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Non-trivial solutions for a semilinear biharmonic problem with critical growth and potential vanishing at infinity

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In this paper, we study the existence of non-trivial solutions for the following class of semilinear biharmonic problem with critical nonlinearity:

$$\Delta^2 u + V(x)u = \mu K(x)f(u) + P(x)|u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N,\ u \in \mathcal{D}^{2,2}(\mathbb{R}^N).$$

Here $\Delta^2 u = \Delta(\Delta u)$, $N \geqslant 5$, $\mu > 0$ is a parameter, $2^{**} = 2N/(N-4)$ is the critical Sobolev exponent, $V(x)$ and $K(x)$ are positive continuous functions that vanish at infinity, f is a function with a subcritical growth and $P(x)$ is a bounded, non-negative continuous function. By working in weighted Sobolev spaces and using a variational method, we prove that the problem has at least one non-trivial solution.

1. Introduction and main results

The main purpose of this paper is to discuss the existence of non-trivial solution for the following class of semilinear biharmonic problem with critical nonlinearity

$$\Delta^2 u + V(x)u = \mu K(x)f(u) + P(x)|u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N,\ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N \geqslant 5$, $\mu > 0$ is a parameter and $2^{**} = 2N/(N-4)$ is the critical Sobolev exponent. The potential V and $K\colon \mathbb{R}^N \to \mathbb{R}$ are positive continuous functions that vanish at infinity, $f\colon \mathbb{R} \to \mathbb{R}$ is a function with a subcritical growth and $P(x) \geqslant 0$ is a bounded continuous function.

Over the last several decades, many authors have shown interest in second-order elliptic differential equations in unbounded domains with critical growth. For example, in the celebrated papers [17,18], Lions established a concentration–compactness principle for some nonlinear elliptic equations in $\mathbb{R}^N$ and studied minimization problems associated with nonlinear elliptic equations in $\mathbb{R}^N$ with critical growth. Following the ideas established by Lions and Brézis [8], a wide class of nonlinear critical elliptic problems have been studied. The reader is referred to [10–13, 16, 23, 25, 26] and references therein.

In particular, Deng *et al.* [13] established a complete non-compact expression for the Palais–Smale (PS) sequences of the variational functional corresponding to

$$-\Delta u - \mu\frac{u}{|x|^2} + V(x)u = |u|^{2^*-2}u + f(x,u), \quad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

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which included all the blow-up bubbles caused by critical exponents, the Hardy term and unbounded domains. By using the non-compact expression for the Palais–Smale sequences of the variational functional corresponding to (1.2), the existence of positive solutions of (1.2) is obtained but they require the potential $V(x)$ to be non-vanishing at infinity.

An important class of problems associated with (1.2) is the zero mass case that occurs with the potentials $V(x)$ vanishing at infinity, that is,

$$\lim_{|x|\to+\infty} V(x) = 0.$$

A typical example is the equation

$$-\Delta u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

with $\lim_{|x|\to+\infty} V(x) = 0$.

In [4], Ambrosetti *et al*. studied (1.3) with the zero mass case when

$$f(s) = s^p \quad \text{with } 2 < p < \frac{N+2}{N-2}$$

and V, K satisfying the following assumptions.

$V, K\colon \mathbb{R}^N \to \mathbb{R}$ are smooth functions and there exist constants $\alpha, \beta, a, A, \kappa > 0$ such that

$$\frac{a}{1+|x|^\alpha} \leqslant V(x) \leqslant A \quad \text{and} \quad 0 < K(x) \leqslant \frac{\kappa}{1+|x|^\beta} \quad \forall x \in \mathbb{R}^N \tag{VK}$$

and such that α and β verify

$$\frac{N+2}{N-2} - \frac{4\beta}{\alpha(N-2)} < p \text{ if } 0 < \beta < \alpha \quad \text{or} \quad p > 1 \text{ when } \beta > \alpha.$$

The condition (VK) is interesting because Opic and Kufner [21] have showed that it can be used to prove that the space E given by

$$E = \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x < +\infty\right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \,\mathrm{d}x$$

is compactly embedded into the weighted Lebesgue space

$$L_K^{p+1}(\mathbb{R}^N) = \left\{u\colon \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^{p+1} \,\mathrm{d}x < +\infty\right\}.$$

In [3], Ambrosetti and Wang also considered the condition (VK) but the inequality on V was assumed only outside of a ball centred at origin.

In [1], Alves and Souto considered a more general condition on $V(x)$ and $K(x)$, from which the space E can be compactly embedded into the weighted space.

In [6], Bonheure and Van Schaftingen introduced a new set of hypotheses on $V(x)$ and $K(x)$ by using the Marcinkiewicz spaces $L^{r,\infty}(\mathbb{R}^N)$ for $r > 1$, which permitted them to show continuous and compact embeddings from E into the weighted space $L^q_K(\mathbb{R}^N)$ for some $q > 1$. Using the compactness results obtained in [1,3,6], one can obtain the existence of positive solutions for (1.3) when $f(s)$ is subcritical under some assumptions on $V(x)$ and $K(x)$. For the critical case, we discussed a general problem

$$-\Delta_p u + V(x)|u|^{p-2}u = K(x)f(u) + P(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N,\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $p^* = Np/(N-p)$, $V(x)$ and $K(x)$ are positive continuous functions that vanish at infinity, f is a function with a subcritical growth and $P(x)$ is a bounded non-negative continuous function. By working in the weighted Sobolev spaces and using a variational method, we prove that the problem has at least one positive solution (see [14]).

However, there seems to be little progress on the existence of a non-trival solution for the biharmonic equation (see, for example, (1.1)) with subcritical growth or critical growth when the potential $V(x)$ vanishes at infinity.

In this paper, we establish the existence of a non-trival solution of (1.1) with critical nonlinearity and the potential $V(x)$ vanishing at infinity. To this end, we need some assumptions on $V(x)$, $K(x)$, $f(s)$ and $P(x)$.

As in [1], we say $(V,K) \in \mathcal{K}$ if the following conditions hold.

(i) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K(x) \in L^\infty(\mathbb{R}^N)$.

(ii) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leqslant R$ for all n and some $R > 0$, we have that

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)\,\mathrm{d}x = 0 \quad \text{uniformly in } n \in \mathbb{N}. \qquad (K_1)$$

(iii) One of the following conditions occurs:

$$\frac{K(x)}{V(x)} \in L^\infty(\mathbb{R}^N) \qquad (K_2)$$

or there is a $p_0 \in (2, 2^{**})$ such that

$$\frac{K(x)}{|V(x)|^{(2^{**}-p_0)/(2^{**}-2)}} \to 0 \quad \text{as } |x| \to +\infty. \qquad (K_3)$$

Related to the function f, we assume the following conditions.

