*J. Appl. Prob.* **58,** 1086–1113 (2021)
*doi:*10.1017/jpr.2021.24

# ON THE FINITENESS AND TAILS OF PERPETUITIES UNDER A LAMPERTI–KIU MAP

LARBI ALILI,* ** AND
DAVID WOODFORD,* *** *University of Warwick*

**Abstract**

Consider a Lamperti–Kiu Markov additive process $(J, \xi)$ on $\{+, -\} \times \mathbb{R} \cup \{-\infty\}$, where $J$ is the modulating Markov chain component. First we study the finiteness of the exponential functional and then consider its moments and tail asymptotics under Cramér's condition. In the strong subexponential case we determine the subexponential tails of the exponential functional under some further assumptions.

*Keywords:* Markov additive process; Lamperti–Kiu representation; self-similar Markov process; perpetuity

2010 Mathematics Subject Classification: Primary 60G18
Secondary 60J45; 91B70; 91B30

## 1. Introduction and preliminaries

Let $E := \{+, -\}$ and suppose $(\mathcal{F}_t)_{t\geq 0}$ is a filtration. A pair of processes $(J, \xi)$ taking values in $E \times \mathbb{R} \cup \{-\infty\}$ with lifetime $\chi$ is a Lamperti–Kiu Markov additive process (MAP) with respect to $(\mathcal{F}_t)_{t\geq 0}$ if, for any continuous bounded function $f \colon E \times \mathbb{R} \to \mathbb{R}^+$, $(z, y) \in E \times \mathbb{R}$ and $s, t \geq 0$, we have

$$\mathbb{E}_{z,y}[f(J_{t+s}, \xi_{t+s} - \xi_t); t + s < \chi \mid \mathcal{F}_t] = \mathbb{E}_{J_t,0}[f(J_s, \xi_s); s < \chi]\mathbb{1}_{\{t<\chi\}}, \tag{1.1}$$

where $\mathbb{P}_{z,y}$ is the law of $(J, \xi)$ started at $(z,y)$ and $\mathbb{E}_{z,y}$ is the corresponding expectation. For all $t > \chi$, the process $\xi$ is in the cemetery state $-\infty$, whilst $J$ continues as a Markov chain. A detailed account of MAPs is given in [2, Chapter XI] whilst a more general definition is given in [1, Section 3, Definition 1]. Note that $(J_t \exp(\xi_t), t \geq 0)$ is a càdlàg multiplicative process taking values in $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$ and so, following [7], we refer to it as a Lamperti–Kiu process.

There is a well-known construction of a Lamperti–Kiu process given in [7, Theorem 6(i)]. Let $\xi^{\pm}$ be two Lévy processes, let $\zeta^{\pm}$ be two exponentially distributed random variables with rates $q^{\pm}$, and let $U^{\pm}$ be two random variables taking values in $\mathbb{R}$. Then consider sequences $(\xi^{\pm,k})_{k\in\{0,1,2,\ldots\}}$, $(\zeta^{\pm,k})_{k\in\mathbb{N}}$, and $(U^{\pm,k})_{k\in\mathbb{N}}$ of independent and identically distributed (i.i.d.) copies of $\xi^{\pm}$, $\zeta^{\pm}$, and $U^{\pm}$, respectively. Under $\mathbb{P}_{\sigma,x}$, i.e. assuming $(J_0, \xi_0) = (\sigma, x)$, for each $k \in \mathbb{N}$ let $\xi^k = \xi^{\gamma,k}$, $\zeta^k = \zeta^{\gamma,k}$, and $U^k = U^{\gamma,k}$, where $\gamma = \sigma(-1)^k$. Finally, let $\chi$ be an exponentially distributed random variable of rate $q \in [0, \infty)$ independent of the rest of the system,

Received 13 December 2019; revision received 18 March 2021.
* Postal address: The University of Warwick, Coventry CV47AL, UK.
** Email address: l.alili@warwick.ac.uk
*** Email address: david.l.woodford@bath.edu

where $q = 0$ is interpreted to mean $\chi = \infty$. Then, for $t \geq 0$ and $x \in \mathbb{R}^* := \mathbb{R} \backslash \{0\}$, set

$$Y_t := xJ_t \exp(\xi_t), \quad t < \chi \tag{1.2}$$

with

$$\xi_t := \xi^{N_t}_{\pi_t} + \sum_{k=0}^{N_t - 1} \left(\xi^k_{\zeta^k} + U^k\right) \quad \text{and} \quad J_t := \sigma(-1)^{N_t},$$

where, for $n \in \mathbb{N}$,

$$T_0 := 0, \quad T_n := \sum_{k=0}^{n-1} \zeta^k, \quad N_t := \max_{m \in \mathbb{N}_0} \{T_m \leq t\} \quad \text{and} \quad \pi_t := t - T_{N_t}$$

using the notation $\mathbb{N}_0 := \{0, 1, 2 \cdots\}$. Then $(Y_t, t \geq 0)$ is a Lamperti–Kiu process and $(J, \xi)$ is the corresponding MAP. Conversely, any Lamperti–Kiu process has such a decomposition, which we will refer to as the Lamperti–Kiu decomposition. We will refer to $\chi$ as the lifetime of the Lamperti–Kiu process.

We study the standard and signed exponential functionals of $(Y_t, t \geq 0)$ defined, respectively, by

$$A_\infty := \int_0^\infty \exp(\xi_t)\, \mathrm{d}t \quad \text{and} \quad B_\infty := \int_0^\infty Y_t\, \mathrm{d}t. \tag{1.3}$$

Recall that a perpetuity is a security where a stream of cashflows is paid indefinitely, such as consols issued by the Bank of England. Under the MAP model, we suppose that the cashflows are paid continuously at a rate $c_t := \exp(\xi_t + rt)$ at time $t \geq 0$, where $r$ is the rate of interest. Each element of $E$ corresponds to a market state (for example, its states may refer to a *bear* or *bull* market), where the state at time $t \geq 0$ is given by $J_t$. The value of the perpetuity is given by $A_\infty$.

It was shown in [7] that $A_\infty$ is the first hitting time of zero by an associated self-similar Markov process (ssMp). Indeed, consider the Lamperti–Kiu MAP $(J, \xi)$ from (1.1), and for $\alpha \in (0, \infty)$ define the time transformation

$$\tau(t) := \inf\left\{u \geq 0 \colon \int_0^u \exp(\alpha \xi_s)\, \mathrm{d}s \geq t\right\}.$$

Then the process $X^{(x)}_t := J_{\tau(t|x|^{-\alpha})} x \exp(\xi_{\tau(t|x|^{-\alpha})})$, for $t < |x|^\alpha \int_0^\infty \exp(\alpha\xi_s)\, \mathrm{d}s$ and $x \in \mathbb{R}^*$, is a self-similar Markov process of index $\alpha$ taking values in $\mathbb{R}^*$ and started at $x$. That is, $X$ is a càdlàg Markov process such that, for all $c > 0$ and $x \in \mathbb{R}^*$, it satisfies

$$\left(cX^{(x)}_{c^{-\alpha}t}, t \geq 0\right) \overset{\mathcal{L}}{=} (X^{(cx)}_t, t \geq 0),$$

where $\overset{\mathcal{L}}{=}$ means equality in law. Moreover, any self-similar Markov process taking values in $\mathbb{R}^*$ of index $\alpha$ can be constructed in this way. The first hitting time of zero by $X^{(x)}$ is $\int_0^\infty \exp(\alpha\xi_s)\, \mathrm{d}s$, which equals $A_\infty$ when $\alpha = 1$. Many papers are devoted to the study of the Lamperti transform of self-similar Markov processes. For example see [5], [7], [17], [20],

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

and [19]. Other applications of MAPs and their exponential functionals include multi-type self-similar fragmentation processes and trees; for example see [27].

The focus of this paper is on the finiteness and right tails of $A_\infty$ and $B_\infty$. The right tails of a distribution determine which positive moments exist and whether it is a member of the classes of heavy-tailed, long-tailed, or subexponential distributions. These classes of heavy-tailed distribution are of particular interest to us because of their applications in finance, risk theory, and insurance (see e.g. [13], [24], and [25]). Empirical data often shows realised market returns to be heavy-tailed (see [8] and [14]). This motivates both considering Lamperti–Kiu processes with Lamperti–Kiu components that are heavy-tailed and studying $A_\infty$ as an example of a heavy-tailed distribution.

For Lévy processes, which are MAPs where $E$ is a singleton set, the theory of the exponential functional is well developed. Several results on the moments and tails of the exponential functional, including random affine equations, are collected in the survey [6]. Under Cramér's condition, with Cramér number $\theta$, it is shown that the right tails are polynomial with order $-\theta$. More recently, [23] provided a complete description at the logarithmic level of the asymptotic of the right tail and, under Cramér's condition, the derivatives of the density. In the heavy-tailed case, Cramér's condition fails and different methods are needed. In this case, the right tails of $A_\infty$ have been studied in, for example, [21], [22], [23], and [26]. The case of a MAP is studied in [18] where, under a Cramér-type condition with Cramér number $\theta \in (0, 1)$, moments of order $s \in (0, 1 + \theta)$ are shown to exist and satisfy a recurrence relation. This leads to polynomial tails similar to the Lévy case. The same recurrence relation is shown in [27] for the case when the additive component is not increasing. Other properties, including the finiteness and integer moments of the exponential functional of non-increasing MAPs, are also given in [27].

We will use the law of large numbers and Erickson's law of large numbers for when the mean is undefined [10], to give a characterisation of the finiteness of $A_\infty$. Then we show that, for a Lamperti–Kiu MAP, both $A_\infty$ and $B_\infty$ satisfy a random affine equation. Under a Cramér-type condition, we show that the conditions of the implicit renewal theorems of [12] and [16] hold, and hence we are able to determine the right tails of $A_\infty$ and $B_\infty$. In the heavy-tailed cases, when Cramér's condition does not hold, a different approach to studying the tails of $A_\infty$ is required. We define a Lamperti–Kiu process to be of strong subexponential type when $Y_{T_2}$ is long-tailed and one of the Lamperti–Kiu components, $\xi_1^{(\pm)}$ or $U^{(\pm)}$, is strong subexponential and has right tails that asymptotically dominate the right tails of the other Lamperti–Kiu components. By careful consideration of an embedded Markov chain, we are able to overcome the lack of independent increments of $\xi$ and provide a generalisation of the subexponential results of [21] to Lamperti–Kiu processes of strong subexponential type. This provides an asymptotic expansion of the right tails of $A_\infty$ to show that $A_\infty$ is long-tailed and $\log (A_\infty)$ is subexponential.

This paper is organised as follows. In Section 2 we give necessary and sufficient conditions for the standard and signed exponential functionals $A_\infty$ and $B_\infty$, respectively, to be finite. In Section 3 we look at the random affine equation approach to studying the moments and tails of $A_\infty$ and $B_\infty$ under Cramér's condition and the assumption that $Y_{T_2}$ does not have a lattice distribution. In Section 4 we study the tails of $A_\infty$ when the Lamperti–Kiu process is of strong subexponential type.

## 2. Finiteness of $A_\infty$ and $B_\infty$

Let us keep the mathematical setting of the Introduction, where $(Y_t, t \geq 0)$ is the Lamperti–Kiu process defined in (1.2) associated with the Lamperti–Kiu MAP $(J, \xi)$ given, for a fixed $t \geq 0$, by $\xi_t := \log |Y_t|$ and $J_t := \text{sgn}\,(Y_t)$.

If possible, define $K \in \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$K := \frac{\mathbb{E}[\xi_{T_2}]}{\mathbb{E}[T_2]} = \frac{q^-}{q^+ + q^-}(\mathbb{E}[\xi_1^+] + q^+\mathbb{E}[U^+]) + \frac{q^+}{q^+ + q^-}(\mathbb{E}[\xi_1^-] + q^-\mathbb{E}[U^-]),$$

where we allow $K$ to take the values $+\infty$ and $-\infty$ but if both $\mathbb{E}[\max(\xi_{T_2}, 0)] = \infty$ and $\mathbb{E}[\max(-\xi_{T_2}^-, 0)] = \infty$ we say that $K$ is undefined.

A Lamperti–Kiu process $Y$ will be called *degenerate* if $Y$ is such that $\limsup_{t\to\infty} |\xi_t| < \infty$. This can be shown to be equivalent to the case when either $Y$ has a finite lifetime or $\xi_t^\pm \equiv 0$ for all $t \geq 0$ and $U^+ = -U^-$ is deterministic, hence

$$Y_t^{(x)} = \begin{cases} x & \text{if } T_{2k} \leq t < T_{2k+1} \text{ for some } k \in \mathbb{N}_0, \\ x\exp(U^{\text{sgn}\,(x)}) & \text{if } T_{2k+1} \leq t < T_{2k+2} \text{ for some } k \in \mathbb{N}_0, \end{cases}$$

for all $t \geq 0$ and $x \in \mathbb{R}^*$.

When $K$ is defined and $Y$ has an infinite lifetime from [2, Proposition 2.10] and [2, Corollary 2.8], it is known that almost surely $\lim_{t\to\infty} t^{-1}\xi_t = K$. Also, if $K = 0$ and $Y$ is non-degenerate, then $\lim_{t\to\infty} t^{-1}\xi_t = 0$, $\liminf_{t\to\infty} \xi_t = -\infty$, and $\limsup_{t\to\infty} \xi_t = \infty$, almost surely.

In the case when $K$ is undefined, we will use Erickson's theorem [10, Theorem 2], which provides a strong law of large numbers for a random walk with an undefined mean. The following lemma provides an analogous result to Erickson's theorem for MAPs.

