of $\mathrm{Diff}^+(\Sigma_g)$ on $\mathrm{Imm}(\Sigma_g, \mathring{M})$ is smooth, proper, and has slices, so $\mathcal{I}_g(M)$ is the coarse space of the infinite-dimensional translation orbifold, and $\mathcal{I}^h_g(M)$ is the homotopy-type of this orbifold.) In order to relate this auxiliary space to the problem at hand, in Section 2 we apply the theory developed by Cervera–Mascaró–Michor to prove the first part of the following theorem.

THEOREM A. *For $g \geqslant 2$, the map $H^*(\mathcal{I}_g(M); \mathbb{Z}[1/(g-1)]) \to H^*(\mathcal{I}_g^h(M); \mathbb{Z}[1/(g-1)])$ is an isomorphism. On the other hand, for every prime p dividing $(g-1)$ the map $H^*(\mathcal{I}_g(M); \mathbb{Z}_{(p)}) \to H^*(\mathcal{I}_g^h(M); \mathbb{Z}_{(p)})$ is not surjective.*

Thus to study the cohomology of $\mathcal{I}_g(M)$ with $\mathbb{Z}[1/(g-1)]$ coefficients it is enough to study the cohomology of $\mathcal{I}_g^h(M)$. We are mainly interested in rational cohomology, so this is no restriction, although our methods will also provide certain torsion information.

Our strategy is to compare the space $\mathcal{I}_g^h(M)$ with a certain universal space which is independent of g, much as in Madsen and Weiss' proof of the Mumford conjecture [**MW07**]. This space is an infinite loop space, and we may describe it as follows.

Let $\mathrm{Gr}_2^+(TM)$ denote the Grassmannian of oriented 2-planes in the tangent bundle of M. That is, it consists of pairs of a point $x \in M$ and an oriented 2-plane $L \subset T_x M$. This space has an evident 2-plane bundle over it (with fibre over the point (x, L) given by the vector space L) which we denote ℓ, and hence a classifying map

$$\theta_M : \mathrm{Gr}_2^+(TM) \longrightarrow BSO(2).$$

Denote by $\mathbf{MT}\theta_M$ the Thom spectrum of the virtual bundle $-\ell \to \mathrm{Gr}_2^+(TM)$, and by $\Omega^\infty\mathbf{MT}\theta_M$ its associated infinite loop space. There is a natural surjective homomorphism

$$E : \pi_0(\Omega^\infty\mathbf{MT}\theta_M) \longrightarrow \mathbb{Z},$$

which we will describe in Section 4, and we denote by $\Omega^\infty_{(n)}\mathbf{MT}\theta_M$ the collection of path components mapping to $n \in \mathbb{Z}$.

THEOREM B. *For any manifold M, there is a map*

$$\alpha_M : \mathcal{I}_g^h(M) \longrightarrow \Omega^\infty_{(1-g)}\mathbf{MT}\theta_M.$$

If M is simply-connected and of dimension at least three, the map $H^*(\alpha_M; \mathbb{Z})$ is an isomorphism for degrees

$$* \leqslant \begin{cases} \dfrac{2g-6}{5} & \text{if } \dim(M) = 3 \\ \dfrac{2g-3}{3} & \text{if } \dim(M) > 3. \end{cases}$$

Remark 1·1. We will refer to the inequality in the above theorem as the "stable range". Thus, if we say that a certain statement about the (co)homology of $\mathcal{I}_g(M)$ or $\mathcal{I}_g^h(M)$ holds "in the stable range", we mean that it holds in all (co)homological degrees $*$ satisfying that inequality.

This theorem will be a consequence of the author's homology stability theorem for moduli spaces of surfaces with tangential structure [**RW09**], along with the Smale–Hirsch theory of immersions. Smale–Hirsch theory relates the space $\mathrm{Imm}(\Sigma_g, M)$ to a space of bundle-theoretic data, which we in turn identify with a space of θ_M-reductions of the bundle $T\Sigma_g$. In order to apply the homology stability theorem of [**RW09**], in Section 3 we give another model of $\mathcal{I}_g^h(M)$, which extends to the case of surfaces with boundary, and in Section 4 we compute the set of path components of these spaces.

Once homology stability is known, the methods of Galatius–Madsen–Tillmann–Weiss [**GMTW09**] identify the stable homology with that of (certain path components of) the infinite loop space $\Omega^\infty \mathbf{MT}\theta_M$. Combining this theorem with Theorem A, we obtain the calculation

COROLLARY C. *There is an isomorphism of algebras*

$$H^*(\mathcal{I}_g(M); \mathbb{Z}[\tfrac{1}{g-1}]) \cong H^*(\Omega^\infty_{(1-g)}\mathbf{MT}\theta_M; \mathbb{Z}[\tfrac{1}{g-1}])$$

in the stable range.

For this theorem to be useful we must be able to compute the right-hand side. In general this is difficult, but for rational cohomology it is easy. Let us write $\Omega^\infty_\bullet \mathbf{MT}\theta_M$ for the basepoint component of the infinite loop space associated to the spectrum $\mathbf{MT}\theta_M$. There is a composition

$$H^{*+2}(\mathrm{Gr}_2^+(TM); \mathbb{Q}) \cong H^*(\mathbf{MT}\theta_M; \mathbb{Q}) \xrightarrow{\sigma} H^*(\Omega^\infty_\bullet \mathbf{MT}\theta_M; \mathbb{Q})$$

where the first map is the Thom isomorphism and the second map is the cohomology suspension. The right-hand side is a graded commutative algebra and so this extends to a map

$$\Lambda(H^{*+2>2}(\mathrm{Gr}_2^+(TM); \mathbb{Q})) \longrightarrow H^*(\Omega^\infty_\bullet \mathbf{MT}\theta_M; \mathbb{Q})$$

from the free graded commutative algebra on the vector space of positive degree elements. This is an isomorphism.

We now give an application of this result, by describing $H^*(\mathcal{I}_g(\mathbb{R}^n); \mathbb{Q})$ in the stable range. In order to do so we first describe a relationship between the cohomology of the moduli space of Riemann surfaces and the cohomology of $\mathcal{I}_g(M)$ for any manifold M.

1·1. *Relation to moduli spaces of curves*

Let M be a manifold equipped with a Riemannian metric. There is then a continuous map

$$r : \mathcal{I}_g(M) \longrightarrow \mathcal{M}_g$$

from the space of unparamerised immersions to the coarse moduli space of Riemann surfaces of genus g. The map is defined as follows.

Pulling back the Riemannian metric on M defines a function $\mathrm{Imm}(\Sigma_g, M) \to \mathrm{Met}(\Sigma_g)$ to the space of Riemannian metrics on Σ_g. If we consider $\mathrm{Met}(\Sigma_g)$ as a subspace of the space of smooth sections of $T^*\Sigma_g \otimes T^*\Sigma_g$, equipped with the Whitney C^∞-topology, then this function is continuous. Riemann's moduli space $\mathcal{M}_g$ is a quotient space of $\mathrm{Met}(\Sigma_g)$, where we first divide by the equivalence relation of conformal equivalence, and then by the action of $\mathrm{Diff}^+(\Sigma_g)$; taking the quotient by $\mathrm{Diff}^+(\Sigma_g)$ on the space of immersions as well gives the continuous map r.

The map r depends on the original choice of Riemannian metric on M, but as the space of metrics is connected (in fact, contractible) it is well-defined up to homotopy, and endows $H^*(\mathcal{I}_g(M); \mathbb{Q})$ with the structure of an algebra over $H^*(\mathcal{M}_g; \mathbb{Q})$. Recall that there are defined the so-called Mumford–Morita–Miller classes ([**Mum83**])

$$\kappa_i \in H^{2i}(\mathcal{M}_g; \mathbb{Q}),$$

and that by the theorem of Madsen and Weiss [**MW07**], the map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \longrightarrow H^*(\mathcal{M}_g; \mathbb{Q})$$

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is an isomorphism in degrees $* \leqslant (2g-2)/3$ (at the time of Madsen and Weiss' proof the best known stability range was not quite as good as this one, which was established later by Boldsen [**Bol12**] and the author [**RW09**] independently). We will typically describe $H^*(\mathcal{I}_g(M); \mathbb{Q})$ as an algebra over the polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$.

1·2. *Immersions in* $\mathbb{R}^3$

Let $i : \Sigma_g \looparrowright \mathbb{R}^3$ be an immersion. Its derivative gives a bundle injection $Di : T\Sigma_g \hookrightarrow \epsilon^3$, and the orientation of $T\Sigma_g$ gives a trivialisation of the complementary bundle: we obtain a canonical isomorphism $T\Sigma_g \oplus \epsilon^1 \cong \epsilon^3$ and so a canonical Spin structure on $T\Sigma_g$; changing i by a regular homotopy does not change the isomorphism class of this Spin structure. Recall ([**Joh80**]) that Spin structures on oriented surfaces are classified up to isomorphism by their $\mathbb{Z}/2$-valued Arf invariant. This construction describes a map

$$\pi_0(\mathcal{I}_g(\mathbb{R}^3)) \longrightarrow \mathbb{Z}/2$$

sending an immersion to the Arf invariant of its associated Spin structure, which we will show in Proposition 4·2 is a bijection for $g \geqslant 1$. Let $\mathcal{I}_g(\mathbb{R}^3)[\lambda]$ denote the path component which maps to $\lambda \in \mathbb{Z}/2$. Then the unit map

$$\mathbb{Q} \longrightarrow H^*(\mathcal{I}_g(\mathbb{R}^3)[\lambda]; \mathbb{Q})$$

is an isomorphism in degrees $5* \leqslant 2g-6$.