(f_1)

$$\limsup_{s\to 0} \frac{f(s)}{s} = 0 \qquad \text{if } (K_2) \text{ holds}$$

or

$$\limsup_{s\to 0} \frac{f(s)}{|s|^{p_0-1}} < +\infty \quad \text{if } (K_3) \text{ holds.}$$

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(f_2) f has a subcritical growth, that is,

$$\limsup_{s\to+\infty}\frac{f(s)}{|s|^{2^{**}-1}}=0.$$

(f_3) There exists a $\theta\in(2,2^{**})$ such that

$$0\leqslant\theta F(s)\leqslant sf(s)\quad\text{for all } s\in\mathbb{R},$$

where $F(u)=\int_0^u f(t)\,\mathrm{d}t$.

Moreover, as for the function $P(x)$, we assume the following.

(P_1) There is a point x_0 such that

$$P(x_0)=\sup_{x\in\mathbb{R}^N}P(x)>0.$$

(P_2) For x close to x_0 we have

$$P(x)=P(x_0)+O(|x-x_0|^\tau)\quad\text{as } x\to x_0,$$

where $\tau\geqslant\min\{4,N-4\}$ is a real number.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $(V,K)\in\mathcal{K}$, f satisfies (f_1)–(f_3) and $P(x)$ satisfies (P_1) and (P_2). Then (1.1) has at least one non-trivial solution if μ, θ and N satisfy one of the following three conditions:*

(A_1) $N\geqslant 8$, $2<\theta<2^{**}$ *and* $\mu>0$;

(A_2) $4<N<8$, $2^{**}-2<\theta<2^{**}$ *and* $\mu>0$;

(A_3) $4<N<8$, $2<\theta\leqslant 2^{**}-2$ *and μ is sufficiently large.*

For the results concerned with fourth-order biharmonic equations involving critical Sobolev exponent on bounded domain, readers are referred to [2,9,15,19,20,22] and references therein.

There are serious difficulties in trying to find the non-trivial solutions of (1.1) by standard variational methods since the space $\mathcal{D}^{2,2}(\mathbb{R}^N)$ can not be embedded into $L^r(\mathbb{R}^N)$ for $r\in(2,2^{**})$ and the embedding $\mathcal{D}^{2,2}(\mathbb{R}^N)\hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ is not compact. Moreover, because the potential $V(x)$ vanishes at infinity, there are also some difficulties to be overcome in dealing with (1.1). In order to prove the existence result, we first define the weight Sobolev space E and $L^q_K(\mathbb{R}^N)$ and then establish a Hardy-type inequality involving V and K (see lemmas 2.1 and 2.2) as in [1]. Since the embedding $E\hookrightarrow L^{p^*}_P(\mathbb{R}^N)$ is still not compact, the method provided in [1] can not be used directly. To overcome this lack of compactness, we imitate the method in [8] by using the mountain pass theorem without (PS) condition, and the existence of a non-trival solution of (1.1) is proved.

The rest of this paper is organized as follow. In § 2 we present some embedding results that generalize the corresponding embedding results of [1]. In § 3 we prove theorem 1.1.

2. Some preliminary lemmas

In this section, we introduce some weighted Sobolev spaces and prove some embedding theorems. To this end, we define the space

$$E := \left\{ u \in \mathcal{D}^{2,2}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)|u|^2) \, \mathrm{d}x \right)^{1/2}.$$

Denote by $L^q_K(\mathbb{R}^N)$ the weighted Lebesgue space

$$L^q_K(\mathbb{R}^N) = \left\{ u \colon \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q \, \mathrm{d}x < +\infty \right\}$$

endowed with the norm

$$\|u\|_{L^q_K(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} K(x)|u|^q \, \mathrm{d}x \right)^{1/q}.$$

E and $L^q_K(\mathbb{R}^N)$ are particular cases of weighted space and are discussed in [21]. The following two lemmas provide the continuous and compact embedding for $E \hookrightarrow L^q_K(\mathbb{R}^N)$.

LEMMA 2.1. *Assume that $(V, K) \in \mathcal{K}$. Then E can be continuously embedded in $L^q_K(\mathbb{R}^N)$ for all $q \in [2, 2^{**}]$ if (K_2) holds. Moreover, E can be continuously embedded in $L^{p_0}_K(\mathbb{R}^N)$ if (K_3) holds.*

Proof. First we assume that (K_2) holds. The proof is trivial if $q = 2$ or 2^{**}. Now we prove that the embedding is true for $q \in (2, 2^{**})$ under the assumption (K_2). For fixed $q \in (2, 2^{**})$, define $\lambda = (2^{**} - q)/(2^{**} - 2)$, and hence $q = 2\lambda + (1 - \lambda)2^{**}$ so we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)|u|^q \, \mathrm{d}x &= \int_{\mathbb{R}^N} K(x)|u|^{2\lambda}|u|^{(1-\lambda)2^{**}} \, \mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{R}^N} |K(x)|^{1/\lambda}|u|^2 \, \mathrm{d}x \right)^{\lambda} \left(\int_{\mathbb{R}^N} |u|^{2^{**}} \, \mathrm{d}x \right)^{1-\lambda} \\
&\leqslant \left(\sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left(\int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x \right)^{\lambda} \left(\int_{\mathbb{R}^N} |u|^{2^{**}} \, \mathrm{d}x \right)^{1-\lambda} \\
&\leqslant C \left(\sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left(\int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x \right)^{\lambda} \left(\int_{\mathbb{R}^N} |\Delta u|^2 \, \mathrm{d}x \right)^{(1-\lambda)2^{**}/2} \\
&\leqslant C \left(\sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left(\int_{\mathbb{R}^N} |\Delta u|^2 + V(x)|u|^2 \, \mathrm{d}x \right)^{\lambda + (1-\lambda)2^{**}/2} \\
&= C \left(\sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left(\int_{\mathbb{R}^N} |\Delta u|^2 + V(x)|u|^2 \, \mathrm{d}x \right)^{q/2}.
\end{aligned}$$

Since $K(x) \in L^\infty(\mathbb{R}^N)$ and $K/V \in L^\infty(\mathbb{R}^N)$, we have that

$$\|u\|_{L^q_K(\mathbb{R}^N)} \leqslant C\|u\| \quad \text{for } q \in (2, 2^{**}).$$