First we define

$$m_-(x) := \int_{-x}^{0} \mathbb{P}(\xi_{T_2} \leq y)\,\mathrm{d}y, \quad m_+(x) := \int_0^x \mathbb{P}(\xi_{T_2} > y)\,\mathrm{d}y$$

and

$$I_+ := \int_0^\infty \frac{x}{m_-(x)}\mathbb{P}(\xi_{T_2} \in \mathrm{d}x), \quad I_- := \int_{-\infty}^0 \frac{|x|}{m_+(|x|)}\mathbb{P}(\xi_{T_2} \in \mathrm{d}x).$$

Then the long-term behaviour of $(\xi_t, t \geq 0)$ is described by the following lemma.

**Lemma 2.1.** *Suppose that $K$ is undefined. Then at least one of $I_+$ and $I_-$ equals infinity, and almost surely we have:*

(i) $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ *if and only if* $I_+ = \infty$,

(ii) $\lim_{t\to\infty} t^{-1}\xi_t = \infty$ *if and only if* $I_+ = \infty$ *and* $I_- < \infty$,

(iii) $\liminf_{t\to\infty} t^{-1}\xi_t = -\infty$ *if and only if* $I_- = \infty$,

(iv) $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$ *if and only if* $I_+ < \infty$ *and* $I_- = \infty$.

*Proof.* Consider the sequence $\{\xi_{T_{2n}}\}_{n\in\mathbb{N}}$ as the random walk

$$\xi_{T_{2n}} = \sum_{k=1}^{n} (\xi_{T_{2k}} - \xi_{T_{2k-2}}),$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

and notice that $\xi_{T_{2n}} - \xi_{T_{2(n-1)}} \stackrel{\mathcal{L}}{=} \xi_{T_2}$ has an undefined mean, for each $n \in \mathbb{N}$. Then Erickson's theorem for random walks with undefined mean [10, Theorem 2], and the remark that follows it, states that either $I_+ = \infty$ or $I_- = \infty$ or both hold, proving the first statement of the lemma. Furthermore, the following statements hold:

(1) $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ a.s. if and only if $I_+ = \infty$,

(2) $\lim_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ a.s. if and only if $I_+ = \infty$ and $I_- < \infty$,

(3) $\liminf_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s. if and only if $I_- = \infty$,

(4) $\lim_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s. if and only if $I_+ < \infty$ and $I_- = \infty$,

and similar statements hold for $\{T_{2n+1}\}_{n\in\mathbb{N}}$.

Since $\mathbb{E}[T_2] < \infty$, it is immediate that if $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ a.s., then $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ a.s. If $\liminf_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s., then $\liminf_{t\to\infty} t^{-1}\xi_t = -\infty$ a.s., hence the 'if' direction of statements (i) and (iii) holds. To prove the 'only if' direction of (i) and (iii), we must first prove (ii) and (iv).

Consider (iv) and notice that the 'only if' direction is an immediate consequence of statement (4) above. Now suppose $I_+ < \infty$ and $I_- = \infty$. Then $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s. and $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n+1}} = -\infty$ a.s., hence $\limsup_{n\to\infty} n^{-1}\xi_{T_n} = -\infty$ a.s. also. Suppose for a contradiction there exists an $R > 0$ such that $\limsup_{t\to\infty} t^{-1}\xi_t > -R > -\infty$ a.s. Since $\lim_{n\to\infty} n^{-1}T_n = \frac{1}{2}\mathbb{E}[T_2]$ a.s. and $\limsup_{n\to\infty} n^{-1}\xi_{T_n} = -\infty$ a.s., there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $T_n > 1$ and

$$\frac{\xi_{T_n}}{T_n} < -2R \quad \text{a.s.}$$

Define a sequence $(\tau_n)_{n\in\mathbb{N}}$ of times and $(x_n)_{n\in\mathbb{N}}$ of values such that, for each $n \in \mathbb{N}$, we have

$$\tau_n = \sup\left\{t \in [T_n, T_{n+1}) \colon \xi_t = \sup_{s\in[T_n,T_{n+1})} \xi_s\right\}$$

and

$$x_n = \sup\{\xi_t \colon t \in [T_n, T_{n+1})\}.$$

Since $\limsup_{t\to\infty} t^{-1}\xi_t > -R$ a.s., there is an increasing sequence of times $(s_n)_{n\in\mathbb{N}}$ such that $s_n^{-1}\xi_{s_n} > -R$ a.s. for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = \infty$. Then we may take a subsequence $(s'_n)_{n\in\mathbb{N}}$ such that there is at most one element of the sequence $(s'_n)_{n\in\mathbb{N}}$ in each interval $[T_m, T_{m+1}]$ and $J_{s'_n}$ is constant.

Let $(\tau_{k_n})_{n\in\mathbb{N}}$ be a subsequence of $(\tau_{n\in\mathbb{N}})$ such that $k_n > N$ and $s'_n \in [T_{k_n}, T_{k_n+1}]$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have

$$x_{k_n} \geq \xi_{s'_n} > -Rs'_n \geq -RT_{k_n+1},$$

whilst $\xi_{T_{k_n+1}} < -2RT_{k_n+1}$ and $T_{k_n+1} > 1$, therefore

$$x_{k_n} - \xi_{T_{k_n}+1} > RT_{k_n+1} > R$$

almost surely. However, for each $n \in \mathbb{N}$, by the Lamperti–Kiu decomposition,

$$\{\xi_{T_{k_n}+t} - \xi_{T_{k_n}} \colon t < T_{k_n+1} - T_{k_n}\}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

is a Lévy process, so, by splitting at the last time of the maximum [4, Chapter VI, Theorem 5], $x_{k_n} - \xi_{T_{k_n+1}}$ is independent of $x_{k_n}$ and has the same distribution as $x_0 - \xi_{T_1}$, hence its distribution does not depend on $R$. This contradicts the fact that it only has support on $(R, \infty)$. Hence $I_+ < \infty$ and $K$ undefined imply that $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$ a.s. and so the 'if' direction of (iv) holds. By applying similar arguments to $-\xi_t$, statement (iii) of the lemma also holds.

To prove the 'only if' direction of (i), suppose $I_+ < \infty$. Since $K$ is undefined, by [10, Theorem 2] and the remark that follows it, we have $I_- = \infty$. Then, by statement (iv), we have $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$ a.s. Hence $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ a.s. only if $I_+ = \infty$. The argument for (iii) is analogous. □

The following theorem shows that in the infinite lifetime case, the convergence and finiteness of $A_\infty$ and $B_\infty$ are fully characterised by $K$ when this is defined and by the finiteness of $I_+$ otherwise.

**Theorem 2.1.** *Suppose $Y$ has an infinite lifetime. Then $A_\infty$ converges a.s. if and only if $B_\infty$ converges a.s. Moreover, $A_\infty$ and $B_\infty$ converge almost surely if and only if either $K$ is defined and $K < 0$ or $K$ is undefined and $I_+ < \infty$.*

Before proving Theorem 2.1, we prove the following preliminary lemma.

**Lemma 2.2.** *If $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s., then both $A_\infty$ and $B_\infty$ diverge a.s.*

*Proof.* If $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s., then there exists a strictly increasing sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\exp(\xi_{T_{2\tau_n}}) \geq 1$ a.s. for all $n \in \mathbb{N}$. First, by considering $A_\infty$ and using the Markov property for the second inequality, we have

$$A_\infty \geq \sum_{n=0}^{\infty} \exp\left(\xi_{T_{2\tau_n}}\right) \int_{T_{2\tau_n}}^{T_{2\tau_n+2}} \exp\left(\xi_t - \xi_{T_{2\tau_n}}\right) \mathrm{d}t \geq \sum_{n=0}^{\infty} \int_0^{T_2^{(n)}} \exp\left(\xi_t^{(n)}\right) \mathrm{d}t.$$

Since the terms of this series form an i.i.d. sequence with strictly positive values, the series must diverge.

Similarly, $B_\infty$ converges a.s. only if the sum $\sum_{n=0}^{\infty} \int_{T_{2n}}^{T_{2n+2}} J_t \exp(\xi_t)\, \mathrm{d}t$ converges a.s., which implies the a.s. convergence to zero of the subsequence

$$\begin{aligned}\left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp(\xi_t)\, \mathrm{d}t\right| &= \exp(\xi_{T_{2\tau_n}})\left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp(\xi_t - \xi_{T_{2\tau_n}})\, \mathrm{d}t\right| \\ &\geq \left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp(\xi_t - \xi_{T_{2\tau_n}})\, \mathrm{d}t\right|.\end{aligned}$$

Then, by using the Markov property, $B_\infty$ converges a.s. only if

$$\left|\int_0^{T_2^{(n)}} J_t^{(n)} \exp(\xi_t^{(n)})\, \mathrm{d}t\right| \to 0 \quad \text{a.s.}$$

as $n \to \infty$. This convergence is impossible since we are dealing with an i.i.d. sequence that does not converge to zero in distribution. □

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

*Proof of Theorem 2.1.* We will consider the cases when $K$ is defined and undefined separately.

1. Suppose that $K$ is defined and $K<0$. Then recall that $\lim_{t\to\infty} t^{-1}\xi_t = K$ a.s. as $t\to\infty$. If $-\infty<K<0$ let $k=\frac{1}{2}K$ and if $K=-\infty$ let $k=-1$. Then a.s. there exists a finite $T\geq 0$ such that $\xi_t < kt < 0$ for all $t>T$. Thus $A_\infty \leq \int_0^T \exp(\xi_t)\,\mathrm{d}t + k^{-1}\,\mathrm{e}^{kT} < \infty$ a.s., and by absolute convergence it is then immediate that $B_\infty$ also converges a.s.

Next we consider the case when either $K>0$ or both $K=0$ and $Y$ is non-degenerate. Then, by [15, Proposition 9.14], $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s., so the result follows from Lemma 2.2. If $K=0$ and $Y$ is degenerate, then, since $Y$ has an infinite lifetime, we have

$$\xi_t = \begin{cases} 0 & \text{if } T_{2k} \leq t < T_{2k+1} \text{ for } k\in\mathbb{N}_0, \\ U^{J_0} & \text{if } T_{2k+1} \leq t < T_{2k+2} \text{ for } k\in\mathbb{N}_0. \end{cases}$$

Thus, for all $t\geq 0$, we have

$$\mathrm{e}^{\xi_t} \geq \min(1, \exp(U^{J_0})) := V > 0,$$

so $A_\infty = \infty$ a.s. Also, $B_\infty$ can be written as the sum

$$B_\infty = \sum_{n=0}^{\infty}\left(\int_{T_{2n}}^{T_{2n+1}} J_t\,\mathrm{e}^{\xi_t}\,\mathrm{d}t + \int_{T_{2n+1}}^{T_{2n+2}} J_t\,\mathrm{e}^{\xi_t}\,\mathrm{d}t\right) = J_0 \sum_{n=0}^{\infty}\left(\zeta^{2n} - \exp\left(U^{J_0}\right)\zeta^{2n+1}\right),$$

and since $\zeta^{2n} - \exp(U^{J_0})\zeta^{2n+1}$ does not converge to zero in distribution, $B_\infty$ must a.s. diverge.

2. Suppose that $K$ is undefined. From [10, Theorem 2] we know that if $I_+ = \infty$ then $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s. Hence, by using Lemma 2.2, both $A_\infty$ and $B_\infty$ diverge a.s.

If $I_+ < \infty$, then since $K$ is undefined, as a consequence of [10, Theorem 2], $I_- = \infty$ and so by Lemma 2.1 we have $\limsup_{t\to\infty} t^{-1}\xi_t = -\infty$ a.s. Then, by the argument of case 1 above, both $A_\infty$ and $B_\infty$ converge a.s. as desired. □

## 3. Moments and tail asymptotics of $A_\infty$ and $B_\infty$

Throughout this section we assume the Lamperti–Kiu process has an infinite lifetime. For $z\in\mathbb{C}$ suppose the characteristic exponents $\psi_\pm(z) := \log(\mathbb{E}[\exp(z\xi_1^\pm)])$ and Laplace transforms $G^\pm(z) := \mathbb{E}[\exp(zU^\pm)]$ exist and are finite. Then the matrix exponent of $Y$ is defined to be

$$F(z) := \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+G^+(z) \\ q_-G^-(z) & -q_- \end{pmatrix},$$

and from [2, XI.2b] it is known that, for $l, j\in\{+,-\}$ and $z\in\mathbb{C}$,

$$\mathbb{E}[\mathrm{e}^{z\xi_t}; J_t = j \mid J_0 = l] = \left(\mathrm{e}^{tF(z)}\right)_{l,j}. \tag{3.1}$$

Then let $\lambda(z)$ denote the eigenvalue of $F(z)$ with largest real part. Using Perron–Frobenius theory, it is shown in [18, Proposition 3.2] that such an eigenvalue is guaranteed to be simple, real, and continuous as a function of $z$. From [18, Proposition 3.4] we also know that $\lambda(z)$ is convex when considered as a map $\lambda\colon \mathbb{R}\to\mathbb{R}$. Provided $F$ exists in a neighbourhood of zero, from [2, Corollary 2.9] it is known that $K=\lambda'(0)\in[-\infty,\infty]$, where the derivative is considered on this restriction.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

It can be shown that the integrals $A_\infty$ and $B_\infty$ are the solutions of random affine equations. We will assume $q_+, q_- > 0$ so that $T_2 < \infty$ and note that $T_2$ is independent of $Y_0$. Then, for any $T \in [0, \infty]$, define the random variables

$$A_T := q \int_0^T \mathrm{e}^{\xi_t}\,\mathrm{d}t \quad \text{and} \quad B_T := q \int_0^T J_t\,\mathrm{e}^{\xi_t}\,\mathrm{d}t,$$

and notice that $A_\infty = \lim_{T\to\infty} A_T$ and $B_\infty = \lim_{T\to\infty} B_T$. Similarly, due to the result in [5, Section 4.3], by the Markov additive property

$$B_\infty = B_{T_2} + Y_{T_2}\hat{B}_\infty,$$

where $\hat{B}_\infty$ is an independent copy of $B_\infty$.

