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RISK-NEUTRAL MEASURES AND PRICING FOR A PURE JUMP PRICE PROCESS

A STOCHASTIC CONTROL APPROACH

ANNA GERARDI AND PAOLA TARDELLI
Department of Electrical and Information Engineering
University of L' Aquila
L' Aquila, Italy
E-mail: {tardelli; gerardi}@ing.univaq.it

This article considers the asset price movements in a financial market when risky asset prices are modeled by marked point processes. Their dynamics depend on an underlying event arrivals process, modeled again by a marked point process. Taking into account the presence of catastrophic events, the possibility of common jump times between the risky asset price process and the arrivals process is allowed. By setting and solving a suitable control problem, the characterization of the minimal entropy martingale measure is obtained. In a particular case, a pricing problem is also discussed.

1. INTRODUCTION

This article deals with the problem of expected utility optimization with an application to the structure of the minimal entropy martingale measure in a financial market for which the assets prices are modeled by marked point processes. As observed, for instance, in Prigent [24], intraday information on financial asset price quotes and the increasing amount of studies on market microstructure show that prices are piecewise constant and jump at irregularly spaced random times in reaction to trades or to significant new information. This is the reason why many authors believe that pure jump processes might be more suitable for modeling the observed price or quantities related to the price. In fact, they believe that models that consider continuous

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trajectory processes, even if the presence of jumps is allowed, and not accounting for the discreteness in the data, could lead to wrong conclusions and might need ad hoc methods for rounding and estimation.

Several models in which the price process is a marked point process are available in the literature; in addition to Prigent [24], we only quote Rogers and Zane [26], Ridberg and Shephard [25], Frey [13], Frey and Runggaldier [14], Centanni and Minozzo [7,8], Gerardi and Tardelli [18], Ceci [5], and the references therein. In order to describe the amount of information received by the traders related to intraday market activity, the activity of other markets, macroeconomics factors, or microstructure rules, some of these references have introduced exogenous stochastic factors. For instance, in Zariphopoulou [29], the fundamental assumption is that the dynamics of the stock price depends on another process, referred to as a stochastic factor, which is generally correlated with the underlying stock price. In all of these cases, the local characteristics of the price process, such as the jump-intensity and the jump-size distribution, depend on a latent process whose behavior has been described in different ways by different authors.

In this article, we consider a financial market with a single risky asset and a bond. As described in Section 2, the price S of the risky asset in units of the numeraire is a stochastic process given by $S_t = S_0 e^{Y_t}$, for $S_0 \in \mathbb{R}^+$. The logreturn price Y is a marked point process whose local characteristics depend on the entire history up to time t of a marked point process X. The processes X and Y are strictly correlated, since we assume that they might have common jump times, which implies that the behavior of X might be affected by trading activity. Furthermore, as in Gerardi and Tardelli [18], we introduce a Markovian assumption.

In Section 3 we deal with the control problem that arises if we assume that the investor seeks to maximize the expected utility from his terminal wealth when an exponential utility function is chosen. Writing down the related Hamilton–Jacobi–Bellmann equation, we provide a solution for it and we find the corresponding feedback optimal strategy. This is a useful result, suitable for many applications. In particular, we are interested in characterizing the minimal entropy martingale measure that plays an important role in the utility indifference approach to valuation of derivatives (Frey [13], Frittelli [16], Grandits and Rheinlander [20], and Delbaen, Grandits, Rheinlander, Samperi et al. [9]).

In Section 4 we recall some general properties and we observe that, since in the model studied in this article S_t is locally bounded, there exists a unique minimal entropy martingale measure P^E, locally equivalent to P, under the assumption that the set of the equivalent martingale measures with finite entropy is nonempty (Frittelli [16]). In the Appendix, we prove that this condition is accomplished. Consequently, in this model, there exists a unique minimal entropy martingale measure P^E, equivalent to P, which we characterize by using a duality relation as the main tool. Duality methods give a relation between the problem of maximization of the expected exponential utility and the minimal entropy martingale measure.

Many articles deal with duality topics—for instance, the work Delbaen et al. [9], Bellini and Frittelli [2], Biagini and Frittelli [3], and the references therein.

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The structure of the model studied in this article leads us to follow the method presented in Biagini and Frittelli [3]. This method allows us to give the main result of this article that is, an explicit representation of the density of the minimal entropy martingale measure in terms of the solution of the Hamilton–Jacobi–Bellmann equation.

Finally, Section 5 is devoted to the problem of pricing of a European contingent claim when the agents can observe only the behavior of the price process. The problem of defining the arbitrage-free price in a model with restricted information arises, and many approaches to a solution are possible. The discussion in Gombani, Jaschke, and Runggaldier [19] suggests a choice similar to that followed in Ceci and Gerardi [6] and in Frey and Runggaldier [15]. This choice consists of characterizing the martingale measures with respect to the filtration $\mathcal{F}_t$ and defining the price of the claim as the expectation conditioned to the observations, under a suitable martingale measure, which, in this article, is the minimal entropy martingale measure. Under an additional assumption on the structure of the model, we reduce the problem of pricing under restricted informations to a filtering problem. The general case appears too abstract and it is briefly discussed in Section 5.3. Thus, in Section 5.2, we restrict ourselves to the case in which Y is discrete-valued. The Kushner–Stratonovitch equation is written down, uniqueness of its solution is proven, and a recursive computation of the filter is provided.

In Section 6 we are concerned about the existence and characterization of risk-neutral measures that is, probability measures equivalent to P, under which the price process is a local martingale. We discuss their characterization in the framework of this model, taking into account the incompleteness of the market due to the presence of infinitely many jumps of the price process.

Because we are mainly interested in the minimal entropy martingale measure, to verify the existence of risk-neutral measures with finite entropy, we characterize the minimal martingale measure for this model, and we prove that it is a probability measure with finite relative entropy with respect to P. The minimal martingale measure has been studied by many authors with regard to the topic of hedging of contingent claims in incomplete markets (Follmer and Schweizer [12], Prigent [24], and Schweizer [28]).

In the literature, the continuous paths processes have been frequently considered. For a marked point process, the minimal martingale measure approach has been discussed, for instance, in Prigent [24]. Sufficient conditions for its uniqueness and existence can be found in Follmer and Schweizer [12], Ansel and Stricker [1], and Schweizer [28].

2. THE MODEL

On a filtered probability space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, we consider a market model with a single risky asset S and a nonrisky asset. The price of the risky asset, discounted with respect to the price of the bond, is a process

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S having the form $S_t = S_0 \exp\{Y_t\}$, with $S_0 \in \mathbb{R}^+$. The logreturn price Y is assumed to be a nonexplosive $\mathbb{R}$-valued marked point process with initial condition $Y_0 = 0$, characterized by the sequence $\{\tau_n^Y, Y_{\tau_n^Y} - Y_{\tau_n^Y -}\}_{n\geq 1}$, where $\{\tau_n^Y\}_{n\geq 1}$ is a nondecreasing sequence of stopping times. This means that the trajectories of the price of the stock are piecewise constant, and changes occur only at random times.

The dynamics of the logreturn price depends on an exogenous process X, representing the amount of news reaching the market. We believe that it is sensible to assume that X is a nonexplosive marked point process, taking values in a finite set $\mathcal{X} \subset \mathbb{R}$, with initial condition $X_0 = 0$ and nonnegative jump sizes. The process X is characterized by the sequence $\{\tau_n, X_{\tau_n} - X_{\tau_n -}\}_{n\geq 1}$ where, again, $\{\tau_n\}_{n\geq 1}$ is a nondecreasing sequence of stopping times.

Next, we will describe the joint dynamics of the processes X and Y. First, we describe the link between trading frequencies and arrivals of news to the market. To this end we define the point process counting the jump times of Y up to time t as

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\tau_n^Y \leq t}. \tag{2.1}$$

We assume that it admits a $(P, \mathcal{F}_t)$-intensity λ_t, whose structure, similar to that given in Centanni and Minozzo [7,8] and Gerardi and Tardelli [18], is

$$\lambda_t = a(t) + bz_0 e^{-kt} + b \sum_{i\geq 1} (X_{\tau_i} - X_{\tau_{i-1}}) e^{-k(t-\tau_i)} \mathbf{1}_{\tau_i \leq t}, \tag{2.2}$$

with b and k real positive parameters and $a(\cdot)$ measurable a $\mathbb{R}^+$-valued deterministic function, verifying

$$0 \leq a(t) \leq \overline{a} < +\infty.$$

Equation (2.2) has a natural and intuitive interpretation. The arrival of a news reaching the market, represented by a positive jump size of X at a random time τ_n, produces a sudden increase in the trading activity. Successively, a progressive normalization of the market occurs, with a speed expressed by k. Finally, $a(\cdot)$ describes the activity of the market in the absence of random perturbations. By adequately choosing the function $a(\cdot)$, we would also be able to take into account deterministic features such as seasonalities.

Assuming z_0 strictly positive, we get that λ_t is strictly positive. In addition, λ_t is bounded, since for Λ suitable positive constant and $\forall t$,

$$\lambda_t \leq \overline{a} + bz_0 + bX_t < \Lambda < +\infty. \tag{2.3}$$

The structure of (2.2) could suggest the introduction of the history process defined by X. To avoid this introduction and the technicalities that it implies, as in Gerardi

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and Tardelli [18], we define the process

$$Z_t := z_0 + \int_0^t e^{ks}\, dX_s$$

and we obtain

$$\lambda_t = a(t) + be^{-kt} Z_t := \lambda(t, Z_t).$$

Then λ is a deterministic measurable function of the time t and of the process Z, which is a nonhomogeneous pure jump process, taking values in a suitable $\mathcal{Z} \subseteq \mathbb{R}^+$, having the same jump times of X and jump sizes given by $Z_t - Z_{t-} = e^{kt}(X_t - X_{t-})$.

The first difference between this model and that proposed in Gerardi and Tardelli [18] is that, in this article, we allow the possibility of common jump times between the latent process X and the logreturn process Y as well as the possibility of catastrophic events. Hence, we denote by $\{\tau_n^X\}_{n\geq 1} \subseteq \{\tau_n\}_{n\geq 1}$ the nondecreasing sequence of stopping times at which $X_{\tau_n^X} \neq X_{\tau_n^X -}$ and $Y_{\tau_n^X} = Y_{\tau_n^X -}$.