Next, we suppose that (K_3) holds. Using the same argument as above, we define $\lambda_0 = (2^{**} - p_0)/(2^{**} - 2)$, and hence $p_0 = 2\lambda_0 + (1-\lambda_0)2^{**}$ so that we have

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|u|^{p_0}\,\mathrm{d}x &= \int_{\mathbb{R}^N} K(x)|u|^{2\lambda_0}|u|^{(1-\lambda_0)2^{**}}\,\mathrm{d}x \\ &\leqslant \biggl(\int_{\mathbb{R}^N} |K(x)|^{1/\lambda_0}|u|^2\,\mathrm{d}x\biggr)^{\lambda_0}\biggl(\int_{\mathbb{R}^N}|u|^{2^{**}}\,\mathrm{d}x\biggr)^{1-\lambda_0} \\ &\leqslant \biggl(\sup_{x\in\mathbb{R}^N}\frac{|K(x)|}{|V(x)|^{\lambda_0}}\biggr)\biggl(\int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x\biggr)^{\lambda_0}\biggl(\int_{\mathbb{R}^N}|u|^{2^{**}}\,\mathrm{d}x\biggr)^{1-\lambda_0} \\ &\leqslant C\biggl(\sup_{x\in\mathbb{R}^N}\frac{|K(x)|}{|V(x)|^{\lambda_0}}\biggr)\biggl(\int_{\mathbb{R}^N}|\Delta u|^2 + V(x)|u|^2\,\mathrm{d}x\biggr)^{p_0/2}.\end{aligned}$$

From (K_3) we deduce that $K(x)/|V(x)|^{(2^{**}-p_0)/(2^{**}-2)} \in L^\infty(\mathbb{R}^N)$. It follows from the above inequality that

$$\|u\|_{L^{p_0}_K(\mathbb{R}^N)} \leqslant C\|u\|.$$

The proof of our lemma is completed. □

Lemma 2.2. *Assume that $(V, K) \in \mathcal{K}$. Then E can be compactly embedded into $L^q_K(\mathbb{R}^N)$ for all $q \in (2, 2^{**})$ if (K_2) holds. Moreover, E can be compactly embedded into $L^{p_0}_K(\mathbb{R}^N)$ if (K_3) holds.*

Proof. The proof of this lemma is divided into two parts.

First we assume that the condition (K_2) holds. For fixed $q \in (2, 2^{**})$ and given $\varepsilon > 0$, there are $0 < s_0 < s_1$ and $C > 0$ such that

$$K(x)|s|^q \leqslant \varepsilon C(V(x)|s|^2 + |s|^{2^{**}}) + CK(x)\chi_{[s_0,s_1]}(|s|)|s|^{2^{**}} \quad \forall s \in \mathbb{R}. \tag{2.1}$$

Hence,

$$\int_{B_r^c(0)} K(x)|u|^q\,\mathrm{d}x \leqslant \varepsilon C Q(u) + C\int_{A\cap B_r^c(0)} K(x)|u|^{2^{**}}\,\mathrm{d}x \quad \forall u \in E, \tag{2.2}$$

where

$$Q(u) = \int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}|u|^{2^{**}}\,\mathrm{d}x$$

and

$$A = \{x \in \mathbb{R}^N : s_0 \leqslant |u(x)| \leqslant s_1\}.$$

If $\{v_n\}$ is a sequence such that $v_n \rightharpoonup v$ in E, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N}(|\Delta v_n|^2 + V(x)|v_n|^2)\,\mathrm{d}x \leqslant M_1 \quad \text{and} \quad \int_{\mathbb{R}^N}|v_n|^{2^{**}}\,\mathrm{d}x \leqslant M_1 \quad \forall n \in \mathbb{N},$$

which gives that $\{Q(v_n)\}$ is bounded. On the other hand, setting

$$A_n = \{x \in \mathbb{R}^N : s_0 \leqslant |v_n(x)| \leqslant s_1\},$$

the last inequality implies that

$$s_0^{2^{**}}|A_n| \leqslant \int_{A_n} |v_n|^{2^{**}}\,\mathrm{d}x \leqslant M_1 \quad \forall n \in \mathbb{N},$$

which gives that $\sup_{n\in\mathbb{N}} |A_n| < +\infty$. Therefore, from (K_1), there is an $r > 0$ such that

$$\int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x < \frac{\varepsilon}{s_1^{2^{**}}} \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

From (2.2) and (2.3) we deduce that

$$\int_{B_r^c(0)} K(x)|v_n|^q\,\mathrm{d}x \leqslant 2\varepsilon C M_1 + C s_1^{2^{**}} \int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x < (2CM_1 + C)\varepsilon \tag{2.4}$$

for all $n \in \mathbb{N}$. Since $q \in (2, 2^{**})$ and K is a continuous function, it follows from Sobolev embeddings on the bounded domain that

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)|v_n|^q\,\mathrm{d}x = \int_{B_r(0)} K(x)|v|^q\,\mathrm{d}x. \tag{2.5}$$

Combining (2.4) and (2.5),

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)|v_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} K(x)|v|^q\,\mathrm{d}x. \tag{2.6}$$

which yields

$$v_n \to v \quad \text{in } L_K^q(\mathbb{R}^N)\ \forall q \in (2, 2^{**}).$$

Next we suppose that (K_3) holds. It is important to observe that for each $x \in \mathbb{R}^N$ fixed, the function

$$g(s) = V(x)s^{2-p_0} + s^{2^{**}-p_0} \quad \forall s > 0$$

has $C_{p_0}V^{(2^{**}-p_0)/P(2^{**}-2)}(x)$ as its minimum value, where

$$C_{p_0} = \left(\frac{2^{**}-2}{2^{**}-p_0}\right)\left(\frac{p_0-2}{2^{**}-p_0}\right)^{(2-p_0)/(2^{**}-2)}.$$

Hence,

$$C_{p_0}V^{(2^{**}-p_0)/(2^{**}-2)}(x) \leqslant V(x)s^{2-p_0} + s^{2^{**}-p_0} \quad \forall x \in \mathbb{R}^N \text{ and } s > 0.$$

It follows from assumption (K_3) that for given $\varepsilon \in (0, C_{p_0})$, there is $r > 0$ large enough such that

$$K(x)|s|^{p_0} \leqslant C\varepsilon(V(x)|s|^2 + |s|^{2^{**}}) \quad \forall s \in \mathbb{R} \text{ and } |x| \geqslant r,$$

which leads to

$$\int_{B_r^c(0)} K(x)|u|^{p_0}\,\mathrm{d}x \leqslant C\varepsilon \int_{B_r^c(0)} (V(x)|u|^2 + |u|^{2^{**}})\,\mathrm{d}x \quad \forall u \in E.$$