Remark 1·2. In this case it is easy to extract a little torsion information as well: in Section 5·1 we will show that there is a surjection

$$\mathbb{Z}/2 \oplus \mathbb{Z}/24 \longrightarrow H_1(\mathcal{I}_g(\mathbb{R}^3)[\lambda]; \mathbb{Z})$$

as long as $g \geqslant 6$, which is an isomorphism after inverting $(g-1)$. It would be interesting to know if this isomorphism is false before localising (for example, when $g = 7$).

1·3. *Immersions in* $\mathbb{R}^4$

Let $i : \Sigma_g \looparrowright \mathbb{R}^4$ be an immersion. Its derivative gives a bundle injection $Di : T\Sigma_g \hookrightarrow \epsilon^4$, with complement V_i. The orientation of $T\Sigma_g$ induces an orientation of V_i, and so there is defined an Euler class $e(V_i) \in H^2(\Sigma_g; \mathbb{Z}) = \mathbb{Z}$. This reduces modulo 2 to $w_2(V_i) = w_2(T\Sigma_g) = 0$, so is even. The assignment $i \mapsto \int_{\Sigma_g} e(V_i)$ gives a map

$$\pi_0(\mathcal{I}_g(\mathbb{R}^4)) \longrightarrow 2\mathbb{Z}$$

which we will show is a bijection for all $g \geqslant 0$, and we write $\mathcal{I}_g(\mathbb{R}^4)[n]$ for the component which maps to $2n \in \mathbb{Z}$. Then the map

$$\mathbb{Q}[\kappa_1] \longrightarrow H^*(\mathcal{I}_g(\mathbb{R}^4)[n]; \mathbb{Q})$$

is an isomorphism in degrees $3* \leqslant 2g-3$ (and all higher κ classes are zero).

1·4. *Immersions in* $\mathbb{R}^{2n+1}$, $2n+1 \geqslant 5$

The space $\mathcal{I}_g(\mathbb{R}^{2n+1})$ is connected for all $g \geqslant 0$, and the map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_{2n-2}] \longrightarrow H^*(\mathcal{I}_g(\mathbb{R}^{2n+1}); \mathbb{Q})$$

is an isomorphism in degrees $3* \leqslant 2g-3$ (and all higher κ classes are zero). Write $\Gamma_g := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ for the mapping class group of the surface Σ_g. In Section 5·5 we show there is a spectral sequence

$$E_2^{p,q} = H^p(\Gamma_g; H^q(\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}); \mathbb{Q})) \Longrightarrow H^{p+q}(\mathcal{I}_g(\mathbb{R}^{2n+1}); \mathbb{Q}) \qquad (1\cdot 1)$$

and we use this calculation to study its behaviour in the stable range, where we find a curious pattern of differentials.

1·5. *Immersions in* $\mathbb{R}^{2n}$, $2n \geqslant 6$

The space $\mathcal{I}_g(\mathbb{R}^{2n})$ is connected for all $g \geqslant 0$. Let $\pi : E \to \mathcal{I}_g(\mathbb{R}^{2n})$ be the universal family of surfaces. It is tautologically equipped with a map $i : E \to \mathbb{R}^{2n}$ which is an immersion on each fibre. The derivative gives a bundle injection $Di : T_\pi E \hookrightarrow \epsilon^{2n}$ of the vertical tangent bundle, and we write $V \to E$ for its $(2n-2)$-dimensional complement, which has an orientation induced by the orientation of $T_\pi E$. Define a cohomology class

$$\Delta := \pi_!(e(V)) \in H^{2n-4}(\mathcal{I}_g(\mathbb{R}^{2n}); \mathbb{Q}).$$

Then the map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_{2n-3}, \Delta] \longrightarrow H^*(\mathcal{I}_g(\mathbb{R}^{2n}); \mathbb{Q})$$

is an isomorphism in degrees $3* \leqslant 2g-3$ (and all higher κ classes are zero).

1·6. *Outline*

In Section 2 we describe the relationship between $\mathcal{I}_g(M)$ and the homotopical version $\mathcal{I}_g^h(M)$, and show that they have isomorphic homology after inverting $(g-1)$. We also show that these spaces have different cohomology when localised at primes dividing $(g-1)$. In Section 3 we relate the space $\mathcal{I}_g^h(M)$ to a moduli space of surfaces with tangential structure, as in [**RW09**], which we define for surfaces with boundary. In Section 4, which is the bulk of the paper, we calculate the set of path components of spaces of immersions of surfaces (possibly with boundary) into a manifold M. This allows us to verify the conditions of the homology stability theorem of [**RW09**], and we then give the stability range and describe the stable homology. In Section 5 we first give the details of the calculations of $\mathcal{I}_g(\mathbb{R}^n)$ described above, and then in Section 5·5 we give a calculation of the spectral sequence (1·1) in the stable range.

2. *The orbifold structure of* $\mathcal{I}_g(M)$ *and proof of Theorem* A

The work of Cervera–Mascaró–Michor [**CMM91**] establishes that the group $\mathrm{Diff}^+(\Sigma_g)$ acts on $\mathrm{Imm}(\Sigma_g, M)$ properly, so in particular with finite stabiliser groups. In fact, they show that stabiliser group $\mathrm{Diff}^+(\Sigma_g)_i$ of an immersion i acts freely and properly discontinuously on Σ_g, so it is finite and

$$2 - 2g = \chi(\Sigma_g) = \#\{\mathrm{Diff}^+(\Sigma_g)_i\} \cdot \chi(\Sigma_g/\mathrm{Diff}^+(\Sigma_g)_i).$$

As $\Sigma_g/\mathrm{Diff}^+(\Sigma_g)_i$ is an orientable surface it has even Euler characteristic, for $g \geqslant 2$ we deduce that

$$\#\{\mathrm{Diff}^+(\Sigma_g)_i\} \mid (g-1).$$

Thus the fibres of $p : \mathcal{I}_g^h(M) \to \mathcal{I}_g(M)$ are all classifying spaces of finite groups of order dividing $(g-1)$, and hence are $\mathbb{Z}[1/(g-1)]$-acyclic. In order for this to imply that p is a $\mathbb{Z}[1/(g-1)]$-cohomology isomorphism we also require that the map p be locally well-behaved in a suitable sense.

Such a sense is provided by the construction in [**CMM91**] of slices for the action of $\mathrm{Diff}^+(\Sigma_g)$ on $\mathrm{Imm}(\Sigma_g, M)$. For each $i \in \mathrm{Imm}(\Sigma_g, M)$ let $\mathcal{N}(i)$ denote the normal bundle of the immersion (formed using a metric on M which we fix once and for all). They construct a submanifold $\mathcal{Q}(i) \subset \mathrm{Imm}(\Sigma_g, M)$ diffeomorphic to a convex open neighbourhood of zero in the space $\Gamma(\mathcal{N}(i))$ of smooth sections of the normal bundle of the immersion i, enjoying the following properties (cf. [**MM05**, section 2·4]):

(i) $\mathcal{Q}(i)$ is invariant under the isotropy group of i;

(ii) $\varphi(\mathcal{Q}(i)) \cap \mathcal{Q}(i) \neq \emptyset$ if and only if φ is in the isotropy group of i;

(iii) $\mathrm{Diff}^+(\Sigma_g) \cdot \mathcal{Q}(i)$ is an open, invariant neighbourhood of the orbit $\mathrm{Diff}^+(\Sigma_g) \cdot \{i\}$, and retracts onto it.

Using that $\mathcal{Q}(i)$ is homeomorphic to a convex open subset of the vector space $\Gamma(\mathcal{N}(i))$, we see that scaling vectors in $\Gamma(\mathcal{N}(i))$ gives a $\mathrm{Diff}^+(\Sigma_g)_i$-equivariant deformation retraction of $\mathcal{Q}(i)$ onto $\{i\}$, and so by ((ii)) a $\mathrm{Diff}^+(\Sigma_g)$-equivariant deformation retraction of $\mathrm{Diff}^+(\Sigma_g) \cdot \mathcal{Q}(i)$ onto $\mathrm{Diff}^+(\Sigma_g) \cdot \{i\}$. Let $U := [\mathrm{Diff}^+(\Sigma_g) \cdot \mathcal{Q}(i)] \subset \mathcal{I}_g(M)$, a contractible open neighbourhood of $[i]$.

For $n > 0$, consider the presheaf $R^n p_* \mathbb{Z}[1/(g-1)]$ on $\mathcal{I}_g(M)$ given by

$$V \mapsto H^n(p^{-1}(V); \mathbb{Z}[1/(g-1)]).$$

The stalk at $[i]$ of this presheaf is computed as a colimit over neighbourhoods of $[i]$, but as $p^{-1}(U)$ deformation retracts onto $p^{-1}([i])$, the stalk is the same as the nth cohomology of $p^{-1}([i])$. This fibre is $E\mathrm{Diff}^+(\Sigma_g) \times_{\mathrm{Diff}^+(\Sigma_g)} \mathrm{Diff}^+(\Sigma_g)_i \simeq B\mathrm{Diff}^+(\Sigma_g)_i$ the classifying space of a finite group of order dividing $(g-1)$, and hence has trivial $\mathbb{Z}[1/(g-1)]$-cohomology in positive degrees. Hence all stalks of the presheaf $R^n p_* \mathbb{Z}[1/(g-1)]$ are trivial, and hence the sheafification is trivial. Thus the Leray spectral sequence for the map p in $\mathbb{Z}[1/(g-1)]$-cohomology collapses, which establishes the first part of Theorem A. We remark that this is entirely analogous to the proof of the same result for finite-dimensional orbifolds.