In addition, in order to better specify the dynamics of the processes X, Y, and Z, let $\xi(t,x,z)$, $\eta_1(t,x,z)$, and $\eta_2(t,x,z)$ be measurable functions such that

$$\xi : [0,T] \times \mathcal{X} \times \mathcal{Z} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

and for any $t \in [0,T]$,

$$x \in \mathcal{X}, \;\; z \in \mathcal{Z}, \;\; x + \xi(t,x,z) \in \mathcal{X},$$
$$\eta_i : [0,T] \times \mathcal{X} \times \mathcal{Z} \longrightarrow \mathbb{R}^+,$$

and for some real constants $\eta_{\max}$ and $\eta_{\min}$,

$$0 < \eta_{\min} \leq \eta_i(t,x,z) \leq \eta_{\max}, \;\; i = 1,2. \tag{2.4}$$

Setting

$$\xi_t := \xi(t, X_{t-}, Z_{t-}), \;\; \eta_t^1 := \eta_1(t, X_{t-}, Z_{t-}), \quad \text{and} \quad \eta_t^2 := \eta_2(t, X_{t-}, Z_{t-}),$$

we are able to give the representation

$$X_t = \int_0^t \xi_u\, [dN_u^0 + dN_u], \;\; Y_t = \int_0^t \eta_u^1\, dN_u^1 - \int_0^t \eta_u^2\, dN_u^2,$$
$$Z_t = z_0 + \int_0^t e^{ku} \xi_u\, [dN_u^0 + dN_u],$$

where

$$N_t^0 = \sum_{n\geq 1} \mathbf{1}_{\tau_n^X \leq t}, \;\; N_t^1 = \sum_{n\geq 1} \mathbf{1}_{\tau_n^Y \leq t} \mathbf{1}_{\{Y_{\tau_n^Y} - Y_{\tau_n^Y -} > 0\}},$$
$$N_t^2 = \sum_{n\geq 1} \mathbf{1}_{\tau_n^Y \leq t} \mathbf{1}_{\{Y_{\tau_n^Y} - Y_{\tau_n^Y -} < 0\}}, \quad \text{and} \quad N_t = N_t^1 + N_t^2.$$

We assume that N^0 admits a $(P, \mathcal{F}_t)$-intensity given by $\lambda_t^0 := \lambda_0(t, X_{t-}, Z_{t-})$, where $\lambda_0(t,x,z)$ is a bounded nonnegative measurable function, such that, for the same Λ

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given in (2.3),

$$\lambda_0(t,x,z) \leq \Lambda. \tag{2.5}$$

For $i = 1,2$, N^i admits a $(P,\mathcal{F}_t)$-intensity $\lambda_t p_t^i$, where $\lambda_t := \lambda(t,Z_{t-})$ and $p_t^i := p_i(t,X_{t-},Z_{t-})$, $i = 1,2$ with $p_i(t,x,z)$ strictly positive measurable functions verifying the condition

$$p_1(t,x,z) + p_2(t,x,z) = 1.$$

Remark 2.1: As we will see in Section 6, if the price process is strictly increasing or strictly decreasing, the model does not admit any equivalent martingale measure. This explains the particular structure of the dynamics of the process Y.

PROPOSITION 2.2: *Under these assumptions,* (X,Y,Z) *is a Markov process and for* $f(t,x,y,z)$ *belonging to a suitable class of real-valued measurable functions,* $t \geq 0$, $x \in \mathcal{X}$, $y \in \mathbb{R}$, *and* $z \in \mathcal{Z}$, *its generator is*

$$Lf(t,x,y,z) = \frac{\partial}{\partial t} f(t,x,y,z) + L_t f(t,x,y,z), \tag{2.6}$$

where

$$\begin{aligned} L_t f(t,x,y,z) = {} & \lambda_0(t,x,z)[f(t,x+\xi(t,x,z),y,z+e^{kt}\xi(t,x,z)) - f(t,x,y,z)] \\ & + \lambda(t,z) \sum_{i=1,2} p_i(t,x,z)[f(t,x+\xi(t,x,z),y+(-1)^{i-1}\eta_i(t,x,z),z \\ & + e^{kt}\xi(t,x,z)) - f(t,x,y,z)]. \end{aligned}$$

PROOF: For a bounded real-valued measurable function f,

$$f(t,X_t,Y_t,Z_t) - f(0,X_0,Y_0,Z_0) - \int_0^t Lf(s,X_{s-},Y_{s-},Z_{s-}))\,ds = (P,\mathcal{F}_t)\text{-martingale.}$$

By Theorem 7.3 in Ethier and Kurtz [11], since the generator L_t that is given in (2.6) is bounded, then the Martingale Problem associated with the operator L and initial condition ($X_0 = 0, Y_0 = 0, Z_0 = z_0$) is well posed and its solution is a Markov process with trajectories in $D_{\mathcal{X}\times\mathbb{R}\times\mathcal{Z}}[0,T]$. ■

From now on, we fix a finite horizon T, and by a little abuse of notations, we set, for $t \leq T$, $\mathcal{F}_t := \sigma\{X_s, Y_s, 0 \leq s \leq t\}$.

3. THE CONTROL PROBLEM

In this section we deal with the control problem that arises if we assume that the investor seeks to maximize the expected value of the exponential utility function from

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his terminal wealth. The control problem is defined by the criterion

$$\inf_{\theta\in\Theta} \mathbb{E}\left[\exp\left(-w_0 - \int_0^T \theta_r \, dS_r\right)\right],$$

where w_0 is a real positive constant and Θ is the class of the $(P, \mathcal{F}_t)$-predictable S-integrable real-valued self-financing processes.

The classical dynamic programming approach leads to the Hamilton–Jacobi–Bellman equation. Herein, we prove that it admits a solution for a.a. t and we give an explicit representation of the solution. To this end, setting

$$\varphi_r = S_{r-}\, \theta_r,$$

the wealth process (Zariphopoulou [29]) reduces to

$$W_t := w_0 + \int_0^t \theta_r \, dS_r = w_0 + \int_0^t \frac{\varphi_r}{S_{r-}} \, dS_r,$$

with values in some $\mathcal{W} \subseteq \mathbb{R}$. By the Itô formula we get

$$W_t = w_0 + \int_0^t \varphi_u (e^{\eta_1(u,X_{u-},Z_{u-})} - 1) \, dN_u^1 + \int_0^t \varphi_u (e^{-\eta_2(u,X_{u-},Z_{u-})} - 1) \, dN_u^2.$$

As we did in Proposition 2.2, we characterize the joint dynamics of (X, Y, Z, W), for every constant control variable $\varphi \in \mathbb{R}$, by the operator L^φ that, for $f(t, x, y, z, w)$ belonging to a suitable class of real-valued functions, $t \geq 0, x \in \mathcal{X}, y \in \mathbb{R}, z \in \mathcal{Z}$, and $w \in \mathcal{W}$, is

$$L^\varphi f(t,x,y,z,w) = \frac{\partial}{\partial t} f(t,x,y,z,w) + L_t^\varphi f(t,x,y,z,w), \tag{3.1}$$

with

$$\begin{aligned} L_t^\varphi f(t,x,y,z,w) = {} & \lambda_0(t,x,z)[f(t,x+\xi(t,x,z),y,z+e^{kt}\xi(t,x,z),w) - f(t,x,y,z,w)] \\ & + \lambda(t,z)[f(t,x+\xi(t,x,z),y+\eta_1(t,x,z),z+e^{kt}\xi(t,x,z),w \\ & + \varphi(e^{\eta_1(t,x,z)}-1)) - f(t,x,y,z,w)]p_1(t,x,z) \\ & + \lambda(t,z)[f(t,x+\xi(t,x,z),y-\eta_2(t,x,z),z+e^{kt}\xi(t,x,z),w \\ & + \varphi(e^{-\eta_2(t,x,z)}-1)) - f(t,x,y,z,w)]p_2(t,x,z). \end{aligned}$$

The cost functional becomes

$$J(\varphi) := \mathbb{E}\left[\exp\left\{-w_0 - \int_0^T \frac{\varphi_r}{S_{r-}} \, dS_r\right\}\right] = \mathbb{E}\left[e^{-W_T}\right]$$

and the Hamilton–Jacobi–Bellman equation joint with its terminal condition is given by

$$\begin{cases} \dfrac{\partial U(t,x,y,z,w)}{\partial t} + \inf_{\varphi\in\mathbb{R}} L_t^\varphi U(t,x,y,z,w) = 0, \\ U(T,x,y,z,w) = e^{-w}. \end{cases} \tag{3.2}$$

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PROPOSITION 3.1: *The Hamilton–Jacobi–Bellman (HJB) equation* (3.2) *reduces to the following linear equation with final condition:*

$$\begin{cases} \dfrac{\partial V(t,x,z)}{\partial t} + L_t^* V(t,x,z) + \lambda(t,z)(G^*(t,x,z) - 1)V(t,x,z) = 0, \\ V(T,x,z) = 1, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} L_t^* V(t,x,z) &:= \lambda^*(t,x,z)[V(t,x+\xi(t,x,z),z+e^{kt}\xi(t,x,z)) - V(t,x,z)], \\ \lambda^*(t,x,z) &:= \lambda_0(t,x,z) + \lambda(t,z)G^*(t,x,z). \end{aligned} \tag{3.4}$$

PROOF: Setting $U(t,x,y,z,w) = e^{-w}V(t,x,z)$, we get that $V(T,x,z) = 1$ and that the generator L^{φ}, (3.1), restricted to a function V that does not depend on y and w, is

$$\begin{aligned} L_t^{\varphi} V(t,x,z) = \lambda_0(t,x,z)\left[V(t,x+\xi(t,x,z),z+e^{kt}\xi(t,x,z)) - V(t,x,z)\right] \\ + \lambda(t,z)V(t,x+\xi(t,x,z),z+e^{kt}\xi(t,x,z))G(\varphi,t,x,z) \\ - \lambda(t,z)V(t,x,z), \end{aligned}$$

where

$$G(\varphi,t,x,z) = e^{-\varphi(e^{\eta_1(t,x,z)}-1)}p_1(t,x,z) + e^{-\varphi(e^{-\eta_2(t,x,z)}-1)}p_2(t,x,z).$$