If $\{v_n\}$ is a sequence such that $v_n \rightharpoonup v$ in E, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^2\,\mathrm{d}x \leqslant M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{2^{**}}\,\mathrm{d}x \leqslant M_1 \quad \forall n \in \mathbb{N}$$

and so

$$\int_{B_r^c(0)} K(x)|v_n|^{p_0}\,\mathrm{d}x \leqslant 2\varepsilon C M_1 \quad \forall n \in \mathbb{N}. \tag{2.7}$$

Since $p_0 \in (2, 2^{**})$ and K is a continuous function, it follows from Sobolev embeddings on the bounded domain that

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)|v_n|^{p_0}\,\mathrm{d}x = \int_{B_r(0)} K(x)|v|^{p_0}\,\mathrm{d}x. \tag{2.8}$$

From (2.7) and (2.8),

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)|v_n|^{p_0}\,\mathrm{d}x = \int_{\mathbb{R}^N} K(x)|v|^{p_0}\,\mathrm{d}x, \tag{2.9}$$

which implies that

$$v_n \to v \quad \text{in } L_K^{p_0}(\mathbb{R}^N).$$

This completes our proof. □

Lemma 2.3. *Suppose that f satisfies (f_1) and (f_2) and $(V, K) \in \mathcal{K}$. Let v_n be a sequence such that $v_n \rightharpoonup v$ in E. Then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)F(v_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} K(x)F(v)\,\mathrm{d}x \tag{2.10}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)f(v_n)v_n\,\mathrm{d}x = \int_{\mathbb{R}^N} K(x)f(v)v\,\mathrm{d}x. \tag{2.11}$$

Proof. We only give the proof of (2.10) but (2.11) can be proved in the same way.

We begin the proof by assuming that (K_2) occurs. From (f_1) and (f_2), we deduce that for fixed $q \in (2, 2^{**})$ and given $\varepsilon > 0$, there is $C > 0$ such that

$$K(x)F(s) \leqslant \varepsilon C(V(x)|s|^2 + |s|^{2^{**}}) + CK(x)|s|^q \quad \forall s \in \mathbb{R}. \tag{2.12}$$

From lemma 2.2,

$$\int_{\mathbb{R}^N} K(x)|v_n|^q\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|v|^q\,\mathrm{d}x$$

and there is $r > 0$ such that

$$\int_{B_r^c(0)} K(x)|v_n|^q\,\mathrm{d}x < \varepsilon \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Since $\{v_n\}$ is bounded in E, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^2\,\mathrm{d}x \leqslant M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{2^{**}}\,\mathrm{d}x \leqslant M_1 \quad \forall n \in \mathbb{N}.$$

It follows from (2.12) and (2.13) that

$$\left|\int_{B_r^c} K(x)F(v_n)\,\mathrm{d}x\right| < (2CM_1 + C)\varepsilon \quad \forall n \in \mathbb{N}. \tag{2.14}$$

Next we assume that (K_3) holds. Repeating the same arguments explored in the proof of lemma 2.2, we have, for given $\varepsilon > 0$ small enough, that there is $r > 0$ large enough such that

$$K(x) \leqslant \varepsilon(V(x)|s|^{2-p_0} + |s|^{2^{**}-p_0}) \quad \forall s \in \mathbb{R} \text{ and } |x| > r.$$

From (f_1) and (f_2), for the given $\varepsilon > 0$, we have

$$F(s) \leqslant C|s|^{p_0} + \varepsilon|s|^{2^{**}} \quad \forall s \in I,$$

where $I = \{x \in \mathbb{R}^N : |s| < s_0 \text{ or } |s| > s_1\}$.

Since $K(x) \in L^\infty(\mathbb{R}^N)$, for all $s \in I$ and $|x| > r$ we have

$$\begin{aligned} K(x)|F(s)| &\leqslant CK(x)|s|^{p_0} + \varepsilon K(x)|s|^{2^{**}} \\ &\leqslant C\varepsilon(V(x)|s|^{2-p_0} + |s|^{2^{**}-p_0})|s|^{p_0} + \varepsilon\|K(x)\|_{L^\infty(\mathbb{R}^N)}|s|^{2^{**}} \\ &\leqslant \varepsilon C(V(x)|s|^2 + |s|^{2^{**}}). \end{aligned}$$

Therefore, for any $u \in E$, we have the following estimate

$$\int_{B_r^c(0)} K(x)F(u)\,\mathrm{d}x \leqslant \varepsilon CQ(u) + C\int_{A\cap B_r^c(0)} K(x)\,\mathrm{d}x,$$

where

$$Q(u) = \int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{2^{**}}\,\mathrm{d}x$$

and

$$A = \{x \in \mathbb{R}^N : s_0 \leqslant |u(x)| \leqslant s_1\}.$$

Since $\{v_n\}$ is bounded in E, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^2\,\mathrm{d}x \leqslant M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{2^{**}}\,\mathrm{d}x \leqslant M_1.$$

Thus,

$$\int_{B_r^c(0)} K(x)F(v_n)\,\mathrm{d}x \leqslant 2CM_1\varepsilon + C\int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x,$$

where

$$A_n = \{x \in \mathbb{R}^N : s_0 \leqslant |v_n(x)| \leqslant s_1\}.$$

Repeating the same arguments used in the proof of lemma 2.2, it follows that

$$\int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x \to 0 \quad \text{as } r \to +\infty$$

and so, for n large enough,

$$\left|\int_{B_r^c(0)} K(x)F(v_n)\,\mathrm{d}x\right| \leqslant C(2M_1 + 1)\varepsilon. \tag{2.15}$$

From (2.14) and (2.15), we need to show that

$$\lim_{n\to+\infty}\int_{B_r(0)} K(x)F(v_n)\,\mathrm{d}x = \int_{B_r(0)} K(x)F(v)\,\mathrm{d}x.$$

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However, this limit follows by using a compactness lemma of Strauss [24, compactness lemma 2, p.156]: $B_r(0)$ is a bounded domain, $|v_n|_{L^{2^{**}}}(B_r(0))$ is bounded and (f_2), together with the convergence almost everywhere, imply the limit as required. □

3. The proof of theorem 1.1

In this section, we prove the existence of a non-trivial solution of (1.1) by the mountain pass lemma [8] without (PS) condition. The basic arguments are adapted from [1, 8, 9, 19].

