REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in *The Journal of Symbolic Logic* from the founding of the Journal in 1936 until the end of 1999. The Association moved the reviews to this Bulletin, beginning in 2000.

Three papers related to the Mitchell order

Omer Ben-Neria, *The structure of the Mitchell order-I*. **Israel Journal of Mathematics** vol. 214 (2016), no. 2, pp. 945–982.

Omer Ben-Neria, *The structure of the Mitchell order-II*. **Annals of Pure and Applied Logic**, vol. 166 (2015), no. 12, pp. 1407–1432.

Sy-David Friedman and Menachem Magidor, *The number of normal measures*. **Journal of Symbolic Logic**, vol. 74 (2009), no. 3, pp. 1069–1080.

The papers under review make significant contributions to our understanding of measurable cardinals and wellfounded partial orders. A regular uncountable cardinal κ is *measurable* if there exists a nonprincipal ultrafilter on the powerset of κ which is κ-complete; i.e., closed under $< \kappa$-sized intersections. If there is such an ultrafilter then there is always a *normal* one; that is, an ultrafilter closed under diagonal intersections. The *Mitchell order on normal ultrafilters*, introduced by Mitchell in the 1970s, provides a stratification of the consistency strength of measurable cardinals, and has proven to be an extremely useful concept in modern set theory. Probably the most famous theorem involving the Mitchell order is Gitik's theorems (from his 1989 and 1991 articles in the **Annals of Pure and Applied Logic**) that: *ZFC plus failure of the Singular Cardinals Hypothesis at $\aleph_\omega$ is equiconsistent with ZFC plus the existence of a measurable cardinal κ of Mitchell order κ^{++}*. Many other combinatorial statements have been shown to be equiconsistent with some statement involving the Mitchell order (e.g., Gitik's 1995 and 1997 **Israel Journal of Mathematics** articles on the nonstationary ideal).

Given two normal ultrafilters $\mathcal{U}$ and $\mathcal{W}$ on a measurable cardinal κ, we say $\mathcal{U}$ *is below* $\mathcal{W}$ *in the Mitchell order*, and write $\mathcal{U} \lhd \mathcal{W}$, iff $\mathcal{U}$ is an element of the ultrapower of the universe by $\mathcal{W}$; intuitively this says that $\mathcal{W}$ is stronger than $\mathcal{U}$. Mitchell proved that this defines a strict partial order on the set of normal ultrafilters of a given measurable cardinal κ, and moreover is always wellfounded. It is called *the Mitchell order on κ* and we will denote it by $\lhd(\kappa)$. Because $\lhd(\kappa)$ is wellfounded, it has a ranking function. If $\mathcal{U}$ is a normal ultrafilter on κ, then $o(\mathcal{U})$ denotes the rank of the ultrafilter $\mathcal{U}$ in $\lhd(\kappa)$. The height of $\lhd(\kappa)$ is usually denoted by $o(\kappa)$. Sometimes $o(\kappa)$ is also called the Mitchell order of κ, but only in situations where $\lhd(\kappa)$ is a linear order, in which case $(o(\kappa), \in)$ and $\lhd(\kappa)$ are isomorphic (so there is really no clash of terminology). Martin, Steel, Neeman, and others have interesting results on the Mitchell order of *extenders* (rather than just normal ultrafilters) but, for the purposes of this review, $\lhd(\kappa)$ refers to the Mitchell order of *only* the normal ultrafilters on κ.

In the canonical inner models such as Mitchell's core model, the Mitchell order $o(\kappa)$ is at most κ^{++}, and $\lhd(\kappa)$ is actually just a *linear* (wellfounded) order, and thus isomorphic to an ordinal. Thus, in Mitchell's core model, if κ is measurable then for each ordinal α there

is at most one normal ultrafilter on κ of Mitchell rank α, and $\lhd(\kappa)$ is isomorphic to some ordinal in the interval $[1, \kappa^{++}]$. These facts were used to provide solutions to the following classic problem in the presence of the Generalized Continuum Hypothesis:

PROBLEM 1. *Given a measurable cardinal κ, what are the possible cardinalities of the following set?*

$$UF_\kappa := \{\mathcal{U} \ : \ \mathcal{U} \text{ is a normal ultrafilter on } \kappa\}.$$

Clearly if κ is measurable then $|\mathrm{UF}_\kappa| \in [1, 2^{2^\kappa}]$. Kunen (*Some applications of iterated ultrapowers in set theory*, Ann. Math. Logic 1970) proved that 1 is a possibility (namely in $L[U]$ models), and Kunen-Paris (*Boolean extensions and measurable cardinals*, **Annals of Mathematical Logic**, 1970/71) proved that 2^{2^κ} was also a possibility. Finally, Mitchell's core model for sequences of measures allows one to obtain any value between 1 and κ^{++} in a model of GCH.