Notice that $Y_{T_2}$ has the same sign as $Y_0$ and that $|Y_{T_2}|$ is independent of $J_0$ because of its symmetry in the components of the decomposition of $Y$. Similarly

$$A_\infty = A_{T_2} + Y_{T_2}\hat{A}_\infty,$$

where $\hat{A}_\infty$ is an independent but identically distributed copy of $A_\infty$ and is independent of $Y_{T_2}$ and $A_{T_2}$.

The following results are generalisations of [5, Corollary 5] to Lamperti–Kiu processes using the implicit renewal theorems given in [12, Theorem 4.1] and [16, Theorem 5]. In the case of a Lévy process, it has been shown in [23] that the coefficient $c_A$ of the proposition below can be found explicitly by evaluating a Bernstein-gamma function.

**Proposition 3.1.** *Suppose $Y$ is an infinite lifetime Lamperti–Kiu process with $K < 0$ and there is a $\kappa > 0$ such that $F(\kappa)$ exists, $\lambda(\kappa) = 0$ and*

$$\mathbb{E}[|Y_{T_2}|^\kappa \log^+ |Y_{T_2}|] < \infty. \tag{3.2}$$

*If $Y_{T_2}$ does not have a lattice distribution, then there exist constants $c_A, c_B^+, c_B^- \in \mathbb{R}$ such that*

$$c_A := \lim_{t\to\infty} t^\kappa \mathbb{P}(A_\infty > t), \quad c_B^+ := \lim_{t\to\infty} t^\kappa \mathbb{P}(B_\infty > t) \quad \textit{and} \quad c_B^- := \lim_{t\to\infty} t^\kappa \mathbb{P}(B_\infty < -t),$$

*and therefore $A_\infty, B_\infty$ have moments of order $s \in \mathbb{C}^+ := \{x \in \mathbb{C} \colon \Re(x) \ge 0\}$ if and only if $0 \le \Re(s) < \kappa$.*

*Proof.* Since the proposition assumes (3.2), the result is an immediate consequence of [12, Theorem 2.3] and [16, Section 4, Theorem 5] provided that

$$\mathbb{E}[\log |Y_{T_2}|] < 0, \tag{3.3}$$
$$\mathbb{E}[|Y_{T_2}|^\kappa] = 1, \tag{3.4}$$
$$0 < \mathbb{E}[|A_{T_2}|^\kappa] < \infty, \tag{3.5}$$
$$0 < \mathbb{E}[|B_{T_2}|^\kappa] < \infty. \tag{3.6}$$

We now prove that under the conditions of the proposition each of these equations holds.

To show (3.3), we expand $\log |Y_{T_2}|$ to get

$$\mathbb{E}[\log |Y_{T_2}|] = \frac{1}{q^+}\mathbb{E}[\xi_1^+] + \frac{1}{q^-}\mathbb{E}[\xi_1^-] + \mathbb{E}[U^+] + \mathbb{E}[U^-] = \left(\frac{1}{q_+} + \frac{1}{q_-}\right)K,$$

and since $q_+^{-1} + q_-^{-1} > 0$, (3.3) follows from the assumption $K < 0$.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

Since $\det(F(z)) = (\psi^+(z) - q^+)(\psi^-(z) - q^-) - q^+q^-G^+(z)G^-(z)$ and by the assumption that 0 is an eigenvalue of $F(\kappa)$, we have

$$1 = \left(\frac{q^+G^+(\kappa)}{\psi^+(\kappa) - q^+}\right)\left(\frac{q^-G^-(\kappa)}{\psi^-(\kappa) - q^-}\right). \tag{3.7}$$

Let $\mu(z)$ be the other eigenvalue of $F(z)$. Then, for all real $z \in (0, \kappa)$, by assumption, $\mu(z) < \lambda(z) < 0$ and so $0 < \mu(z)\lambda(z) = \det(F(z))$. Rearranging this gives $(\psi^+(z) - q^+)(\psi^-(z) - q^-) > 0$, so $\psi^\pm(z) - q^\pm$ has no roots in $(0, \kappa)$. Since $\psi^\pm(0) - q^\pm < 0$, by continuity, $\psi^\pm(z) - q^\pm < 0$ for all $z \in (0, \kappa)$. By independence and using (3.7), we get

$$\mathbb{E}[|Y_{T_2}|^\kappa] = \mathbb{E}[\exp(\psi^+(\kappa)\zeta^+)]G^+(\kappa)\,\mathbb{E}[\exp(\psi^-(\kappa)\zeta^-)]G^-(\kappa) = 1,$$

and hence (3.4) holds.

Using independence and the inequality $(a+b)^x \le 2^x(a^x + b^x)$ for $a, b, x > 0$, we get

$$2^{-\kappa}\mathbb{E}[|A_{T_2}|^\kappa] \le \mathbb{E}\left[\left(\int_0^{\zeta^1} \exp(\xi_s^1)\,\mathrm{d}s\right)^\kappa\right] + \mathbb{E}\big[\exp\big(\kappa\big(\xi^1_{\zeta^1} + U^1\big)\big)\big]\,\mathbb{E}\left[\left(\int_0^{\zeta^2} \exp(\xi_s^2)\,\mathrm{d}s\right)^\kappa\right].$$

From [5] it is known that

$$\mathbb{E}\left[\left(\int_0^{\zeta_\pm} \exp(\xi_s^\pm)\,\mathrm{d}s\right)^x\right] < \infty$$

for all $x \in (0, \infty)$ such that $\psi^\pm(x) - q^\pm < 0$. This follows from the fact that $\int_0^{\zeta^\pm} \exp(\xi_s^\pm)\,\mathrm{d}s$ is the exponential functional of the Lévy process $\xi^{(\pm)}$ sent to the cemetery state $-\infty$ at an independent, exponentially distributed time of rate $q^\pm$. Then, since we have already seen that $\psi^\pm(\kappa) - q^\pm < 0$, it follows that

$$\mathbb{E}\left[\left(\int_0^{\zeta^\pm} \exp(\xi_s^\pm)\,\mathrm{d}s\right)^\kappa\right] < \infty.$$

By the assumption that $F(\kappa)$ exists, we have $\mathbb{E}[\exp(\kappa U^\pm)] < \infty$, whilst $\mathbb{E}[\exp(\kappa\xi^\pm_{\zeta_\pm})] < \infty$ follows easily from $\mathbb{E}[\exp(\kappa\xi_t^\pm)] = \exp(t\psi_\pm(\kappa))$ for $t \ge 0$. Hence $\mathbb{E}[|B_{T_2}|^\kappa] \le \mathbb{E}[|A_{T_2}|^\kappa] < \infty$ and so equations (3.5) and (3.6) hold. □

**Remark 3.1.** In the case when $G^\pm$ are continuous, (3.2) is automatic. Indeed, by continuity, we can pick $\epsilon > 0$ such that $\psi^\pm(\kappa+\epsilon) - q^\pm < 0$ and $G^\pm(\kappa+\epsilon) < \infty$. Then, from the proof of (3.4), we obtain

$$\mathbb{E}[|Y_{T_2}|^{\kappa+\epsilon}] = \left(\frac{q^+G^+(\kappa+\epsilon)}{q^+ - \psi^+(\kappa+\epsilon)}\right)\left(\frac{q^-G^-(\kappa+\epsilon)}{q^- - \psi^-(\kappa+\epsilon)}\right) < \infty.$$

Since $\log(x)^+ < x^\epsilon$ for all $x \ge R$, for some sufficiently large $R > 0$ we have

$$\begin{aligned}\mathbb{E}[|Y_{T_2}|^\kappa \log Y_{T_2}^+] &= \mathbb{E}[|Y_{T_2}|^\kappa \log Y_{T_2}^+; |Y_{T_2}| \le R] + \mathbb{E}[|Y_{T_2}|^\kappa \log Y_{T_2}^+; |Y_{T_2}| > R]\\ &\le R^{\kappa+\epsilon} + \mathbb{E}[|Y_{T_2}|^{\kappa+\epsilon}]\\ &< \infty.\end{aligned}$$

Hence (3.2) holds.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

## 4. Subexponential tails of $A_\infty$

When the conditions of Proposition 3.1 do not hold, a different approach to the investigation of the tails and moments of $A_\infty$ is required. In Proposition 3.1 it is assumed that $F(\kappa)$ exists, which requires that positive exponential moments of $\xi$ must exist. This is a condition that does not necessarily hold in general, and in particular, when $\xi_{T_2}$ is heavy-tailed.

In this section we will define Lamperti–Kiu processes of strong subexponential type and study the right tails of the exponential functional of such processes. The main result of this section is Theorem 4.1, which shows, under some conditions, that the exponential functional, $A_\infty$, is itself long-tailed and $\log(A_\infty)$ is subexponential.

First we will prove two preliminary lemmas in Section 4.1. In Section 4.2 a framework is presented for bounding $\log(A_\infty)$ by considering a piecewise linear bound for $\{\xi_t : t \geq 0\}$ with a.s. finitely many discontinuities. Finally, in Section 4.3, we define Lamperti–Kiu processes of strong subexponential type and use the framework from Section 4.2, in conjunction with the subexponential properties, to obtain the right tails of $A_\infty$. Interestingly, the resulting tails are of a very different nature to those considered under Cramér's condition.

### 4.1. Preliminary results

We will need the following lemma, which bounds the probability distribution of the supremum of a Lévy process over an exponentially distributed interval of time in terms of the distribution of the Lévy process at the end of the interval. It is a variation of [28, Lemma 1], where the time interval considered was fixed.

**Lemma 4.1.** *Let $X$ be a Lévy process, let $\tau$ be an independent exponentially distributed random variable, and suppose $0 < u_0 < u$. Then*

$$\mathbb{P}\left(\sup_{0\leq s<\tau} X_s > u\right) \leq \frac{\mathbb{P}(X_\tau \geq u-u_0)}{\mathbb{P}(X_\tau \geq -u_0)}. \tag{4.1}$$

*Proof.* Let $S_u := \inf\{t \geq 0 : X_t > u\}$. Then, since $X_{S_u} \geq u$, by independence of increments of $X$ and the memoryless property of $\tau$, we have

$$\begin{aligned}\mathbb{P}(S_u < \tau; X_\tau < u-u_0) &\leq \mathbb{P}(S_u < \tau; X_\tau - X_{S_u} < -u_0)\\ &= \mathbb{P}(S_u < \tau; \tilde{X}_{\tilde{\tau}} < -u_0)\\ &= \mathbb{P}(S_u < \tau)\,\mathbb{P}(X_\tau < -u_0),\end{aligned}$$

where $\tilde{X}$ and $\tilde{\tau}$ are independent and identically distributed copies of $X$ and $\tau$, respectively. Then (4.1) can be obtained by rearranging the inequality

$$\begin{aligned}\mathbb{P}(S_u < \tau) &\leq \mathbb{P}(X_\tau \geq u-u_0) + \mathbb{P}(S_u < \tau; X_\tau < u-u_0)\\ &\leq \mathbb{P}(X_\tau \geq u-u_0) + \mathbb{P}(S_u < \tau)\,\mathbb{P}(X_\tau < -u_0).\end{aligned}$$ □

We now consider long-tailed distributions. Let $Q : \mathbb{R}^+ \to [0, 1]$ be a probability distribution and $\bar{Q}(x) := 1 - Q(x)$ for all $x \in \mathbb{R}^+$. Then $Q$ is a long-tailed distribution if $Q(x+y)/Q(x) \to 1$ as $x \to \infty$, for any $y \in \mathbb{R}^+$. For any two functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, we will write $f \sim g$ if $\lim_{x\to\infty} f(x)/g(x) = 1$ and say that $f$ and $g$ are asymptotically equivalent.

For a random variable $X$, we define the functions

$$G_X(x) := \int_x^\infty \mathbb{P}(X > u)\,\mathrm{d}u \quad \text{and} \quad H_X(x) := \min(1, G_X(x))$$

and refer to $H_X$ as the integrated tail of $X$.

In the next lemma we show that the integrated tail of a long-tailed random variable is asymptotically equivalent to an infinite series. This will be used in Lemma 4.6 to show the asymptotic equivalence of the integrated tail $G_X$ and an infinite series that we will construct.

