Robotica (2010) volume 28, pp. 975–987. © Cambridge University Press 2010
doi:10.1017/S0263574709990816

Output feedback sliding mode control for robot manipulators

Shafiqul Islam* and Peter X. Liu

Department of Systems and Computer Engineering, Carleton University, 1125 Colonel By Drive, Ottawa, ON, Canada.
E-mail: xpliu@sce.carleton.ca

(Received in Final Form: December 3, 2009. First published online: January 22, 2010)

SUMMARY

In this work, we propose an output feedback sliding mode control (SMC) method for trajectory tracking of robotic manipulators. The design process has two steps. First, we design a stable SMC controller by assuming that all state variables are available. Then, an output feedback version of this SMC design is presented, which incorporates a model-free linear observer to estimate unknown velocity signals. We then show that the tracking performance under the output feedback design can asymptotically converge to the performance achieved under state-feedback-based SMC design. A detailed stability analysis is given, which shows semi-global uniform ultimate boundedness property of all the closed-loop signals. The proposed method is implemented and evaluated on a robotic system to illustrate the effectiveness of the theoretical development.

KEYWORDS: Output feedback; Sliding mode control; Observer; Robotics; Perturbation.

1. Introduction

Over the past decades, the sliding mode control (SMC) techniques have been studied extensively for robotic control (see refs. [1,5,6,8,10,12,13,15,18,20], to name a few). Most SMC (detailed definition can be found in refs. [8,18]) designs reported in the literature, however, assume that velocity signals are available for feedback. But most advanced industrial robotic systems (e.g., those from CRS Robotics and Applied AI[1,2]) provide only joint position measurements. So, the required joint velocity signals have to be derived by differentiating the position measurements from encoders or resolvers. As a consequence, the obtained velocity signals are often contaminated by severe noise, and the performance of the designed system is limited, since the disturbance noise is amplified with the increase of controller gains. Specifically, the control system under SMC design may exhibit excessive chattering and infinitely fast switching, causing poor tracking performance (see for example refs. [1, 2]). To deal with the above-mentioned practical problem, we propose to estimate the velocity signals in the classical SMC design framework. Results in this direction can be found in refs. [4, 7, 14] and the references therein. The nonlinear dynamic observer was proposed in ref. [7] for a planar flexible manipulator. The observer dynamics is based on the use of system dynamics, uncertain model parameters, and nonlinear control inputs. This method guarantees local stability of the closed-loop system. Moreover, the design is not in adaptive framework demanding very large control effort to cope with the modeling error uncertainty.

Recently, the authors of ref. [14] proposed output feedback SMC based on sliding mode observer (SMO) for a certain class of nonlinear dynamics. The design and stability analysis of their technique replicate the output feedback method earlier reported in ref. [9] for robotic systems. The theoretical concepts and the critical assumptions of such SMO can be found for uncertain robotic manipulators.[9] Further difficulties associated with the SMO design synthesis are clearly illustrated via using simulation example. The SMO structure,[14] however, requires system dynamics, nonlinear control input, switching control input terms, and the undesirable switching function $sign()$. The narrow stability range of initial conditions with the assumption of the zero initial position estimation error and the difficulties associated with the chattering activity of the switching controls as well as the switching function in SMO structure are also discussed. In the face of large-scale parametric uncertainty, such a nonadaptive SMC[14] demands very large control effort, as nonlinear control terms and high-frequency switching function intensify the control chattering activity in the sliding surface. The method, however, imposes global growth conditions on the system nonlinearities. In addition, the observer-controller only ensure local stability of the closed-loop design. More recently, the dynamic observer based full-order output feedback sliding mode design was introduced for a certain class of nonlinear systems.[4] The idea of this approach is twofold. The first part comprises the introduction of an auxiliary dynamics whose input is the system output-input, and the other part is the transformation of the augmented system into an observable linear system with an injection term that contains the system output as well as the state of the auxiliary dynamics. The parameters of the controller-observer are chosen using complex matrix–vector transformations. As a matter of fact, such a large number of tuning parameters may not be realistic for highly uncertain robotic systems. Moreover, the designed controller is not robust, as it cannot cope with parametric uncertainty. This is mainly because the design uses fixed system dynamics, which may cause poor tracking performance in the face of large uncertainty.