Relatively recently, however, Friedman and Magidor provided a uniform way to answer Problem 1. In fact, their proof uses only a *single measure*—a significantly weaker assumption than Mitchell's measures of high order—yet still manages to provide models where the set UF_κ from Problem 1 can have any cardinality up to κ^{++}:

THEOREM 2 (Friedman-Magidor). *Assume GCH. Suppose κ is measurable and let λ be a cardinal at most κ^{++}. Then in a cofinality-preserving forcing extension, there are exactly λ many normal ultrafilters over κ.*

Although the proof of Theorem 2 doesn't use or refer to the Mitchell order, it is relevant for subsequent theorems of Ben-Neria about the Mitchell order, so we briefly sketch it. By preliminary class forcing and "coding the universe into a real", we may assume that $V = L[U]$; by classic theory of $L[U]$, U is the unique normal measure on κ. Let $j : V = L[U] \to_U N$ be the ultrapower. A cofinality-preserving forcing iteration $\mathbb{P}$ of length $\kappa + 1$ is defined in V, where at each inaccessible stage α, a generalized Sacks forcing followed by a coding forcing is used. The forcing is designed so that if G is $(V, \mathbb{P})$-generic, then in $V[G]$ there are exactly λ-many objects G' which are $(N, j(\mathbb{P}))$-generic and extend $j"G$. This is equivalent to saying that in $V[G]$, there are exactly λ-many ways to lift j to an elementary embedding on $V[G]$. Furthermore, distinct liftings of j give rise to distinct derived normal ultrafilters. It follows that

$$|\mathrm{UF}_\kappa|^{V[G]} \geq \lambda.$$

Finally, using the fact that $V = L[U]$, together with classic theory of $L[U]$, every normal measure on κ in $V[G]$ is derived from such a lifting of j, which yields the other inequality

$$|\mathrm{UF}_\kappa|^{V[G]} \leq \lambda.$$

One novel feature of the iteration $\mathbb{P}$ is the use of *nonstationary support*; i.e., at inaccessible limit stages, conditions are allowed to be nontrivial at nonstationarily many coordinates. This allows for certain fusion properties of generalized Sacks forcing to be preserved by the iteration.

In what follows, *wellfounded order* will mean a wellfounded relation which is antisymmetric and transitive. Recall from above that $\lhd(\kappa)$ is always a wellfounded order, and that in Mitchell's core model and other canonical inner models, $\lhd(\kappa)$ is typically a linear order and thus just isomorphic to an ordinal. Let us say that a wellfounded order $(S, <_S)$ is *realized by a Mitchell order* iff there exists a κ such that $(S, <_S)$ is isomorphic to $\lhd(\kappa)$. A natural problem is:

PROBLEM 3. *Is it consistent that every wellfounded order is realized by a Mitchell order?*

Partial answers to Problem 3 were provided by Mitchell, Baldwin, Cummings, and Witzany. In particular, Cummings (*Possible behaviours for the Mitchell ordering II*, **Journal of Symbolic Logic**, 1994) provided a model where every so-called *tame* wellfounded order can be embedded into some Mitchell order, and Witzany (*Any behaviour of the Mitchell ordering of normal measures is possible*, **Proceedings of the American Mathematical Society**, 1996) provided a model where every wellfounded order can be embedded into some Mitchell order. Recently, Ben-Neria provided a complete solution to Problem 3:

THEOREM 4 (Part II of the Ben-Neria articles under review, Corollary 3.31). *Relative to the consistency of a large cardinal assumption between a strong and a Woodin cardinal, it is consistent that* every *wellfounded order is realized by a Mitchell order.*

Part I of Ben-Neria's two papers under review, starting from a weaker large cardinal assumption, provided a model where every *tame* wellfounded order was realized by a Mitchell order.