The minimum value of $G(\varphi,t,x,z)$, with respect to φ, is attained at

$$\begin{aligned} \varphi^*(t,x,z) &= -\log\left\{\left(\frac{1-e^{-\eta_2(t,x,z)}}{e^{\eta_1(t,x,z)}-1}\,\frac{p_2(t,x,z)}{p_1(t,x,z)}\right)^{(e^{\eta_1(t,x,z)}-e^{-\eta_2(t,x,z)})^{-1}}\right\}, \\ G^*(t,x,z) &= \min_{\varphi} G(\varphi,t,x,z) \\ &= \exp\{-\varphi^*(t,x,z)(e^{\eta_1(t,x,z)}-1)\}p_1(t,x,z) \\ &\quad + \exp\{-\varphi^*(t,x,z)(e^{-\eta_2(t,x,z)}-1)\}p_2(t,x,z), \end{aligned} \tag{3.5}$$

and the thesis is reached by a direct computation. ∎

PROPOSITION 3.2: *Problem* (3.3) *admits a unique measurable bounded solution that is absolutely continuous with respect to t. Then, for any* (x,z) *and for a.a. t, there exists* $\partial V(t,x,z)/\partial t$ *and it is bounded.*

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PROOF: We note that, taking into account (3.4), (3.3) can be written as

$$\begin{cases} \dfrac{\partial V(t,x,z)}{\partial t} - (\lambda(t,z) + \lambda_0(t,x,z))V(t,x,z) + \lambda^*(t,x,z)V(t,x+\xi(t,x,z),z \\ \quad + e^{kt}\xi(t,x,z)) = 0, \\ V(T,x,z) = 1, \end{cases} \tag{3.6}$$

which is equivalent to

$$V(t,x,z) = e^{-\int_t^T (\lambda(s,z)+\lambda_0(s,x,z))\,ds} + \int_t^T \lambda^*(s,x,z)V(s,x+\xi(s,x,z),z + e^{ks}\xi(s,x,z))e^{-\int_t^s (\lambda(r,z)+\lambda_0(r,x,z))\,dr}\,ds. \tag{3.7}$$

In fact, differentiating both sides of (3.7) with respect to t, we obtain an equation that, joint with (3.7), reproduces (3.6).

Equation (3.7) has a unique bounded solution. If V_1 and V_2 are two bounded solutions, setting

$$\Gamma(t) = \sup_{x,z} |V_1(t,x,z) - V_2(t,x,z)|,$$

we get

$$\Gamma(t) \leq \Lambda^* \int_t^T \Gamma(s)\,ds,$$

and the assertion follows by a slight modification of the Gronwall Lemma.

Finally, by a classical recursive method, we obtain the existence of a bounded solution absolutely continuous with respect to t. Setting

$$\beta(t,u,x,z) = e^{-\int_t^u (\lambda(r,z)+\lambda_0(r,x,z))\,dr} \leq 1,$$

we define, recursively, for $k \geq 0$,

$$V_0(t,x,z) = \beta(t,T,x,z) + \int_t^T \lambda^*(s,x,z)\beta(t,s,x,z)\,ds,$$

$$V_{k+1}(t,x,z) = \beta(t,T,x,z) + \int_t^T \lambda^*(s,x,z)V_k(s,x+\xi(s,x,z),z + e^{ks}\xi(s,x,z))\beta(t,s,x,z)\,ds.$$

Since

$$0 < G^*(t,x,z) \leq \frac{e^{\eta_{\max}} - e^{-\eta_{\min}}}{1 - e^{-\eta_{\min}}} \quad \text{and}$$

$$\lambda^*(t,x,z) \leq \Lambda(1 + G^*(t,x,z)) \leq \Lambda\left(1 + \frac{e^{\eta_{\max}} - e^{-\eta_{\min}}}{1 - e^{-\eta_{\min}}}\right) = \Lambda^*,$$

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we have that

$$\|V_1 - V_0\| \le (2 + \Lambda^* T)\Lambda^* T,$$

$$|V_{k+1}(t,x,z) - V_k(t,x,z)| \le \frac{(\Lambda^*)^k\,(T-t)^k}{k!}\,\|V_1 - V_0\| \le \frac{(\Lambda^*)^k\,T^k}{k!}\,\|V_1 - V_0\|$$

and the conclusion by standard arguments. ■

We give the main result of this section.

THEOREM 3.3: *The HJB equation* (3.2) *admits a unique measurable bounded solution that is absolutely continuous with respect to t. Then, for any x and for a.a. t, there exists $\partial U(t,x,z,w)/\partial t$ and it is bounded. Moreover,*

$$\theta_t^* = \frac{\varphi^*(t, X_{t-}, Z_{t-})}{S_{t-}} \tag{3.8}$$

is the optimal strategy with φ^* given in (3.5).

The Martingale Problem for the operator L_t^*, with initial condition (t,x,z), is well posed. Denoted by $(\widetilde{X},\widetilde{Z})$ as its solution, $(\widetilde{X},\widetilde{Z})$ is a Markov process with trajectories in $D_{\mathcal{X}\times\mathcal{Z}}[0,T]$ and let $\widetilde{P}_{(t,x,z)}$ be its law. Then the value function of the control problem discussed in this section is given by

$$U(t,x,z,w) := e^{-w}\widetilde{\mathbb{E}}_{(t,x,z)}\left[\exp\left\{\int_t^T \lambda(s,\widetilde{Z}_s)\left(G^*(s,\widetilde{X}_s,\widetilde{Z}_s) - 1\right) ds\right\}\right]. \tag{3.9}$$

PROOF: The first claim is reached by setting

$$U(t,x,z,w) = e^{-w}\, V(t,x,z),$$

with V provided by Proposition 3.2.

For the second claim, we get that the generator L_t^*, given in (3.4), is bounded and, again, by Theorem 7.3 in Ethier and Kurtz [11], the Martingale Problem is well posed. By Proposition 3.2, the solution to (3.3) belongs to the domain of L_t^*, and we can write

$$V(u,\widetilde{X}_u,\widetilde{Z}_u) = V(t,x,z) + \int_t^u \left(\frac{\partial V(s,\widetilde{X}_s,\widetilde{Z}_s)}{\partial s} + L_s^* V(s,\widetilde{X}_s,\widetilde{Z}_s)\right) ds + m_u - m_t,$$

where m_u is a zero-mean $(\widetilde{P}_{(t,x,z)}, \mathcal{F}_t)$-martingale.

Finally, by the product formula and by (3.3), we get

$$V(t,x,z) = \widetilde{\mathbb{E}}_{(t,x,z)}\left[\exp\left\{\int_t^T \lambda(s,\widetilde{Z}_s)\left(G^*(s,\widetilde{X}_s,\widetilde{Z}_s) - 1\right) ds\right\}\right]. \tag{3.10}$$

Thus, the function U, defined in (3.9), is a solution to (3.2) and the optimal Markovian strategy is given by θ_t^* in (3.8). ■

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Remark 3.4: Looking at the generator L_t^* given by (3.4), we are able to characterize explicitly the structure of the probability measure $\widetilde{P}$ such that $\widetilde{P}_{(t,x,z)}(\cdot) = \widetilde{P}(\cdot\ |X_t = x, Z_t = z)$. Actually, under $\widetilde{P}$, the point process N^0 admits the same intensity $\lambda^0(t, X_{t-}, Z_{t-})$ as under the original measure P. For $i = 1, 2$, the point process N^i admits the new intensity $\lambda(t, Z_{t-})p_i(t, X_{t-}, Z_{t-})G^*(t, X_{t-}, Z_{t-})$.

Then, by Brémaud [4], we are able to write

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\mathcal{F}_T} = \exp\left\{\int_0^T \log\left(1 + G^*(t, X_{t-}, Z_{t-})\right) dN_t - \int_0^T G^*(t, X_{t-}, Z_{t-})\lambda(t, Z_{t-})\, dt\right\}.$$

4. MINIMAL ENTROPY MARTINGALE MEASURE

Several criteria have been proposed in the literature in order to find, among all equivalent martingale measures, the one that is closest to P, in some sense (Follmer and Schweizer [12], Ansel and Stricker [1], Schweizer [27,28]).

In this section we investigate the probability measure that minimizes the relative entropy with respect to the probability measure P and we give its density, with respect to P in terms of the value function of the control problem discussed in Section 3. Note that, as observed in Frittelli [16], the notion of relative entropy is used intuitively as a measure of the "distance" between two probability measures, even if it is not a metric and it does not define a topology.

4.1. General Properties

For an introduction and a review of the applications of the notion of relative entropy to economics and finance, see Frittelli [16] and the references therein. To discuss the existence of the minimal entropy martingale measure for this model, we need some preliminaries.

DEFINITION 4.1: *On a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$*, given a probability measure* P'*, the relative entropy of* P' *with respect to* P *is defined as*

$$H(P'|P) = \begin{cases} \mathbb{E}^P\left[\dfrac{dP'}{dP}\log\left(\dfrac{dP'}{dP}\right)\right], & P' \ll P \\ +\infty, & \textit{otherwise} \end{cases}$$

and it is a nonnegative strictly convex function, vanishing only if P *coincides with* P'.

DEFINITION 4.2: *Let us denote by* $\mathcal{M}$ *the set of probability measures* Q *absolutely continuous with respect to* P *such that* S *is a* $(Q, \mathcal{F}_t)$*-local martingale and by* $\mathcal{M}_e$ *the set of* $Q \in \mathcal{M}$ *such that* Q *is locally equivalent (and then equivalent on a finite horizon) with respect to* P *(risk-neutral or equivalent martingale measures; see the Appendix for further details).*

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Definition 4.3: *The minimal entropy martingale measure is a probability measure $P^E \in \mathcal{M}$ such that*

$$H(P^E|P) = \min_{Q\in\mathcal{M}} H(Q|P).$$

Since the functional defining the relative entropy, $P' \longrightarrow H(P'|P)$, is strictly convex, if the minimal entropy martingale measure exists, it is unique and equivalent to P (Frittelli [16, Thm 2.2] and Grandits and Rheinlander [20, Thm 3.1]) under the assumption

$$\inf_{Q\in\mathcal{M}_e} H(Q|P) < +\infty. \tag{4.1}$$

A known generalization of Theorem 2.1 in Frittelli [16] enables us to assert that if the price process S is locally bounded, a necessary and sufficient condition for the existence (and, obviously, uniqueness) of the minimal entropy martingale measure is

$$\inf_{Q\in\mathcal{M}} H(Q|P) < +\infty. \tag{4.2}$$

Concerning the structure of the density of P^E with respect to P, we give the following theorem, which summarizes the results reached in Frittelli [16, Thm 2.3] and Grandits and Rheinlander [20, Prop 3.4].