In the current work, we propose linear-observer-based output feedback design SMC for robotic systems. The

* Corresponding author. E-mail: sislam@sce.carleton.ca

observer structure is simple in the sense that it does not require system dynamics, control inputs, and undesirable switching terms. The proof of the stability of this method has two parts. In the first part, we derive the SMC as a state feedback approach with the assumption that all the state variables are available for feedback. Using the Lyapunov method, we show that the state-feedback-based SMC design ensures the global stability property for all the signals in the closed-loop system. This property is established by assuming that the parameters are unknown but belong to a known compact set that is relatively large. In the second part of the proof, we replace the unknown velocity signals with the output of the linear estimator. For the given set of initial conditions, we first define the estimated region of interest for the state-feedback-based SMC design. Then, we saturate the controller outside the estimated region of interest, ensuring that the output feedback controller remains bounded over the estimated region. The idea of using saturation mechanism is to deal with large-scale uncertainty and disturbance. The bounded control allows the designer to increase the speed of the observer dynamic without sacrificing the transient tracking performance. In convergence analysis, we show that the stability region under the SMC design based on position and velocity can be recovered asymptotically by the output feedback SMC design. The proposed method can be used to formulate an output feedback form of any state-feedback-based classical SMC design reported in the literature. In contrast with the existing design, the method proposed in the current paper is adaptive, while the output feedback method for a certain class of nonlinear systems[4,7,14] is not in adaptive control framework. In addition, the output feedback design proposed in refs. [4,14] only ensures local stability, while our approach guarantees semi-global stability of the observer–controller closed-loop system.

The rest of the paper is organized as follows: In Section 2, we first describe model dynamics and its dynamical properties. Then, we formulate the SMC and its stability property by assuming that position–velocity signals are available for feedback design. The output feedback form of this classical SMC is also introduced in Section 2. A detailed stability analysis of the proposed output feedback method is given. Section 3 presents an adaptive output feedback SMC design. In Section 4, we implement and evaluate the proposed method on a robotic system that demonstrates the theoretical development of the current paper. Finally, Section 5 concludes the paper.

2. Output Feedback Sliding Mode Control

In this section, we first formulate the stability criterion for the SMC as a state feedback (position–velocity) control approach. Then, an output feedback version of this SMC that incorporates a linear observer in order to remove the demand of the velocity signals from SMC design is presented. Let us first consider the equation of motion for an n-link rigid robot:[9,16,17,19,20]

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \qquad (1)$$

where $q \in \Re^n$ is the joint position vector; $\dot{q} \in \Re^n$ is the joint velocity vector; $\ddot{q} \in \Re^n$ is the joint acceleration vector; $\tau \in \Re^n$ is the input torque vector; $M(q) \in \Re^{n\times n}$ is the symmetric positive definite inertia matrix; $C(q,\dot{q})\dot{q} \in \Re^n$ is the coriolis and centrifugal loading vector; and $G(q) \in \Re^n$ is the gravitational loading vector. We now represent the robot model (1) in error–state space form as follows:

$$\dot{e}_1 = e_2, \dot{e}_2 = \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau - \ddot{q}_d, \qquad (2)$$

