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How Mathematics is Rooted in Life

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Mathematics is almost always an insider's affair. But sometimes things happen within the mathematical community that have a relevance, and perhaps also an interest, beyond the tribe itself. The *Grundlagenstreit* of the 1920s is such an example. In this review essay we tell this story with a focus on the main actors involved, David Hilbert in Göttingen and L.E.J. Brouwer in Amsterdam. We shall see how fine points concerning the existence of mathematical objects, the question of the editorship of the *Mathematische Annalen*, and the attempts to resume normal scientific contacts between French and German scientists after the First World War led to an unusually bitter conflict within the tribe and beyond. But even if the effects of the fight were at the time negative, the long range outcome was positive. Hilbert's work on the foundation of mathematics is still a powerful influence on current research, and Brouwer's view on the constructive foundation of mathematics, which at the time inspired both Husserl and Wittgenstein, is today of increasing importance in the evolving science of logic and computing.

The current article borrows its title from the subtitle of Dirk van Dalen's recent biography of the great Dutch mathematician, logician and philosopher, Luitzen Egbertus Jan Brouwer, known to his friends and foes alike as Bertus. A history of mathematics is almost always an insider project with little relevance to those outside the tribe itself; Van Dalen's book is an exception. We shall explain why.

Mathematics in the nineteenth century was very explicit, algebra was basically symbolic calculations, a function had to be given by an explicit formula, and if you asserted that something existed, e.g. a root of an equation or the value of a function for given arguments, it meant that you could actually compute the answer. Cantor's *Allgemeine Mengenlehre* changed this. Algebra became the study of abstract structures, functions changed from formulas to sets of ordered pairs, and existence came to mean that non-existence was contradictory.

Cantor's set theory was, for most mathematicians, a welcome expansion of the mathematical universe. In particular, it was important for the early development of general topology. However, difficulties soon emerged. First in the more distant and not too important parts of logic and mathematics, e.g. the Russell paradox of the set X of all sets x such that x is not an element of x itself: is X an element of X?

But soon the uncertainty about the soundness of set-theoretical reasoning moved closer to the central parts of mathematics. Dedekind had in the 1880s presented his axiomatization of the set *N* of natural numbers: 1, 2, 3, … And by stipulating that the set *N* is the least set that satisfies the axioms, he could 'prove' that *N* is completely characterized by his axioms. But this means that *N* is defined in terms of itself, since *N* is one of the sets that satisfies the axioms. This goes against our intuition that *N* is stepwise constructed by starting with the number 1 and then adding 1 successively to generate the full number series.

Objects defined in terms of themselves are called impredicative. Some saw impredicativity as the villain of set theory. Prominent among these was the great French mathematician Henri Poincaré. His range as a mathematician was formidable. Here we take note of him, relevant for our later story, as one of the fathers of modern topology. His views inspired a French school of 'semi-intuitionists', including leading mathematicians such as Borel, Baire and Lebesgue. They wanted to understand the nature of an arbitrary subset of the real continuum, not by implicit impredicative definitions, but by stepwise building up an extensive hierarchy of definable subsets, now known as the Borel hierarchy, as an approximation to the full family of subsets of the reals. The resulting theory, today known as descriptive set theory, is an important part of current mathematics, both in praxis and in foundational studies.

Paris and Göttingen were the two main centres of mathematics. This was recognized by all in the mathematical profession in the late nineteenth century. The two communities lived side by side in a fruitful collaboration and competition and hardly noticed the war of 1870–1871. This changed with the First World War, when mathematics became part of a larger political game. This will be part of our history, but first we return to more peaceful times. At the turn of the century, Paris became semi-intuitionistic, Göttingen remained set-theoretic. Hilbert's *Grundlagen der Geometrie* was an extraordinary success, and the axiomatic method became the chosen Göttingen method of how mathematics was to be done. Zermelo made an important step forward by his axiomatization of set theory in 1904. The next step would be to prove the axioms 'consistent', which means that the set theoretic universe exists, and finally that they were 'categorical', which means that the set theoretic universe is uniquely described by Zermelo's axioms.