Theorem 4.4: *On a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$*, under* (4.1)*, a probability measure P^E is the minimal entropy martingale measure if and only if it is defined by the Radon–Nikodym derivative*

$$\frac{dP^E}{dP} = e^{H(P^E|P)+F^E},$$

where F^E is an $\mathcal{F}_T$-measurable random variable, such that, for any $Q \in \mathcal{M}$, F^E is Q-integrable, $\mathbb{E}^Q[F^E] \geq 0$, and $\mathbb{E}^{P^E}[F^E] = 0$.

In addition, there exists a predictable process θ_t^E such that $F^E = \int_0^T \theta_t^E \, dS_t$.

Therefore,

$$H(P^E|P) = -\log \mathbb{E}\left[\exp\left(\int_0^T \theta_r^E dS_r\right)\right] \quad \text{and}$$

$$\frac{dP^E}{dP} = \frac{\exp\left(\int_0^T \theta_r^E \, dS_r\right)}{\mathbb{E}\left[\exp\left(\int_0^T \theta_r^E \, dS_r\right)\right]}.$$

Furthermore, the $(P, \mathcal{F}_t)$-predictable process θ^E can be characterized by a duality result, exhaustively discussed in Delbaen et al. [9], Bellini and Frittelli [2], Biagini

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and Frittelli [3], and the references therein. The Duality Principle can be written as

$$\sup_{\theta\in\Theta} E\left[-\exp\left\{-\int_0^T \theta_r\, dS_r\right\}\right] = -\exp\left\{-\inf_{Q\in\mathcal{M}} H(Q|P)\right\} = -\exp\left\{-H(P^E|P)\right\}, \tag{4.3}$$

where Θ is a suitable family of admissible strategies and the supremum on the left-hand side is attained by choosing the predictable process θ^E mentioned in Theorem 4.4.

4.2. The Minimal Entropy Martingale Measure in This Model

We will prove in the Appendix that (4.1) is verified in this model by introducing the minimal martingale measure and showing that it has finite entropy. Then, since the price process is locally bounded, we claim the existence of the minimal entropy martingale measure. To give an explicit representation of its structure, we have to find the $(P, \mathcal{F}_t)$-predictable process θ^E.

The main result of this subsection consists in proving that we can identify θ^E with the optimal control θ^* given in (3.8). The main tool to this end is the Duality Principle as presented in Biagini and Frittelli [3], choosing the exponential utility function $u(x) = -e^{-x}$.

First, to take into account that an agent can accept higher risk but only within a certain degree, we define the set $\mathcal{D}$ of loss random variables (i.e. the family of random variables $D \geq 1$, P-a.s., such that Assumption 4.5 and Assumption 4.6 hold).

ASSUMPTION 4.5: *The random variable D is called S-suitable if there exists $\theta \in \Theta$ such that, $\forall t \in [0, T]$,*

$$\theta_t \neq 0 \quad P-a.s. \quad and \quad \left|\int_0^t \theta_r\, dS_r\right| \leq D.$$

ASSUMPTION 4.6: *The random variable D is u-compatible if, for any real constant $c > 0$, $\mathbb{E}[e^{cD}] < +\infty$.*

Assumption 4.5 implies that all of the investments θ and $-\theta$ are admissible and D controls the loss admitted in trading. Assumption 4.6 assures that the admissible trading strategies are compatible with the preferences of the investors.

Next, we set

$$\Theta^E = \left\{\theta \in \Theta, \text{such that } \exists D \in \mathcal{D}, c > 0 \text{ and } \int_0^T \theta_r\, dS_r \geq -cD\right\}.$$

Then Theorem 4.7 gives us the Duality Principle written in the frame of this article. The proof can be obtained by looking at Theorem 1 in Biagini and Frittelli [3], which holds in a more general setting.

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THEOREM 4.7: *Let us assume that there exists a random variable* $D_0 \in \mathcal{D}$ *such that*

$$\sup_{\theta \in \Theta_0} \mathbb{E}\left[-\exp\left\{-\int_0^T \theta_r \, dS_r\right\}\right] < 0, \tag{4.4}$$

where $\Theta_0 = \{\theta \in \Theta : \exists c \geq 0$ *and* $\int_0^T \theta_r \, dS_r \geq -cD_0\}$. *Then*

$$\sup_{\theta \in \Theta^E} \mathbb{E}\left[-\exp\left\{-\int_0^T \theta_r dS_r\right\}\right] = -\exp\left\{-\min_{Q \in \mathcal{M}_f} H(Q|P)\right\} \tag{4.5}$$

and $\mathcal{M}_f$ *is the set of the risk-neutral measures equivalent to P with finite entropy.*

Remark 4.8: Assumption 4.4 holds true. In fact, in this model the price of the risky asset S is locally bounded and $1 \in \mathcal{D}$, (Biagini and Frittelli [3]). Furthermore, we know that P^E exists and is equivalent to P. Then the results given in Bellini and Frittelli [2, Corol. 2.3] assures that, for $D_0 \equiv 1$,

$$\sup_{\theta \in \Theta_0} \mathbb{E}\left[-\exp\left\{-\int_0^T \theta_r dS_r\right\}\right] = -\exp\left\{-H(P^E|P)\right\} < 0.$$

Finally, Proposition 4.9 allows us to reach our goal.

PROPOSITION 4.9: *Under the additional assumption*

$$p_i(t,x,z) \geq \varepsilon > 0, \quad i = 1,2, \tag{4.6}$$

the optimal control θ^* *given in* (3.8) *belongs to* Θ^E.

PROOF: Under (4.6), $|\varphi^*|$ is bounded, since

$$0 \leq |\varphi^*(t,x,z)| \leq \frac{1}{e^{\eta_{\min}} - e^{-\eta_{\min}}} \max\left\{1, \frac{e^{\eta_{\max}} - 1}{1 - e^{-\eta_{\min}}} \frac{1-\varepsilon}{\varepsilon}\right\} := C \tag{4.7}$$

and

$$\int_0^T \theta_r^* \, dS_r = \sum_{u \leq t} \frac{\varphi_u^*}{S_{u-}}(S_u - S_{u-}) = \sum_{u \leq t} \varphi_u^*(e^{Y_u - Y_{u-}} - 1).$$

Thus, we obtain that

$$\left|\int_0^t \theta_r^* dS_r\right| \leq C\,(e^{\eta_{max}} - 1)N_t.$$

Moreover, for $\theta_t \equiv 1/S_{t-}$,

$$\left|\int_0^t \theta_r \, dS_r\right| = \left|\int_0^t \frac{1}{S_{r-}} \, dS_r\right| = \left|\int_0^t (e^{\eta_r^1} - 1)\, dN_r^1 + \int_0^t (e^{-\eta_r^2} - 1)\, dN_r^2\right|$$
$$\leq (e^{\eta_{\max}} - 1)N_t.$$

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If we choose

$$D_t^* = \hat{C}(N_t + 1) \quad \text{with } \hat{C} > \max\{1, C(e^{\eta_{\max}} - 1), (e^{\eta_{\max}} - 1)\},$$

we get that D^* is S-suitable. Next, we will prove that $D^* \in \mathcal{D}$ and $\theta^* \in \Theta^E$, since $\mathbb{E}[e^{cD^*}] < +\infty, \forall c > 0$.

For any real constant $c > 0$,

$$M_t^{\bar{f}} = e^{cN_t} - 1 - \int_0^t [e^{c(N_s+1)} - e^{cN_s}]\lambda_s\, ds = e^{cN_t} - 1 - (e^c - 1)\int_0^t e^{cN_s}\lambda_s\, ds$$

is a $(P, \mathcal{F}_t)$-local martingale. Let $\{t_n\}_{n\geq 1}$ be a localizing sequence of stopping times, such that $M_{t\wedge t_n}^{\bar{f}}$ is a $(P, \mathcal{F}_t)$-uniformly integrable martingale and $\mathbb{E}[M_{t\wedge t_n}^{\bar{f}}] = 0$,

$$\mathbb{E}[e^{cN_{t\wedge t_n}}] = 1 + (e^c - 1)\mathbb{E}\left[\int_0^{t\wedge t_n} e^{cN_{s\wedge t_n}}\lambda_s\, ds\right] \leq 1 + \Lambda(e^c - 1)\int_0^t \mathbb{E}\left[e^{cN_{s\wedge t_n}}\right] ds.$$

By the Gronwall lemma we get $\mathbb{E}[e^{cN_{t\wedge t_n}}] \leq \exp\{\Lambda(e^c - 1)\}$, and a monotone convergence argument yields the thesis. ■

COROLLARY 4.10: *Under the assumptions of this article, setting $\varphi_s^* = \varphi^*(s, \widetilde{X}_s, \widetilde{Z}_s)$,*

$$\sup_{\theta\in\Theta} \mathbb{E}\left[-\exp\left\{-\int_0^T \theta_r\, dS_r\right\}\right] = \sup_{\theta\in\Theta^E} \mathbb{E}\left[-\exp\left\{-\int_0^T \theta_r\, dS_r\right\}\right]$$
$$= -\exp\left\{-H(P^E|P)\right\}$$

and

$$H(P^E|P) = -\log\left\{\widetilde{\mathbb{E}}_{(0,x_0,z_0)}\left[\exp\left(\int_0^T \lambda(s, \widetilde{Z}_s)\left(G(\varphi_s^*, s, \widetilde{X}_s, \widetilde{Z}_s) - 1\right)\, ds\right)\right]\right\}.$$

PROOF: The first equality follows by considering the stochastic control problem discussed in Section 3. Moreover, setting $t = 0$, $x = x_0$, $z = z_0$, and $w = 0$ in (3.10),

$$U(0, x_0, z_0, 0) := V(0, x_0, z_0) = \widetilde{\mathbb{E}}_{(0,x_0,z_0)}\left[\exp\left\{\int_0^T \lambda(s, \widetilde{Z}_s)\left(G(\varphi_s^*, s, \widetilde{X}_s, \widetilde{Z}_s) - 1\right) ds\right\}\right]$$

and for the Duality Principle (4.5), we have the second claim. ■

As a conclusion, we are able to identify the predictable process θ^E, mentioned in Theorem 4.4, with the optimal strategy θ^* given in (3.8), where φ^* is defined in (3.5).

