

On some Loci connected with Conics.

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1. For a proof of the following theorem due to Frégier, see Salmon's Conic Sections, p. 175, or Gergonne's *Annales*, VI. 231 (1816).

“If two straight lines at right angles be drawn through any point on a conic, the line joining their other points of section will pass through a fixed point on the normal.”

If the conic be $x^2/a^2 + y^2/b^2 = 1$ A, and the point be $a\cos\theta$, $b\sin\theta$ then the lines $x = a\cos\theta$, $y = b\sin\theta$ will be a pair of lines through the point at right angles. The chord joining their other points of section is

$$x/a\cos\theta + y/b\sin\theta = 0,$$

which intersects the normal in the point

$$x = \frac{a^2 - b^2}{a^2 + b^2} a\cos\theta, \quad y = -\frac{a^2 - b^2}{a^2 + b^2} b\sin\theta,$$

the locus of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 \quad \dots \quad \dots \quad \dots \quad \text{B.}$$

If we take this conic as the original, the corresponding locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^4 \quad \dots \quad \dots \quad \dots \quad \text{C.}$$

Now the polar of $a\cos\theta$, $b\sin\theta$ with respect to B is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2,$$

which is a tangent to C.

We could in this manner obtain an infinite series of concentric similar and similarly placed ellipses, such that of any three consecutive ones the external one would be the polar reciprocal of the internal one with respect to the middle one.

It can be shown that this is true of concentric similar and similarly situated ellipses whose major axes are in Geometrical Progression.

The same theorem holds for the Hyperbola.

If the parabola $y^2 = 4ax$ be taken, the lines $x = am^2$, $y = 2am$ will be a pair of lines at right angles through the point am^2 , $2am$. The line joining their other points of section is $y = -2am$, which intersects the normal in the point $4a + am^2$, $-2am$, the locus of which is

$$y^2 = 4a(x - 4a),$$

a parabola having the same latus rectum and axis as the original one.

If we find the corresponding locus for this parabola we obtain

$$y^2 = 4a(x - 8a).$$

Now the polar of $(am^2, 2am)$ with respect to $y^2 = 4a(x - 4a)$ is

$$my = x - 8a + am^2,$$

which is a tangent to $y^2 = 4a(x - 8a)$.

It can be shown that this theorem is true of a series of parabolas having the same latus rectum and axis and similarly placed, provided that their vertices are distributed at equal intervals along the axis.

One example of the last series is found as an answer to the following question :—

If parallel chords of a parabola be drawn, the locus of the point of intersection of the tangents at their extremities is a straight line, and the locus of the point of intersection of the normals at their extremities another straight line. To find the locus of the point of intersection of these two lines.

If $y = 2am$ be the diameter on which the tangents intersect the normals will intersect on the normal at $4am^2$, $-4am$, which is

$$-8am(x - 4am^2) + 4a(y + 4am) = 0,$$

and the locus reduces to $y^2 = 4a(x - 3a)$.

This can be shown geometrically to be the locus of the point of intersection of normals at right angles.

2. If from a point h , k three normals be drawn to the parabola $y^2 = 4ax$ the equation $m^3 - m(h - 2a)/a - k/a = 0$ determines their intersection with the curve.

If tangents be drawn at these points the orthocentre of the triangle so formed is

$$-a, a\{m_1 + m_2 + m_3 + m_1m_2m_3\},$$

or $-a \quad k.$

That is the intersection of the diameter through h, k with the directrix determines the orthocentre.

3. In the central conic it is geometrically proved that (Fig. 26)

$$PG.PF = BC^2, Pg.PF = AC^2.$$

Now from the circles $NGFK$ and $ngFk$

$$\begin{aligned} \angle FGK = \angle FNK = \angle FnP, \text{ since } P, N, C, F, n \text{ are concyclic.} \\ = \angle Fgk. \end{aligned}$$

Hence GK is parallel to gk and the circles $FGNK, gFnk$ touch at F .

If CL be perpendicular to TPt the points C, n, L, P, N, F are concyclic, and the centre of the corresponding circle is the mid point of CP .

If this point be O , then $\angle OFP = \angle OPF = \angle CNF$.

Therefore OF touches the circle $NGFK$, and therefore $nFgk$.

As $On = OF = ON$, On and ON are also tangents. But NON is a straight line. Therefore nON is the other common tangent to the two circles.

These two circles are therefore cut by $CnLPNF$ orthogonally.

This theorem is quite independent of the ellipse except for the direction of TPt . If this be taken at pleasure, the theorem still holds, and from it a number of geometrical derivatives can be obtained.

(i.) The circle about F, N, K passes through G . This can be extended to similar circles derivable from the cyclic hexagon $CnLPNF$ whose opposite sides are parallel. Thus the point of intersection of Pn and CL is on the circle nL .

(ii.) Six circles of the series $FGNK$ are obtainable, each of which touches the two adjacent ones.

(iii.) Two sets of three circles corresponding to $knFg$ are obtainable. The circles of each set touch each other and four circles of group (ii.).

(iv.) O is the radical centre of all these circles.

Taking the pair of circles $FGNK, Fgkn$ we shall get another circle, the image of PNC in their axis, and on it a point P' the image of P .

If now CB, CA and P be fixed, the locus of F will be a circle on CP as diameter, and the locus of P' a cardioid with P as pole.