**Lemma 4.2.** *Suppose $K+\epsilon < 0$ and that $X$ is a long-tailed random variable, which is independent of $(T_{2n})_{n\in\mathbb{N}}$. Then the integrated tail of $X$ has the asymptotics*

$$\int_x^\infty \mathbb{P}(X > u)\,\mathrm{d}u \sim \mathbb{E}[T_2]|K+\epsilon| \sum_{n=0}^\infty \mathbb{P}(X > x - (K+\epsilon)T_{2n}). \tag{4.2}$$

*Proof.* By splitting the interval $(0, \infty)$ into a disjoint union, we can write

$$G_X(x) = \sum_{n=0}^\infty \int_{-T_{2n}(K+\epsilon)}^{-T_{2(n+1)}(K+\epsilon)} \mathbb{P}(X > u_1 + x)\,\mathrm{d}u_1.$$

Then, by using the change of variables $u = -(K+\epsilon)^{-1}u_1$, we get

$$G_X(x) = \sum_{n=0}^\infty \int_{T_{2n}}^{T_{2(n+1)}} \mathbb{P}(X > -(K+\epsilon)u + x)\,|K+\epsilon|\,\mathrm{d}u.$$

By independence of $\{T_{2n}\}_{n\in\mathbb{N}}$ and $X$, we can shift the domain of integration to obtain

$$G_X(x) = \sum_{n=0}^\infty \int_0^{T_{2(n+1)}-T_{2n}} \mathbb{P}(X > x - (K+\epsilon)(u + T_{2n}) \mid T_{2n})\,|K+\epsilon|\,\mathrm{d}u.$$

Taking expectations and noting that the left-hand side is not random, that $T_{2(n+1)} - T_{2n} \stackrel{\mathcal{L}}{=} T_2$, and that $T_{2(n+1)} - T_{2n}$ is independent of $T_{2n}$ gives

$$G_X(x) = |K+\epsilon| \sum_{n=0}^\infty \mathbb{E}\left[\int_0^{\tilde{T}_2} \mathbb{P}(X > x - (K+\epsilon)(u+T_{2n}) \mid T_{2n})\,\mathrm{d}u\right],$$

where $\tilde{T}_2$ is an independent and identically distributed copy of $T_2$. This can be written in the integral form

$$G_X(x) = |K+\epsilon| \sum_{n=0}^\infty \int_0^\infty \mathbb{P}(\tilde{T}_2 \in \mathrm{d}s) \int_0^s \int_0^\infty \mathbb{P}(T_{2n} \in \mathrm{d}v)\,\mathbb{P}(X > x - (K+\epsilon)(u+v))\,\mathrm{d}u. \tag{4.3}$$

Let $\delta > 0$; then, since $X$ is long-tailed, for all $s > 0$ there exists $R(s) > 0$ such that, whenever $z > R(s)$ and $y \in [0, -s(K+\epsilon)]$,

$$(1-\delta) \le \frac{\mathbb{P}(X > z + y)}{\mathbb{P}(X > z)} \le (1+\delta),$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

and since $-v(K+\epsilon)\geq 0$ for $v\geq 0$, we have for all $x>R(s)$ and $u\in[0, s]$

$$(1-\delta)\leq \frac{\mathbb{P}(X>x-(K+\epsilon)(v+u))}{\mathbb{P}(X>x-(K+\epsilon)v)}\leq(1+\delta).$$

To show the lower bound we use this inequality within the last two integrals of (4.3) to obtain, for $x>R(s)$,

$$\int_0^s\int_0^\infty \mathbb{P}(T_{2n}\in \mathrm{d}v)\,\mathbb{P}(X>x-(K+\epsilon)(u+v))\,\mathrm{d}u$$
$$\geq \int_0^s\int_0^\infty \mathbb{P}(T_{2n}\in \mathrm{d}v)(1-\delta)\,\mathbb{P}(X>x-(K+\epsilon)v)\,\mathrm{d}u.$$

Then, evaluating the integrals and noticing that the integrand is constant with respect to $u$ gives, for $x>R(s)$,

$$\int_0^s\int_0^\infty \mathbb{P}(T_{2n}\in \mathrm{d}v)\,\mathbb{P}(X>x-(K+\epsilon)(u+v))\,\mathrm{d}u\geq s(1-\delta)\,\mathbb{P}(X>x-(K+\epsilon)T_{2n}).$$

Now consider some $l>0$ and suppose $x>R(l)$, so that $x>R(s)$ for any $s\in[0, l]$. Then

$$\int_0^\infty \mathbb{P}(T_2\in \mathrm{d}s)\int_0^s\int_0^\infty \mathbb{P}(T_{2n}\in \mathrm{d}v)\,\mathbb{P}(X>x-(K+\epsilon)(u+v))\,\mathrm{d}u$$
$$\geq \int_0^l \mathbb{P}(T_2\in \mathrm{d}s)s(1-\delta)\,\mathbb{P}(X>x-(K+\epsilon)T_{2n})$$
$$=(1-\delta)\,\mathbb{P}(X>x-(K+\epsilon)T_{2n})\,\mathbb{E}[T_2; T_2<l].$$

Since $l>0$ and $\delta>0$ are arbitrary and $T_2$ is integrable, we can take $l$ sufficiently large to obtain $\mathbb{E}[T_2; T_2<l]\geq(1-\delta)\,\mathbb{E}[T_2]$ and thus

$$\int_0^\infty \mathbb{P}(T_2\in \mathrm{d}s)\int_0^s\int_0^\infty \mathbb{P}(T_{2n}\in \mathrm{d}v)\,\mathbb{P}(X>x-(K+\epsilon)(u+v))\,\mathrm{d}u$$
$$\geq(1-\delta)^2\,\mathbb{P}(X>x-(K+\epsilon)T_{2n})\,\mathbb{E}[T_2].$$

If this is substituted into the expression for $G_X$ for $x>R(l)$, we have

$$G_X(x)\geq(1-\delta)^2|K+\epsilon|\,\mathbb{E}[T_2]\sum_{n=0}^{\infty}\mathbb{P}(X>x-(K+\epsilon)T_{2n}).$$

For the upper bound, since $-(K+\epsilon)u>0$ for $u>0$,

$$\mathbb{E}\Bigg[\int_0^{\tilde{T}_2}\mathbb{P}(X>x-(K+\epsilon)(u+T_{2n})\mid\sigma(T_{2n}))\,\mathrm{d}u\Bigg]\leq\mathbb{E}[\tilde{T}_2]\,\mathbb{P}(X>x-(K+\epsilon)T_{2n}),$$

which, substituted into the expression for $G_X$, gives, for all $x>0$,

$$G_X(x)\leq\mathbb{E}[T_2]|K+\epsilon|\sum_{n=0}^{\infty}\mathbb{P}(X>x-(K+\epsilon)T_{2n}).$$

Combining the upper and lower bounds gives (4.2). □

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

### 4.2. Framework for an upper bound of log $(A_\infty)$

We now develop a framework for bounding log $(A_\infty)$ whenever $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$. This will be used in the strong subexponential setting of Section 4.3 to obtain the right tails of $A_\infty$. This framework consists of constructing a series of stopping times between which $\xi$ is bounded from above. By studying the properties of these stopping times and the behaviour of $\xi$ at those times, we can begin to understand log $(A_\infty)$.

For $\epsilon \in (0, -K)$ and sufficiently large $A \in \mathbb{R}$, define a sequence of stopping times by $\sigma_0 := 0$ and

$$\sigma_n := \inf\{t > \sigma_{n-1} : \xi_t - \xi_{\sigma_{n-1}} \geq (K+\epsilon)(t - \sigma_{n-1}) + A\},$$

for each $n \in \mathbb{N}$, with the convention $\inf(\emptyset) = \infty$, and setting $\sigma_n = \infty$ if $\sigma_{n-1} = \infty$. Then also define

$$N := \max\{n \in \mathbb{N}_0 \mid \sigma_n < \infty\} \quad \text{and} \quad \rho_n := \mathbb{P}(\sigma_n < \infty \mid \sigma_{n-1} < \infty).$$

The next lemma concerns the finiteness of $N$.

**Lemma 4.3.** *If $\mathbb{E}[\xi_{T_2}] < 0$, then there exists an $A^* > 0$ such that, for all $A > A^*$, $N$ is a.s. finite.*

*Proof.* Define a new MAP $\{(J_t, \tilde{\xi}_t) : t \geq 0\}$ by setting $\tilde{\xi}_t := \xi_t - (K+\epsilon)t$. For each $n \in \mathbb{N}$,

$$1 - \rho_n = \mathbb{P}\left(\sup_{t > \sigma_{n-1}} \tilde{\xi}_t - \tilde{\xi}_{\sigma_{n-1}} < A \mid \sigma_{n-1} < \infty\right),$$

then, since $\sigma_{n-1}$ is a stopping time, using the Markov additive property and summing over the events $\{J_{\sigma_{n-1}} = j\}$ for $j \in \{+, -\}$,

$$1 - \rho_n = \sum_{j \in \{+,-\}} \mathbb{P}_j\left(\sup_{t \geq 0} \tilde{\xi}_t < A\right) \mathbb{P}(J_{\sigma_{n-1}} = j).$$

By the strong law of large numbers,

$$\lim_{t\to\infty} t^{-1}\tilde{\xi}_t = \lim_{t\to\infty} t^{-1}(\xi_t - (K+\epsilon)t) = K - (K+\epsilon) = -\epsilon < 0 \quad \text{a.s.}$$

and hence there a.s. exists $T > 0$ such that if $t > T$ then $\tilde{\xi}_t < 0$. From this we can conclude that $\sup_{t\geq 0} \tilde{\xi}_t = \max(0, \sup_{t \in [0,T]} \tilde{\xi}_t) < \infty$, since the supremum of a càdlàg process over a compact interval is bounded.

This implies that there exists an $A^* > 0$ such that for all $A > A^*$ we have $\mathbb{P}(\sup_{t\geq 0} \tilde{\xi}_t > A) < 1$. From this, we conclude $N < \infty$ a.s., since

$$\mathbb{P}(N > n) = \prod_{k=1}^{n} \mathbb{P}(\sigma_k < \infty \mid \sigma_{k-1} < \infty) \leq \max_{j \in \{+,-\}} \mathbb{P}_j\left(\sup_{t\geq 0} \tilde{\xi}_t < A\right)^n. \tag{4.4}$$

□

Since there are conditions for $N$ to be finite, we can bound log $(A_\infty)$ by using the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ to split the process $\{\xi_t : t \geq 0\}$ into a finite number of bounded sections.

Define the constant

$$C := \log\left(\frac{e^A}{|K+\epsilon|}\right),$$

and by taking $A > A^*$ sufficiently large we can ensure $e^C > 2$. Then we have the following upper bound for log $(A_\infty)$.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

**Lemma 4.4.** *If* $\mathbb{E}[\xi_{T_2}] < 0$*, then*

$$\log A_\infty \leq (N+1)C + \sum_{n=1}^{N} (\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+,$$

*where* $(\cdot)^+$ *denotes the positive part.*

*Proof.* Following the approach of [21, Lemma 4.1], $A_\infty$ may be expanded as

$$A_\infty = \int_0^{\sigma_1} \mathrm{e}^{\xi_t}\, \mathrm{d}t + \mathrm{e}^{\xi_{\sigma_1}} \left( \int_{\sigma_1}^{\sigma_2} \mathrm{e}^{\xi_t - \xi_{\sigma_1}}\, \mathrm{d}t + \mathrm{e}^{\xi_{\sigma_2} - \xi_{\sigma_1}} \left( \int_{\sigma_2}^{\sigma_3} \mathrm{e}^{\xi_t - \xi_{\sigma_2}}\, \mathrm{d}t + \cdots \right.\right.$$
$$\left.\left. \cdots + \mathrm{e}^{\xi_{\sigma_N} - \xi_{\sigma_{N-1}}} \left( \int_{\sigma_N}^{\sigma_{N+1}} \mathrm{e}^{\xi_t - \xi_{\sigma_N}}\, \mathrm{d}t \right) \right) \right),$$

noting that $\sigma_{N+1} = \infty$. By the definition of $\sigma_n$, we have

$$\int_{\sigma_n}^{\sigma_{n+1}} \mathrm{e}^{\xi_t - \xi_{\sigma_n}}\, \mathrm{d}t \leq \int_{\sigma_n}^{\sigma_{n+1}} \exp\left((K+\epsilon)(t-\sigma_n) + A\right) \mathrm{d}t \leq \mathrm{e}^C,$$

which, when substituted into the expression for $A_\infty$, gives

$$A_\infty \leq \mathrm{e}^C + \mathrm{e}^{\xi_{\sigma_1}} (\mathrm{e}^C + \mathrm{e}^{\xi_{\sigma_2} - \xi_{\sigma_1}} (\mathrm{e}^C + \cdots + \mathrm{e}^{\xi_{\sigma_N} - \xi_{\sigma_{N-1}}} (\mathrm{e}^C))). \tag{4.5}$$

Then, by considering the logarithms of both sides of (4.5) and repeatedly using the property $\log(A+B) \leq \log(A) + \log(B)$ whenever $A, B > 2$,

$$\log(A_\infty) \leq (N+1)C + \sum_{n=1}^{N} (\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+,$$

where we have made use of the fact that $\mathrm{e}^{(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+} \geq 1$ and $\mathrm{e}^C > 2$ in order to use the log inequality. □

As a consequence of this lemma, the right tails of $\log(A_\infty)$ can be studied by considering the evolution of the MAP between the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$. First we consider the $J$ component.

Let $(K_n)_{n\in\mathbb{N}_0}$ be the sequence of random variables, taking values in $\{+, -, \infty\}$, such that for each $n \in \mathbb{N}$, if $\sigma_n < \infty$ then $K_n = J_{\sigma_n}$, otherwise $K_n = \infty$. For each $\alpha, \beta \in \{+, -\}$ we will be interested in the number of times that $\{K_n\}_{n\in\mathbb{N}}$ transitions from $\alpha$ to $\beta$. For this purpose, define the random variable $N(\alpha, \beta) := \sum_{k=1}^{\infty} \mathbb{1}_{\{K_{k-1}=\alpha, K_k=\beta\}}$. We also make use of the notation $f = \mathrm{o}(g)$ for any two functions $f, g \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to\infty} f(x)/g(x) = 0$.