We will now prove the second part of Theorem A. Let $G = \mathbb{Z}/(g-1)$, which has a free action on Σ_g with quotient diffeomorphic to Σ_2 (by viewing Σ_g as a torus with $(g-1)$ smaller tori glued on at regular intervals around a meridian, where G acts by rotation). We then have a map

$$E = EG \times_G \Sigma_g \longrightarrow \{*\} \times_G \Sigma_g \cong \Sigma_2$$

which is a submersion on each fibre over BG, so choosing an immersion $\Sigma_2 \looparrowright M$ gives a fibrewise immersion of the surface bundle $E \to BG$ into M, which is classified by a map $f : BG \to \mathcal{I}_g^h(M)$. Its composition with $p : \mathcal{I}_g^h(M) \to \mathcal{I}_g(M)$ is constant, taking value the equivalence class of the immersion $\Sigma_g \to \Sigma_g/G \cong \Sigma_2 \looparrowright M$.

Let $P \subset G$ be a subgroup of order p. We claim that the map $BP \to BG \to \mathcal{I}_g^h(M)$ is non-trivial on integral cohomology in positive degrees: as the cohomology of P is p-local in positive degrees, it follows that there are classes in $H^*(\mathcal{I}_g^h(M); \mathbb{Z}_{(p)})$ which do not come from $H^*(\mathcal{I}_g(M); \mathbb{Z}_{(p)})$. To establish this claim we consider the composition

$$BP \longrightarrow BG \stackrel{f}{\longrightarrow} \mathcal{I}_g^h(M) \longrightarrow */\!\!/\mathrm{Diff}^+(\Sigma_g) = B\mathrm{Diff}^+(\Sigma_g) \longrightarrow B\Gamma_g$$

where $\Gamma_g = \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the *mapping class group* of Σ_g. On fundamental groups the homomorphism $G \to \Gamma_g$ is injective (which may be seen, for example, by considering the action of G on the first homology of Σ_g). It is well known that Γ_g has a torsion-free normal

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subgroup of finite index, say Δ (see e.g. [**BL83**, theorem 4·3]), so $P \to G \to \Gamma_g \to \Gamma_g/\Delta$ is also injective. It follows from a theorem of Swan [**Swa60**, theorem 1] that this composition is non-trivial on integral cohomology in infinitely many positive degrees, which proves the claim.

3. *Other homotopical models and surfaces with boundary*

Let us write $\mathrm{Bun}(T\Sigma_g, \ell)$ for the set of bundle maps $T\Sigma_g \to \ell$, i.e. those continuous maps which are linear isomorphisms on each fibre, and equip it with the compact-open topology. We will construct a slightly different homotopical model to $\mathcal{I}_g^h(M)$. Define

$$\mathcal{M}^{\theta_M}(\Sigma_g) := \mathrm{Bun}(T\Sigma_g, \ell)/\!\!/\mathrm{Diff}^+(\Sigma_g)$$

to be the homotopy quotient, or Borel construction.

LEMMA 3·1. *As long as M has dimension at least three, there is a (naïve)* $\mathrm{Diff}^+(\Sigma_g)$-*equivariant weak homotopy equivalence*

$$\mathrm{Imm}(\Sigma_g, \mathring{M}) \simeq \mathrm{Bun}(T\Sigma_g, \ell).$$

Here, by a *naïve equivariant weak homotopy equivalence* we mean that the two $\mathrm{Diff}^+(\Sigma_g)$-spaces are connected by a zig-zag of equivariant maps which are (non-equivariant) weak homotopy equivalences. This relation is too coarse for many applications of equivariant homotopy theory, but is sufficient to guarantee that the homotopy quotients are weakly equivalent.

Proof. We proceed in two steps. The first step is to note that Smale–Hirsch immersion theory (cf. [**Ada93**, theorem 3·9]) implies that the $\mathrm{Diff}^+(\Sigma_g)$-equivariant "derivative map"

$$\mathrm{Imm}(\Sigma_g, \mathring{M}) \longrightarrow \mathrm{Bun}_{inj}(T\Sigma_g, TM),$$

to the subspace of $\mathrm{Bun}(T\Sigma_g, TM)$ consisting of injective bundle maps, is a weak homotopy equivalence.

The second step is to note that there is a $\mathrm{Diff}^+(\Sigma_g)$-equivariant map

$$\mathrm{Bun}_{inj}(T\Sigma_g, TM) \longrightarrow \mathrm{Bun}(T\Sigma_g, \ell)$$

sending the bundle injection $e : T\Sigma_g \hookrightarrow TM$ to the bundle map

$$\begin{aligned} e' : T\Sigma_g &\longrightarrow \ell \\ (x, v) &\mapsto ((e(x), e(T_x\Sigma_g) \subset T_{e(x)}M), e(v)). \end{aligned}$$

This is easily seen to be a homeomorphism, and the claim follows.

By taking homotopy quotients, this lemma shows that there is a weak homotopy equivalence $\mathcal{I}_g^h(M) \simeq \mathcal{M}^{\theta_M}(\Sigma_g)$. Along with Theorem A this implies the zig-zag

$$\mathcal{I}_g(M) \overset{\simeq_{H_*}}{\longleftarrow} \mathcal{I}_g^h(M) \overset{\simeq}{\longrightarrow} \mathcal{M}^{\theta_M}(\Sigma_g)$$

where the leftwards map is a $\mathbb{Z}[1/(g-1)]$-homology equivalence and the rightwards map is a weak homotopy equivalence.

3·1. *Surfaces with boundary*

The definition of the spaces $\mathcal{M}^{\theta_M}(\Sigma_g)$ extends easily to the case of surfaces with boundary. Let us write $\Sigma_{g,b}$ for a surface of genus g with b boundary components. We fix a bundle map $\delta : T\Sigma_{g,b}|_{\partial\Sigma_{g,b}} \to \ell$ and let $\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)$ be the space of those bundle maps which restrict to δ on the boundary. The group $\mathrm{Diff}^+_\partial(\Sigma_{g,b})$ of diffeomorphisms which are the identity near $\partial\Sigma_{g,b}$ acts on this space, and we define

$$\mathcal{M}^{\theta_M}(\Sigma_{g,b}; \delta) := \mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta) /\!\!/ \mathrm{Diff}^+_\partial(\Sigma_{g,b}).$$

The main results of [**RW09**] reduce the problem of establishing a homology stability theorem for the moduli spaces $\mathcal{M}^\theta(\Sigma_{g,b}; \delta)$ to the problem of understanding the sets $\pi_0(\mathcal{M}^\theta(\Sigma_{g,b}; \delta))$ and the gluing maps between them sufficiently well. We will explain what this means in Section 4.

3·2. *Spaces of bundle maps as spaces of lifts*

In order to understand these sets of path components, we require a further model for the space $\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)$. Let us pick a map $\tau : \Sigma_{g,b} \to BSO(2)$ classifying the tangent bundle (i.e. we have a given bundle isomorphism $\varphi : \tau^*\gamma_2^+ \cong T\Sigma_{g,b}$).

Definition 3·2. For a fibration $\theta : X \to BSO(2)$ let $\mathrm{Lifts}(\tau, \theta)$ denote the space of maps $l : \Sigma_{g,b} \to X$ such that $\tau = \theta \circ l$. If a lift $b : \partial\Sigma_{g,b} \to X$ of $\tau|_{\partial\Sigma_{g,b}}$ is already given, let $\mathrm{Lifts}(\tau, \theta; b)$ denote the subspace of those l which restrict to b on the boundary.

If θ is any map, we define $\mathrm{Lifts}(\tau, \theta; b)$ to be $\mathrm{Lifts}(\tau, \theta^f; b^f)$ where $\theta^f : X^f \to BSO(2)$ is the canonical replacement of θ by a fibration, and b^f is the composition of b with the canonical map $X \to X^f$. If θ is already a fibration then the spaces $\mathrm{Lifts}(\tau, \theta; \delta)$ and $\mathrm{Lifts}(\tau, \theta^f; b^f)$ are homotopy equivalent.

Suppose θ is a fibration (by replacing it if necessary). Given a lift l of τ, we have a bundle map

$$\mathcal{B}(l) : T\Sigma_{g,b} \overset{\varphi}{\cong} \tau^*\gamma_2^+ = (\theta \circ l)^*\gamma_2^+ \cong l^*(\theta^*\gamma_2^+) \longrightarrow \theta^*\gamma_2^+,$$

and this construction defines a map

$$\mathcal{B} : \mathrm{Lifts}(\tau, \theta; b) \longrightarrow \mathrm{Bun}_\partial(T\Sigma_{g,b}, \theta^*\gamma_2^+; \mathcal{B}(b)).$$

This map is a weak homotopy equivalence: this may be proved by directly showing that the induced map on homotopy groups is a bijection, using the homotopy lifting property for θ.