where $e_1 = (q_1 - q_d) \in \Re^n$ is the vector of joint position tracking error with $q_1 = q$; $e_2 = (q_2 - \dot{q}_d) \in \Re^n$ is the vector of joint velocity tracking error with $q_2 = \dot{q}$; $e = [e_1^T, e_2^T]^T$; $\phi_1(e, q_d, \dot{q}_d) = -M^{-1}(q)[C(q,\dot{q})\dot{q} + G(q)]$; and $\phi_2(e_1, q_d) = M^{-1}(q)$. We first assume that the desired trajectory $q_d(t)$ and its first and second derivatives are bounded as $Q_d \in \Omega_d = [q_d, \dot{q}_d, \ddot{q}_d]^T \subset \Re^{3n}$ with compact set Ω_d. Then, we consider the following well-know properties of the robot dynamics:[1,8,9,12,20] (i) $M(q) \in R^{n\times n}$ is a symmetric, bounded, and positive definite matrix that satisfies the inequalities $\|M(q)\| \le M_M$ and $\|M^{-1}(q)\| \le M_{MI}$, where M_M and M_{MI} are bounded positive constants. (ii) The matrix $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric. (iii) The norm of the gravity and centripetal-coriolis forces are bounded and can be represented as $\|C(q,\dot{q})\| \le C_M\|\dot{q}\|$ and $\|C(q,\dot{q}_d)\| \le k_{cd}\|\dot{q}_d\| \le k_c$, where C_M, k_{cd}, and k_c are bounded positive constants.

Let us define the reference state as $\dot{q}_r = (\dot{q}_d - \lambda e_1)$, where $\lambda = diag[\lambda_1, \lambda_2, \ldots, \lambda_n]$ with $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_n > 0$. Then we define the sliding surface as $S = e_2 + \lambda e_1$. The control objective is to drive the joint position $q(t)$ to the desired position $q_d(t)$. To obtain such control objective, we consider the following control law for the robot system (1):

$$\tau(e, Q_d, \hat{\theta}) = \hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{G}(q) - \mathcal{K}S - K\,sgn(S), \qquad (3)$$

where $\ddot{q}_r = (\ddot{q}_d - \lambda e_2)$; $\hat{M}$ and $\hat{C}$ are the estimates of $M(q)$ and $C(q,\dot{q})$, respectively; $\mathcal{K} = diag[\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n,]$, $K = diag[K_1, K_2, \ldots, K_n]$ with $\mathcal{K}_1 > 0, \mathcal{K}_2 > 0, \ldots, \mathcal{K}_n > 0$ and $K_1 > 0, K_2 > 0, \ldots, K_n > 0$. Now, using (3), we simplify the closed-loop dynamics as follows:

$$M\dot{S} + (C + \mathcal{K})S = \triangle\beta - K\,sgn(S), \qquad (4)$$

with $\triangle\beta = (\hat{M} - M)\ddot{q}_r + (\hat{C} - C)\dot{q}_r + (\hat{G} - G) = \triangle M\ddot{q}_r + \triangle C\dot{q}_r$, where $\triangle M = \hat{M} - M$, $\triangle C = \hat{C} - C$, and $\triangle G = \hat{G} - G$. We now explore the convergence condition of the closed-loop system (4) in the Lyapunov sense. To do that, let us define the following positive definite Lyapunov-like candidate function:

$$V = \frac{1}{2}S^T M S, \qquad (5)$$

where M is a symmetric and positive definite matrix. As M is symmetric and positive definite, $V > 0$ for $S \ne 0$. The function V can be considered as an indicator of the energy for S. Let us show that the energy V decays as long as $S \ne 0$. To do that, we take the derivative along the closed-loop trajectory

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(4) and then use property (ii) to obtain $\dot{V}$ as

$$\dot{V} = \frac{1}{2}S^T\dot{M}S + S^T M\dot{S}$$
$$= -S^T\mathcal{K}S - \sum_{i=1}^{n}(S_i\,[K_i sgn(|S_i|) - \triangle\beta_i]). \qquad (6)$$