**Proposition 4.1.** *Suppose* $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$*. Then the sequence* $(K_n)_{n\in\mathbb{N}_0}$ *is a discrete-time homogeneous Markov chain, with* $\infty$ *as an absorbing state. Moreover, if* $\eta$ *is the stochastic matrix of* $\{K_n\}_{n\in\mathbb{N}}$ *and* $\alpha, \beta, \gamma \in \{+, -\}$ *with* $\alpha \neq \beta$*, then*

$$\eta_{\alpha,\gamma} \to 0, \quad \mathbb{E}_\alpha[N(\alpha, \gamma)] \sim \eta_{\alpha,\gamma} \quad \textit{and} \quad \mathbb{E}_\beta[N(\alpha, \gamma)] = \mathrm{o}(\eta_{\alpha,\gamma})$$

*as* $A \to \infty$*.*

*Proof.* First we show that $\{K_n\}_{n\in\mathbb{N}}$ is a Markov chain. If $K_{n-1} \neq \infty$, then by the Markov additive property, since $\sigma_{n-1}$ is a stopping time, $\{\xi_{\sigma_{n-1}+t} - \xi_{\sigma_{n-1}} \mid t \geq 0\}$ is independent of $\mathcal{F}_{\sigma_{n-1}}$ given $K_{n-1}$. Moreover, the random variable

$$\Delta\sigma_n := \sigma_n - \sigma_{n-1} = \inf\{t \geq 0 \colon \xi_{t+\sigma_{n-1}} - \xi_{\sigma_{n-1}} \geq t(K+\epsilon) + A\}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

is a function of $\{\xi_{t+\sigma_{n-1}} - \xi_{\sigma_{n-1}} : t \geq 0\}$. Thus the event $\{K_n = \infty\} = \{\Delta\sigma_n = \infty\}$ is independent of $\mathcal{F}_{\sigma_{n-1}}$ given $K_{n-1}$, and has the same law as the event $\{K_1 = \infty\}$ given $K_0$.

Moreover, if $\Delta\sigma_n < \infty$ then $K_n = J_{\sigma_n} = J_{\sigma_{n-1}+\Delta\sigma_n}$, hence $K_n$ is a function of

$$\{(\xi_{\sigma_{n-1}+t} - \xi_{\sigma_{n-1}}, J_{\sigma_{n-1}+t}) \colon t \geq 0\},$$

and so by the Markov additive property is independent of $\mathcal{F}_{\sigma_{n-1}}$ given $K_{n-1}$ and has the same distribution as $K_1$ given $K_0$. Hence the sequence $(K_n)_{n\in\mathbb{N}}$ is a time homogeneous Markov chain. By definition of $\sigma_n$, $\infty$ is clearly an absorbing state for $(K_n)_{n\in\mathbb{N}}$.

Now we consider the limiting behaviour of $\eta$ as $A \to \infty$. Let $\alpha, \gamma \in \{+, -\}$. From the proof of Lemma 4.3 we know that $\sup_{t\geq 0} \tilde{\xi}_t < \infty$ a.s., where $\tilde{\xi}_t := \xi_t - (K+\epsilon)t$. Thus

$$\lim_{A\to\infty} \mathbb{P}\left(\sup_{t\geq 0}\{\xi_t - (K+\epsilon)t\} > A\right) = \lim_{A\to\infty} \mathbb{P}\left(\sup_{t\geq 0} \tilde{\xi}_t > A\right) = 0.$$

However,

$$\eta_{\alpha,+} + \eta_{\alpha,-} = \mathbb{P}_\alpha(\sigma_1 < \infty) = \mathbb{P}_\alpha\left(\sup_{t\geq 0}\{\xi_t - (K+\epsilon)t\} > A\right)$$

and, since $\eta_{\alpha,\gamma}$ is non-negative, this implies $\lim_{A\to\infty} \eta_{\alpha,\gamma} = 0$.

Further assume that $\gamma \in \{+, -\}$ and $\alpha \neq \beta$. Then it is easily seen that

$$\mathbb{E}_\theta[N(\alpha, \gamma)] = \sum_{n=1}^{\infty} \mathbb{P}_\theta(K_n = \gamma \mid K_{n-1} = \alpha)\, \mathbb{P}_\theta(K_{n-1} = \alpha) = \eta_{\alpha,\gamma}(\mathbb{1}_{\{\theta=\alpha\}} + \phi_\sigma(\alpha)),$$

where $\phi_\theta(\alpha) := \sum_{n=1}^{\infty} \mathbb{P}_\theta(K_n = \alpha)$. Since $\infty$ is an absorbing state of the Markov chain $(K_n)_{n\in\mathbb{N}}$ and $\alpha \neq \beta$, for each $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}_\theta(K_n = \alpha) &= \mathbb{P}_\theta(K_n = \alpha \mid K_{n-1} = \alpha)\, \mathbb{P}_\theta(K_{n-1} = \alpha)\\ &\quad + \mathbb{P}_\theta(K_n = \alpha \mid K_{n-1} = \beta)\, \mathbb{P}_\theta(K_{n-1} = \beta)\\ &= \eta_{\alpha,\alpha}\mathbb{P}_\theta(K_{n-1} = \alpha) + \eta_{\beta,\alpha}\mathbb{P}_\theta(K_{n-1} = \beta),\end{aligned}$$

then summing up over $n \in \mathbb{N}$ we have

$$\phi_\theta(\alpha) = \eta_{\alpha,\alpha}(\mathbb{1}_{\{\theta=\alpha\}} + \phi_\theta(\alpha)) + \eta_{\beta,\alpha}(\mathbb{1}_{\{\theta=\beta\}} + \phi_\theta(\beta)),$$

and by symmetry

$$\phi_\theta(\beta) = \eta_{\beta,\beta}(\mathbb{1}_{\{\theta=\beta\}} + \phi_\theta(\beta)) + \eta_{\alpha,\beta}(\mathbb{1}_{\{\theta=\alpha\}} + \phi_\theta(\alpha)).$$

Solving this system gives

$$\phi_\theta(\alpha) = \frac{\mathbb{1}_{\{\theta=\alpha\}}(\eta_{\alpha,\alpha}(1-\eta_{\beta,\beta}) + \eta_{\beta,\alpha}\eta_{\alpha,\beta}) + \mathbb{1}_{\{\theta=\beta\}}\eta_{\beta,\alpha}}{(1-\eta_{\alpha,\alpha})(1-\eta_{\beta,\beta}) - \eta_{\beta,\alpha}\eta_{\alpha,\beta}},$$

and thus

$$\mathbb{E}_\theta[N(\alpha, \gamma)] = \frac{\eta_{\alpha,\gamma}(\mathbb{1}_{\{\theta=\alpha\}}(1-\eta_{\beta,\beta}) + \mathbb{1}_{\{\theta=\beta\}}\eta_{\beta,\alpha})}{(1-\eta_{\alpha,\alpha})(1-\eta_{\beta,\beta}) - \eta_{\beta,\alpha}\eta_{\alpha,\beta}},$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

from which the two asymptotic results for $\mathbb{E}_\theta[N(\alpha,\gamma)]$ are a consequence of the limiting behaviour of $\eta$. □

In the next lemma we consider the evolution of $\xi$ between the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$, conditioned on the values of $\{K_n\}_{n\in\mathbb{N}}$.

**Lemma 4.5.** *Suppose that $\mathbb{E}[\xi_{T_2}]\in(-\infty,0)$ and $m,n\in\mathbb{N}$ with $m<n$. Then, conditionally on $K_{m-1}$, $K_m$, $K_{n-1}$, and $K_n$, the increments $\xi_{\sigma_m}-\xi_{\sigma_{m-1}}$ and $\xi_{\sigma_n}-\xi_{\sigma_{n-1}}$ are independent. If $\alpha,\beta\in\{+,-\}$, then conditional on the event $\{K_{n-1}=K_{m-1}=\alpha; K_n=K_m=\beta\}$, the increments $\xi_{\sigma_m}-\xi_{\sigma_{m-1}}$ and $\xi_{\sigma_n}-\xi_{\sigma_{n-1}}$ are equal in distribution and independent. Furthermore, for any $l\in\mathbb{N}$ such that $m\neq l$ and any bounded continuous function $f:\mathbb{R}\to\mathbb{R}^+$, we have*

$$\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m,K_{l-1},K_l)]=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m)].$$

*Proof.* First suppose that $m<l$. Then we have

$$\begin{aligned}&\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m,K_{l-1},K_l)]\\&\quad=\sum_{\gamma,\delta\in\{+,-\}}\mathbb{1}_{\{K_{l-1}=\gamma,K_l=\delta\}}\frac{\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}});K_{l-1}=\gamma,K_l=\delta\mid\sigma(K_{m-1},K_m)]}{\mathbb{P}(K_{l-1}=\gamma,K_l=\delta\mid\sigma(K_{m-1},K_m))}.\end{aligned}$$

It follows that, by using the tower property and the fact that $(K_k)_{k\in\mathbb{N}}$ is a Markov chain, we have

$$\begin{aligned}&\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}});K_{l-1}=\gamma,K_l=\delta\mid\sigma(K_{m-1},K_m)]\\&\quad=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\,\mathbb{E}[\mathbb{1}_{\{K_{l-1}=\gamma,K_l=\delta\}}\mid\mathcal{F}_{\sigma_m}]\mid\sigma(K_{m-1},K_m)]\\&\quad=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\,\mathbb{E}[\mathbb{1}_{\{K_{l-1}=\gamma,K_l=\delta\}}\mid\sigma(K_m)]\mid\sigma(K_{m-1},K_m)]\\&\quad=\mathbb{P}(K_{l-1}=\gamma,K_l=\delta\mid\sigma(K_{m-1},K_m))\,\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m)],\end{aligned}$$

which, when substituted into the previous equation, gives

$$\begin{aligned}&\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m,K_{l-1},K_l)]\\&\quad=\sum_{\gamma,\delta\in\{+,-\}}\mathbb{1}_{\{K_{l-1}=\gamma,K_l=\delta\}}\,\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m)]\\&\quad=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m)].\end{aligned}$$

Now suppose $m>l$, and through a direct application of the Markov additive property we have

$$\begin{aligned}&\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{l-1},K_l,K_{m-1},K_m)]\\&\quad=\sum_{\alpha\in\{+,-\}}\mathbb{1}_{\{K_m=\alpha\}}\frac{\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}});K_m=\alpha\mid\sigma(K_{l-1},K_l,K_{m-1})]}{\mathbb{P}(K_m=\alpha\mid\sigma(K_{l-1},K_l,K_{m-1}))}\\&\quad=\sum_{\alpha\in\{+,-\}}\mathbb{1}_{\{K_m=\alpha\}}\frac{\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}});K_m=\alpha\mid\sigma(K_{m-1})]}{\mathbb{P}(K_m=\alpha\mid\sigma(K_{m-1}))}\\&\quad=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(K_{m-1},K_m)].\end{aligned}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

To see the independence of increments, suppose that $f, g\colon \mathbb{R} \to \mathbb{R}^+$ are bounded continuous functions, and then

$$\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(K_{m-1}, K_m, K_{n-1}, K_n)]$$
$$= \sum_{\alpha \in \{+,-\}} \mathbb{1}_{\{K_n=\alpha\}} \frac{\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}); K_n = \alpha \mid \sigma(K_{m-1}, K_m, K_{n-1})]}{\mathbb{P}(K_n = \alpha \mid \sigma(K_{m-1}, K_m, K_{n-1}))}.$$

Then, by the tower property, we get

$$\begin{aligned}
&\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}); K_n = \alpha \mid \sigma(K_{m-1}, K_m, K_{n-1})] \\
&\quad = \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}})\, \mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}); K_n = \alpha \mid \mathcal{F}_{\sigma_{n-1}}] \mid \sigma(K_{m-1}, K_m, K_{n-1})] \\
&\quad = \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}})\, \mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}); K_n = \alpha \mid \sigma(K_{\sigma_{n-1}})] \mid \sigma(K_{m-1}, K_m, K_{n-1})] \\
&\quad = \mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}); K_n = \alpha \mid \sigma(K_{\sigma_{n-1}})]\, \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(K_{m-1}, K_m)].
\end{aligned}$$

Plugging this into the previous equation yields

$$\begin{aligned}
&\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(K_{m-1}, K_m, K_{n-1}, K_n)] \\
&\quad = \sum_{\alpha \in \{+,-\}} \mathbb{1}_{\{K_n=\alpha\}} \frac{\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}); K_n = \alpha \mid \sigma(K_{\sigma_{n-1}})]}{\mathbb{P}(K_n = \alpha \mid \sigma(K_{m-1}, K_m, K_{n-1}))} \\
&\qquad \times \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(K_{m-1}, K_m)] \\
&\quad = \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \times \sigma(K_{m-1}, K_m)] \\
&\qquad \times \sum_{\alpha \in \{+,-\}} \mathbb{1}_{\{K_n=\alpha\}} \frac{\mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}); K_n = \alpha \mid \sigma(K_{\sigma_{n-1}})]}{\mathbb{P}(K_n = \alpha \mid \sigma(K_{n-1}))} \\
&\quad = \mathbb{E}[g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(K_{m-1}, K_m)]\, \mathbb{E}[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}) \mid \sigma(K_{\sigma_{n-1}}, K_{\sigma_n})].
\end{aligned}$$

□

### 4.3. Lamperti–Kiu processes of strong subexponential type

Strong subexponential distributions are a widely studied class of heavy-tailed distributions, because of their mathematical tractability and their appearance in empirical data. We will use [11] as a reference to the background theory of subexponential distributions, within which further discussion of the use of these distributions can be found.

For a probability distribution $Q\colon \mathbb{R}^+ \to [0, 1]$, define $\bar{Q}(x) := q1 - Q(x)$ for all $x \in \mathbb{R}^+$. Then, if $\overline{Q * Q}(x)/\overline{Q}(x) \to 2$ as $x \to \infty$, we say that $Q$ is a subexponential distribution. It can be shown (see e.g. [11]) that all subexponential distributions are also long-tailed. The distribution $Q$ is a strong subexponential distribution if it also satisfies the property that

$$\lim_{x\to\infty} \frac{1}{\bar{Q}(x)} \int_0^x \bar{Q}(x-y)\, \bar{Q}(y)\, \mathrm{d}y = 2m,$$

where $m = \mathbb{E}[X^+]$ and $X$ is a random variable with distribution $Q$. We will refer to a random variable with a (strong) subexponential distribution as a (strong) subexponential random variable.