3·3. *Gluing*

Suppose we are given a surface Σ, a collar $c : [0, 1) \times \partial\Sigma \to \Sigma$ and a boundary condition $\delta : T\Sigma|_{\partial\Sigma} \to \ell$, and similar data (Σ', c', δ'). Suppose we have embeddings

$$\partial\Sigma \overset{i}{\longleftarrow} \partial_0 \overset{i'}{\longrightarrow} \partial\Sigma'$$

such that

$$\epsilon^1 \oplus T\partial_0 \overset{\epsilon^1 \oplus Di}{\longrightarrow} \epsilon^1 \oplus T\partial\Sigma = T([0, 1) \times \partial\Sigma)|_{\{0\}\times\partial\Sigma} \overset{Dc}{\longrightarrow} T\Sigma|_{\partial\Sigma} \overset{\delta}{\longrightarrow} \ell$$

and the analogous map for $(\Sigma', c', \delta', i')$ are equal. Then there is a gluing map

$$\mathrm{Bun}_\partial(T\Sigma, \ell; \delta) \times \mathrm{Bun}_\partial(T\Sigma', \ell; \delta') \longrightarrow \mathrm{Bun}_\partial(T(\Sigma \cup_{\partial_0} \Sigma'); \delta \cup \delta')$$

and an associated gluing map

$$\mathcal{M}^{\theta_M}(\Sigma;\delta)\times\mathcal{M}^{\theta_M}(\Sigma';\delta')\longrightarrow\mathcal{M}^{\theta_M}(\Sigma\cup_{\partial_0}\Sigma';\delta\cup\delta'). \tag{3·1}$$

4. *Isotopy classes of immersions and homology stability*

Fix an immersion $\delta : [0,1)\times\partial\Sigma_{g,b}\looparrowright M$ and let us write δ for the associated bundle map $T\Sigma_{g,b}|_{\partial\Sigma_{g,b}}\to\ell$ as well as for the underlying map $\partial\Sigma_{g,b}\to\mathrm{Gr}_2^+(TM)$; the precise meaning will be clear from the context. We first aim to compute

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b},\ell;\delta))=\pi_0(\mathrm{Lifts}(\tau,\theta_M;\delta))$$

as then $\pi_0(\mathcal{M}^{\theta_M}(\Sigma_{g,b};\delta))$ will be the quotient of this set by the evident action of the *mapping class group*

$$\pi_0(\mathrm{Diff}_\partial^+(\Sigma_{g,b}))=:\Gamma_{g,b}.$$

From now on we make the assumption that M is simply-connected and of dimension at least three, which implies that all boundary conditions δ are regularly homotopic, and we can hence take them to be in some standard position.

There is an action of the group $\pi_2(M)\cong H_2(M;\mathbb{Z})$ on the set $\pi_0(\mathrm{map}_\partial(\Sigma_{g,b},M;\delta))$ defined by composing with the map $\Sigma_{g,b}\to S^2\vee\Sigma_{g,b}$ which collapses a small embedded circle in the interior of $\Sigma_{g,b}$. An easy obstruction theory argument shows that this action makes $\pi_0(\mathrm{map}_\partial(\Sigma_{g,b},M;\delta))$ into a $H_2(M;\mathbb{Z})$-torsor, so a choice of immersed discs with boundary condition δ gives a bijection $\pi_0(\mathrm{map}_\partial(\Sigma_{g,b},M;\delta))\cong H_2(M;\mathbb{Z})$. Let us choose once and for all such a collection of immersed discs with boundary condition δ.

PROPOSITION 4·1. *Let M be simply-connected and of dimension $d\geqslant 5$. Then the natural map*

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b},\ell;\delta))\longrightarrow\pi_0(\mathrm{map}_\partial(\Sigma_{g,b},M;\pi_M\circ\delta))\cong H_2(M;\mathbb{Z})$$

induces a bijection. The action of the mapping class group is trivial.

Proof. Consider the diagram

$$\begin{array}{ccccc} S^{d-2} & & & & \\ \downarrow & & & & \\ \mathrm{Fr}_2(\mathbb{R}^d) & \longrightarrow & \mathrm{Gr}_2^+(TM) & \xrightarrow{\theta_M\times\pi_M} & BSO(2)\times M \\ \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \pi_{BSO(2)}} \\ S^{d-1} & & BSO(2) & = & BSO(2) \end{array} \tag{4·1}$$

where the first column and middle row are homotopy fibre sequences, and π_X denotes the natural projection to X. By the first column, if $d\geqslant 5$ then $\mathrm{Fr}_2(\mathbb{R}^d)$ is 2-connected, and hence the map $\theta_M\times\pi_M$ is 3-connected. Fixing a map $\tau:(\Sigma_{g,b},\partial\Sigma_{g,b})\to(BSO(2),*)$ classifying the tangent bundle, the map

$$\mathrm{Lifts}(\tau,\theta_M;\delta)\longrightarrow\mathrm{Lifts}(\tau,\pi_{BSO(2)};(\theta_M\times\pi_M)\circ\delta)\simeq\mathrm{map}_\partial(\Sigma_{g,b},M;\pi_M\circ\delta)$$

is then 1-connected and in particular a bijection on π_0.

The cases of 3- and 4-dimensional background manifolds are rather more complicated. In these cases an immersion of a surface into such a manifold endows the surface with additional geometric structure which cannot be recovered from the homotopy class of the map alone. In dimension 3 this is a Spin structure, and in dimension 4 it is a choice of oriented rank 2 vector bundle with Euler class satisfying a certain congruence condition.

PROPOSITION 4·2. *Let M be a simply-connected 3-manifold; such a manifold admits a Spin structure. A choice of Spin structure on M gives a bijection*

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)) \cong \mathrm{Spin}(\Sigma_{g,b}; *) \times H_2(M; \mathbb{Z}),$$

where $\mathrm{Spin}(\Sigma_{g,b}; *)$ *denotes the set of isomorphism classes of Spin structures on $\Sigma_{g,b}$, with the trivial Spin structure around the boundary. The action of the mapping class group is given by its usual action on the set of Spin structures.*

Proof. An orientable 3-manifold admits a Spin structure as there is a relation $w_2 = w_1^2$ among its Stiefel–Whitney classes by Wu's formula. We choose one, $\mathfrak{s}$, once and for all.

From diagram (4·1) in the proof of Proposition 4·1, we see there are homotopy cartesian squares

$$\begin{array}{ccc} & \mathrm{Lifts}(\tau, \theta_M; \delta) & \longleftarrow & \mathrm{Lifts}(\tau \times f, \theta_M \times \pi_M; \delta) \\ & \downarrow & & \downarrow \\ \mathrm{map}_\partial(\Sigma_{g,b}, M; \pi_M \circ \delta) \simeq & \mathrm{Lifts}(\tau, \pi_{BSO(2)}; \pi_M \circ \delta) & \longleftarrow & \{f : \Sigma_{g,b} \to M\} \end{array}$$

for each point $\{f\} \in \mathrm{Lifts}(\tau, \pi_{BSO(2)}; \pi_M \circ \delta)$.

The space $\mathrm{Lifts}(\tau \times f, \theta_M \times \pi_M; \delta)$ is always non-empty. Finding a point in it is the same as finding a bundle injection $T\Sigma_{g,b} \hookrightarrow f^*TM$ extending δ. The bundle f^*TM is always trivial (it is a Spin vector bundle on a 2-manifold), and choosing a trivialisation $\eta : f^*TM \to \epsilon^3$, we see that finding such a bundle injection is the same as finding a lift in a diagram

$$\begin{array}{ccc} \partial\Sigma_{g,b} & \xrightarrow{\eta \circ \delta} & \mathrm{Gr}_2^+(\mathbb{R}^3) \simeq S^2 \\ \cap\downarrow & & \downarrow \\ \Sigma_{g,b} & \xrightarrow{\tau} & BSO(2). \end{array}$$

By obstruction theory there is a unique obstruction $w_2(\tau) \in H^2(\Sigma_{g,b}, \partial\Sigma_{g,b}; \mathbb{Z}/2)$ to the existence of such a lift, but this is of course zero, as all orientable surfaces are Spin. Hence there is a surjection

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)) \longrightarrow \pi_0(\mathrm{map}_\partial(\Sigma_{g,b}, M; \pi_M \circ \delta)) \cong H_2(M; \mathbb{Z})$$

and the preimage of $[f]$ is the quotient of $\pi_0(\mathrm{Lifts}(\tau \times f, \theta_M \times \pi_M; \delta))$ by the action of $\pi_1(\mathrm{map}_\partial(\Sigma_{g,b}, M; \pi_M \circ \delta), \{f\})$.

Let $s : \mathrm{Gr}_2^+(\gamma_3^{\mathrm{Spin}}) \to BSO(2)$ classify the canonical oriented 2-plane bundle, and $t : \mathrm{Gr}_2^+(\gamma_3^{\mathrm{Spin}}) \to B\mathrm{Spin}(3)$ classify the canonical Spin 3-plane bundle. In the fibration

$$\mathrm{Fr}_2(\mathbb{R}^3) \longrightarrow \mathrm{Gr}_2^+(\gamma_3^{\mathrm{Spin}}) \xrightarrow{s \times t} BSO(2) \times B\mathrm{Spin}(3),$$

the space $\mathrm{Gr}_2^+(\gamma_3^{\mathrm{Spin}})$ has a sequence of bundles

$$0 \longrightarrow s^*\gamma_2^+ \longrightarrow t^*\gamma_3^{\mathrm{Spin}} \longrightarrow \epsilon^1 \longrightarrow 0$$