From (6), we can see that $\dot{V} \leq 0$ holds if $\sum_{i=1}^{n}(S_i[K_i sgn(|S_i|) - |\triangle\beta_i|]) \geq 0$. This can be shown when $K_i \geq |\triangle\beta_i|_{\max}$ with upper bound $|\triangle\beta_i|_{\max}$ that satisfies $|\triangle\beta_i|_{\max} > |\triangle\beta_i|$. If $S_i > 0$ and $K_i \geq |\triangle\beta_i|_{\max}$, then we have $\triangle\beta_i - K_i sgn(|S_i|) = \triangle\beta_i + K_i \leq 0$. This implies that $S_i[\triangle\beta_i - K_i sgn(|S_i|)] \leq 0$. Similarly, if $S_i < 0$ and $K_i \geq |\triangle\beta_i|_{\max}$, then we can write $\triangle\beta_i - K_i sgn(|S_i|) = \triangle\beta_i - K_i \geq 0$ such that $S_i[\triangle\beta_i - K_i sgn(|S_i|)] \leq 0$. Hence $\sum_{i=1}^{n}[S_i(\triangle\beta_i - K_i sgn(|S_i|))] \leq 0$. As $\mathcal{K}$ is a positive definite matrix, the first term of (6) can be written as $-S^T\mathcal{K}S \leq 0$. Based on our above analysis, we can write (6) as

$$\dot{V} = \sum_{i=1}^{n}(S_i[\triangle\beta_i - K_i sgn(|S_i|)]) - S^T\mathcal{K}S \leq 0. \qquad (7)$$

Then Eq. (5) can be viewed as an energy indicator for S. This implies the decay of the energy of S as long as $S \neq 0$. Thus, the sufficient condition $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}S_i^2 \leq -\eta_i|S_i|$, where η_i is a positive constant, is satisfied.[8,13] To reduce the control chattering activity, we estimate the switching function $sgn(.)$ by using a smooth bounded saturation function $sat(.)$.

2.1. *Output feedback sliding mode control design*

The analysis presented above is based on the strict assumption that all the state variables are available for feedback. However, these algorithms cannot be applied directly, as most advanced robotic systems remove the velocity sensors to reduce the weight and cost of the system. To obtain the velocity signals, the common practical approach is to differentiate the position measurements obtained from encoders or resolvers, which are often contaminated by noise (see for example refs. [1, 2]). As a consequence, the performance under state-feedback-based SMC approach is limited, as in practice, the measurement noise is amplified with the increase of the values of controller gains. In the face of the large parametric uncertainty, the demand of high control gain makes the design even more complex, as high control gain intensifies high-frequency control chattering activity. In fact, such a large control effort may not be available in real-time application, as control inputs are restricted in most control system technologies.

To deal with the above-mentioned practical control problem, we propose to estimate the velocity signals by the output of the observer. We consider a model-free linear observer to formulate an output feedback form of the classical SMC approach. The observer–controller algorithm is given as

$$\tau(\hat{e}, Q_d, \hat{\theta}) = \hat{M}(q)\ddot{\hat{q}}_r + \hat{C}(q, \hat{e}_2 + \dot{q}_d)\dot{\hat{q}}_r + \hat{G}(q) - \mathcal{K}(\hat{e}_2 + \lambda\hat{e}_1) - K sat\left(\frac{\hat{S}}{\phi}\right), \qquad (8)$$

where $sat(.)$ is a bounded smooth saturation function; $\dot{\hat{q}}_r = (\dot{q}_d - \lambda\hat{e}_1)$; $\ddot{\hat{q}}_r = (\ddot{q}_d - \lambda\hat{e}_2)$; $\hat{S} = (\hat{e}_2 + \lambda\hat{e}_1)$; and the unknown velocity signals $\hat{e}_2$ is replaced by the output of the following linear estimator designed as follows:

$$\dot{\hat{e}}_1 = \hat{e}_2 + \frac{H_1}{\epsilon}\tilde{e}_1, \quad \dot{\hat{e}}_2 = \frac{H_2}{\epsilon^2}\tilde{e}_1, \qquad (9)$$