Hilbert's proof theory was gradually constructed in the early part of the twentieth century to achieve these goals. We shall return to this story, but let us at this point pause and review the situation in the years before the Great War. Cantor's Mengenlehre had made a great impact on mathematical praxis and understanding, but there were difficulties. Two ways out were suggested, Hilbert's axiomatization with proofs of consistency and categoricity, and Poincaré's limitation to predicative methods of proof. Both ways were in a sense conservative, the classical kernel of mathematics was to be preserved, perhaps with certain restrictions and refinements. Hermann Weyl was a star student of Hilbert's in Göttingen. It is possible to view his 1910 paper *Über die Definitionen der mathematischen Grundbegriffe* as an attempt to build a bridge between Paris and Göttingen. Thus, there could be a harmonious way forward. But mathematics never fails to surprise, there will always be a new young voice confounding the wisdom of the elders: enter Brouwer in 1907, the year of his thesis *Over de grondlagen der wiskunde*.

Brouwer had the confidence of youth and talent. Hilbert had at the world congress of mathematics in Paris in 1900 given his great lecture on *Mathematische Probleme*, intended as a guide to the future development of mathematics. The challenge was accepted by the community, and your reputation as a mathematician was assured if you solved or contributed significantly to the solution of one of these problems. Brouwer claimed to solve three of these problems in his thesis: problem one in Hilbert's list on the nature of the continuum, problem two in the list on the consistency of the arithmetical axioms, and problem five in the list on a certain question of topology connected with Lie groups. Let problems one and two rest for a moment; we turn to problem five.

What Brouwer did was to solve a special, but significant, case of the fifth problem. This was mainstream mathematics and was instantly recognized. And it led to further work in topology. In a short period from 1907 to 1913 Brouwer, continuing the early work by Poincaré, created to a large extent modern topology, the Brouwer fixed-point theorem being a noteworthy example. Brouwer had a powerful geometric intuition and made little use of algebraic tools, which later replaced his more direct geometric approach. But his results stand and are the foundation for modern work. Hilbert at once recognized the new star, and in 1914 Brouwer was invited to be an editor of the *Mathematische Annalen*, the leading journal of mathematics at the time, and he became an honorary member of the Göttingen mathematical community. But this was before Hilbert recognized the 'subversive' views of Brouwer on the foundation of mathematics. We now turn to questions one and two in Hilbert's list, the nature of the continuum and the consistency of the arithmetical axioms.

Brouwer's view of mathematics was deeply influenced by German idealistic philosophy, mainly Kant, as can be seen from his first major published work *Life, Art, and Mysticism* from 1905. In his thesis from 1907 he characterized mathematics as the free activity of the human mind; thus, mathematical truth is neither to be secured by reference to some abstract platonic realm beyond space and time, nor by reference to a formal game of axiomatics and consistency proofs. Mathematics was a meaningful activity of human thinking, it was primarily rooted in the intuition of structure and patterns. Language, formalism and axioms were secondary and could never be seen as a proper foundation. According to Brouwer's views, a mathematical proposition can be recognized as true only when the thinking subject has experienced its truth through a mental construction; in particular, an existential proposition can be true only if you have an appropriate mental construction of an object satisfying the proposition. This led to the rejection of the ancient Aristotelian law of the excluded middle, that for every proposition A, either A or not-A is true. Accepting this principle of classical logic means, in our case, that to prove the consistency of arithmetic, which is Hilbert's problem two, it is sufficient to prove that inconsistency is impossible. This was rejected by Brouwer, existence means construction, not the impossibility of non-existence. He also rejected Hilbert's problem one concerning the nature of the continuum as ill-posed and meaningless. The notion of a continuum as a completed and uncountable pointset was denied by Brouwer and the core part of his intuitionism consisted of developing methods to 'construct' points on the real line and to build a 'sound' mathematical praxis on this foundation. Through this work he became a pioneer in the development of constructive mathematics. We shall below return to constructivity and its impact on mathematics and computing today; here, we continue with the *Grundlagenstreit* of the 1920s. This foundational *Streit* inside mathematics became, perhaps as a surprise to many university professors, part of the broader cultural and political conflicts of the turbulent years between the First World War and the coming of Hitler's Germany.