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which splits to give an isomorphism $s^*\gamma_2^+ \oplus \epsilon^1 \cong t^*\gamma_3^{\mathrm{Spin}}$. Thus $s^*\gamma_2^+$ in fact has a canonical Spin structure, and the map s factors canonically through $B\mathrm{Spin}(2)$. Given a map $f : \Sigma_{g,b} \to M$ with boundary condition $\pi_M \circ \delta$, there is a diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Fr}_2(\mathbb{R}^3) & = & \mathrm{Fr}_2(\mathbb{R}^3) & \longrightarrow & \mathbb{RP}^\infty \\
 & & \downarrow & & \downarrow & & \downarrow \\
\partial\Sigma_{g,b} & \xrightarrow{\delta} & \mathrm{Gr}_2^+(TM) & \longrightarrow & \mathrm{Gr}_2^+(\gamma_3^{\mathrm{Spin}}) & \longrightarrow & B\mathrm{Spin}(2) \\
\big\downarrow & \overset{l}{\nearrow} & \big\downarrow {\scriptstyle \theta_M \times \pi_M} & & \big\downarrow {\scriptstyle s \times t} & & \big\downarrow \\
\Sigma_{g,b} & \xrightarrow{\tau \times f} & BSO(2) \times M & \xrightarrow{\mathrm{Id} \times \mathfrak{s}} & BSO(2) \times B\mathrm{Spin}(3) & \xrightarrow{\pi_{BSO(2)}} & BSO(2)
\end{array}$$

and hence a lift l produces a Spin structure on $\Sigma_{g,b}$, and we have defined a map

$$\pi_0(\mathrm{Lifts}(\tau \times f, \theta_M \times \pi_M)) \longrightarrow \pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \gamma_2^{\mathrm{Spin}}; *)) =: \mathrm{Spin}(\Sigma_{g,b}; *).$$

This is easily seen to be a bijection (as $\mathrm{Fr}_2(\mathbb{R}^3) \simeq \mathbb{RP}^3$, and the induced map $\mathrm{Fr}_2(\mathbb{R}^3) \to \mathbb{RP}^\infty$ between fibres of the rightmost two vertical maps in the diagram is the standard inclusion, hence 3-connected), and the action of a self-homotopy of f, that is, an element of $\pi_1(\mathrm{map}_\partial(\Sigma_{g,b}, M; \pi_M \circ b), \{f\})$, is trivial. Thus there is the exact sequence of sets

$$* \longrightarrow \mathrm{Spin}(\Sigma_{g,b}; *) \longrightarrow \pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)) \longrightarrow H_2(M; \mathbb{Z}) \longrightarrow *,$$

but a bundle map $T\Sigma_{g,b} \to \ell$ gives in particular a map f and a lift of $\tau \times f$, so there is a function

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)) \longrightarrow \mathrm{Spin}(\Sigma_{g,b}; *)$$

which splits it.

PROPOSITION 4·3. *Let M be a simply-connected 4-manifold. Then there is a surjection*

$$\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta)) \longrightarrow H_2(M; \mathbb{Z})$$

with preimage of f in natural bijection with the set

$$P(f) := \{a \in \mathbb{Z} \mid a \equiv \langle w_2(M), f\rangle \bmod 2\}.$$

Thus $\pi_0(\mathrm{Bun}_\partial(T\Sigma_{g,b}, \ell; \delta))$ may be identified with a subset of $\mathbb{Z} \times H_2(M; \mathbb{Z})$. The action of the mapping class group is trivial.

Proof. Recall that $\mathrm{Fr}_2(\mathbb{R}^4)$ is the homogenous space $SO(2)\backslash SO(4)$, and that $\mathrm{Gr}_2^+(\gamma_4^+)$ is the two-sided homotopy quotient

$$\mathrm{Gr}_2^+(\gamma_4^+) := SO(2)\backslash\!\!\backslash(SO(2)\backslash SO(4))/\!\!/SO(4) \simeq BSO(2) \times BSO(2).$$

A map giving this homotopy equivalence is given as follows: let $s : \mathrm{Gr}_2^+(\gamma_4^+) \to BSO(2)$ classify the canonical oriented 2-plane bundle, and $t : \mathrm{Gr}_2^+(\gamma_4^+) \to BSO(4)$ classify the canonical oriented 4-plane bundle. The space $\mathrm{Gr}_2^+(\gamma_4^+)$ carries a sequence of bundles

$$0 \longrightarrow s^*\gamma_2^+ \longrightarrow t^*\gamma_4^+ \longrightarrow V \longrightarrow 0$$

so there is a map $c : \mathrm{Gr}_2^+(\gamma_4^+) \to BSO(2)$ classifying the oriented 2-plane bundle V. The map $s \times c : \mathrm{Gr}_2^+(\gamma_4^+) \to BSO(2) \times BSO(2)$ provides the homotopy equivalence.

Given a map $f : \Sigma_{g,b} \to M$ with boundary condition $\pi_M \circ \delta$, there is a diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Fr}_2(\mathbb{R}^4) & = & \mathrm{Fr}_2(\mathbb{R}^4) & & \\
 & & \downarrow & & \downarrow & & \\
\partial\Sigma_{g,b} & \xrightarrow{\delta} & \mathrm{Gr}_2^+(TM) & \longrightarrow & \mathrm{Gr}_2^+(\gamma_4^+) & \xrightarrow{c} & BSO(2) \\
\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle \theta_M \times \pi_M} & & \downarrow{\scriptstyle s \times t} & & \\
\Sigma_{g,b} & \xrightarrow{\tau \times f} & BSO(2) \times M & \xrightarrow{\mathrm{Id} \times \tau_M} & BSO(2) \times BSO(4) & &
\end{array}$$

and hence composing a lift l with c produces an element $c(l) \in H^2(\Sigma_{g,b}, \partial\Sigma_{g,b}; \mathbb{Z})$, that is, an integer. It reduces modulo 2 to $f^*w_2(M) \in H^2(\Sigma_{g,b}, \partial\Sigma_{g,b}; \mathbb{Z}/2) = \mathbb{Z}/2$, which is a homotopy invariant of the map f. Furthermore, the set of homotopy classes of lifts is a $\pi_2(\mathrm{Fr}_2(\mathbb{R}^4))$-torsor, and the image of

$$\mathbb{Z} \cong \pi_2(\mathrm{Fr}_2(\mathbb{R}^4)) \longrightarrow \pi_2(\mathrm{Gr}_2^+(\gamma_4^+)) \xrightarrow{c_V} \pi_2(BSO(2)) = \mathbb{Z},$$

is the even integers. Thus we have defined an injective map

$$\begin{aligned}
\pi_0(\mathrm{Lifts}(\tau \times f, \theta_M \times \pi_M; \delta)) &\longrightarrow P(f) := \begin{cases} 1 + 2\mathbb{Z} & \text{if } f^*w_2(M) \neq 0 \\ 2\mathbb{Z} & \text{if } f^*w_2(M) = 0 \end{cases} \subset \mathbb{Z} \qquad (4{\cdot}2) \\
l &\longmapsto c(l).
\end{aligned}$$

Given an oriented 2-plane bundle $V \to \Sigma_{g,b}$ trivialised over the boundary, standard obstruction theory shows that the only obstruction to finding an isomorphism $T\Sigma_{g,b} \oplus V \cong f^*TM$ (extending the standard isomorphism of trivial bundles on the boundary) is the class

$$f^*w_2(TM) - w_2(V) - w_2(T\Sigma_{g,b}) \in H^2(\Sigma_{g,b}, \partial\Sigma_{g,b}; \mathbb{Z}/2).$$

This shows that the map (4·2) is also surjective.

COROLLARY 4·4. *Let M be simply-connected. If* $\dim(M) = 3$ *then choosing a Spin structure on M gives a bijection*

$$\pi_0(\mathcal{M}^{\theta_M}(\Sigma_{g,b}; \delta)) \cong \pi_0(\mathcal{M}^{\mathrm{Spin}}(\Sigma_{g,b}; *)) \times H_2(M; \mathbb{Z})$$

for all g and b. Gluing surfaces along boundary components corresponds to adding homology classes and gluing Spin structures.

If $\dim(M) = 4$ *then there is a surjection*

$$\pi_0(\mathcal{M}^{\theta_M}(\Sigma_{g,b}; \delta)) \longrightarrow H_2(M; \mathbb{Z})$$

with preimage of f given by the set $P(f)$, for all g and b. Gluing surfaces $(a \in P(f), f)$ and $(b \in P(g), g)$ along boundary components has result $(a + b \in P(f + g), f + g)$.

If $\dim(M) \geqslant 5$ *then there is a bijection*

$$\pi_0(\mathcal{M}^{\theta_M}(\Sigma_{g,b}; \delta)) \cong H_2(M; \mathbb{Z})$$

for all g and b. Gluing surfaces along boundary components corresponds to adding homology classes.

Proof. The previous three propositions provide the descriptions of the sets. Gluing together immersed surfaces certain adds the homology classes they represent. In dimension 3 the Spin structure induced on the union of two immersed surfaces is the union of the Spin structures on each, and in dimension 4 the Euler number of the normal bundle of an immersion of a union of two surfaces is the sum of the Euler numbers of the individual immersions.

Using these calculations and the methods of [**RW09**], we establish the following homology stability theorem for the spaces $\mathcal{M}^{\theta_M}(\Sigma_{g,b};\delta)$. Recall from Section 3·3 that for a pair of surfaces with boundary condition and partially identified boundaries, we have defined a gluing map (3·1). Let us call a *pair of pants* any genus zero surface with three boundary components, and write

$$\alpha(g) : \mathcal{M}^{\theta_M}(\Sigma_{g,b};\delta) \longrightarrow \mathcal{M}^{\theta_M}(\Sigma_{g+1,b-1};\delta')$$

for any gluing map which adds on a pair of pants along two boundary components (such maps can only exist if $b \geqslant 2$). Similarly, write

$$\beta(g) : \mathcal{M}^{\theta_M}(\Sigma_{g,b};\delta) \longrightarrow \mathcal{M}^{\theta_M}(\Sigma_{g,b+1};\delta')$$

for any gluing map which adds on a pair of pants along a single boundary component (which can only exist if $b \geqslant 1$), and write

$$\gamma(g) : \mathcal{M}^{\theta_M}(\Sigma_{g,b};\delta) \longrightarrow \mathcal{M}^{\theta_M}(\Sigma_{g,b-1};\delta')$$

for any gluing map which adds on a disc along some boundary component (which can only exist if $b \geqslant 1$). We call these *stabilisation maps*.