where $\tilde{e}_1 = (e_1 - \hat{e}_1)$; $\tilde{e}_2 = (e_2 - \hat{e}_2)$; $\hat{e}_1$ and $\hat{e}_2$ are the estimates of e_1 and e_2; ϵ is a small positive constant that needs to be specified; and H_1 and H_2 are the positive constant matrices. The proposed observer dynamics (9) for the SMC approach can be viewed as a simple chain of integrator plus correction terms injected by the output error term. Since the observer is linear, its dynamics can be made exponentially fast of the form $\frac{1}{\epsilon}\exp^{\frac{-at}{\epsilon}}$ with $a > 0$. The observer structure (9) is independent of the system dynamics, uncertain model parameters, nonlinear switching control input, and nonlinear control inputs. In comparison, the performance with a nonadaptive SMC design[4,14] relies on the fact that there exists a known system dynamics. For a robotic manipulator, however, it is very difficult to obtain an exact system dynamics that ensures robust reconstruction of unknown states in order to guarantee asymptotic tracking error convergence. This is mainly because robot dynamics is associated with many structured and unstructured uncertainties that cannot be exactly modeled. To analyze the convergence rate for the proposed output feedback SMC design, we use singularly perturbation method. To begin with that, we first define an observer error dynamics. Using (8), one obtains $\dot{e}_2$ as follows:

$$\dot{e}_2 = \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau(\hat{e}, Q_d, \hat{\theta}) - \ddot{q}_d. \qquad (10)$$

Then, the observer error becomes

$$\left.\begin{aligned} \dot{\tilde{e}}_1 &= \tilde{e}_2 - \frac{H_1}{\epsilon}\tilde{e}_1, \\ \dot{\tilde{e}}_2 &= -\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau(\hat{e}, Q_d, \hat{\theta}) \\ &\quad - \frac{H_2}{\epsilon^2}\tilde{e}_1. \end{aligned}\right\} \qquad (11)$$

Now replace the observer errors by the scaled estimation error to form a singularly perturbed system as $\eta_1 = \frac{e_1 - \hat{e}_1}{\epsilon} = \frac{\tilde{e}_1}{\epsilon} \Rightarrow \epsilon\eta_1 = \tilde{e}_1$ and $\eta_2 = e_2 - \hat{e}_2 = \tilde{e}_2$ with a small positive parameter ϵ. Using this scaled estimation error, one gets

$$\epsilon\dot{\eta} = B\epsilon[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d) \times \tau(e - \zeta(\epsilon)\eta, Q_d, \hat{\theta})] + A_o\eta, \qquad (12)$$

where $A_o = \begin{bmatrix} -H_1 & I \\ -H_2 & 0_{n\times n} \end{bmatrix}$, $\zeta(\epsilon) = \begin{bmatrix} \epsilon I_{n\times n} & 0_{n\times n} \\ 0 & I_{n\times n} \end{bmatrix}$ and H_1 and H_2 are positive constant matrices such that the matrix A_o is Hurwitz. In output feedback design, we show that the performance achieved under state-feedback-based SMC (3) can be recovered asymptotically by the output feedback SMC design (8). This performance recovery analysis has been shown by using singular perturbation method, where the closed-loop system has the following standard singularly

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perturbed form:

$$\dot{e} = B[\phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta}) - \ddot{q}_d] + Ae, \quad (13)$$

$$\epsilon\dot{\eta} = B\epsilon[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d) \times \tau(e - \zeta(\epsilon)\eta, Q_d, \hat{\theta})] + A_o\eta, \quad (14)$$