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Then the final result of this section is an explicit expression of the density of the minimal entropy martingale measure. Setting

$$\gamma^1(t,x,z) = \exp\{\varphi^*(t,x,z)(e^{\eta^1(t,x,z)} - 1)\} - 1,$$
$$\gamma^2(t,x,z) = \exp\{\varphi^*(t,x,z)(e^{-\eta^2(t,x,z)} - 1)\} - 1,$$

and

$$\gamma_t^i = \gamma^i(t, X_{t^-}, Z_{t^-}) \quad \text{for } i = 1,2,$$

since by (A.5),

$$\int_0^T \theta_r^*\, dS_r = \int_0^t \frac{\varphi_r^*}{S_{r^-}}\, dS_r = \int_0^t \sum_{i=1,2} \log(1+\gamma_s^i)\, dN_s^i, \tag{4.8}$$

we get

$$\frac{dP^E}{dP} = \frac{\exp\left\{\int_0^T \theta_r^*\, dS_r\right\}}{\mathbb{E}\left[\exp\left\{\int_0^T \theta_r^*\, dS_r\right\}\right]} = \frac{\exp\left\{\int_0^t \sum_{i=1,2} \log(1+\gamma_r^i)\, dN_r^i\right\}}{\mathbb{E}\left[\exp\left\{\int_0^t \sum_{i=1,2} \log(1+\gamma_r^i)\, dN_r^i\right\}\right]}. \tag{4.9}$$

5. PRICING UNDER RESTRICTED INFORMATION

In this section we assume that the process (X, Z) is unobservable by the agents, who can only observe the behavior of the price process S or, equivalently, the behavior of the logreturn process Y. Given a European contingent claim with maturity T, referred to as the option, with payoff $B = B(S_T)$, a bounded $\mathcal{F}_T^S$-measurable random variable, we deal with the problem of pricing that is to determine the value of B at each time $t \in [0, T]$ in order to avoid arbitrage opportunities. Thus, we face the problem of pricing in a partially observed model under incompleteness of the market.

It is well known that in the full information case and in a complete market, there exists a unique risk-neutral measure. The arbitrage-free price of a contingent claim is defined as the expectation of B conditioned with respect to $\mathcal{F}_t$ under this measure. However, we are studying a model in which the market is incomplete, hence there exist many martingale measures and a choice must be made. The minimal entropy martingale measure appears as a good choice, as a consequence of an asymptotic result given in Mania and Schweizer [23, Thm. 17].

Furthermore, we must define the arbitrage-free price in a model with restricted information. The discussion in Gombani et al. [19] suggests a choice similar to that followed in Ceci and Gerardi [6] and Frey and Runggaldier [15]. Thus, we consider martingale measures with respect to the filtration $\mathcal{F}_t$ and we will define the price of the claim as the expectation conditioned to the observations, under the minimal entropy martingale measure.

To accomplish this program, we need a better knowledge of the structure of this measure. In the next subsection, we obtain such knowledge at the cost of a strong condition on the parameters defining the dynamics of the model (see (5.2)).

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5.1. A Particular Model

Throughout this entire section we assume that (4.6) holds true and we define the process

$$M_t^E = \int_0^t \sum_{i=1,2} \gamma_s^i \,(dN_s^i - \lambda_s^i \,ds).$$

Under (4.6), φ^* is bounded (see (4.7)), $|\gamma^i|$ and λ^i are bounded for $i = 1, 2$. This implies that M_t^E is a $\{P, \mathcal{F}_t\}$-martingale. Moreover, $\gamma_t^i + 1 > 0$ and we define the process

$$\mathcal{L}_t^E = \mathcal{E}(M_t^E) = \exp\left\{\sum_{i=1,2} \int_0^t \log\left(1 + \gamma_s^i\right) dN_s^i - \sum_{i=1,2} \int_0^t \gamma_s^i \lambda_s^i \,ds\right\}. \quad \mathbf{(5.1)}$$

LEMMA 5.1: *The process $\mathcal{L}_t^E$ is a $\{P, \mathcal{F}_t\}$-martingale.*

PROOF: For all $t \geq 0$, $\mathcal{L}_t^E$ is the solution to

$$\mathcal{L}_t^E = 1 + \int_0^t \mathcal{L}_{s-}^E \,dM_s^E = 1 + \int_0^t \mathcal{L}_{s-}^E \gamma_s^1 \left[dN_s^1 - \lambda_s^1 \,ds\right] + \int_0^t \mathcal{L}_{s-}^E \gamma_s^2 \left[dN_s^2 - \lambda_s^2 \,ds\right]$$

and is a strictly positive supermartingale (Ansel and Stricker [1] and Doléans-Dade [10]). Consequently (Brémaud [4]), since

$$\mathbb{E}\left[\int_0^t \mathcal{L}_{s-}^E |\gamma_s^1| \lambda_s^1 \,ds + \int_0^t \mathcal{L}_{s-}^E |\gamma_s^2| \lambda_s^2 \,ds\right] \leq K\Lambda \int_0^t \mathbb{E}[\mathcal{L}_{s-}^E] \,ds < +\infty,$$

where K is a positive constant depending on $\eta_{\min}$ and $\eta_{\max}$, such that $|\gamma_t^1| \vee |\gamma_t^2| \leq K$, the required result follows. ■

PROPOSITION 5.2: *Setting*

$$\frac{dQ^E}{dP} = \mathcal{L}_T^E,$$

we define a probability measure equivalent to P, which is a risk-neutral measure.

PROOF: The measure Q^E preserves the Markovianity of the model (see the Appendix), and then we apply the sufficient condition (A.12). A direct computation provides

$$\begin{aligned}
&(e^{\eta_1(t,x,z)} - 1)[1 + \gamma^1(t,x,z)]p_1(t,x,z) + (e^{-\eta_2(t,x,z)} - 1)[1 + \gamma^2(t,x,z)]p_2(t,x,z) \\
&\quad = (e^{\eta_1(t,x,z)} - 1)e^{\varphi^*(t,x,z)(e^{\eta_1(t,x,z)} - 1)} p_1(t,x,z) \\
&\qquad + (e^{-\eta_2(t,x,z)} - 1)e^{\varphi^*(t,x,z)(e^{-\eta_2(t,x,z)} - 1)} p_2(t,x,z) = 0.
\end{aligned}$$

■

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THEOREM 5.3: *Assume that there exists a deterministic measurable function* $h(t)$, *integrable on* $[0,T]$, *such that*

$$\sum_{i=1,2} \gamma^i(t,x,z)\lambda(t,z)p^i(t,x,z) = -h(t). \tag{5.2}$$

In such a case, the measures Q^E *and* P^E *coincide.*

PROOF: By (4.8), we have that

$$\mathcal{L}_t^E = \mathcal{E}(M_t^E) = \exp\left\{\int_0^t \frac{\varphi_r^*}{S_{r-}}\, dS_r + \int_0^t h(r)\, dr\right\},$$

and recalling that $\mathbb{E}[\mathcal{L}_t^E] = 1$,

$$\mathbb{E}\left[\exp\left\{\int_0^t \theta_r^*\, dS_r\right\}\right] = \mathbb{E}\left[\exp\left\{\int_0^t \frac{\varphi_r^*}{S_{r-}}\, dS_r\right\}\right] = \exp\left\{-\int_0^t h(r)\, dr\right\}.$$

Moreover, by Proposition 5.2, $\mathbb{E}^{Q^E}[\int_0^t \theta_r^*\, dS_r] = 0$ and

$$H(Q^E|P) = \mathbb{E}^{Q^E}\left[\log \frac{dQ^E}{dP}\right] = \mathbb{E}^{Q^E}\left[\log \mathcal{L}_T^E\right] = \int_0^T h(r)\, dr.$$

Finally, for any martingale measure Q,

$$\begin{aligned} H(Q|P) \geq \mathbb{E}^Q\left[\log \frac{dP^E}{dP}\right] &= \mathbb{E}^Q\left[\int_0^t \theta_r^*\, dS_r\right] - \log \mathbb{E}\left[\exp\left\{\int_0^t \theta_r^*\, dS_r\right\}\right] \\ &= \int_0^T h(r)\, dr = H(Q^E|P). \end{aligned}$$

■

Remark 5.4: When all of the parameters defining the structure of the system depend only on time, as it happens, for instance, in Fujiwara and Miyahara [17], condition (5.2) is trivial and reduces to the definition of the function $h(t)$. This is not the case in the model described in this article, where (5.2) represents a true restriction on the dynamics of the model.

5.2. Filtering in the Discrete Case

We want to compute

$$\mathbb{E}^{P^E}[B(S_T)|\mathcal{F}_t^Y] = \mathbb{E}^{P^E}[\mathbb{E}^{P^E}[B(S_T)|\mathcal{F}_t]|\mathcal{F}_t^Y]. \tag{5.3}$$

Since P^E preserves the Markovianity of the process (X,Y,Z), there exists a measurable function $h(t,x,y,z)$ such that $\mathbb{E}^{P^E}[B(S_T)|\mathcal{F}_t] = h(t,X_t,Y_t,Z_t)$, where h solves

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the system

$$\begin{cases} L^E h(t,x,y,z) = \dfrac{\partial}{\partial t} h(t,x,y,z) + L_t^E h(t,x,y,z) = 0, \\ h(T,x,y,z) = B(S_0 e^y), \end{cases} \tag{5.4}$$

with

$$L_t^E h(t,x,y,z) = L_{0,t}^E h(t,x,y,z) + L_{1,t}^E h(t,x,y,z) + L_{2,t}^E h(t,x,y,z), \tag{5.5}$$

$$L_{0,t}^E h(t,x,y,z) = \lambda_0(t,x,z)[h(t,x+\xi(t,x,z),y,z+e^{kt}\xi(t,x,z)) - h(t,x,y,z)],$$

and, for $i = 1, 2$,

$$\begin{aligned} L_{i,t}^E h(t,x,y,z) = \lambda(t,z)p_i(t,x,z)(1+\gamma^i(t,x,z))[h(t,x+\xi(t,x,z),y \\ + (-1)^{i-1}\eta_i(t,x,z), z + e^{kt}\xi(t,x,z)) - h(t,x,y,z)]. \end{aligned}$$

The system (5.4) can be treated with a procedure similar to that used in Proposition 3.2, and (5.3) reduces to

$$\mathbb{E}^{P^E}[B(S_T)|\mathcal{F}_t^Y] = \mathbb{E}^{P^E}[h(t,X_t,Y_t,Z_t)|\mathcal{F}_t^Y].$$

Finally, for any bounded measurable F, we consider the filter $\pi_t(F(t,\cdot,Y_t,\cdot)) = \mathbb{E}^{P^E}[F(t,X_t,Y_t,Z_t)|\mathcal{F}_t^Y]$ (i.e., the cadlag version of the law of the process (X,Y,Z), given the σ-algebra $\mathcal{F}_t^S = \mathcal{F}_t^Y$, under P^E). Hence, by the classical innovation method (Brémaud [4]), we write down the Kushner–Stratonovich equation, which the filter has to satisfy. This equation will be derived in this subsection under the assumption that the process Y is discrete-valued. The more abstract general case is briefly presented in the next subsection.