THEOREM 4·5. *Any stabilisation map $\alpha(g)$ is a homology epimorphism in degrees*

$$* \leqslant \begin{cases} \dfrac{2g-1}{5} & \textit{if } \dim(M) = 3 \\ \dfrac{2g}{3} & \textit{if } \dim(M) > 3 \end{cases}$$

and a homology isomorphism in one degree lower.

Any stabilisation map $\beta(g)$ is a homology epimorphism in degrees

$$* \leqslant \begin{cases} \dfrac{2g-2}{5} & \textit{if } \dim(M) = 3 \\ \dfrac{2g-1}{3} & \textit{if } \dim(M) > 3 \end{cases}$$

and a homology isomorphism in one degree lower.

Any stabilisation map $\gamma(g)$ is a homology isomorphism in the same range that $\beta(g)$ is an epimorphism.

Proof. By Corollary 4·4 if $\dim(M) \geqslant 4$ then all stabilisation maps induce bijections on sets of path components, and if $\dim(M) = 3$ stabilisation maps induce surjections or bijections on sets of path components precisely when Spin structures do. Thus by [**RW09**, section 7] these cases have the same stability ranges as the tangential structures $BSO(2)\times M$ and $B\mathrm{Spin}(2) \times M$ respectively. The range in the first case has been calculated in [**RW09**, section 7.5], and the range in the second case follows from [**RW10**, section 2·4–2·6].

To identify the stable homology, we apply the theorem of Galatius–Madsen–Tillmann–Weiss [**GMTW09**, section 7]. Write $\mathbf{MTSO}(2)$ for the Thom spectrum of the virtual bundle $-\gamma_2 \to BSO(2)$. The map of Thom spectra induced by $\theta_M : \mathrm{Gr}_2^+(TM) \to BSO(2)$ gives a map

$$\Omega^\infty \mathbf{MT}\theta_M \longrightarrow \Omega^\infty \mathbf{MTSO}(2)$$

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and it has been calculated in [**MT01**] that there is a natural isomorphism

$$E : \pi_0(\Omega^\infty \mathbf{MTSO}(2)) \cong \mathbb{Z}$$

under which the point represented by an oriented surface of genus g maps to $1-g$. The evident composition defines a map

$$E : \pi_0(\Omega^\infty \mathbf{MT}\theta_M) \longrightarrow \mathbb{Z}$$

and we let $\Omega^\infty_{(n)}\mathbf{MT}\theta_M$ be those path components which map to $n \in \mathbb{Z}$ under E. In [**GMTW09**, section 5] the authors show that Pontrjagin–Thom theory provides a map

$$\alpha_M : \mathcal{M}^{\theta_M}(\Sigma_g) \longrightarrow \Omega^\infty_{(1-g)}\mathbf{MT}\theta_M,$$

and by Theorem 4·5 and [**RW09**, section 11], this map induces a homology isomorphism in degrees

$$* \leqslant \begin{cases} \dfrac{2g-6}{5} & \text{if } \dim(M) = 3 \\ \dfrac{2g-3}{3} & \text{if } \dim(M) > 3, \end{cases}$$

which establishes Theorem B.

5. *Calculations in Euclidean space*

In this section we prove the results of Sections 1·2–1·5.

5·1. *Immersions in* $\mathbb{R}^3$

That $\mathcal{I}_g(\mathbb{R}^3)$ has two components, distinguished by the Arf invariant of the associated Spin structures, follows from Proposition 4·2. In this case $\mathrm{Gr}_2^+(\mathbb{R}^3) \simeq S^2$ and the tautological bundle corresponds to TS^2. This is stably trivial and so $\mathbf{MT}\theta_{\mathbb{R}^3} \simeq \mathbf{S}^{-2} \vee \mathbf{S}^0$. Hence the associated infinite loop space is $Q_0(S^0) \times \Omega^2_0 Q(S^0)$, which has trivial rational cohomology in positive degrees. The result now follows from Corollary C.

The torsion calculation may be seen as follows. The fundamental group of this infinite loop space is $\mathbb{Z}/2 \oplus \mathbb{Z}/24$, by the well known homotopy groups of spheres in low degrees. This is its first integral homology too, by Hurewicz' theorem, and it follows from Corollary C that this is the first homology of $\mathcal{I}_g^h(\mathbb{R}^3)$ as long as $g \geqslant 6$. As the fibres of $\mathcal{I}_g^h(\mathbb{R}^3) \to \mathcal{I}_g(\mathbb{R}^3)$ are all connected, it follows that the map is surjective on first homology, and the statement about localisation follows from Theorem A.

5·2. *Immersions in* $\mathbb{R}^4$

That $\mathcal{I}_g(\mathbb{R}^4)$ has components indexed by $\mathbb{Z}$, distinguished by the Euler number of the normal bundle of the immersion, follows from Proposition 4·3. $\mathrm{Gr}_2^+(\mathbb{R}^4)$ is a simply-connected 4-manifold (in fact, it is $S^2 \times S^2$ but we do not require this). Thus the cohomology of the spectrum $\mathbf{MT}\theta_{\mathbb{R}^4}$ has a unique class in positive degree, $[\mathrm{Gr}_2^+(\mathbb{R}^4)]^* \cdot u_{-2} \in H^2(\mathbf{MT}\theta_{\mathbb{R}^4}; \mathbb{Q})$, and so the rational cohomology of $\Omega^\infty_\bullet \mathbf{MT}\theta_{\mathbb{R}^4}$ is

$$H^*(\Omega^\infty_\bullet \mathbf{MT}\theta_{\mathbb{R}^4}; \mathbb{Q}) \cong \mathbb{Q}[a_2]$$

a polynomial algebra on a single generator in degree 2. One can easily check that the natural map $\mathrm{Gr}_2^+(\mathbb{R}^4) \to \mathrm{Gr}_2^+(\mathbb{R}^\infty)$ pulls back the square of the Euler class to a non-trivial top-dimensional class, and so a_2 can be taken to be κ_1. The result now follows from Corollary C.

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5·3. *Immersions in* $\mathbb{R}^{2n+1}$, $2n+1 \geqslant 5$

That $\mathcal{I}_g(\mathbb{R}^{2n+1})$ is connected follows from Proposition 4·1. We have the calculation

$$H^*(\mathrm{Gr}_2^+(\mathbb{R}^{2n+1}); \mathbb{Q}) = \mathbb{Q}[e]/(e^{2n})$$

where $e \in H^2(\mathrm{Gr}_2^+(\mathbb{R}^{2n+1}); \mathbb{Q})$ is the Euler class. Thus

$$H^*(\Omega^\infty_\bullet \mathbf{MT}\theta_{\mathbb{R}^{2n+1}}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_{2n-2}]$$

and so the result now follows from Corollary C.

5·4. *Immersions in* $\mathbb{R}^{2n}$, $2n \geqslant 6$

That $\mathcal{I}_g(\mathbb{R}^{2n+1})$ is connected follows from Proposition 4·1. By [**Lai74**, theorem 2] we have the calculation

$$H^*(\mathrm{Gr}_2^+(\mathbb{R}^{2n}); \mathbb{Q}) = \mathbb{Q}[e, \delta]/(\delta^2, e^n - 2\delta e)$$

where e is the Euler class and δ is Poincaré dual to the fundamental class of the submanifold $\mathbb{CP}^{n-1} \hookrightarrow \mathrm{Gr}_2^+(\mathbb{R}^{2n})$. In terms of these classes, the Euler class of the orthogonal complement to the tautological bundle is

$$\bar{e} = 2\delta - e^{n-1}.$$

We define a spectrum cohomology class $\bar{e} \cdot u_{-2} \in H^{2n-4}(\mathbf{MT}\theta_{\mathbb{R}^{2n}}; \mathbb{Q})$ and let $\Delta \in H^{2n-4}(\Omega^\infty_\bullet \mathbf{MT}\theta_{\mathbb{R}^{2n}}; \mathbb{Q})$ denote its cohomology suspension. Then

$$H^*(\Omega^\infty_\bullet \mathbf{MT}\theta_{\mathbb{R}^{2n}}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_{2n-3}, \Delta]$$

(and $\kappa_i = 0$ for $i > 2n-3$), so the result now follows from Corollary C.

5·5. *The spectral sequence for immersions in* $\mathbb{R}^{2n+1}$, $2n+1 \geqslant 5$

In order to emphasise the nontriviality of the fibration

$$\mathrm{Imm}(\Sigma_g, M) \longrightarrow \mathcal{I}_g^h(M) \longrightarrow B\mathrm{Diff}^+(\Sigma_g), \tag{5·1}$$

we will describe a computation of its Leray–Serre spectral sequence when $M = \mathbb{R}^{2n+1}$ with $2n+1 \geqslant 5$, for rational cohomology in the stable range. We must first understand the cohomology of the space $\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1})$, and the coefficient system it describes over $B\mathrm{Diff}^+(\Sigma_g)$.

Recall that the *mapping class group* of a closed genus g surface is defined to be

$$\Gamma_g := \pi_0(\mathrm{Diff}^+(\Sigma_g)),$$

and for $g \geqslant 2$ the map $B\mathrm{Diff}^+(\Sigma_g) \to B\Gamma_g$ is a homotopy equivalence [**EE69**]. The action of a diffeomorphism of Σ_g on the homology of the surface makes $H_1(\Sigma_g; \mathbb{Z})$ into a Γ_g-module, which we call H. We write $H_\mathbb{Q} = H \otimes \mathbb{Q}$. The naturality of Poincaré duality and the Universal Coefficient Theorem shows that $H \cong H^*$ as Γ_g-modules.