At first Hilbert did not pay much attention to the critical Brouwer. For him, Brouwer was the new star of topology, to be welcomed as editor of the *Mathematische Annalen* and to be a valued associate member of the Göttingen community. In fact, Hilbert at one point wanted Brouwer as the successor to Felix Klein. But when his star student Hermann Weyl in 1921 announced in a paper on the new foundational crisis in mathematics his conversion to Brouwer's intuitionism – this is the revolution – the old master was not amused.

Hilbert had after his Paris lecture argued for an axiomatic approach to the foundations of mathematics. Existing mathematics should be codified in an all-inclusive axiomatic system with a well-defined syntax and exact rules of proof. The challenge was to prove such a system consistent. What was added in the 1920s in response to the critiques of Brouwer and Weyl, was a more profound analysis of what it means to prove a system consistent. To prove consistency of Mengenlehre by unrestricted set theoretic methods would carry no conviction. Geometry was proved consistent by Hilbert by reduction to an arithmetical foundation. But how to prove the consistency of number theory itself? Hilbert's proposal was to restrict consistency proofs to finitary methods, with no use of quantifiers (such as the universal 'for all' and the existential 'there is') over infinite domains, such as the set of all natural numbers.

Hilbert presented his ideas in a number of lectures during the 1920s, most famously in his lecture in Hamburg in 1927 on *Die Grundlagen der Mathematik*. Brouwer did not directly respond to the sequence of Hilbert lectures; he presented his ideas to wider audiences in Berlin in 1927 and Vienna in 1928. The fight between Hilbert and Brouwer engaged the mathematical community. In a larger context it is possible to see the *Grundlagenstreit* inside mathematics as part of the larger struggle between the ancient regime and the newer revolutionary ideas emerging after the First World War. Hilbert spoke for the status quo, Brouwer championed something new, which could be a revolution in how to do mathematics. We know that Wittgenstein was in Brouwer's audience in Vienna, and it is suggested that his return to philosophy was inspired by Brouwer. Wittgenstein went forward and never looked back. Hermann Weyl is an interesting example of the mood of uncertainty of the times. After joining the Brouwerian revolution in 1921, he moved back closer to Hilbert's views after the Hamburg address in 1927. And in the early 1930s he was for a short time Hilbert's successor in Göttingen.

International cooperation also suffered after the First World War. There were strong feelings on both sides. *A Conseil International de Recherches* was formed in 1919. Germany was excluded. Indeed, in the words of the French mathematician Picard, Germany was a nation that had placed itself beyond humanity, 'there is too much blood, and there are too many crimes, separating us'. As an early example of the activities of the new *Conseil*, Picard took the initiative to arrange an international congress of mathematics in Strasbourg in 1920. The choice of location was not arbitrary; the Germans were barred from attending. German scientists responded by calling their own *Naturforscherversammlung* in Nauheim. Brouwer attended and gave a lecture, 'Does every real number have a decimal expansion?', explaining the fine structure of the real numbers from an intuitionistic point of view.

The absurdity of boycotts and counter-boycotts could not continue. The situation inside Germany was not easy: should one accept the French initiated *Conseil* as a satisfactory framework for international cooperation, or was a new start needed? Hilbert became a strong spokesman for re-establishing 'normal' relations within mathematics accepting the existing mechanisms. Brouwer argued that a new start was necessary and was in this respect more German (*Deutschfreundlich*) than most Germans themselves. When an international mathematics congress was organized in Bologna in 1928 with the cooperation of the *Conseil*, and the Germans were invited, Brouwer ended up by demanding a German boycott, much to the annoyance of Hilbert, who was invited to give the opening address at the meeting. There is an end to this story. The *Conseil* was dissolved and a new International Council of Scientific Unions was founded in 1931. It took longer to bring order in the ranks of mathematicians: the present day International Mathematical Union dates from 1950.