The variational functional associated with (1.1) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + V(x)u^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N}K(x)F(u)\,\mathrm{d}x - \frac{1}{2^{**}}\int_{\mathbb{R}^N}P(x)|u|^{2^{**}}\,\mathrm{d}x \quad \forall u \in E. \tag{3.1}$$

From the conditions on $f(s)$ and lemmas 2.1 and 2.2, the functional I is well defined and $I \in \mathcal{C}^1(E, \mathbb{R})$. Its Gateaux derivative is given by

$$I'(u)v = \int_{\mathbb{R}^N}(\Delta u\Delta v + V(x)uv)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N}K(x)f(u)v\,\mathrm{d}x - \int_{\mathbb{R}^N}P(x)|u|^{2^{**}-2}uv\,\mathrm{d}x \tag{3.2}$$

for all $u, v \in E$. It is then easy to check that the critical points of I are weak solutions of (1.1).

Since E can be embedded into $L^q_K(\mathbb{R}^N)$ continuously for some q (see lemma 2.1), we can verify that the functional I exhibits the mountain pass geometry.

Lemma 3.1. *The functional I satisfies the following two conditions:*

(i) *there exist $\alpha, \rho > 0$ such that $I(u) > \alpha$ for all $\|u\| = \rho$;*

(ii) *there exists an $e \in E$ such that $\|u\| > \rho$ and $I(e) < 0$.*

As a consequence of lemma 3.1 and the mountain pass lemma [8], for the constant

$$c_0 = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)) > 0, \tag{3.3}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], E),\ \gamma(0) = 0,\ \gamma(1) \neq 0,\ I(\gamma(1)) < 0\}$$

there exists a $(\mathrm{PS})_{c_0}$ sequence $\{u_n\}$ in E at the level c_0, that is,

$$I(u_n) \to c_0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.4}$$

Lemma 3.2. *The sequence $\{u_n\}$ in (3.4) is bounded in E.*

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Proof. From (f_3) we have

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \frac{1}{\theta}\mu \int_{\mathbb{R}^N} K(x)(f(u_n)u_n - \theta F(u_n))\,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} P(x)|u_n|^{2^{**}}\,\mathrm{d}x \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2.$$

Since $I(u_n) \to c_0$ and $I'(u_n) \to 0$ as $n \to +\infty$, we obtain that $\{u_n\}$ is bounded in E. □

Using a standard argument, it follows that there is a $u \in E$ such that (up to a subsequence)

$$\left.\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E,\\ &u_n \to u \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } 2 \leqslant r < 2^{**},\\ &u_n \to u \quad \text{almost everywhere in } \mathbb{R}^N. \end{aligned}\right\} \tag{3.5}$$

In the following we prove that u must be a non-trivial solution of (1.1). To this end, we exploit the fact that the critical equation

$$\Delta^2 u = |u|^{2^{**}-2}u \quad \text{in } \mathbb{R}^N$$

has positive solutions

$$u_\varepsilon(x) = \frac{C_N \varepsilon^{(N-4)/2}}{(\varepsilon^2 + |x - x_0|^2)^{(N-4)/2}}, \quad C_N = [(N-4)(N-2)N(N+2)]^{(N-4)/8},$$

for any $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$. Furthermore,

$$\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |u_\varepsilon|^{2^{**}}\,\mathrm{d}x = S^{N/4},$$

where S denotes the best constant for the embedding $\mathcal{D}^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$, namely,

$$S := \inf_{u \in \mathcal{D}^{2,2}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\Delta u|^2\,\mathrm{d}x,\ \int_{\mathbb{R}^N} |u|^{2^{**}}\,\mathrm{d}x = 1 \right\}.$$

For small enough $R > 0$, define a cut-off function $\psi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supt}\{\psi\} \in B_{2R}(x_0)$, $\psi(x) \equiv 1$ in $B_R(x_0)$, $0 \leqslant \psi(x) \leqslant 1$ in $B_{2R}(x_0)$ and $|\nabla\psi| \leqslant C/R$. Define

$$w_\varepsilon(x) = \psi(x)u_\varepsilon(x), \tag{3.6}$$

$$v_\varepsilon(x) = w_\varepsilon(x)\left[\int_{\mathbb{R}^N} P(x) w_\varepsilon^{2^{**}}(x)\,\mathrm{d}x\right]^{-1/2^{**}}, \tag{3.7}$$

$$V_{\max} := \max_{x \in B_{2R}(x_0)} V(x),$$

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and

$$K_{\min} := \min_{x\in B_{2R}(x_0)} K(x).$$

By direct calculation, we have the inequality

$$\begin{aligned}\int_{B_R(x_0)} |\Delta w_\varepsilon|^2 \,\mathrm{d}x &= \int_{B_R(x_0)} |\Delta u_\varepsilon|^2 \,\mathrm{d}x \\ &= \int_{B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x - \int_{\partial B_R(x_0)} \frac{\partial(\Delta u_\varepsilon)}{\partial \boldsymbol{n}} u_\varepsilon \,\mathrm{d}S + \int_{\partial B_R(x_0)} \frac{\partial u_\varepsilon}{\partial \boldsymbol{n}}(\Delta u_\varepsilon)\,\mathrm{d}S \\ &\leqslant \int_{B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x\end{aligned}$$

and by the assumption on $P(x)$, we also have

$$P(x_0)\int_{B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x \leqslant \int_{B_R(x_0)} P(x)|u_\varepsilon|^{2^{**}} \,\mathrm{d}x + O(\varepsilon^\tau). \tag{3.8}$$

Simple calculations also show that

$$\begin{gathered}\int_{\mathbb{R}^N\setminus B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x = O(\varepsilon^N), \\ A_\varepsilon := \int_{\mathbb{R}^N\setminus B_R(x_0)} |\Delta w_\varepsilon|^2 \,\mathrm{d}x = O(\varepsilon^{N-4}), \\ \int_{\mathbb{R}^N} |v_\varepsilon|^2 \,\mathrm{d}x = \begin{cases} O(\varepsilon^\gamma), & \text{if } N\geqslant 5,\ N\neq 8, \\ O(\varepsilon^\gamma|\ln\varepsilon|), & \text{if } N=8\end{cases}\end{gathered} \tag{3.9}$$

as $\varepsilon\to 0$, where $\gamma=\min\{4, N-4\}$. Since

$$S = \left[\int_{\mathbb{R}^N} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x\right]^{4/N},$$

we have that

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta w_\varepsilon|^2 \,\mathrm{d}x &= \int_{B_R(x_0)} |\Delta w_\varepsilon|^2 \,\mathrm{d}x + A_\varepsilon \\ &\leqslant \int_{B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x + A_\varepsilon \\ &\leqslant S\left[\int_{B_R(x_0)} |u_\varepsilon|^{2^{**}} \,\mathrm{d}x\right]^{2/2^{**}} + A_\varepsilon \\ &\leqslant S[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{-2/2^{**}}\left[\int_{B_R(x_0)} P(x)|u_\varepsilon|^{2^{**}} \,\mathrm{d}x\right]^{2/2^{**}} \\ &\quad + O(\varepsilon^\tau) + O(\varepsilon^{N-4}).\end{aligned} \tag{3.10}$$

Note that the last inequality is from (3.8).