Let $\mathcal{S}$ denote the set of real-valued subexponential random variables and let $\mathcal{S}^*$ denote the subset of $\mathcal{S}$ comprising strong subexponential random variables.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

For a random variable $X$, recall the definitions

$$G_X(x) := \int_x^\infty \mathbb{P}(X > u)\,\mathrm{d}u \quad \text{and} \quad H_X(x) := \min(1, G_X(x)).$$

If $H_X$ is a subexponential distribution then we write $X \in \mathcal{S}_I$, and from [11, Theorem 3.27] we have $\mathcal{S}^* \subset \mathcal{S}_I$.

For ease of notation, we also define the integrated tails $H(x) := H_{\xi_{T_2}}(x)$ and, for each $\alpha \in \{+, -\}$, define

$$H_{\xi_\alpha} := H_{\xi^{(\alpha)}_{\zeta_\alpha}} \quad \text{and} \quad H^{(\alpha)} := H_{\xi_\alpha} + H_{U_{-\alpha}}.$$

Let us introduce a subset of components of the Lamperti–Kiu decomposition given by $L := \{\xi^{(+)}_{\zeta_+}, \xi^{(-)}_{\zeta_-}, U^+, U^-\}$.

For any functions $f, g \colon \mathbb{R}^+ \to \mathbb{R}^+$, we write $f = \mathrm{O}(g)$ if $\limsup_{x\to\infty} f(x)/g(x) \in \mathbb{R}$.

**Definition 4.1.** We will say that a Lamperti–Kiu process is of *strong subexponential type* if $\xi_{T_2}$ is long-tailed and there exists $X \in L$ such that $X \in \mathcal{S}^*$ and for all $W \in L \setminus \{X\}$ we have $\mathbb{P}(W > x) = \mathrm{O}(\mathbb{P}(X > x))$.

If a Lamperti–Kiu process is of this type, there is a heaviest-tailed component of the Lamperti–Kiu decomposition and it is strong subexponential. Denote this component by $X \in L$. Let $B \subseteq \{+, -\}$ be the set of all $\beta \in \{+, -\}$ such that

$$\limsup_{x\to\infty} H_X(x)^{-1} H^{(\beta)}(x) \neq 0.$$

Then, for any $b \in B$ and $\beta \in \{+, -\} \setminus B$, we have $H^{(\beta)}(x) = \mathrm{o}(H^{(b)}(x))$ as $x \to \infty$.

By the closure properties of $\mathcal{S}^*$ [11, Corollary 3.16], it follows that $\xi_{T_2}$ is also strong subexponential with tails and integrated tails, respectively, given, as $x \to \infty$, by

$$\mathbb{P}(\xi_{T_2} > x) \sim \sum_{\beta\in\{+,-\}} \left(\mathbb{P}\left(\xi^{(\beta)}_{\zeta_\beta} > x\right) + \mathbb{P}(U^\beta > x)\right)$$

and

$$H(x) \sim \sum_{\beta\in\{+,-\}} H^{(\beta)}(x) \sim \sum_{\beta\in B} H^{(\beta)}(x).$$

Recall from Section 2 that $K := \mathbb{E}[\xi_{T_2}]/\mathbb{E}[T_2]$, and hence, when $\mathbb{E}[\xi_{T_2}] \neq 0$, it follows that $K \neq 0$. The main result of this section, which extends [21, Section 4, p. 166] to Lamperti–Kiu processes, is the following result.

**Theorem 4.1.** *Suppose that $Y$ is a Lamperti–Kiu process of strong subexponential type such that $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$. Then*

$$\mathbb{P}(A_\infty > x) \sim \frac{H(\log(x))}{\mathbb{E}[T_2]|K|}, \quad \textit{as } x \to \infty. \tag{4.6}$$

*Furthermore, $A_\infty$ is long-tailed and $\log(A_\infty)$ is subexponential.*

The proof of Theorem 4.1 requires a number of intermediary lemmas. These lemmas are stated with proofs below, followed by the proof of Theorem 4.1 at the end of this section.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

Set $Z_n := \xi_{\sigma_n} - \xi_{\sigma_{n-1}}$ for each $n \geq 1$. We are now in a position to consider the asymptotic behaviour of the survival function of $Z_n$ conditioned on $K_{n-1}$ and $K_n$, under the assumptions of Theorem 4.1. Recall that for $\alpha \in \{+, -\}$, $H^{(\alpha)} = H_{\xi_\alpha}(x) + H_{U_{-\alpha}}(x)$.

**Lemma 4.6.** *Suppose that* $Y$ *is a Lamperti–Kiu process of strong subexponential type such that* $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$*, and fix* $\alpha, \beta \in \{+, -\}$*. Then, if* $\beta \in B$*, for each* $n \in \mathbb{N}$*,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{H^{(\beta)}(x)} \leq \frac{1}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]}.$$

*Furthermore, if* $\beta \in \{+, -\} \setminus B$ *and* $b \in B$*, then, for each* $n \in \mathbb{N}$*,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{H^{(b)}(x)} = 0.$$

*Proof.* Suppose that $x > A$, let $u_0 \in (0, x - A)$, and fix $\alpha, \beta \in \{+, -\}$. For ease of notation, let $\sigma := \sigma_1$. Then, since

$$\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta) = \mathbb{P}_\alpha(\xi_\sigma > x \mid \sigma < \infty,\ J_\sigma = \beta),$$

we have

$$\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta) = \frac{1}{\eta^{(\alpha,\beta)}} \sum_{m=0}^{\infty} \mathbb{P}_\alpha(\xi_\sigma > x; T_m \leq \sigma < T_{m+1}; J_\sigma = \beta).$$

To bound the elements of the sum, first consider the strict inequality $T_m < \sigma < T_{m+1}$ for some $m \in \mathbb{N}$; then

$$\mathbb{P}_\alpha(\xi_\sigma > x; T_m < \sigma < T_{m+1}; J_\sigma = \beta) \leq \mathbb{P}_\alpha\left(\sup_{T_m<u<T_{m+1}} \xi_u > x; \xi_{T_m} < (K+\epsilon)T_m + A; J_{T_m} = \beta\right),$$

and using the Lamperti–Kiu decomposition followed by Lemma 4.1, we have

$$\begin{aligned}\mathbb{P}_\alpha(\xi_\sigma > x; T_m < \sigma < T_{m+1}; J_\sigma = \beta) &\leq \mathbb{P}_\alpha\left(\sup_{0<u<\tilde{\zeta}_\beta} \tilde{\xi}^{(\beta)}_u > x - (K+\epsilon)T_m - A; J_{T_m} = \beta\right)\\ &\leq \frac{\mathbb{P}_\alpha\big(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \geq x - (K+\epsilon)T_m - A - u_0; J_{T_m} = \beta\big)}{\mathbb{P}\big(\xi^{(\beta)}_{\zeta_\beta} \geq -u_0\big)}\end{aligned}$$

where $\tilde{\xi}^{(\beta)}$ and $\tilde{\zeta}_\beta$ are independent copies of the Lévy process $\xi^{(\beta)}$ and the exponential random variable $\zeta_\beta$, respectively.

In the case when $\sigma = T_m$,

$$\begin{aligned}\mathbb{P}_\alpha(\xi_\sigma > x; T_m = \sigma; J_\sigma = \beta) &\leq \mathbb{P}_\alpha(\xi_{T_m} > x; \xi_{T_m-} \leq (K+\epsilon)T_m + A; J_{T_m} = \beta)\\ &\leq \mathbb{P}_\alpha(\xi_{T_m} - \xi_{T_m-} > x - (K+\epsilon)T_m - A; J_{T_m} = \beta)\\ &= \mathbb{P}_\alpha(U^{(-\beta)} > x - (K+\epsilon)T_m - A; J_{T_m} = \beta).\end{aligned}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

If $\alpha=\beta$ then there must be an even number of changes of $J$ before $\sigma$, so there exists $m\in\mathbb{N}$ such that $\sigma\in[T_{2m}, T_{2m+1})$. Hence combining the two results above gives

$$\mathbb{P}(Z_n>x\mid K_{n-1}=\alpha,\, K_n=\beta)\leq\frac{1}{\eta^{(\alpha,\beta)}}\Bigg(\sum_{m=0}^{\infty}\frac{\mathbb{P}_\alpha\big(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}\geq x-(K+\epsilon)T_{2m}-A-u_0\big)}{\mathbb{P}\big(\xi^{(\beta)}_{\zeta_\beta}>-u_0\big)} + \sum_{m=0}^{\infty}\mathbb{P}_\alpha(U^{(-\beta)}\geq x-(K+\epsilon)T_{2m}-A-u_0)\Bigg).$$

If $\alpha\neq\beta$ then there is an odd number of changes in $J$ before time $\sigma$. However, $T_{2m+1}\geq T_{2m}$, so the inequalities can be weakened to give the same result as the $\alpha=\beta$ case.

For ease of notation define

$$Q_\beta(u_0):=\mathbb{P}\big(\xi^{(\beta)}_{\zeta_\beta}\geq -u_0\big)\leq 1.$$

If $\xi^{(\beta)}_{\zeta_\beta}$ is long-tailed, we can use Lemma 4.2 to obtain the asymptotic approximation

$$\sum_{m=0}^{\infty}\mathbb{P}_\alpha\big(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}\geq x-(K+\epsilon)T_{2m}-A-u_0\big)\sim\frac{G_{\xi_\beta}(x)}{|K+\epsilon|\,\mathbb{E}[T_2]},$$

as $x\to\infty$. Similarly, if $U^{(-\beta)}$ is long-tailed we have

$$\sum_{m=0}^{\infty}\mathbb{P}_\alpha(U^{(-\beta)}>x-(K+\epsilon)T_{2m}-A-u_0)\sim\frac{G_{U^{(-\beta)}}(x)}{|K+\epsilon|\,\mathbb{E}[T_2]},$$

as $x\to\infty$. We will consider separately the cases where both of the asymptotics hold, exactly one holds, or neither hold.

In the case when both $\xi^{(\beta)}_{\zeta_\beta}$ and $U^{(-\beta)}$ are subexponential (and hence are long-tailed) for all $\delta>0$, there exists an $R>0$ such that, for all $x>R$,

$$\begin{aligned}\frac{\mathbb{P}(Z^{(\alpha,\beta)}>x)}{G_{\xi_\beta}(x)+G_{U^{(-\beta)}}(x)}&\leq\frac{1}{(G_{\xi_\beta}(x)+G_{U^{(-\beta)}}(x))\eta^{(\alpha,\beta)}}\\&\quad\times\left(\frac{(1+\delta)G_{\xi_\beta}(x)}{Q_\beta(u_0)|K+\epsilon|\,\mathbb{E}[T_2]}+\frac{(1+\delta)G_{U^{(-\beta)}}(x)}{|K+\epsilon|\,\mathbb{E}[T_2]}\right)\\&\leq\frac{(1+\delta)}{\eta^{(\alpha,\beta)}Q_\beta(u_0)|K+\epsilon|\,\mathbb{E}[T_2]},\end{aligned}$$

where the second inequality holds since $Q_\beta(u_0)<1$. Since $\delta$ was arbitrary, taking the lim sup as $x\to\infty$ yields

$$\limsup_{x\to\infty}\frac{\mathbb{P}(Z_n>x\mid K_{n-1}=\alpha,\,K_n=\beta)}{G_{\xi_\beta}(x)+G_{U^{(-\beta)}}(x)}\leq\frac{1}{\eta^{(\alpha,\beta)}Q_\beta(u_0)|K+\epsilon|\,\mathbb{E}[T_2]}.$$

In the case when exactly one of $\xi^{(\beta)}_{\zeta_\beta}$ and $U_{-\beta}$ is subexponential, it asymptotically dominates the other as $x\to\infty$, since $Y$ is of strong subexponential type. Suppose that it is $\xi^{(\beta)}_{\zeta_\beta}$ that is

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

subexponential and note that the following argument is symmetric in $\xi^{(\beta)}_{\zeta_\beta}$ and $U^{(-\beta)}$. For all $\delta > 0$, there exists $\hat{\delta} > 0$ such that $\hat{\delta}(1+\hat{\delta}) < \delta/2$, and an $R > 0$ such that for all $x > R$ and $n \in \mathbb{N}$,

$$\mathbb{P}_\alpha(U^{(-\beta)} > x - (K+\epsilon)T_{2n} - A - u_0) \le \hat{\delta}\mathbb{P}_\alpha\big(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \ge x - (K+\epsilon)T_{2n} - A - u_0\big).$$

Thus, for all $x > R$, for suitably large $R$,

$$\sum_{m=0}^{\infty} \mathbb{P}_\alpha(U^{(-\beta)} > x - (K+\epsilon)T_{2m} - A - u_0) \le \frac{\hat{\delta}(1+\hat{\delta})G_{\xi_\beta}(x)}{|K+\epsilon|\,\mathbb{E}[T_2]},$$

which gives for $x > R$

$$\begin{aligned}\frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{G_{\xi_\beta}(x) + G_{U^{(-\beta)}}(x)} &\le \frac{1}{\eta^{(\alpha,\beta)}G_{\xi_\beta}(x)}\bigg(\frac{(1+\delta/2)G_{\xi_\beta}(x)}{|K+\epsilon|\,\mathbb{E}[T_2]Q_\beta(u_0)} + \frac{(\delta/2)G_{\xi_\beta}}{|K+\epsilon|\,\mathbb{E}[T_2]}\bigg)\\ &\le \frac{1+\delta}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]Q_\beta(u_0)}.\end{aligned}$$

Therefore, since $\delta > 0$ was arbitrary, we may take the lim sup first as $\delta \to 0$ and then as $x \to \infty$ to obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{G_{\xi_\beta}(x) + G_{U^{(-\beta)}}(x)} \le \frac{1}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]Q_\beta(u_0)}.$$