The conflict between Hilbert and Brouwer had a sad end. Brouwer always had a high respect for the mathematician Hilbert, and Hilbert had, around 1910, recognized Brouwer as the new star of topology and had, as noted above, tried to get him as the successor of the great Felix Klein in Göttingen. But *Grundlagenstreit* and *Deutschfreundlichkeit* led to a bitter end to their former friendship. The old master acted in an imperial fashion, Brouwer was dismissed from favour; in fact, he was dismissed as a member of the editorial board of the *Mathematische Annalen*. This was at the time no small matter, and many – including Albert Einstein – were asked to intervene. But the old man could not be moved. Yet there were limits to his 'victory'. With Hitler came the collapse of Göttingen and German mathematics, and with Gödel came the collapse of Hilbert's foundational programme as it was originally conceived. Gödel proved in 1931 his famous incompleteness theorem, which in particular implies that the consistency of arithmetic cannot be proved by finitary means. But whereas Hitler and his Germany forever have disappeared, proof theory in the Hilbert tradition prospered and is today a substantial part of constructive and computational mathematics.

In the 1920s Brouwer published a number of papers on constructivism. In particular he obtained far reaching results on the continuum and total functions. This contradicted classical mathematics and showed that Brouwer's programme was incompatible with the traditional Hilbert approach. Constructivity in mathematics is, however, far more than mathematics in the style of Brouwer; we may point to the work by E. Bishop in his *Foundations of Constructive Analysis* and to P. Martin-Löf and his *Notes on Constructive Mathematics*. These studies develop a praxis of constructive mathematics matching in scope a large part of the 'classical' theories. Of equal interest is the study of the philosophy and logic of constructive reasoning; a wide-ranging (but somewhat technical) study is the survey *Foundations of Constructive Mathematics* by M. Beeson. But whereas the praxis has had limited impact on how mathematics is done, the philosophy has flourished.

Brouwer was no logician, hence not interested in developing a formal system of axioms and proofs for his intuitionistic approach. For him, intuition and structure were primary and, he would insist, independent of language and formalism. The early 1930s saw, however, the development of a formal system of intuitionistic logic. The pioneers were A. Kolmogorov in Moscow and A. Heyting, a student of Brouwer's in Amsterdam. At the same time, algorithms and computability were given a precise and general theory through the work of Alan Turing. The community soon noticed the close connections between general algorithms and proofs in logical systems, and in particular in constructive proof theory, where existence means that you can actually 'compute' an example. Again, Martin-Löf is our example; extending the original Heyting logic he developed a powerful intuitionistic type theory as a framework for constructive mathematics. And he followed up with an influential lecture on 'Constructive mathematics and computer programming' at the 1979 International Congress of Logic, Methodology and Philosophy of Science. His ideas are very much alive today. In the academic year 2012–2013, the Institute of Advanced Study in Princeton acted as host to a programme in 'Homotopy type theory'. This is, at first, a rather surprising link between homotopy theory, a branch of topology, and type theory and theoretical computer science. We shall not explain why, just quote the promise made by the organizers of the programme of what is to be expected: '[The programme] suggests a new conception of the foundation of mathematics, with intrinsic homotopical content, an "invariant" conception of the objects of mathematics – and convenient machine implementations, which can serve as a practical aid to the working mathematician'.

We cannot know, but it would have been interesting to see, how Brouwer, equally an expert in topology and constructivity, would have reacted to the fusion of types and homotopy. What we do know is that deep insights will never disappear. Brouwer's achievements will be an integral part of a cultural heritage that will continue to enrich – in new and unexpected ways – our understanding of what knowledge is.

It is time to return to this paper's title and to discuss how mathematics is rooted in life.

The title is actually a quote from a letter Brouwer wrote to his thesis advisor in 1906 and is an early expression of how Brouwer saw mathematics as a free activity of the human mind and that mathematical truth was neither to be secured by reference to some abstract platonic realm, nor by reference to language, axioms and proofs of consistency. For Brouwer, the creative activity of the individual mind was the focal point. There are also strong connections between Brouwer's philosophy and the phenomenology of Husserl. They met in Amsterdam in 1928 and had, as Husserl later wrote to Heidegger, long conversations which, in Husserl's words, 'made a most significant impression on me'. Brouwer on his side reported in a letter, 'here, at the moment, Husserl is going around, I am also strongly attracted'.