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Set $V_\varepsilon \equiv \int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2 \,\mathrm{d}x$ since, for small $\varepsilon > 0$, say $\varepsilon \leqslant \varepsilon_0$, it is easy to see that

$$\int_{B_R(x_0)} P(x)|w_\varepsilon|^{2^{**}} \,\mathrm{d}x \geqslant C_{\varepsilon_0}$$

for some positive constant C_{ε_0}. It follows from (3.10) and the definition of V_ε that

$$V_\varepsilon \leqslant S[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{-2/2^{**}} + O(\varepsilon^\tau) + O(\varepsilon^{N-4}). \tag{3.11}$$

LEMMA 3.3. *Assume that $(V, K) \in \mathcal{K}$, that f satisfies (f_1)–(f_3) and that $P(x)$ satisfies (P_1) and (P_2). There then exists a $u_0 \in E \setminus \{0\}$ such that*

$$0 < \sup_{t\geqslant 0} I(tu_0) < \frac{2}{N} S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} \tag{3.12}$$

if one of the assumptions (A_1), (A_2) and (A_3) holds.

Proof. We now consider

$$\begin{aligned} I(tv_\varepsilon) &= \frac{t^2}{2}\int_{\mathbb{R}^N} (|\Delta v_\varepsilon|^2 + V(x)|v_\varepsilon|^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)F(tv_\varepsilon)\,\mathrm{d}x \\ &\quad - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} P(x)|v_\varepsilon|^{2^{**}}\,\mathrm{d}x \\ &= \frac{t^2}{2}V_\varepsilon + \frac{t^2}{2}\int_{B_{2R}(x_0)} V(x)|v_\varepsilon|^2\,\mathrm{d}x - \mu\int_{B_{2R}(x_0)} K(x)F(tv_\varepsilon)\,\mathrm{d}x - \frac{t^{2^{**}}}{2^{**}}. \end{aligned}$$

By assumptions (f_1) and (f_2), we can easily verify that $\lim_{t\to+\infty} I(tv_\varepsilon) = -\infty$ for all $\varepsilon > 0$ and $\sup_{t\geqslant 0} I(tv_\varepsilon) > 0$ is attained by some $t_\varepsilon > 0$.

We claim that there are two positive constants A_1, A_2 independent of ε such that $A_1 < t_\varepsilon < A_2$ for small $\varepsilon > 0$.

In fact, since $I(t_\varepsilon v_\varepsilon) = \sup_{t\geqslant 0} I(tv_\varepsilon)$, and hence $\mathrm{d}I(tv_\varepsilon)/\mathrm{d}t|_{t=t_\varepsilon} = 0$, we have that

$$\begin{aligned} t_\varepsilon \int_{B_{2R}(x_0)} (|\Delta v_\varepsilon|^2 + V(x)|v_\varepsilon|^2)\,\mathrm{d}x - \mu\int_{B_{2R}(x_0)} K(x)f(t_\varepsilon v_\varepsilon)v_\varepsilon\,\mathrm{d}x & \\ - t_\varepsilon^{2^{**}-1}\int_{B_{2R}(x_0)} P(x)|v_\varepsilon|^{2^{**}}\,\mathrm{d}x = 0. & \end{aligned} \tag{3.13}$$

If there is a sequence $t_{\varepsilon_n} \to +\infty$ as $\varepsilon_n \to 0^+$, by (3.13) we get

$$t_{\varepsilon_n}\int_{B_{2R}(x_0)} (|\Delta v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2)\,\mathrm{d}x \geqslant (t_{\varepsilon_n})^{2^{**}-1}\int_{B_{2R}(x_0)} P(x)|v_{\varepsilon_n}|^{2^{**}}\,\mathrm{d}x.$$

This is impossible because $2^{**} - 1 > 1$.

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Similarly, we suppose that there is a sequence $t'_{\varepsilon_n} \to 0$ as $\varepsilon_n \to 0^+$. Firstly, if (K_2) holds, from (f_1) and (f_2), for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} K(x) f(t'_{\varepsilon_n} v_{\varepsilon_n}) v_{\varepsilon_n} \,\mathrm{d}x &\leqslant \delta t'_{\varepsilon_n} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^2 \,\mathrm{d}x + C_\delta (t'_{\varepsilon_n})^{2^{**}-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x \\ &\leqslant \delta C t'_{\varepsilon_n} \int_{\mathbb{R}^N} (|\Delta v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2) \,\mathrm{d}x + C_\delta (t'_{\varepsilon_n})^{2^{**}-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x.\end{aligned}$$

Taking $\delta = 1/2C$, it follows from (3.13) that

$$\begin{aligned}\frac{t'_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} (|\Delta v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2) \,\mathrm{d}x &\leqslant C_\delta (t'_{\varepsilon_n})^{2^{**}-1} \mu \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x + (t'_{\varepsilon_n})^{2^{**}-1} \int_{\mathbb{R}^N} P(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x.\end{aligned}$$

This is also impossible because $2^{**} - 1 > 1$.