The Leray–Serre spectral sequence for the fibration (5·1) has the form

$$E_2^{p,q} = H^p(\Gamma_g; H^q(\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}); \mathbb{Q})) \Longrightarrow H^{p+q}(\mathcal{I}_g^h(\mathbb{R}^{2n+1}); \mathbb{Q}), \tag{5·2}$$

where we can equally well write $H^*(\mathcal{I}_g(\mathbb{R}^{2n+1}); \mathbb{Q})$ for the abutment, by Theorem A, to obtain the spectral sequence (1·1).

PROPOSITION 5·1. *There is an isomorphism of* Γ_g*-modules*

$$H^*(\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}); \mathbb{Q}) \cong \Lambda[x_{4n-3}, x_{4n-1}] \otimes \mathrm{Sym}^*(H_\mathbb{Q}[4n-2])$$

where $H_{\mathbb{Q}}[4n-2]$ *denotes* $H_{\mathbb{Q}}$ *considered as a vector space of grading* $(4n-2)$*, and* Sym^* *denotes the symmetric algebra on this graded vector space.*

Proof. We first decompose $\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1})$ using the fibration

$$\mathrm{Imm}_\partial(D^2, \mathbb{R}^{2n+1}) \longrightarrow \mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}) \overset{\text{restrict}}{\longrightarrow} \mathrm{Imm}(\Sigma_g \setminus D^2, \mathbb{R}^{2n+1}), \qquad (5{\cdot}3)$$

which restricts an immersion to the complement of a disc. It is far from obvious that this restriction map is a fibration: this follows from Smale's "fibration theorem" [**Sma59**, theorem 1·1], which is the fundamental ingredient of Smale–Hirsch theory. We then consider the fibration

$$\mathrm{Imm}_*(\Sigma_g \setminus D^2, \mathbb{R}^{2n+1}) \longrightarrow \mathrm{Imm}(\Sigma_g \setminus D^2, \mathbb{R}^{2n+1}) \overset{\text{restrict}}{\longrightarrow} \mathrm{Fr}_2(\mathbb{R}^{2n+1}), \qquad (5{\cdot}4)$$

where the base is the space of linearly-independent 2-frames in $\mathbb{R}^{2n+1}$, which takes the derivative of an immersion at a point $x_0 \in \partial(\Sigma_g \setminus D^2)$. The notation $\mathrm{Imm}_*(\Sigma_g \setminus D^2, \mathbb{R}^{2n+1})$ means the space of immersions which agree with a fixed germ near x_0.

Both fibrations admit an action of $\mathrm{Diff}^+(\Sigma_{g,1}, \partial)$. In the first case the action is trivial on each fibre, and in the second case it is trivial on the base. If we write $\Gamma_{g,1} := \pi_0(\mathrm{Diff}^+(\Sigma_{g,1}, \partial))$, this makes the Serre spectral sequence a spectral sequence of $\Gamma_{g,1}$-modules in both cases.

We now apply Smale–Hirsch theory. There is a map

$$\begin{aligned} \mathrm{Imm}_\partial(D^2, \mathbb{R}^{2n+1}) &\longrightarrow \mathrm{map}_\partial(D^2, \mathrm{Fr}_2(\mathbb{R}^{2n+1})) \\ i &\longmapsto \epsilon^2 = TD^2 \overset{Di}{\to} T\mathbb{R}^{2n+1} \end{aligned}$$

to the space of maps which have some fixed behaviour on the boundary, and by Smale–Hirsch theory this is an equivalence. As $\mathrm{Fr}_2(\mathbb{R}^{2n+1})$ is simply-connected, we can suppose that the boundary condition is the constant map to a basepoint, and so there is a homotopy equivalence $\mathrm{Imm}_\partial(D^2, \mathbb{R}^{2n+1}) \simeq \Omega^2 \mathrm{Fr}_2(\mathbb{R}^{2n+1})$. The fibration

$$S^{2n-1} \longrightarrow \mathrm{Fr}_2(\mathbb{R}^{2n+1}) \longrightarrow S^{2n}$$

is equivalent to the sphere bundle of TS^{2n}. As such it has non-trivial Euler class, and so its Serre spectral sequence has a non-trivial differential. Thus we have $H^*(\mathrm{Fr}_2(\mathbb{R}^{2n+1}); \mathbb{Q}) = \Lambda[x_{4n-1}]$, and we deduce that

$$H^*(\mathrm{Imm}_\partial(D^2, \mathbb{R}^{2n+1}); \mathbb{Q}) = \Lambda[x_{4n-3}].$$

We now study the cohomology of $\mathrm{Imm}_*(\Sigma_{g,1}, \mathbb{R}^{2n+1})$, which is a little more complicated. Choosing a trivialisation $\varphi : T\Sigma_{g,1} \cong \epsilon^2$ defines a map

$$\begin{aligned} T_\varphi : \mathrm{Imm}_*(\Sigma_{g,1}, \mathbb{R}^{2n+1}) &\longrightarrow \mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1})) \\ i &\longmapsto \epsilon^2 \cong_{\varphi^{-1}} T\Sigma_{g,2} \overset{Di}{\to} T\mathbb{R}^{2n+1} \end{aligned}$$

and by Smale–Hirsch theory this is a homotopy equivalence. Both sides have an action of the group $\mathrm{Diff}^+(\Sigma_{g,1}, \partial)$, but the map T_φ is not equivariant for this action. To describe the failure of equivariance, consider the map

$$\begin{aligned} \psi : \mathrm{Diff}^+(\Sigma_{g,1}, \partial) &\longrightarrow \mathrm{map}_\partial(\Sigma_{g,1}, GL_2^+(\mathbb{R})) \\ f &\longmapsto \epsilon^2 \cong_{\varphi^{-1}} T\Sigma_{g,1} \overset{f}{\to} T\Sigma_{g,1} \cong_{\varphi} \epsilon^2. \end{aligned}$$

Let us write $f \cdot -$ for the action of f on $\mathrm{Imm}_*(\Sigma_{g,1}, \mathbb{R}^{2n+1})$, $f * -$ for the action on

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$\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1}))$, and $\psi(f) \circ -$ for the action of $\psi(f)$ on $\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1}))$ induced by the action of $GL_2^+(\mathbb{R})$ on $\mathrm{Fr}_2(\mathbb{R}^{2n+1})$. Then we have the relationship $T_\varphi(f \cdot i) = \psi(f) \circ (f * T_\varphi(i))$ between these actions.

We have the homotopy equivalence

$$\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1})) \simeq [\Omega \mathrm{Fr}_2(\mathbb{R}^{2n+1})]^{2g}$$

and so $H^*(\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1})); \mathbb{Q}) = \mathbb{Q}[a^1_{4n-2}, b^1_{4n-2}, \dots, a^g_{4n-2}, b^g_{4n-2}]$ and the action $f * -$ of a diffeomorphism is the usual symplectic action on the variables a^i, b^i. Thus $H^*(\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1})); \mathbb{Q}) = \mathrm{Sym}^*(H_\mathbb{Q}[4n-2])$ as a $\Gamma_{g,1}$-module.

This computes $H^*(\mathrm{Imm}_*(\Sigma_{g,1}, \mathbb{R}^{2n+1}); \mathbb{Q})$ as a ring, but we must compute the action $f \cdot -$ as well. By the formula above, this corresponds to computing the action $\psi(f) \circ -$ on $H^*(\mathrm{map}_*(\Sigma_{g,1}, \mathrm{Fr}_2(\mathbb{R}^{2n+1})); \mathbb{Q})$ induced by $\mu : GL_2^+(\mathbb{R}) \times \mathrm{Fr}_2(\mathbb{R}^{2n+1}) \to \mathrm{Fr}_2(\mathbb{R}^{2n+1})$. However for dimension reasons the action μ is trivial on homology and it is easy to deduce from this that $\psi(f) \circ -$ acts trivially too. Thus despite T_φ not being an equivariant map, the map $(T_\varphi)^*$ is a map (in fact, an isomorphism) of $\Gamma_{g,1}$-modules.

Consider the Serre spectral sequence for the fibration (5·4), which has the form

$$\mathrm{Sym}^*(H_\mathbb{Q}[4n-2]) \otimes \Lambda[x_{4n-1}] \Longrightarrow H^*(\mathrm{Imm}(\Sigma_g \setminus D^2, \mathbb{R}^{2n+1}); \mathbb{Q}).$$

The only possible differential is d_{4n-1}, and it is determined by

$$d_{4n-1} : E^{0,4n-2} = H_\mathbb{Q} \longrightarrow E^{4n-1,0} = \mathbb{Q}$$

but is also a map of $\Gamma_{g,1}$-modules, so must be zero (it corresponds to an invariant vector in $H^*_\mathbb{Q} \cong H_\mathbb{Q}$). Thus the spectral sequence collapses, and one may check that there are no extensions.

Next we consider the Serre spectral sequence for the fibration (5·3), which has the form

$$\Lambda[x_{4n-3}] \otimes \mathrm{Sym}^*(H_\mathbb{Q}[4n-2]) \otimes \Lambda[x_{4n-1}] \Longrightarrow H^*(\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}); \mathbb{Q}).$$

The only possible differential is d_{4n-2}, and it is determined by

$$d_{4n-2} : E^{0,4n-1} = \mathbb{Q} \longrightarrow E^{4n-2,0} = H_\mathbb{Q}$$

but this must again be trivial as it corresponds to a $\Gamma_{g,1}$-invariant vector in $H_\mathbb{Q}$. Thus this spectral sequence also collapses, and again one may check that there are no extensions. This determines $H^*(\mathrm{Imm}(\Sigma_g, \mathbb{R}^{2n+1}); \mathbb{Q})$ as a $\Gamma_{g,1}$-module, and the final step is to observe that the natural homomorphism $\Gamma_{g,1} \to \Gamma_g$ is surjective, and so we have determined the Γ_g-module structure.