Then, we assume, without loss of generality that the process Y takes value in the set $\mathbb{Z}$ of integer numbers and, as in Gerardi and Tardelli [18], we introduce the multivariate point process $U = (U^1, U^2, \ldots)$ defined as

$$U_t^j := \sum_{i \geq 1} \mathbf{1}_{\{\tau_i^Y \leq t\}} \mathbf{1}_{\{Y_{\tau_i^Y} = j\}}, \ \ j \in \mathbb{Z}.$$

For any $j \in \mathbb{Z}$, U_t^j counts the number of jumps bringing Y_s on j, for $s \leq t$, and the relation

$$Y_t = Y_0 + \int_0^t \sum_{j \in \mathbb{Z}} [j - Y_{s-}]\, dU_s^j \tag{5.6}$$

implies that $\mathcal{F}_t^Y = \mathcal{F}_t^U := \sigma\{U_s^1, U_s^2, \ldots, s \leq t\}$. Thus, the filtering problem reduces to finding the conditional law of (X,Y,Z) given $\mathcal{F}_t^U$, under P^E. We note that in (5.6) just one term of the integrand is not null, almost surely.

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The joint generator of (X, Y, Z) and U^j, $j \in \mathbb{Z}$, for $F(t, x, y, u, z)$ belonging to a suitable class of real-valued measurable functions, $t \geq 0$, $x \in \mathcal{X}$, $y \in \mathbb{Z}$, $z \in \mathcal{Z}$, and $u \in \mathbb{N}$, is given by

$$L^{E,j}F(t,x,y,u,z) = \frac{\partial}{\partial t}F(t,x,y,u,z) + L_t^{E,j}F(t,x,y,u,z), \tag{5.7}$$

for

$$L_t^{E,j}F(t,x,y,u,z) = L_{0,t}^{E,j}F(t,x,y,u,z) + L_{1,t}^{E,j}F(t,x,y,u,z) + L_{2,t}^{E,j}F(t,x,y,u,z), \tag{5.8}$$

where

$$\begin{aligned} L_{0,t}^{E,j}F(t,x,y,u,z) = \lambda_0(t,x,z)[&F(t,x+\xi(t,x,z),y,u,z+e^{kt}\xi(t,x,z)) \\ &- F(t,x,y,u,z)], \end{aligned}$$

and, for $i = 1, 2$,

$$\begin{aligned} L_{i,t}^{E,j}F(t,x,y,u,z) = \lambda(t,z)p_i(t,x,z)(1+\gamma^i(t,x,z)) \cdot \mathbf{1}_{y+(-1)^{i-1}\eta_i(t,x,z)=j} \\ \times [F(t,x+\xi(t,x,z),y+(-1)^{i-1}\eta_i(t,x,z),u+e^j,z \\ + e^{kt}\xi(t,x,z)) - F(t,x,y,u,z)]. \end{aligned}$$

Then we give the main result of this subsection.

THEOREM 5.5: *The $(P, \mathcal{F}_t)$-intensity of U^j, for a fixed $j \in \mathbb{Z}$, is given by the process*

$$\begin{aligned} &\lambda_j(t, X_{t-}, Y_{t-}, Z_{t-}) \\ &= \lambda(t, Z_{t-}) \sum_{i=1,2} p_i(t, X_{t-}, Z_{t-})(1+\gamma^i(t, X_{t-}, Z_{t-}))\mathbf{1}_{Y_{t-}+(-1)^{i-1}\eta_i(t,X_{t-},Z_{t-})=j}. \end{aligned} \tag{5.9}$$

For any real-valued bounded measurable function F defined on $[0, T] \times \mathcal{X} \times \mathbb{R} \times \mathcal{Z}$, the Kushner–Stratonovich equation can be written as

$$\begin{aligned} \pi_t(F) = \pi_0(F) &+ \int_0^t \pi_s(L_s^E F(s,\cdot,Y_{s-},\cdot))\,ds \\ &+ \sum_{j\in\mathbb{Z}} \int_0^t \Psi_{s-}^j(F(s,\cdot,Y_{s-},\cdot))\big(dU_s^j - \pi_{s-}(\lambda_j(s,\cdot,Y_{s-},\cdot))\,ds\big), \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \Psi_{s-}^j(F(s,\cdot,Y_{s-},\cdot)) = \pi_{s-}(\lambda_j(s,\cdot,Y_{s-},\cdot))^+\{&\pi_{s-}(\lambda_j(s,\cdot,Y_{s-},\cdot)F) \\ &- \pi_{s-}(\lambda_j(s,\cdot,Y_{s-},\cdot))\pi_{s-}(F) + \pi_{s-}(L_{1,s}^{E,j}F(s,\cdot,Y_{s-},\cdot) \\ &+ L_{2,s}^{E,j}F(s,\cdot,Y_{s-},\cdot))\} \end{aligned}$$

and $a^+ := (1/a)\mathbf{1}_{\{a>0\}}$. Moreover, the filtering equation has a unique strong solution.

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PROOF: The first claim is obtained by taking into account the joint dynamics of X_t, Y_t, Z_t, and U_t^j given in (5.7). We get the Kushner–Stratonovich equation by applying the classical innovation method as described in Brémaud [4]. In particular, the last term in $\Psi^j_{s-}(F(s,\cdot,Y_{s-},\cdot))$, which arises when common jump times between the state and the observations are allowed, is related with $\langle F(t,X,Z), U^j\rangle_t$.

As far as the strong uniqueness of the solutions of (5.10), we observe that at any jump time $t = \tau_k^Y$, the filter is uniquely determined by the knowledge of π_{t-}. In fact, for $j = Y_t$, $\pi_{t-}(\lambda_j(t,\cdot,\cdot)) \neq 0$, and we write

$$\begin{aligned}\pi_t(F) &= \pi_{t-}(F) + \Psi^j_{t-}(F(t,\cdot,Y_{t-},\cdot))\Big|_{j=Y_t}\\ &= \frac{\pi_{t-}\big(\lambda_j(t,\cdot,Y_{t-},\cdot)F + L^{E,j}_{1,t}F(t,\cdot,Y_{t-},\cdot) + L^{E,j}_{2,t}F(t,\cdot,Y_{t-},\cdot)\big)}{\pi_{t-}\big(\lambda_j(t,\cdot,Y_{t-},\cdot)\big)}\Bigg|_{j=Y_t}.\end{aligned}$$

For $t \in \big[\tau_k^Y, \tau_{k+1}^Y\big)$, the behavior of the filter is defined by

$$\pi_t(F) = \pi_{\tau_k^Y}(F) + \int_{\tau_k^Y}^t \Big[\pi_s\left(L^E_{0,s}F(s,\cdot,Y_{s-},\cdot)\right) + \pi_s(\hat{\lambda}(s,\cdot,\cdot))\pi_s(F) - \pi_s(\hat{\lambda}(s,\cdot,\cdot)F)\Big]\,ds, \tag{5.11}$$

where

$$\begin{aligned}\hat{\lambda}(t,X_t,Z_t) &= \sum_{j\in\mathbb{Z}} \lambda_j(t,X_{t-},Y_{t-},Z_{t-})\\ &= \lambda(t,Z_{t-}) \sum_{i=1,2} p_i(t,X_{t-},Z_{t-})(1+\gamma^i(t,X_{t-},Z_{t-})).\end{aligned}$$

For any two solutions π_t^1 and π_t^2 of (5.11), such that $\pi^1_{\tau_k^Y}(F) = \pi^2_{\tau_k^Y}(F)$, we can find a suitable positive constant C depending on $\|F_t\| = \sup_{x,z}|F(t,x,Y_t,z)|$, such that

$$|\pi_t^1(F) - \pi_t^2(F)| \leq C \int_{\tau_k^Y}^t \|\pi_s^1 - \pi_s^2\|\,ds,$$

where $\|\cdot\|$ denotes the bounded variation norm of the signed measure $\pi_s^1 - \pi_s^2$. The last inequality guarantees uniqueness for $t \in \big[\tau_k^Y, \tau_{k+1}^Y\big)$ and the thesis follows by induction. ■

Finally, a representation for the filter via a classical linearized method can be performed, as in Gerardi and Tardelli [18] for example, showing that the computation of the filter between two consecutive jump times can be reduced to the evaluation of an ordinary expectation. As a matter of fact, (5.11) is a nonlinear one. Thus, an explicit expression for its solutions is not usually available, but it is possible to provide a representation for it with a method that is a modification of that proposed in Kliemann,

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Koch, and Marchetti [22]. First, we introduce the linearized equation

$$\rho_t(F) = \rho_0(F) + \int_0^t \left[\rho_s \left(L^E_{0,s} F(s,\cdot,Y_{s-},\cdot) \right) - \rho_{s-}\left(\hat{\lambda}(s,\cdot,\cdot) F \right) \right] ds, \tag{5.12}$$

which is obtained by (5.11) dropping out the nonlinear terms. Equation (5.12) not only admits a unique solution in the weak sense, but, by a Lipschitz argument, it also admits a unique pathwise solution that is necessarily $\mathcal{F}^U_t$-adapted. This last claim can be proven with a procedure similar to that used in Theorem 5.5.

PROPOSITION 5.6: *Equation* (5.12) *admits at least one solution that is* $\mathcal{F}^U_t$*-adapted. In addition, this solution is a finite positive measure and, for all* $t \in [\tau^Y_i, \tau^Y_{i+1})$, $i \geq 1$, $0 < e^{-(t-\tau^Y_i)C} \leq \rho_t(1) \leq 1$,

$$\pi_t(F) = \frac{\rho_t(F)}{\rho_t(1)}.$$

PROOF: First we claim that for ρ_t any solution of (5.12), $\rho_t(F)/\rho_t(1)$ provides a solution of (5.11) and then coincides with the filter up to time $t_0 = \inf\{t \geq 0 : \rho_t(1) = 0\}$. Hence, we construct a solution for (5.12) that has the required properties, and such that $\rho_t(1) > 0$.