Next, we suppose that (K_3) holds. By (f_1), (f_2), there is a constant $\tilde{C} > 0$, such that

$$\begin{aligned}\int_{\mathbb{R}^N} K(x) f(t'_{\varepsilon_n} v_{\varepsilon_n}) v_{\varepsilon_n} \,\mathrm{d}x &\leqslant (t'_{\varepsilon_n})^{p_0 - 1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{p_0} \,\mathrm{d}x + \tilde{C} (t'_{\varepsilon_n})^{2^{**}-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x.\end{aligned}$$

It again follows from (3.13) that

$$\begin{aligned}t'_{\varepsilon_n} \int_{\mathbb{R}^N} (|\Delta v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2) \,\mathrm{d}x &\leqslant (t'_{\varepsilon_n})^{p_0 - 1} \mu \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{p_0} \,\mathrm{d}x + \tilde{C} (t'_{\varepsilon_n})^{2^{**}-1} \mu \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x \\ &\quad + (t'_{\varepsilon_n})^{2^{**}-1} \int_{\mathbb{R}^N} P(x)|v_{\varepsilon_n}|^{2^{**}} \,\mathrm{d}x,\end{aligned}$$

which is also impossible because $p_0 > 2$ and $2^{**} - 1 > 1$. So the proof of our claim is completed.

Since $0 < A_1 < t_\varepsilon < A_2 < \infty$, together with the definitions of $V_{\max}$ and $K_{\min}$, we have,

$$\begin{aligned}I(tv_\varepsilon) &= \frac{t^2}{2} \int_{\mathbb{R}^N} (|\Delta v_\varepsilon|^2 + V(x)|v_\varepsilon|^2) \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x) F(tv_\varepsilon) \,\mathrm{d}x \\ &\quad - \frac{t^{2^{**}}}{2^{**}} \int_{\mathbb{R}^N} P(x)|v_\varepsilon|^{2^{**}} \,\mathrm{d}x \\ &= \frac{t^2}{2} V_\varepsilon + \frac{t^2}{2} \int_{B_{2R}(x_0)} V(x)|v_\varepsilon|^2 \,\mathrm{d}x - \mu \int_{B_{2R}(x_0)} K(x) F(tv_\varepsilon) \,\mathrm{d}x - \frac{t^{2^{**}}}{2^{**}}\end{aligned}$$

$$\leqslant \frac{t_\varepsilon^2}{2}V_\varepsilon + \frac{t_\varepsilon^2}{2}\int_{B_{2R}(x_0)} V(x)|v_\varepsilon|^2\,\mathrm{d}x - \mu\int_{B_{2R}(x_0)} K(x)F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x - \frac{t_\varepsilon^{2^{**}}}{2^{**}}$$
$$\leqslant \frac{t_\varepsilon^2}{2}V_\varepsilon + \frac{t_\varepsilon^2}{2}V_{\max}\int_{B_{2R}(x_0)} |v_\varepsilon|^2\,\mathrm{d}x - \mu K_{\min}\int_{B_{2R}(x_0)} F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x - \frac{t_\varepsilon^{2^{**}}}{2^{**}}.$$

Since $(t^2/2)V_\varepsilon - t^{2^{**}}/2^{**} \leqslant (2/N)V_\varepsilon^{N/4}$, for all $t \geqslant 0$, we deduce from (3.9) and (3.11) that

$$\begin{aligned}\sup_{t\geqslant 0} I(tv_\varepsilon) &= I(t_\varepsilon v_\varepsilon)\\ &\leqslant \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} + O(\varepsilon^\tau) + O(\varepsilon^{N-4})\\ &\quad + \frac{t_\varepsilon^2}{2}V_{\max}\int_{B_{2R}(x_0)} |v_\varepsilon|^2\,\mathrm{d}x - \mu K_{\min}\int_{B_{2R}(x_0)} F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x\\ &\leqslant \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} - \mu K_{\min}\int_{B_{2R}(x_0)} F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x\\ &\quad + \begin{cases} O(\varepsilon^\gamma), & \text{if } N \geqslant 5,\ N \neq 8,\\ O(\varepsilon^\gamma|\ln\varepsilon|), & \text{if } N = 8,\end{cases} \qquad \gamma = \min\{4, N-4\}. \qquad (3.14)\end{aligned}$$

By (f_3), we have that

$$F(s) \geqslant Cs^\theta$$

for $s > 0$. Therefore,

$$\int_{B_{2R}(x_0)} F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x \geqslant C\int_{B_{2R}(x_0)} (t_\varepsilon v_\varepsilon)^\theta\,\mathrm{d}x \geqslant CA_1^\theta\int_{B_R(x_0)} (v_\varepsilon)^\theta\,\mathrm{d}x.$$

It follows from (3.14) that

$$\begin{aligned}\sup_{t\geqslant 0} I(tv_\varepsilon) &\leqslant \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} - \mu K_{\min}CA_1^\theta\int_{B_R(x_0)} (v_\varepsilon)^\theta\,\mathrm{d}x\\ &\quad + \begin{cases} O(\varepsilon^\gamma), & \text{if } N \geqslant 5,\ N \neq 8,\\ O(\varepsilon^\gamma|\ln\varepsilon|), & \text{if } N = 8,\end{cases} \qquad \gamma = \min\{4, N-4\}\\ &\leqslant \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4}\\ &\quad - \mu C\varepsilon^{N-(N-4)\theta/2}\int_0^{R/\varepsilon}\left(\frac{1}{(1+t^2)^{(N-4)/2}}\right)^\theta t^{N-1}\,\mathrm{d}t\\ &\quad + \begin{cases} O(\varepsilon^\gamma), & \text{if } N \geqslant 5,\ N \neq 8,\\ O(\varepsilon^\gamma|\ln\varepsilon|), & \text{if } N = 8,\end{cases} \qquad \gamma = \min\{4, N-4\}. \qquad (3.15)\end{aligned}$$

Case 1 (assumption (A_1) holds). For $N \geqslant 8$, $\theta > 2$ and $\mu > 0$, we deduce that $N - (N-4)\theta/2 < \gamma = 4$ and

$$0 < C_1 \leqslant \int_0^{R/\varepsilon}\left(\frac{1}{(1+t^2)^{(N-4)/2}}\right)^\theta t^{N-1}\,\mathrm{d}t < \infty.$$

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It follows from (3.15) that

$$\sup_{t\geqslant 0} I(tv_\varepsilon) < \frac{2}{N} S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} \tag{3.16}$$

if $\varepsilon > 0$ is sufficiently small.

Case 2 (assumption (A_2) holds). For $4 < N \leqslant 8$, $\theta > 2^* - 2 = 8/(N-4)$ and $\mu > 0$, we deduce that $N - (N-4)\theta/2 < \gamma = N - 4$ and

$$0 < C_1 \leqslant \int_0^{R/\varepsilon} \left(\frac{1}{(1+t^2)^{(N-4)/2}}\right)^\theta t^{N-1}\,\mathrm{d}t < \infty.$$

The inequality (3.16) again follows from (3.15) if $\varepsilon > 0$ is sufficiently small.