where $A = \left[\begin{smallmatrix} 0 & I_{n\times n} \\ 0_{n\times n} & 0 \end{smallmatrix}\right]$. To begin with the recovery analysis, we first consider that the system output and its derivatives are available for feedback. Then, we design an SMC as a state feedback control law (3) such that the design meets the desired tracking objectives. We then replace the unknown velocity state vectors in the SMC by the output of the observer (9). If we consider a situation in which the initial-state estimates as well as the initial parameter estimates become large, then the observer speed will be required to increase for robust reconstruction of unknown velocity states. The problem with high-speed observer is that it may cause large control effort during transient phase, which may enter into the closed-loop system, resulting in an unstable control system. To protect the plant from large transient control effect, we saturate the control input within the estimated region of interest.[3] This saturation function will only be active during the transient period. To obtain such a bounded control, let us consider that $\Omega_c = \{e \mid e^T Q_{sm} e \leq c\}$, $c > 0$, is an estimate of the region of attraction of the state-feedback-based SMC design, $\tau(e, Q_d, \hat{\theta})$, where $Q_{sm} = 0.5\left[\begin{smallmatrix} \lambda^2 M & \lambda M \\ I & M \end{smallmatrix}\right]$. Since $\tau(e, Q_d, \hat{\theta})$ is a continuous function over e, Q_d, and $\hat{\theta}$, there always exists a maximum control $\tau_i\max = max|\tau_i(e, Q_d, \hat{\theta})|$, $1 \leq i \leq n$, such that the input can be saturated over the compact set Ω_{cr} that satisfies $\tau_i^s(e, Q_d, \hat{\theta}) = \tau_i\max\, sat(\frac{\tau_i(e, Q_d, \hat{\theta})}{\tau_i\max}) = \tau_i(e, Q_d, \hat{\theta})$, $\forall e(0) \in \Omega_{co} = \{e(0) \mid e(0)^T Q_{sm} e(0) \leq c_4\}$, where $c_4 < c$, $\theta \in \Omega$, $\hat{\theta} \in \Omega$, $Q_d \in \Omega_d$, and $\tau_i\max$ is taken over for all $e \in \Omega_{c_r}$ with $c_r > c$ and $sat(.)$ is the smooth bounded saturation function. Then, $\forall e(t) \in \Omega_c$ and $\hat{\theta}(t) \in \Omega$, and one has $|\tau_i(e, Q_d, \hat{\theta})| \leq \tau_i\max$ $\forall t \geq 0$ and $\tau_i^s(e, Q_d, \hat{\theta}) = \tau_i(e, Q_d, \hat{\theta})$. Hence, the saturation function will not be effective when short transient time period is over. We now replace the state vectors e in the control law by the state estimator $\hat{e}$. Then, the bounded output feedback SMC can also be achieved via saturating outside of the region of interest, Ω_c, as follows: $\tau_i^s(\hat{e}, Q_d, \hat{\theta}) = \tau_i\max\, sat(\frac{\tau_i(\hat{e}, Q_d, \hat{\theta})}{\tau_i\max}) = \tau_i(\hat{e}, Q_d, \hat{\theta})$ $\forall \hat{e}(0) \in \Omega_{co}$, $\forall \hat{e} \in \Omega_c$, $\forall e(0) \in \Omega_{co}$, $\forall e \in \Omega_c$, $\forall \hat{\theta} \in \Omega$, $\forall \theta \in \Omega$, and $Q_d \in \Omega_d$. Let us now summarize our main results for the output feedback SMC design.

Theorem 1: *Consider the closed-loop system (13)–(14). Then, for any given compact set of $e(0) \in \Omega_{co}$, $\hat{e}(0) \in \Omega_{co}$, and $\hat{\theta}(0) \in \Omega$, there exists a small ϵ_1^* such that for all $0 < \epsilon < \epsilon_1^*$, all the state variables of the closed-loop systems are bounded by a bound that can be made arbitrary small by making ϵ_1^* small.*

Proof: See the Appendix.

3. Adaptive Sliding Mode Control

The level of uncertainty in the classical SMC design ((3) and (8)) can be reduced by adding an adaptation term. For this purpose, we first propose to introduce an estimation algorithm to develop an adaptive SMC (ASMC) algorithm as a state feedback as

$$\tau(e, Q_d, \hat{\theta}) = Y(e, \dot{q}_r, \ddot{q}_r)\hat{\theta} - \mathcal{K}S - K sat\left(\frac{S}{\phi}\right), \quad (15)$$

$$\dot{\hat{\theta}} = \Gamma Y^T(e, \dot{q}_r, \ddot{q}_r)S, \quad (16)$$

where $\Gamma = diag[\Gamma_1, \Gamma_2, \ldots, \Gamma_n]$ with constant diagonal elements $\Gamma_n > 0$; $Y(e, \dot{q}_r, \ddot{q}_r)$ is the regressor matrix;[19] and $\hat{\theta}$ denotes the estimation of the manipulator parameters and the masses of the working loads. To construct $Y(e, \dot{q}_r, \ddot{q}_r)$, one requires the model structure of the system. Such a parameter adaptation law may exhibit discontinuous property even after the learning estimate converges to the actual parameter.[9] Therefore, the parameter estimates $\hat{\theta}$ are required to adjust with the smooth parameter projection scheme[11] as