Finally, we consider the case when neither is subexponential. Since $Y$ is of strong subexponential type, the tails of $\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}$ and $U^{(-\beta)}$ are dominated by the tails of at least one of either $\tilde{\xi}^{(b)}_{\tilde{\zeta}_b}$ or $U^{(-b)}$. Denote the dominating random variable by $X$ and let $W \in \big\{\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}, U^{(-\beta)}\big\}$. Following the above calculation, for all $\delta > 0$ there exists an $R > 0$ such that for any $x > R$ and $n \in \mathbb{N}$,

$$\mathbb{P}_\alpha(W \ge x - (K+\epsilon)T_{2n} - A - u_0) \le \delta\mathbb{P}_\alpha(X > x - (K+\epsilon)T_{2n} - A - u_0).$$

Then, by using the results of the previous two cases, for suitably large $R > 0$ we have

$$\sum_{n=0}^{\infty} \mathbb{P}_\alpha(W \ge x - (K+\epsilon)T_{2n} - A - u_0) \le \frac{\delta(1+\delta)G_X(x)}{|K+\epsilon|\,\mathbb{E}[T_2]H_b(u_0)}.$$

Hence, for $x > R$,

$$\frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{G_{\xi_{-b}}(x) + G_{U^{(b)}}(x)} \le \frac{2\delta(1+\delta)}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]Q_b(u_0)}$$

so, as $x \to \infty$, since $\delta > 0$ was arbitrary,

$$\mathbb{P}\big(Z^{(\alpha,\beta)} > x\big) = \mathrm{o}(G_{\xi_b}(x) + G_{U^{(-b)}}(x)).$$

Since all the components of the Lamperti–Kiu decomposition are finite, for sufficently large $x$ we have $G_{(\cdot)}(x) = H_{(\cdot)}(x)$. Hence, in the first two cases, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{H_{\xi_\beta}(x) + H_{U^{(-\beta)}}(x)} \le \frac{1}{\eta^{(\alpha,\beta)}Q(u_0)|K+\epsilon|\,\mathbb{E}[T_2]}.$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

As we are taking the limit $x \to \infty$, we may also take $u_0 \to \infty$ and use the fact that $Q(u_0) \to 1$ to obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{H_{\xi_\beta}(x) + H_{U^{(-\beta)}}(x)} \le \frac{1}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]},$$

whilst in the third case

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid K_{n-1} = \alpha, K_n = \beta)}{H_{\xi_{-\beta}}(x) + H_{U^{(\beta)}}(x)} = 0.$$

This completes the proof. □

The following lemma provides a way to use a bound on a random variables distribution function to produce an inequality between random variables.

**Lemma 4.7.** *Suppose $Z$ is a real-valued random variable with tail $\mathbb{P}(Z \ge x)$ that is bounded above by some function $F(x)$ such that $1 - F(x)$ is a true distribution function. Then there exists a random variable $X$, which is a function of $Z$ and an independent uniformly distributed random variable, such that $Z \le X$ and $\mathbb{P}(X \ge x) = F(x)$.*

*Proof.* Let $P(x) := \mathbb{P}(Z \ge x)$ and $V \sim \mathrm{Unif}(0, 1)$ be independent of $Z$. We will use the notation $P(x^+) := \lim_{y\downarrow x} P(y)$, which exists for all $x \ge 0$ since $P$ is non-increasing and bounded from below. Define the random function $U : \mathbb{R}^+ \to [0, 1]$ by setting $U(x) := P(x) - V(P(x) - P(x^+))$. Let $x_1 < x_2$; then since $P$ is non-increasing, $P(x^+) \le U(x) \le P(x)$ for all $x \in \mathbb{R}$ and

$$U(x_2) \le P(x_2) \le P(x_1^+) = P(x_1) - 1(P(x_1) - P(x_1^+)) \le U(x_1),$$

hence $U$ is also non-increasing.

Furthermore, suppose $U(x_1) = U(x_2)$ for some $x_1 < x_2$. Then $P(x_1^+) \le U(x_1) = U(x_2) \le P(x_2) \le P(x_1^+)$, where the last inequality is because $P$ is non-increasing and so $P(x_1^+) = P(x_2)$. If $P(x_1^+) = P(x_1)$, then we have $P(x_2) = P(x_1)$. Otherwise we have

$$U(x_1) > P(x_1^+) \ge P(x_2) > U(x_2).$$

This is a contradiction. Hence, if $x_1 < x_2$ and $U(x_1) = U(x_2)$, then $P(x_1^+) = P(x_1) = P(x_2)$ a.s., so $\mathbb{P}(x_1 \le Z < x_2) = 0$.

From this we can now conclude that, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}(U(z) \le U(x)) &= \mathbb{P}(Z \ge x) + \mathbb{P}(Z < x;\, U(Z) = U(x)) \\ &= P(x) + \mathbb{P}(Z < x;\, U(Z) = U(x)).\end{aligned}$$

However, by the above calculation, we get

$$\mathbb{P}(Z < x;\, U(Z) = U(x)) \le \mathbb{P}(Z < x;\, P(Z) = P(x)) = 0$$

and hence, for all $x \in \mathbb{R}$, we have $\mathbb{P}(U(z) \le U(x)) = P(x)$.

Now let $q \in [0, 1]$ and suppose that there exists $x \in \mathbb{R}^+$ with $P(x^+) = P(x) = q$, so

$$\begin{aligned}\mathbb{P}(U(Z) \le q) &= \mathbb{P}(U(Z) \le P(x)) \\ &= \mathbb{P}(U(Z) \le U(x)) = q.\end{aligned}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

If there is not such an $x$, then, since $\lim_{x\to-\infty} P(x) = 1$ and $\lim_{x\to\infty} P(x) = 0$, there exists $x \in \mathbb{R}^+$ such that $q \in [P(x^+), P(x))$. For any $y > x$, we have $U(y) \le P(y) \le P(x^+)$, so $U(y) \notin (P(x^+), P(x))$ and similarly, for any $y < x$, we have $U(y) \ge U(y^+) \ge P(x)$, so $U(y) \notin (P(x^+), P(x))$. Hence $U(Z) \in (q, P(x)) \subset (P(x^+), P(x))$ implies $Z = x$.

Next we want to consider $\mathbb{P}(P(x) = U(Z); Z \neq x)$. Notice that if $z > x$ then $U(z) \le P(x^+) < P(x)$ and so $\mathbb{P}(P(x) = U(z); Z > x) = 0$. If $z < x$ and $P(x) = U(z)$, then $P(x) = U(z) \ge P(z^+) \ge P(x)$ and so $P(x) = P(z^+)$. However, if there is a discontinuity point $y \in [z, x)$, then $U(z) \ge P(z^+) \ge P(y) > P(y^+) \ge P(x)$ so $\mathbb{P}(P(x) = U(z); Z < x) = 0$, and we conclude that $P(z) = P(z^+) = P(x)$ and hence $\mathbb{P}(P(x) = U(Z); Z \neq x) = 0$.

From this and since $V$ is uniformly distributed on [0,1], we have

$$\begin{aligned}\mathbb{P}(U(Z) \in (q, P(x)]) &= \mathbb{P}(Z = x)\,\mathbb{P}(U(x) \in (q, P(x)]) + \mathbb{P}(Z \neq x; U(Z) = P(x))\\ &= \mathbb{P}(Z = x)\,\mathbb{P}\left(\frac{q - P(x^+)}{P(x) - P(x^+)} < V \le \frac{P(x) - P(x^+)}{P(x) - P(x^+)}\right) + 0\\ &= (P(x) - P(x^+))\frac{q - P(x^+)}{P(x) - P(x^+)},\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{P}(U(Z) \le q) &= \mathbb{P}(U(Z) \le P(x)) - \mathbb{P}(U(Z) \in (q, P(x)])\\ &= P(x) - (P(x) - P(x^+))\frac{(P(x) - q)}{P(x) - P(x^+)} = q.\end{aligned}$$

Hence $U(Z)$ is uniformly distributed on [0,1]. Using [9, Proposition 3.1], the random variable defined by $X := U^-(Z) := \inf\{x \in \mathbb{R}^+ \mid F(x) < U(Z)\}$ has the distribution $F$. Moreover, since $F(x) \ge P(x)$ for all $x \in \mathbb{R}^+$, we have

$$F(Z) \ge P(Z) \ge P(Z) - V(P(Z) - P(Z^+)) = U(Z)$$

and, since $F$ is non-increasing, we have $X = \inf\{x \in \mathbb{R}^+ \mid F(x) < U(Z)\} \ge Z$, as required. □

We are now able to derive the upper bound of Theorem 4.1.

**Lemma 4.8.** *If $Y$ is a Lamperti–Kiu process of strong subexponential type, then the right tail of $A_\infty$ satisfies*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(\log(A_\infty) > x)}{H(x)} \le \frac{1}{|K|\,\mathbb{E}[T_2]}, \tag{4.7}$$

*where $H$ is the integrated tail from Theorem 4.1.*

*Proof.* Fix $\sigma \in \{+, -\}$ and let $\delta_2 > 0$. For sufficiently large $A > 0$, by Proposition 4.1, we know $\mathbb{E}_\sigma[N(\alpha, \beta)]/\eta^{(\alpha,\beta)} \le \mathbb{1}_{\{\sigma=\alpha\}} + \delta_2$. Now fix such an $A > 0$ and let $\delta_1 > 0$.

From Lemma 4.4 we have $\log(A_\infty) \le (N+1)C + \sum_{i=1}^N Z_i^+$.

For each $i \in \mathbb{N}$, we have a tail estimate for $Z_i^+$, given $(K_n)_{n\in\mathbb{N}_0}$, from Lemma 4.6, which, used in conjunction with Lemma 4.7, gives the existence of random variables $X_i(k)$ for each $k \in \cup_{n\in\mathbb{N}}\{+, -\}^n$ with $k_0 = \sigma$ such that:

(1) each $X_i(k)$ is a function of $Z_i$ and a random variable independent of the rest of the system,

(2) $X_i(k) \ge (Z_i^+ + C)\mathbb{1}_{\{N=n;\,(K_0,\ldots,K_n)=k\}}$,

(3) $X_i(k)$ has tails given by $\min(1, H^{(k_i)}(x)(\eta^{(k_{i-1},k_i)}|K + \epsilon|\,\mathbb{E}[T_2]|)^{-1})$ if $k_i \in B$ and tails which are $o(\min(1, H^{(b)}(x)))$ for $b \in B$ if $k_i \notin B$.

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

Then, summing up over the sample paths of $(K_n)_{n\in\mathbb{N}_0}$, we have the upper bound

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{N} Z_i^+ + C > x\right) \le \sum_{n\in\mathbb{N}} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N=n; (K_0, \ldots, K_n)=k\right).$$

For ease of notation, let $\bar{\eta} = \max_{\alpha,\beta\in\{+,-\}} \eta_{\alpha,\beta}$, $\underline{\eta} = \min_{\alpha,\beta\in\{+,-\}} \eta_{\alpha,\beta}$, and $d \in (0, (1-2\bar{\eta})/2\bar{\eta})$.

For each $\alpha, \beta \in \{+,-\}$, $n\in\mathbb{N}$ and $k\in\{+,-\}^{n+1}$ such that $k_0=\sigma$, let

$$n_{\alpha,\beta}(k) := \sum_{i=0}^{n} \mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}}.$$

Then, by Lemma 4.5, given the event $\{N=n; (K_0, \ldots, K_n)=k\}$, the sum

$$Y_{\alpha,\beta}(k) := \sum_{i=1}^{n} X_i \mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}}$$

is a sum of $n_{\alpha,\beta}(k)$ i.i.d. random variables, and hence, in the case $\beta\in B$, from Kesten's bound [11, Theorem 3.34],

$$\begin{aligned}
&\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k)\mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}} > x \mid N=n; (K_0, \ldots, K_n)=k\right)\\
&\quad\le c(d)(1+d)^{n_{\alpha,\beta}(k)}\mathbb{P}_\sigma(X_1((\alpha,\beta)) > x)\\
&\quad\le \frac{c(d)(1+d)^n F(x)}{\underline{\eta}|K+\epsilon|\,\mathbb{E}[T_2]}.
\end{aligned}$$

In the case $\beta\notin B$, since $Y$ is of strong subexponential type, for any $b\in B$,

$$\begin{aligned}
&\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k)\mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}} > x \mid N=n; (K_0, \ldots, K_n)=k\right)\\
&\quad\le \mathbb{P}_\sigma\left(\sum_{i=1}^{n} W_i\mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}} > x \mid N=n; (K_0, \ldots, K_n)=k\right)\\
&\quad\le \frac{c(d)(1+d)^n H(x)}{\underline{\eta}|K+\epsilon|\,\mathbb{E}[T_2]},
\end{aligned}$$

where for each $i\in\mathbb{N}$ the random variable $W_i$ depends only on $X_i(k)$ and has the distribution of $X_1((\alpha, b))$. We can now use Corollaries 3.16 and 3.18 in [11, Chapter 3, p. 52] to sum the $Y_{\alpha,\beta}$ and obtain the bound

$$\begin{aligned}
&\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \mid N=n; (K_0, \ldots, K_n)=k\right)\\
&\quad= \mathbb{P}_\sigma\left(\sum_{\alpha,\beta\in\{+,-\}} \sum_{i=1}^{n} X_i(k)\mathbb{1}_{\{k_{i-1}=\alpha, k_i=\beta\}} > x \mid N=n; (K_0, \ldots, K_n)=k\right)\\
&\quad\le \frac{4c(d)(1+d)^n H(x)}{\underline{\eta}|K+\epsilon|\,\mathbb{E}[T_2]}.
\end{aligned}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

Using the bound on the distribution of $N$ from (4.4), we see that there is an $M \in \mathbb{N}$ such that $\mathbb{E}[\underline{\eta}^{-1} 4c(d)(1+d)^N; N > M] \le \delta_1$ for sufficiently small $d$. This then gives