Case 3 (assumption (A_3) holds). For $4 < N \leqslant 8$, $2 < \theta \leqslant 2^* - 2 = 8/(N-4)$, we take $\mu = \varepsilon^{-4}$ so that $N - (N-4)\theta/2 - 4 < \gamma = N - 4$. Since

$$\int_0^{R/\varepsilon} \left(\frac{1}{(1+t^2)^{(N-4)/2}}\right)^\theta t^{N-1}\,\mathrm{d}t \geqslant C > 0$$

it follows from (3.15) that

$$\begin{aligned}\sup_{t\geqslant 0} I(tv_\varepsilon) &\leqslant \frac{2}{N} S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} \\ &\quad - \varepsilon^{-4} C \varepsilon^{N-(N-4)\theta/2} \int_0^{R/\varepsilon} \left(\frac{1}{(1+t^2)^{(N-4)/2}}\right)^\theta t^{N-1}\,\mathrm{d}t + O(\varepsilon^{N-4}) \\ &\leqslant \frac{2}{N} S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} - C\varepsilon^{N-(N-4)\theta/2-4} + O(\varepsilon^{N-4}) \\ &< \frac{2}{N} S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4} \end{aligned}\tag{3.17}$$

if $\varepsilon > 0$ is sufficiently small.

Now the inequality (3.12) follows from (3.16) and (3.17) by taking $u_0 = v_\varepsilon$ with $\varepsilon > 0$ sufficiently small. □

The proof of theorem 1.1. The conditions for the mountain pass lemma [8] are satisfied by lemma 3.1. By (3.1)–(3.4), we have

$$\begin{aligned} I(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x)F(u_n)\,\mathrm{d}x \\ &\quad - \frac{1}{2^{**}} \int_{\mathbb{R}^N} P(x)|u_n|^{2^{**}}\,\mathrm{d}x \\ &= c_0 + o_n(1) \end{aligned}\tag{3.18}$$

and

$$\begin{aligned} I'(u_n)u_n &= \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x)f(u_n)u_n\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} P(x)|u_n|^{2^{**}}\,\mathrm{d}x \\ &= o_n(1)\|u_n\|. \end{aligned}\tag{3.19}$$

Define $v_n = u_n - u$. Then, from (3.5), lemma 2.3 and the Brézis–Lieb lemma [7],

$$I(u_n) = I(u) + \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta v_n|^2 + V(x)|v_n|^2)\,\mathrm{d}x - \frac{1}{2^{**}}\int_{\mathbb{R}^N} P(x)|v_n|^{2^{**}}\,\mathrm{d}x \tag{3.20}$$

and

$$\begin{aligned} I'(u_n)u_n &= \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} P(x)|u_n|^{2^{**}}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)|u|^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)f(u)u\,\mathrm{d}x - \int_{\mathbb{R}^N} P(x)|u|^{2^{**}}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} (|\Delta v_n|^2 + V(x)|v_n|^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} P(x)|v_n|^{2^{**}}\,\mathrm{d}x + o_n(1). \end{aligned} \tag{3.21}$$

Since $I'(u_n) \to 0$ as $n \to \infty$ and by (3.5) again, we obtain

$$\begin{aligned} I'(u_n)u = \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)u^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)f(u)u\,\mathrm{d}x \\ - \int_{\mathbb{R}^N} P(x)|u|^{2^{**}}\,\mathrm{d}x + o_n(1). \end{aligned} \tag{3.22}$$

By (3.21) and (3.22), we have

$$\int_{\mathbb{R}^N} (|\Delta v_n|^2 + V(x)|v_n|^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} P(x)|v_n|^{2^{**}}\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty \tag{3.23}$$

and

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)u^2)\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)F(u)\,\mathrm{d}x - \frac{1}{2^{**}}\int_{\mathbb{R}^N} P(x)|u|^{2^{**}}\,\mathrm{d}x \\ &= \frac{1}{2}\mu\int_{\mathbb{R}^N} K(x)f(u)u\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} K(x)F(u)\,\mathrm{d}x \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^{**}}\right)\int_{\mathbb{R}^N} P(x)|u|^{2^{**}}\,\mathrm{d}x \\ &\geqslant 0. \end{aligned} \tag{3.24}$$

Without loss of generality we can suppose that

$$\int_{\mathbb{R}^N} (|\Delta v_n|^2 + V(x)|v_n|^2)\,\mathrm{d}x \to \ell \quad \text{as } n \to \infty \tag{3.25}$$

and from (3.23)

$$\int_{\mathbb{R}^N} P(x)|v_n|^{2^{**}}\,\mathrm{d}x \to \ell \quad \text{as } n \to \infty. \tag{3.26}$$

We also have by Sobolev's inequality that

$$\int_{\mathbb{R}^N} |\Delta v_n|^2 \, dx \geqslant S\left(\int_{\mathbb{R}^N} |v_n|^{2^{**}} \, dx\right)^{2/2^{**}}$$
$$\geqslant S[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{-2/2^{**}}\left(\int_{\mathbb{R}^N} P(x)|v_n|^{2^{**}} \, dx\right)^{2/2^{**}}. \tag{3.27}$$

Combining (3.25)–(3.27), if $\ell > 0$, we have

$$\ell \geqslant S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4}. \tag{3.28}$$

Taking the limit in (3.20) as $n \to +\infty$, we have

$$c_0 \geqslant \frac{2}{N}\ell \geqslant \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4}. \tag{3.29}$$

On the other hand, from (3.3) and lemma 3.3, we have

$$c_0 < \frac{2}{N}S^{N/4}[\|P(x)\|_{L^\infty(\mathbb{R}^N)}]^{(4-N)/4}.$$

For a contradiction, show that $\ell = 0$. Thus,

$$I(u) = c_0 > 0 \quad \text{and} \quad I'(u) = 0,$$

i.e. u is a non-trivial solution of (1.1). The proof is complete. □

REMARK 3.4. Applying theorem 1.1 to the case in which $P(x) = 1$, we obtain the following corollary.

COROLLARY 3.5. *Assume that $(V, K) \in \mathcal{K}$ and that f satisfies (f_1)–(f_3). Then the biharmonic problem*

$$\Delta^2 u + V(x)u = \mu K(x)f(u) + |u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N, \ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \ N \geqslant 5, \tag{3.30}$$

has at least one non-trivial solution u such that $0 < I(u) < (2/N)S^{N/4}$ if one of the assumptions (A_1), (A_2) and (A_3) holds.

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