$$\dot{\hat{\theta}}_i = [Proj(\hat{\theta}, \Psi)]_i = \begin{cases} \gamma_{ii}\Psi_i & \text{if } a_i \leq \hat{\theta}_i \leq b_i \text{ or} \\ & \text{if } \hat{\theta}_i > b_i \text{ and } \Psi_i \leq 0 \text{ or} \\ & \text{if } \hat{\theta}_i < a_i \text{ and } \Psi_i \geq 0, \\ \gamma_{ii}\bar{\Psi}_i & \text{if } \hat{\theta}_i > b_i \text{ and } \Psi_i > 0, \\ \gamma_{ii}\breve{\Psi}_i & \text{if } \hat{\theta}_i < a_i \text{ and } \Psi_i < 0 \end{cases} \quad (17)$$

for $\theta \in \Omega = \{\theta \mid a_i \leq \theta_i \leq b_i\}, 1 \leq i \leq p\}$, where $\bar{\Psi}_i = [1 + \frac{b_i - \hat{\theta}_i}{\delta}]\Psi_i$; $\breve{\Psi}_i = [1 + \frac{\hat{\theta}_i - a_i}{\delta}]\Psi_i$; Ψ_i is the ith element of the column vector $Y^T(e, \dot{q}_r, \ddot{q}_r)S$; γ_{ii} is the ith element of Γ; and $\delta > 0$ is chosen such that $\Omega \subset \Omega_\delta$ with $\Omega_\delta = \{\theta \mid a_i - \delta \leq \theta_i \leq b_i + \delta\}, 1 \leq i \leq p\}$. Then the closed-loop error model can be written as

$$M\dot{S} = Y(e, \dot{q}_r, \ddot{q}_r)\tilde{\theta} - (C + \mathcal{K})S - K sat\left(\frac{S}{\phi}\right), \quad (18)$$

where $\tilde{\theta} = (\theta - \hat{\theta})$ and θ denotes the actual manipulator parameters and the masses of the working loads. The proposed adaptive control law is designed by using the following control Lyapunov function:

$$V = \frac{1}{2}S^T M S + \frac{1}{2}\tilde{\theta}\Gamma^{-1}\tilde{\theta}. \quad (19)$$

Using property (ii), the time derivative of (19) along the closed-loop error trajectories (18) can be simplified as

$$\dot{V}(e, \tilde{\theta}) \leq -\lambda_{\min}(\mathcal{K})\|S\|^2 - K\|S\| \leq 0, \quad (20)$$

$\forall e \in \Omega_c$, $\forall \hat{\theta}(0) \in \Omega$, $\forall \theta(0) \in \Omega$ and $\hat{\theta}(t) \in \Omega_\delta$.

3.1. Adaptive output feedback sliding mode control

The above-given design is implementable only when all the process states are measurable. To relax this strict assumption, we now replace the unknown state vectors e in (15)–(17) by

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the output of estimator (9):

$$\tau(\hat{e}, Q_d, \hat{\theta}) = Y(\hat{e}, \dot{\hat{q}}_r, \ddot{\hat{q}}_r)\hat{\theta} - \mathcal{K}\hat{S} - K sat\left(\frac{\hat{S}}{\phi}\right), \quad (21)$$

$$\dot{\hat{\theta}} = \Gamma Y^T(\hat{e}, \dot{\hat{q}}_r, \ddot{\hat{q}}_r)S. \quad (22)$$