Let $\Xi_{s,x,z}(t)$ be a process with initial condition (s,x,z), $s \geq 0$, $x \in \mathcal{X}$, $z \in \mathcal{Z}$, and generator L^E_t, given by (5.5). Let $P_{s,x,z}$ be its law on $D_{\mathcal{X}\times\mathcal{Z}}[s,T]$. Then, by the Feynman–Kac's formula, $\forall t \in [\tau^Y_i, \tau^Y_{i+1})$,

$$\rho_t(F) = \int_{\mathcal{X}\times\mathcal{Z}} \mathbb{E}_{s,x,z}\left[F(\Xi_{s,x,z}(t)) \exp\left\{ -\int_s^t \widehat{\lambda}(\Xi_{s,x,z}(u))\, du \right\} \right]\Bigg|_{s=\tau^Y_i} \frac{\rho_{\tau^Y_i}(dx,dz)}{\rho_{\tau^Y_i}(1)},$$

$$\rho_t(1) = \int_{\mathcal{X}\times\mathcal{Z}} \mathbb{E}_{s,x,z}\left[\exp\left\{ -\int_s^t \widehat{\lambda}(\Xi_{s,x,z}(u))\, du \right\} \right]\Bigg|_{s=\tau^Y_i} \frac{\rho_{\tau^Y_i}(dx,dz)}{\rho_{\tau^Y_i}(1)},$$

and, under Assumption (4.6), there exists a real constant $C := C(\Lambda, \varepsilon, \eta_{\min}, \eta_{\max}) > 0$, such that

$$0 < e^{-(t-\tau^Y_i)C} \leq \rho_t(1) \leq 1. \qquad \blacksquare$$

5.3. Filtering in the Continuous Case

We derive the filtering equation with a procedure quite similar to that provided in Ceci [5]; thus, the proof will be omitted. Preliminarily, we define the integer-valued random measure associated to Y:

$$m(dt,dy) = \sum_{n\geq 1} \delta_{\{\tau^Y_n, Y_{\tau^Y_n} - Y_{\tau^Y_n -}\}}(dt,dy)\, \mathbb{1}_{\tau^Y_n < +\infty},$$

whose $\{P^E, \mathcal{F}_t\}$-predictable projection is $\nu^E(dy)\, dt$, and

$$\nu^E(dy) = \lambda^1_t\,(1+\gamma^1_t)\,\delta_{\{\eta^1_t\}}(dy) + \lambda^2_t\,(1+\gamma^2_t)\,\delta_{\{-\eta^2_t\}}(dy).$$

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PROPOSITION 5.7: *Under the assumptions of this section, for any real-valued, bounded, measurable function $f(x, z)$, the Kushner–Stratonovich equation is given by*

$$\pi_t(f) = f(0, z_0) + \int_0^t \pi_s(L_t^E f)\, ds + \int_0^t \int_{\mathbb{R}} \left\{ \frac{d\pi_{s-}(f\ \nu_s^E)}{d\pi_{s-}(\nu_s^E)} - \pi_{s-}(f) + \frac{d\pi_{s-}(B_s f)}{d\pi_{s-}(\nu_s^E)} \right\} \left(m(ds, dy) - \pi_{s-}(\nu_s^E(dy))\, ds \right), \tag{5.13}$$

where

$$\frac{d\mu_1}{d\mu_2}$$

denotes the Radon–Nikodym derivative of the measure μ_1 with respect to the measure μ_2 and $B_s f$ is measure defined as

$$B_s f(dy) = \left[f\left(X_{s-} + \xi_s, Z_{s-} + e^{ks}\xi_s\right) - f\left(X_{s-}, Z_{s-}\right) \right] \nu_s^E(dy).$$

Again, at any jump time τ_n^Y, the filter is uniquely determined by its behavior for $t \in [\tau_{n-1}^Y, \tau_n^Y)$. For $t \in [\tau_{n-1}^Y, \tau_n^Y)$, the filtering equation reduces to an equation that verifies a Lipschitz condition with respect to the bounded variation norm.

Thus, the discussion about the uniqueness of the solutions (as well as the linearization method, which, jointly with the Feynmann–Kac formula, provides an expression of the filter between two jump times in terms of ordinary expectations) can be performed as in the previous case.

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APPENDIX

Risk-Neutral Measures and the Minimal Martingale Measure

In the model considered in this article, the price process is a semimartingale, as we will prove in Lemma A.1. Successively, we will characterize the equivalent martingale measures that is, the probability measures Q, equivalent to P, under which S is a $(Q, \mathcal{F}_t)$-local martingale.

LEMMA A.1: *The stock price process S_t is a special $(P, \mathcal{F}_t)$-local semimartingale with canonical decomposition*

$$S_t = S_0 + A_t^S + M_t^S, \tag{A.1}$$

where A_t^S is a locally bounded variation predictable process,

$$A_t^S = \int_0^t \lambda_u S_{u-} \left[e^{\eta_u^1} p_u^1 + e^{-\eta_u^2} p_u^2 - 1 \right] du, \tag{A.2}$$

M_t^S is a locally square integrable $(P, \mathcal{F}_t)$-local-martingale,

$$M_t^S = \int_0^t S_{u-}(e^{\eta_u^1} - 1)[dN_u^1 - \lambda_u p_u^1 \, du] + \int_0^t S_{u-}(e^{-\eta_u^2} - 1)[dN_u^2 - \lambda_u p_u^2 \, du], \tag{A.3}$$

and

$$\langle M^S \rangle_t = \int_0^t \lambda_u S_{u-}^2 \left[(e^{\eta_u^1} - 1)^2 \, p_u^1 + (e^{-\eta_u^2} - 1)^2 \, p_u^2 \right] du. \tag{A.4}$$

PROOF: By the Itô formula

$$\begin{aligned} S_t &= S_0 + \int_0^t S_{u-}(e^{\eta_u^1} - 1) \, dN_u^1 + \int_0^t S_{u-}(e^{-\eta_u^2} - 1) \, dN_u^2 \\ &= S_0 + \int_0^t S_{u-}(e^{\eta_u^1} - 1)[dN_u^1 - \lambda_u p_u^1 \, du] + \int_0^t S_{u-}(e^{\eta_u^1} - 1) \, \lambda_u p_u^1 \, du \\ &\quad + \int_0^t S_{u-}(e^{-\eta_u^2} - 1) \, [dN_u^2 - \lambda_u p_u^2 \, du] + \int_0^t S_{u-}(e^{-\eta_u^2} - 1) \, \lambda_u p_u^2 \, du. \end{aligned} \tag{A.5}$$

Thus, (A.2) and (A.3) follow. Moreover, M_t^S is a locally square integrable $(P, \mathcal{F}_t)$-local martingale since

$$\int_0^t \lambda_u \, S_u^2 \left[(e^{\eta_u^1} - 1)^2 p_u^1 + (e^{-\eta_u^2} - 1)^2 p_u^2 \right] du < +\infty \quad P - \text{a.s.}$$

In order to compute $\langle M^S \rangle_t$, we note that, for $\bar{f}(y) = S_0 e^y$,

$$\langle M^S \rangle_t = \int_0^t \left[L_u(\bar{f}(Y_{u-})^2) - 2\bar{f}(Y_{u-}) L_u \bar{f}(Y_{u-}) \right] du. \qquad \blacksquare$$

A.1. Equivalent Martingale Measures, General Properties

The main tool for characterization of the risk-neutral measures is a suitable version of the Girsanov theorem. The choice of the internal filtration allows us to claim that any probability measure Q equivalent to P is a solution to the exponential equation $\mathcal{L}_t = 1 + \int_0^t \mathcal{L}_{s-} dM_s$, for M $(P, \mathcal{F}_t)$-local martingale.

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As is well known, Doléans-Dade [10] and Jacod [21], this equation has a unique solution. Furthermore, if M is a $(P, \mathcal{F}_t)$-local martingale such that $M_t - M_{t-} > -1$, then $\mathcal{L}$ is a $(P, \mathcal{F}_t)$-local martingale and a strictly positive supermartingale, which turns to be a $(P, \mathcal{F}_t)$-martingale when $\mathbb{E}[\mathcal{L}_T] = 1$. In this last case, the measure Q defined by the Radon–Nykodim derivative $dQ/dP|_{\mathcal{F}_T} = \mathcal{L}_T$ is a probability measure equivalent to P.

On the other hand, any $(P, \mathcal{F}_t)$-local martingale M_t admits the representation

$$M_t = M_0 + \sum_{i=0,1,2} \int_0^t g_s^i \left[dN_s^i - \lambda_s^i \, ds \right], \tag{A.6}$$

where g_s^i, for $i = 0, 1, 2$, are $(P, \mathcal{F}_s)$-predictable processes. Under the assumption that

$$\sum_{i=0,1,2} \int_0^t |g_s^i| \lambda_s^i \, ds < +\infty \quad P\text{-a.s.} \quad \text{or} \quad \mathbb{E}\left[\sum_{i=0,1,2} \int_0^t |g_s^i| \lambda_s^i \, ds \right] < +\infty,$$

M_t is a $(P, \mathcal{F}_t)$-local martingale or a $(P, \mathcal{F}_t)$-martingale, respectively. In this last case, necessarily, M_t is uniformly integrable, since we are working on a finite horizon.

Remark A.2: By (A.6),

$$M_t - M_{t-} = \sum_{i=0,1,2} g_t^i \left(N_t^i - N_{t-}^i \right).$$

Therefore, at any jump time of the process (X, Y, Z),

$$M_t - M_{t-} > -1 \quad \Longleftrightarrow \quad g_t^i > -1 \quad \text{for} \quad i = 0, 1, 2, \tag{A.7}$$

and the density of Q with respect to P can be written as

$$\mathcal{L}_T = \prod_{i=0,1,2} \exp \left\{ \int_0^T \log\left(1 + g_s^i\right) \, dN_s^i - \int_0^T g_s^i \lambda_s^i \, ds \right\}.$$

Moreover, if

$$\sum_{i=0,1,2} \int_0^T (g_t^i + 1) \, \lambda_t^i \, dt < +\infty \quad P\text{-a.s.} \tag{A.8}$$

the processes N^0, N^1, and N^2 admit $(Q, \mathcal{F}_t)$-intensity given by $(g_t^0 + 1)\lambda_t^0$, $(g_t^1 + 1)\lambda_t^1$, and $(g_t^2 + 1)\lambda_t^2$, respectively.