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{N} X_i((K_n)_{n\in\mathbb{N}}) > x; N > M\right) \le \frac{\delta_1 H(x)}{|K+\epsilon|\,\mathbb{E}[T_2]}.$$

Moreover, by [11, Corollary 3.16], for all $n \le M$ and $k \in \{+,-\}^{n+1}$ with $k_0 = \sigma$ there exists $R_{n,k} > 0$ such that, for all $x > R_{n,k}$,

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \mid N = n; (K_0, \ldots, K_n) = k\right)$$
$$\le \left(\left(1+\frac{\delta_1}{2}\right) \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} n_{\alpha,\beta}(k) \frac{H^\beta(x)}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]}\right) \bar{*} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} \mathbb{1}_{\{k_i \notin B\}} X_i(k) > x\right)(x),$$

where for two survival functions $\bar{F}$ and $\bar{G}$ we write $\bar{F} \bar{*} \bar{G}$ for the survival function of the convolution of the distribution functions $F := 1 - \bar{F}$ and $G := 1 - \bar{G}$. Then, since

$$\mathbb{P}_\sigma(\mathbb{1}_{\{k_i \notin B\}} X_i(k)) > x) = \mathrm{o}(H^\beta(x)) \quad \text{for any } \beta \in B,$$

by [11, Corollary 3.18] we have

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \mid N = n; (K_0, \ldots, K_n) = k\right)$$
$$\le (1+\delta_1) \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} n_{\alpha,\beta}(k) \frac{H^\beta(x)}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]}.$$

Since there are finitely many such pairs $(n,k)$ with $n \le M$, we can take $R := \max_{n\le M} R_{n,k}$ so that for all $x > R$ we get

$$\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1} \\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N = n; (K_0, \ldots, K_n) = k\right)$$
$$\le (1+\delta_1) \sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1} \\ k_0=\sigma}} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} n_{\alpha,\beta}(k) \frac{H^{(\beta)}(x)}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]}$$
$$\times \mathbb{P}_\sigma(N = n; (K_0, \ldots, K_n) = k)$$
$$= (1+\delta_1) \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} \mathbb{E}_\sigma[n_{\alpha,\beta}(k); N \le M] \frac{H^{(\beta)}(x)}{\eta^{(\alpha,\beta)}|K+\epsilon|\,\mathbb{E}[T_2]}$$
$$\le \frac{(1+\delta_1)}{|K+\epsilon|\,\mathbb{E}[T_2]} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} \frac{\mathbb{E}_\sigma[n_{\alpha,\beta}(k)] H^{(\beta)}(x)}{\eta^{(\alpha,\beta)}}.$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

Next, by our choice of $A$ we have $\mathbb{E}_\sigma[n(\alpha,\beta)]/\eta^{(\alpha,\beta)} \le \mathbb{1}_{\sigma=\alpha} + \delta_2$ and thus

$$\begin{aligned}\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1}\\ K_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N=n; (K_0,\ldots,K_n)=k\right) &\le \frac{(1+\delta_1)}{|K+\epsilon|\,\mathbb{E}[T_2]} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} (\mathbb{1}_{\{\sigma=\alpha\}}+\delta_2) H^{(\beta)}(x)\\ &= \frac{(1+\delta_1)}{|K+\epsilon|\,\mathbb{E}[T_2]}(1+2\delta_2)\sum_{\beta\in B} H^{(\beta)}(x)\\ &\le \frac{(1+\delta_1)^2}{|K+\epsilon|\,\mathbb{E}[T_2]}(1+2\delta_2)H(x),\end{aligned}$$

where the last inequality holds for sufficiently large $x$ since $Y$ is of strong subexponential type. Hence, for all $x > R$, we have

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{N}(Z_i^+ + C) > x\right) \le \frac{(1+\delta_1)^2(1+2\delta_2)}{|K+\epsilon|\,\mathbb{E}[T_2]} F(x) + \frac{\delta_1 H(x)}{|K+\epsilon|\,\mathbb{E}[T_2]},$$

and so from the definition of lim sup,

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma(C+\sum_{i=1}^{N}(Z_i^+ + C) > x)}{H(x-C)} \le \frac{1+2\delta_2}{|K+\epsilon|\,\mathbb{E}[T_2]}.$$

However, because $H$ is long-tailed, $\lim_{x\to\infty} H(x-C)/H(x) = 1$ and therefore

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma(C+\sum_{i=1}^{N}(Z_i^+ + C) > x)}{H(x)} \le \frac{1+2\delta_2}{|K+\epsilon|\,\mathbb{E}[T_2]}.$$

Then, by comparison,

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma(\log(A_\infty) > x)}{H(x)} \le \frac{1+2\delta_2}{|K+\epsilon|\,\mathbb{E}[T_2]},$$

and since both $\epsilon$ and $\delta_2$ were arbitrary, the result follows. □

It remains to show that the lower bound for $\liminf_{x\to\infty} \mathbb{P}(A_\infty > x)$ also holds. To this end, for each $x\in\mathbb{R}$, define the stopping time

$$\tau_d(x) := \inf\{T_{2n} \mid n\in\mathbb{N}, \xi_{T_{2n}} \ge x\}$$

and notice that $\tau_d(x) < \infty$ if and only if $\sup_{n\in\mathbb{N}} \xi_{T_{2n}} \ge x$. Furthermore, $J_{\tau_d(x)} = J_0$ whenever $\tau_d(x) < \infty$. Since $\xi_{T_2}$ is strong subexponential, its integrated tail, $H$, is subexponential and thus, by [29, Theorem 1(ii)], $\mathbb{P}(\tau_d(x) < \infty)$ is also subexponential. Then, by considering the random walk $(\xi_{T_{2n}})_{n\in\mathbb{N}}$ in the place of $(\xi_n)_{n\in\mathbb{N}}$ in the proof of [21, Lemma 4.3], we find that for every $y > 0$

$$\lim_{x\to\infty} \mathbb{P}(\xi_{\tau_d(x)} - x > y \mid \tau_d(x) < \infty) = 1. \tag{4.8}$$

https://doi.org/10.1017/jpr.2021.24 Published online by Cambridge University Press

We are now able to prove Theorem 4.1.

*Proof of Theorem 4.1.* Equation (4.6) of Theorem 4.1 follows from inequality (4.7) of Lemma 4.8 and the inequality

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\log(A_\infty) > x)}{H(x)} \geq \frac{1}{|\mathbb{E}[\xi_{T_2}]|} = \frac{1}{\mathbb{E}[T_2]|K|},$$

which we will now prove.

The following inequality is immediate:

$$\mathbb{P}(\log A_\infty > x) \geq \mathbb{P}\bigg(\log\bigg(\int_{\tau_d(x)}^{\infty} |Y_t|\,\mathrm{d}t\bigg) > x;\, \tau_d(x) < \infty\bigg).$$

Applying the Markov additive property and recalling $J_{\tau_d(x)} = J_0$ gives

$$\mathbb{P}(\log A_\infty > x) \geq \mathbb{P}(\xi_{\tau_d(x)} + \log(\hat{A}_{\infty,J_0}) > x \mid \tau_d(x) < \infty)\,\mathbb{P}(\tau_d(x) < \infty),$$

where $\hat{A}_{\infty,j}$ is an independent and identically distributed copy of $A_\infty$ with $\hat{J}_0 = j$. Then, by applying (4.8), we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\log A_\infty > x)}{H(x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}(\tau_d(x) < \infty)}{H(x)} = \liminf_{x\to\infty} \frac{\mathbb{P}(\sup_{n\in\mathbb{N}} \xi(T_{2n}) \geq x)}{H(x)}.$$

However, $\xi_{T_{2n}}$ is a sum of the random variables $\xi_{T_{2m}} - \xi_{T_{2(m-1)}}$, which are i.i.d. copies of $\xi_{T_2}$. Since $Y$ is of strong subexponential type, the integrated tail, $H$, of $\xi_{T_2}$ is long-tailed, and we can apply Veraverbeke's theorem [29, Theorem 1(i)] to conclude that

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\sup_{n\in\mathbb{N}} \xi(T_{2n}) \geq x)}{H(x)} \geq \frac{1}{|\mathbb{E}[\xi_{T_2}]|} = \frac{1}{|K|\,\mathbb{E}[T_2]}.$$ □

**Remark 4.1.** The results of this paper are presented only for settings where $|E| = 2$. However, they can easily be extended to the case of any finite $E$ provided the modulating Markov chain $J$ is irreducible. In the proofs, extensive use is made of the fact that $J_{T_{2n}} = J_0$ for all $n \in N$. To extend this to the case $|E| > 2$, we replace $\{T_{2n}\}_{n\in N}$ with the sequence of return times to $J_0$ of $J$, which have finite expectation. In the case $|E| = \infty$, two further difficulties arise, which may prevent an extension of the results. Firstly, even if $J$ is a recurrent Markov chain, it may be the case that the expected return time of $J$ is infinite. Secondly, arguments that rely on taking maxima or sums over the elements of $E$ are no longer valid. The reader may also be interested by the work in [3], where a necessary and sufficient condition for finiteness of $\int_0^\infty \mathrm{e}^{\xi_s}\,\mathrm{d}\eta_s$ is given, where $(\xi, \eta)$ is a bivariate Markov additive process with some modulating Markov chain $J$.

## Acknowledgements

We are grateful to the anonymous referees for their careful reading of our manuscript and their valuable comments which helped to improve the paper.

## References


[1] ALILI, L., CHAUMONT, L., GRACZYK, P. AND ŻAK, T. (2017). Inversion, duality and Doob $h$-transforms for self-similar Markov processes. *Electron. J. Prob.* **22**, 1–18.

[2] ASMUSSEN, S. (2003). *Applied Probability and Queues* (Applications of Mathematics: Stochastic Modelling and Applied Probability **51**). Springer.

[3] BEHME, A. AND SIDERIS, A. (2020). Exponential functionals of Markov additive processes. *Electron. J. Prob.* **25**, 1–25.

[4] BERTOIN, J. (1998). *Lévy Processes* (Camb. Tracts Math). Cambridge University Press.

[5] BERTOIN, J. AND YOR, M. (2005). Exponential functionals of Lévy processes. *Prob. Surveys* **2**, 191–212.

[6] CARMONA, P., PETIT, F. AND YOR, M. (1994). Sur les fonctionnelles exponentielles de certains processus de Lévy [Exponential functionals of certain Lévy processes]. *Stoch. Stoch. Reports* **47**, 71–101.

[7] CHAUMONT, L., PANTÍ, H. AND RIVERO, V. (2013). The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli* **19**, 2494–2523.

[8] CONT, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quant. Finance* **1**, 223–236.

[9] EMBRECHTS, P. AND HOFERT, M. (2013). A note on generalized inverses. *Math. Methods Operat. Res.* **77**, 423–432.

[10] ERICKSON, K. B. (1973). The strong law of large numbers when the mean is undefined. *Trans. Amer. Math. Soc.* **185**, 371–381.

[11] FOSS, S., KORSHUNOV, D. AND ZACHARY, S. (2013). *An Introduction to Heavy-Tailed and Subexponential Distributions* (Springer Series in Operations Research and Financial Engineering). Springer, New York.

[12] GOLDIE, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Prob.* **1**, 126–166.

[13] IBRAGIMOV, M., IBRAGIMOV, R. AND WALDEN, J. (2015). *Heavy-Tailed Distributions and Robustness in Economics and Finance* (Lecture Notes Statist. **214**). Springer International Publishing.

[14] JUNG, R. C. AND LIESENFELD, R. (2000). Stochastic volatility models: conditional normality versus heavy-tailed distributions. *J. Appl. Econometrics* **15**, 137–160.

[15] KALLENBERG, O. (2002). *Foundations of Modern Probability* (Probability Theory and Stochastic Modelling **99**). Springer, New York.

[16] KESTEN, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.

[17] KIU, S. W. (1980). Semi-stable Markov processes in $\mathbb{R}^n$. *Stoch. Process. Appl.* **10**, 183–191.

[18] KUZNETSOV, A., KYPRIANOU, A. E., PARDO, J. C. AND WATSON, A. R. (2014). The hitting time of zero for a stable process. *Electron. J. Prob.* **19**, 1–26.

[19] LAMPERTI, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc* **104**, 62–78.

[20] LAMPERTI, J. (1972). Semi-stable Markov processes, I. *Z. Wahrscheinlichkeitsth.* **22**, 205–225.

[21] MAULIK, K. AND ZWART, B. (2006). Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Process. Appl.* **116**, 156–177.

[22] PALAU, S., PARDO, J. C. AND SMADI, C. (2016). Asymptotic behaviour of exponential functionals of Lévy processes with applications to random processes in random environment. *ALEA Lat. Am. J. Prob. Math. Statist.* **13**, 1235–1258.

[23] PATIE, P. AND SAVOV, M. (2018). Bernstein-gamma functions and exponential functionals of Lévy processes. *Electron. J. Prob.* **23**, 1–101.

[24] RACHEV, S. (2003). *Handbook of Heavy Tailed Distributions in Finance* (Handbooks in Finance **1**). Elsevier Science.

[25] RESNICK, S. (2007). *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling* (Springer Series in Operations Research and Financial Engineering). Springer, New York.

[26] RIVERO, V. (2012). Tail asymptotics for exponential functionals of Lévy processes: the convolution equivalent case. *Ann. Inst. H. Poincaré Prob. Statist.* **48**, 1081–1102.

[27] STEPHENSON, R. (2018). On the exponential functional of Markov additive processes, and applications to multi-type self-similar fragmentation processes and trees. *ALEA Lat. Am. J. Prob. Math. Statist.* **15**, 1257–1292.

[28] WILLEKENS, E. (1987). On the supremum of an infinitely divisible process. *Stoch. Process. Appl.* **26**, 173–175.

[29] ZACHARY, S. (2004). A note on Veraverbeke's theorem. *Queueing Systems* **46**, 9–14.