But with this exclusive focus on the individual mind there is a difficulty, shared with the formalists, of explaining the applicability of mathematica to the world around us. Mathematicians, as hired hands, are happy to apply their tools to analyse and sometimes solve an expanding set of problems about nature, society and mind. Mathematicians, as philosophers, have their difficulties, and there are a number of different schools of thought about the foundation of mathematics. It seems fair to say that neither Hilbert nor Brouwer had a satisfactory and comprehensive philosophy of truth and applicability. We may take Hermann Weyl as our witness. In questions of philosophy and mathematical truth he sided with Brouwer. In his work on general relativity and quantum theory, he came quite close to Hilbert and his views on classical mathematics. But Weyl's long efforts to work out a unified philosophy of mathematics and its applicability remained unfinished. The *Grundlagenstreit* never produced a winner.

Perhaps there is a way out. Between the individual mind and the platonic realm of structures outside time and space, there is culture, or – in the language of some anthropologists – there is the collective mind of the species. In a recent paper 'On what there is – infinitesimals and the nature of numbers' I have tried to argue for a cultural approach to the foundation of mathematics. Let me briefly quote one paragraph:

> P. Suppes has in a recent paper, *Why the effectiveness of mathematics in the natural sciences is not surprising*, discussed the emergence of geometry. He sees an early beginning in the perceptual processes necessary for humans to survive. He further notes in his paper how human culture developed the skill to represent 'the structure of the external world in remarkable paintings and drawings', and concludes that the key to the early development of geometry lies in the gradually emerging 'structural isomorphism between, in one case a perception and an object or a process in the world, and in another, between a mental image and an abstract structure'. Modern geometry is thus a complex product of nature, mind and culture, and as such a 'product' obviously applicable to what is seen ….

At this point we should return to Brouwer. It is possible to see a connection between the quote above and Brouwer's views on how mathematics is linked to the outer world, in particular his *Mathematik, Wissenschaft und Sprache* from 1929. The philosophers will, however, have to argue whether Brouwer in his 1929 paper is closer to Husserls phenomenology than to a possible cultural foundation for mathematics.

In the introductory paragraph it was claimed that van Dalen's biography had an interest beyond the mathematical community. And I promised to explain why. The reader will judge if I have succeeded in presenting Brouwer not just as a mathematician and philosopher, but also as a deeply engaged participant in the intellectual and political activities at a European level in the first part of the twentieth century. I have also tried to trace the importance of his work for current research. The van Dalen biography runs to a total of 875 pages. In addition to the public life, it also tells the story of the private life of a strong, old-fashioned, and at times difficult professor. And for those who think of Brouwer in terms of dry and sometimes forbidding mathematical papers, his somewhat bohemian lifestyle in the little village Blaricum on the outskirts of Amsterdam may come as a surprise. The book is an engaging account of a great life.

Further Reading

The list of references here contains a few items of possible interest to anyone who wants a deeper understanding of constructivity and its applications. The main reference is the book by van Dalen (2013), which is a rich and well-documented source for the life and work of Brouwer and his time. This book has also a complete listing of all the published works of Brouwer. The classical textbook by Kleene (1952) is a thorough introduction to the ideas and the technical work of Hilbert, Brouwer, Gödel and Turing on foundational issues in logic and computing; some of their fundamental papers are reprinted in Van Heijenoort (1967). Beeson (1985) and Troelstra and Van Dalen (1988) are more recent and commendable updates. Feferman (1998) is recommended for its analysis both of Weyl's original work and of current ideas on predicativity in mathematics. Martin-Löf (1982) was an important step in linking constructive type theory and programming languages This has been carried forward in the current IAS project on homotopy type theory. The link between Brouwer and Husserl is discussed in Van Atten (2007). Fenstad (2014) discusses some further philosophical issues related to the cultural approach to the foundation of mathematics.

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