The heart of Theorem 4 is the solution to the following local variant of the problem: given an appropriate large cardinal κ and a wellfounded order $(S, <_S)$ of size $\leq \kappa$, can one force to modify the structure of $\lhd(\kappa)$ so that it becomes isomorphic to $(S, <_S)$? Suppose $V = L[E]$ is an extender model, κ is an appropriate large cardinal and $(S, <_S)$ is a wellfounded order of size $\leq \kappa$. For the tame case something weaker than $o(\kappa) = \kappa^+$ suffices. For the general case he assumes a large cardinal hypothesis in the region of "almost linear iterations" as in Schindler (*The core model for almost linear iterations*, **Annals of Pure and Applied Logic**, 2002), which lies between strong and Woodin cardinals in consistency strength. The very rough outline involves two major steps: (1) First force $\lhd(\kappa)$ to be sufficiently rich to absorb $(S, <_S)$, but in such a way that the ultrafilters on κ are nicely separated; i.e., so that each ultrafilter has some measure one set that uniquely identifies it. This is accomplished using a 3-step iteration $\mathcal{P}^0 * \mathcal{P}^1 * \mathcal{P}^2$. The first step $\mathcal{P}^0$ is the Friedman-Magidor iteration described above, the second step $\mathcal{P}^1$ is a Magidor iteration of Prikry type forcings, and the third step $\mathcal{P}^2$ is a "collapse and coding" forcing which again uses some of the Friedman-Magidor techniques. (2) Then perform what he calls a "final cut" poset $\mathcal{P}_S^{\text{cut}}$ to eliminate exactly those ultrafilters on κ which don't correspond to a member of S.

Some explanation is in order regarding the global statement in Theorem 4. The only part of either of the Ben-Neria articles under review that claims to prove such a global statement is Corollary 3.31 of part II, whose explanation is somewhat incomplete, although the implicit argument is probably familiar to those already well-versed in the topic. The author cites the well-known fact that forcings of size $< \kappa$ do not affect the large cardinal properties of κ; however it is not clear what kind of iteration of these forcings the author has in mind, and more importantly why the iteration would not alter the Mitchell order of measurable cardinals from earlier stages of the iteration which were used to realize some previously considered wellfounded order. Based on correspondence with the author, here is a brief sketch of the missing details. In the ground model, let $\langle \dot{S}_\alpha : \alpha \in \text{ORD} \rangle$ be a bookkeeping function and $\vec{\kappa} = \langle \kappa_\alpha : \alpha \in \text{ORD} \rangle$ an increasing sequence of the appropriately spaced large cardinals. The goal is to define a class iteration $\langle \mathbb{R}_\alpha, \dot{Q}_\beta : \alpha, \beta \in \text{ORD} \rangle$ so that if $\dot{S}_\alpha$ is a wellfounded order in $V^{\mathbb{R}_\alpha}$ of size $\leq \kappa_\alpha$, then $\dot{Q}_\alpha$ will force over $V^{\mathbb{R}_\alpha}$ that $\dot{S}_\alpha$ is realized by $\lhd(\kappa_\alpha)$, *and* so that $\lhd(\kappa_\alpha)$ remains unchanged from the model $V^{\mathbb{R}_{\alpha+1}} = V^{\mathbb{R}_\alpha * \dot{Q}_\alpha}$ onward. Working in $V^{\mathbb{R}_\alpha}$, to deal with $\dot{S}_\alpha$ define a 4-step iteration $\dot{Q}_\alpha = \mathcal{P}_\alpha^0 * \mathcal{P}_\alpha^1 * \mathcal{P}_\alpha^2 * \mathcal{P}_\alpha^{\text{cut}}$ similar to that described in the "local problem" of the previous paragraph, but only start iterating on cardinals above some fixed inaccessible $\lambda > \sup\{\kappa_\xi : \xi < \alpha\}$ such that $\lambda < \kappa_\alpha$ (the $\vec{\kappa}$ sequence is discontinuous, so this will be possible). This ensures that $\dot{Q}_\alpha$ will not add new subsets of 2^λ, and in particular $\lhd(\kappa_\xi)$ won't be affected for any $\xi < \alpha$ when going from $V^{\mathbb{R}_\alpha}$ to $V^{\mathbb{R}_{\alpha+1}}$. At limit stages one can use either Easton supports, or mixed supports where nonstationary support is used on the $\mathcal{P}_-^0$, $\mathcal{P}_-^2$, and $\mathcal{P}_-^{\text{cut}}$ components, and Magidor supports on the $\mathcal{P}_-^1$ component. A good reference for this topic is Gitik's chapter in the *Handbook of Set Theory*.

In short, the reviewed papers have provided a complete solution of the possible behavior of the Mitchell order on normal ultrafilters, by showing that the class of Mitchell orders can consistently equal the class of wellfounded orders. The reviewed papers mention a few open problems left in the area, such as the exact large cardinal strength of this phenomenon.

SEAN D. COX

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, 1015 Floyd Ave, Richmond, VA 23284, USA. scox9@vcu.edu.