Hence (for $g \geqslant 2$) the Leray–Serre spectral sequence for the fibration (5·1) has the form

$$H^*(\Gamma_g; \Lambda[x_{4n-3}, x_{4n-1}] \otimes \mathrm{Sym}^*(H_\mathbb{Q}[4n-2])) \Longrightarrow H^*(\mathcal{I}^h_g(\mathbb{R}^{2n+1}); \mathbb{Q}).$$

To determine the E_2-term of this spectral sequence we must compute the cohomology of Γ_g with coefficients in $\mathrm{Sym}^q(H_\mathbb{Q})$, and it is very useful to have the multiplicative structure, induced by $\mathrm{Sym}^p(H_\mathbb{Q}) \otimes \mathrm{Sym}^q(H_\mathbb{Q}) \to \mathrm{Sym}^{p+q}(H_\mathbb{Q})$, available to us too.

In [**ERW12**], Ebert and the author constructed certain cohomology classes $\ell_{2i+1} \in H^{2i+1}(\Gamma_g; H_\mathbb{Q})$ (denoted $x_{i,1}$ in that paper) for each $i \geqslant 1$. Considering $H_\mathbb{Q}$ to be $\mathrm{Sym}^1(H_\mathbb{Q})$, using the classes $\kappa_i \in H^{2i}(\Gamma_g; \mathbb{Q})$, and using the bigraded-commutative algebra structure on $\oplus_{p,q} H^p(\Gamma_g; \mathrm{Sym}^q(H_\mathbb{Q}))$, we obtain a map of bigraded algebras

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \otimes \Lambda[\ell_3, \ell_5, \ell_7, \dots] \longrightarrow \bigoplus_{p,q} H^p(\Gamma_g; \mathrm{Sym}^q(H_\mathbb{Q})). \tag{5·5}$$

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PROPOSITION 5·2. *The map (5·5) is an isomorphism in bidegrees (p, q) such that $p + q \leqslant (2g-3)/3$.*

Proof. By a theorem of Looijenga [**Loo96**, example 1] there is an isomorphism

$$H^*(\Gamma_g; \mathrm{Sym}^s(H_{\mathbb{Q}})) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \Sigma^{s(s+2)}\mathbb{Q}[x_2, \ldots, x_{2s}]$$

of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-modules in the stable range, i.e. the stable cohomology is a free $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-module on one generator for each monomial of $\mathbb{Q}[x_2, \ldots, x_{2s}]$, with suitably shifted degrees. Some dilligent work with generating functions—which we omit—shows that the two sides of (5·5) are isomorphic as $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-modules: there are the correct number of free generators in each degree.[1]

What remains is to show that the map we constructed induces an isomorphism. This uses many of the same techniques as [**ERW12**], where Ebert and the author computed the bigraded algebra $H^*(\Gamma_g; \Lambda^* H_{\mathbb{Q}})$ in the stable range, so we will just sketch the argument briefly.

We consider the moduli space $\mathcal{S}_{g,b} := \mathrm{map}_\partial(\Sigma_{g,b}, K(\mathbb{Z}, 3); *)/\!\!/\mathrm{Diff}^+_\partial(\Sigma_{g,b})$ which classifies surface bundles with a third integral cohomology class on the total space. By a theorem of Cohen and Madsen [**CM09**, **CM11**], and the extension to closed surfaces by the author [**RW09**], these spaces have homological stability with homology independent of g in degrees $* \leqslant (2g-3)/3$. The stable homology is that of the infinite loop space $\Omega^\infty_\bullet(\mathbf{MTSO}(2) \wedge K(\mathbb{Z}, 3)_+)$, and standard methods give the calculation

$$H^*(\Omega^\infty_\bullet(\mathbf{MTSO}(2) \wedge K(\mathbb{Z}, 3)_+); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \Lambda[l_1, l_3, l_5, \ldots],$$

for some classes l_i of degree i. Next we observe that there is a decomposition

$$\mathrm{map}(\Sigma_g, K(\mathbb{Z}, 3)) \simeq K(\mathbb{Z}, 3) \times K(H, 2) \times K(\mathbb{Z}, 1),$$

as the space of maps into an Eilenberg–MacLane space is again a product of Eilenberg–MacLane spaces, and it is easy to calculate its homotopy groups. Thus the Serre spectral sequence for the fibration

$$\mathrm{map}(\Sigma_g, K(\mathbb{Z}, 3)) \longrightarrow \mathcal{S}_g \longrightarrow B\mathrm{Diff}^+(\Sigma_g)$$

has the form

$$H^*(\Gamma_g; \mathrm{Sym}^*(H_{\mathbb{Q}}[2])) \otimes \Lambda[x_1, x_3] \Longrightarrow \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \Lambda[l_1, l_3, l_5, \ldots]$$

in total degrees $* \leqslant (2g-3)/3$. By counting dimensions using Looijenga's theorem, we see that this spectral sequence must collapse in the stable range, and that the assiciated filtration on $\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \Lambda[l_1, l_3, l_5, \ldots]$ must be by word length in the l_i. The proposition follows.

Thus the spectral sequence (5·2) has the form

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \Lambda[x_{4n-3}, x_{4n-1}, \ell_{4n+1}, \ell_{4n+3}, \ldots] \Longrightarrow H^*(\mathcal{I}^h_g(\mathbb{R}^{2n+1}); \mathbb{Q}),$$

in the stable range, where $\ell_{4n+1+2k}$ has bidegree $(p, q) = (3 + 2k, 4n - 2)$. A chart of the

[1] As a hint to the reader, after a change of variables the generating functions for the ranks as a $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-module of Looijenga's and our description become the left- and right-hand sides respectively of the identity $\sum_{k=0}^\infty q^{k(k-1)/2} 1/(1-q) \cdots 1/(1-q^k) \cdot q^k x^k = \prod_{k=1}^\infty (1 + q^k x)$, which holds as the right-hand side is nothing but the q-Pochhammer symbol $(-x; q)_\infty$ and the left-hand side is its well-known series expansion.

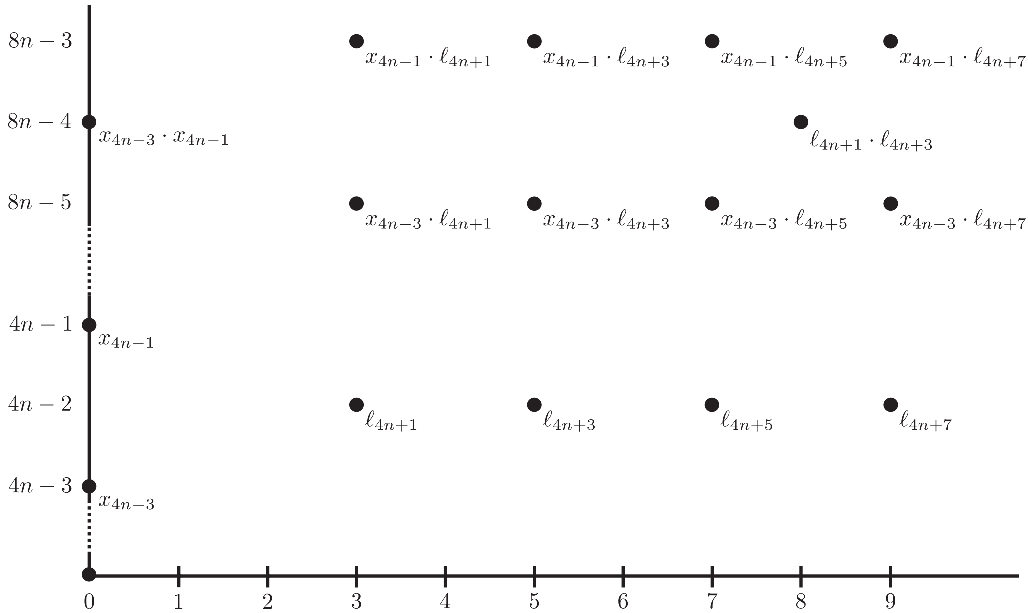

Fig. 1. Chart of the spectral sequence (5·2).

spectral sequence is then as shown in Figure 1, where each dot represents a free $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-module generator.

By the result of Section 1·4, we know that

$$H^*(\mathcal{I}_g^h(\mathbb{R}^{2n+1}); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \ldots, \kappa_{2n-2}]$$

in degrees $* \leqslant (2g-3)/3$. If we suppose that $g \geqslant n$, then the classes $\kappa_{2n-1}, \kappa_{2n}, \ldots$ cannot survive the spectral sequence. We see from the chart that the only possible pattern of differentials in the stable range is

$$\begin{aligned} d_{4n-2}(x_{4n-3}) &= \kappa_{2n-1} \\ d_{4n-1}(\ell_{4n+1+2k}) &= \kappa_{2n+1+k} \quad \text{for } k \geqslant 0 \\ d_{4n}(x_{4n-1}) &= \kappa_{2n} \end{aligned}$$

up to units, which then determines the remaining structure of the spectral sequence in the stable range by multiplicativity.

Remark 5·3. *It is curious that* κ_{2n-1} *and* κ_{2n} *do not survive the spectral sequence for what seem like less systematic reasons than the higher* κ_i. *From the point of view of Section* 5·3 *there is nothing special about them: all of the vanishing is deduced from the relation* $e^{2n} = 0$ *in the cohomology of* $\mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$. *It would be interesting to find a geometric interpretation of this fact.*

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