With a procedure similar to that used in the proof of Lemma A.1 we have that, under (A.7), the price of the risky asset, S, is a special $(Q, \mathcal{F}_t)$-local semimartingale, such that

$$S_t = S_0 + A_t^Q + M_t^Q,$$

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where

$$A_t^Q = \int_0^t \lambda_u S_{u-}[(e^{\eta_u^1} - 1)(g_u^1 + 1)p_u^1 + (e^{-\eta_u^2} - 1)(g_u^2 + 1)p_u^2]\, du$$

and

$$M_t^Q = \int_0^t S_{u-}(e^{\eta_u^1} - 1)[dN_u^1 - (g_u^1 + 1)\lambda_u^1\, du] + \int_0^t S_{u-}(e^{-\eta_u^2} - 1)[dN_u^2 - (g_u^2 + 1)\lambda_u^2\, du].$$

As a consequence, we give a characterization of the risk-neutral measures.

PROPOSITION A.3: *The measure Q is risk-neutral if and only if*

$$A_t^Q = 0 \quad P\text{−a.s.} \quad \text{and} \quad M_t^Q = (Q, \mathcal{F}_t)\text{−local martingale.} \qquad \textbf{(A.9)}$$

We only observe that if

$$\int_0^t \lambda_u S_{u-}\left[(e^{\eta_u^1} - 1)(g_u^1 + 1)p_u^1 + (1 - e^{-\eta_u^2})(g_u^2 + 1)p_u^2\right] du < +\infty \quad P\text{−a.s.},$$

then M_t^Q is a $(Q, \mathcal{F}_t)$-local martingale that turns to be a $(Q, \mathcal{F}_t)$-martingale if

$$\mathbb{E}\left[\int_0^t \lambda_u S_{u-}\left\{(e^{\eta_u^1} - 1)(g_u^1 + 1)p_u^1 + (1 - e^{-\eta_u^2})(g_u^2 + 1)p_u^2\right\} du\right] < +\infty.$$

Let us finally note that if the price process is strictly increasing or strictly decreasing, (A.9) cannot be satisfied and the model does not admit any equivalent martingale measure.

Remark A.4: Recalling Lemma A.1, we observe that when $A_t^S = 0$, P-a.s., the original measure P is a risk-neutral measure.

A.2. Equivalent Martingale Measures, Markov Property

Next, in the set of the risk-neutral measures, when it is not empty, we can find an element preserving the Markovianity of the process (X, Y, Z), assuming the existence of the real-valued measurable deterministic functions $g^i(t, x, y, z)$, $i = 0, 1, 2$, such that

$$g_t^i = g^i(t, X_{t-}, Y_{t-}, Z_{t-}). \qquad \textbf{(A.10)}$$

In this case, defining

$$L^Q f(t, x, y, z) = \frac{\partial}{\partial t} f(t, x, y, z) + L_t^Q f(t, x, y, z), \qquad \textbf{(A.11)}$$

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where

$$L_t^Q f(t,x,y,z) = \lambda_0(t,x,z)(1+g^0(t,x,y,z))$$
$$\times [f(t,x+\xi(t,x,z),y,z+e^{kt}\xi(t,x,z)) - f(t,x,y,z)]$$
$$+ \lambda(t,z)\sum_{i=1,2} p_i(t,x,z)(1+g^i(t,x,y,z))\,[f(t,x+\xi(t,x,z),y$$
$$+ (-1)^{i-1}\eta_i(t,x,z), z+e^{kt}\xi(t,x,z)) - (t,x,y,z)]$$

for any bounded, real-valued, measurable function f, under (A.8) with g_t^i given by (A.10), we have that the process

$$M_f^Q(t) = f(t,X_t,Y_t,Z_t) - f(0,x_0,0,z_0) - \int_0^t L^Q f(s,X_s,Y_s,Z_s)\,ds$$

is a $(Q,\mathcal{F}_t)$-local martingale.

The Martingale Problem for the operator L^Q given in (A.11) and initial condition $(0,0,0,z_0)$ is well posed since such is the Martingale Problem for the operator L given in (2.6) with the same initial conditions. This implies that the process (X_t,Y_t,Z_t) is Markovian under the measure Q. We observe that when (A.10) holds true, the condition

$$(e^{\eta_1(t,x,z)}-1)[1+g^1(t,x,y,z)]p_1(t,x,z) + (e^{-\eta_2(t,x,z)}-1)$$
$$\times [1+g^2(t,x,y,z)]p_2(t,x,z) = 0 \qquad \textbf{(A.12)}$$

provides a sufficient condition for (A.9).

A.3. The Minimal Martingale Measure

To verify condition (4.2) in the frame of this article, we introduce the minimal martingale measure. Note that, herein, the properties of the minimal martingale measure provide the main tool to prove that (4.2) holds true. This allows us to claim that, in this model, the minimal entropy martingale measure exists, is unique, and is equivalent to P and that its density with respect to P is given by (4.9).

The minimal martingale measure $\hat{P}$, as observed in Prigent [24] was introduced in Follmer and Schweizer [12], to obtain hedging strategies that are optimal in a suitable sense. In Schweizer [28], the author shows that the value process can be computed as the conditional expectation with respect to $\hat{P}$ and then a risk-neutral approach to option valuation is provided.

DEFINITION A.5: *An equivalent martingale measure $\hat{P}$ is called minimal if each $(P,\mathcal{F}_t)$-local martingale, R, strictly orthogonal to M^S,* (A.3)*, is a $(\hat{P},\mathcal{F}_t)$-local martingale.*

For any initial probability P, there exists at most one minimal martingale measure, (Ansel and Stricker [1, Thm. 2.1]). In this subsection we prove its existence and describe its structure for this model. The main tool is the following theorem whose proof can be found in Schweizer [27, Prop. 2].

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THEOREM A.6: *Let us assume that there exists a $(P, \mathcal{F}_t)$-predictable process c_u such that, P-a.s.*

(i) $A_t^S = \int_0^t c_u \, d\langle M^S \rangle_u$,

(ii) $\int_0^t |c_u|^2 \, d\langle M^S \rangle_u < +\infty$,

(iii) $1 - c_t(M_t^S - M_{t-}^S) > 0$,

then, $\hat{\mathcal{L}}_t := \mathcal{E}(-\int_0^t c_u \, dM_u^S)$ is a $(P, \mathcal{F}_t)$-strictly positive local martingale. When $\hat{\mathcal{L}}_t$ is a $(P, \mathcal{F}_t)$-martingale, the probability measure $\hat{P}$ defined by $\hat{\mathcal{L}}_t = d\hat{P}/dP|_{\mathcal{F}_t}$ is the minimal martingale measure. On the other hand, when $P(1 - c_t(M_t^S - M_{t-}^S) \leq 0) > 0$, the minimal martingale measure does not exist.

From now on, we take into account the particular structure of the model studied in this article. In this way we are able to prove results stronger than in a more general setting.

Recalling (A.2)–(A.4), condition (i) implies that

$$c_t = \frac{(e^{\eta_t^1} - 1)p_t^1 + (e^{-\eta_t^2} - 1)p_t^2}{S_{t-}[(e^{\eta_t^1} - 1)^2 p_t^1 + (e^{-\eta_t^2} - 1)^2 p_t^2]}$$

and c_t is a predictable process as required. Condition (ii) holds true since λ_u is bounded:

$$\int_0^t |c_u|^2 \, d\langle M^S \rangle_u \leq \int_0^t \lambda_u \, du < +\infty \quad \text{P-a.s.}$$

and, by an easy computation, we get that condition (iii) is always verified.

Summing up, the process $\hat{M}_t := -\int_0^t c_u \, dM_u^S$ is a martingale that has the structure given in (A.6), with

$$\hat{g}_t^0 = 0, \quad \hat{g}_t^1 = -c_t \, S_{t-} \, (e^{\eta_t^1} - 1), \quad \hat{g}_t^2 = -c_t \, S_{t-} \, (e^{-\eta_t^2} - 1), \tag{A.13}$$

and $\hat{\mathcal{L}}_t := \mathcal{E}(-\int_0^t c_u \, dM_u^S)$ is a $(P, \mathcal{F}_t)$-strictly positive local martingale.

PROPOSITION A.7: *The probability measure $\hat{P}$ is equivalent to P. It is a risk-neutral measure and coincides with the minimal martingale measure.*

PROOF: By (2.4), we have that

$$|\hat{g}_t^1| \vee |\hat{g}_t^2| \leq K, \tag{A.14}$$

for a real constant $K > 0$, depending on $\eta_{\min}$ and $\eta_{\max}$. Then, as in Lemma 5.1, $\mathbb{E}[\hat{\mathcal{L}}_T] = 1$. ■

PROPOSITION A.8: *The minimal martingale measure, $\hat{P}$, preserves the Markovianity.*

PROOF: The claim is true if (A.10) and (A.8) hold. Assumption (A.10) is an easy consequence of (A.13).

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Furthermore, by (A.14), we have that

$$\mathbb{E}\left[\sum_{i=0,1,2}\int_0^T (1+\hat{g}_t^i)\lambda_t^i\,dt\right] \le \mathbb{E}\left[\int_0^T (\lambda_t^0 + \lambda_t K)\,dt\right] \le T\Lambda(1+K) < +\infty,$$

which implies (A.8). ■

Finally, we get the main property.

PROPOSITION A.9: *The minimal martingale measure $\hat{P}$ has finite entropy.*

PROOF: By (A.14) and recalling that $\lambda_s \le \Lambda$, we get

$$H(\hat{P}|P) = \mathbb{E}^{\hat{P}}\left[\log\left(\frac{d\hat{P}}{dP}\right)\right] \le K\left(\mathbb{E}^{\hat{P}}\,[N_T] + \Lambda T\right)$$

and

$$\mathbb{E}^{\hat{P}}[N_t] = \mathbb{E}^{\hat{P}}\left[\int_0^t \lambda_s\left[\hat{g}_s^1 p_s^1 + \hat{g}_s^2 p_s^2\right] ds\right] \le \Lambda T K,$$

which provides the required result. ■

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