To smooth the parameter estimates, we may use parameter projection scheme (17), where the state estimates have been replaced by the output of observer (9). Then the closed-loop model under the adaptive output feedback SMC (AOFBSMC) scheme has the following form:

$$\dot{e} = B[\phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta}) - \ddot{q}_d] + Ae, \quad (23)$$

$$\dot{\hat{\theta}} = Proj(\hat{\theta}, \Psi(e - \zeta(\epsilon))\eta, Q_d, \hat{\theta})),$$

$$\epsilon\dot{\eta} = B\epsilon[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d) \times \tau(e - \zeta(\epsilon)\eta, Q_d, \hat{\theta})] + A_o\eta. \quad (24)$$

For any given $e(0) \in \Omega_{co}$, $\hat{e}(0) \in \Omega_{co}$, $\theta(0) \in \Omega$, and $\hat{\theta}(0) \in \Omega$, we then have the inequality

$$\|[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})]\| \leq k_{2o}$$

for $k_{2o} > 0$, $\forall e(t) \in \Omega_c$, $\forall \hat{\theta}(t) \in \Omega_\delta$.

We now state the main results for the AOFBSMC design in the following theorem.

Theorem 2: *Let us consider the closed-loop system (23)–(24) with the output feedback control laws (9), (17), (21), and (22). Then, for any given $\hat{\theta}(0) \in \Omega$ and $(e(0), \hat{e}(0)) \in \Omega_{co} \subseteq \Omega_c$, there exists ϵ_1^* such that for all $0 < \epsilon < \epsilon_1^*$, all the signals in the closed-loop system will be bounded by a bound that can be made arbitrarily small for small value of the observer design constant ϵ.*

Proof: The proof of stability of Theorem 2 is similar to Theorem 1. So, we removed the details stability analysis for brevity, and it can be obtained from the authors on request.

Remark 1: It is worth noting that the bounded inequalities $\|[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})]\| \leq k_1$ and $\|[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})]\| \leq k_{2o}$ used in observer error dynamics analysis do not impose any restriction either in theory or in practice. Note also that these inequalities do not enforce growth condition on systems nonlinearities. More specifically, the constants k_1 and k_{2o} are used to establish the semi-global stability property of Theorems 1 and 2. For the given set of initial conditions of interest, the designer can calculate the values of k_1 and k_{2o} over the domain of attraction Ω_c. Their values can be obtained as follows: For the given $e(0) \in \Omega_{co}$ and $\theta(0) \in \Omega$, one calculates the terms $\phi_1(e, q_d, \dot{q}_d)$ and $\phi_2(e_1, q_d)$. Then, for the given $\ddot{q}_d$ and initial conditions of interest, we define the saturation level $\tau_{\max}$ (maximum bound on $\tau(e, Q_d, \hat{\theta})$). For a small value of ϵ and the initial error estimates $\hat{e}(0) \in \Omega_{co}$, we then calculate the bound on the output feedback controller $\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})$ as well as the bound on the term $-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})$.

However, in practice, the designer does not require the calculation of the maximum bound on $\tau_{\max}$, as it is predefined by a manufacturer.

Remark 2: The design parameter ϵ represents the bandwidth of the observer dynamics. For the given set of initial interests, it is not hard to obtain the minimum bound on ϵ from the combined Lyapunov function candidate as $V_C = \frac{(1-d)}{2}S^T MS + \frac{d}{2}\eta^T P\eta$ with the design constant $d > 0$. To calculate the bound on ϵ, let us differentiate V_C with respect to time along the perturbed closed-loop trajectory to simplify $\dot{V}_C$ as $\dot{V}_C \leq -(1-d)[\alpha_1\|e\|^2 + \zeta_1\|e\|\|\eta\|] - d\frac{\alpha_2}{\epsilon}\|\eta\|^2 + 2d\eta^T PBf_o(e, Q_d, \theta, \eta, \epsilon)$, where $\zeta_1 = \varsigma_o(k_{\epsilon 1} + k_{sm_1})$ with $\varsigma_o = \|\lambda^2\ \lambda; \lambda I\|$, $\alpha_1 = \lambda_{\min}(\mathcal{V})$ with $\mathcal{V} = \varsigma_o\mathcal{K}$ and $\alpha_2 = \|Q_o\|$, Q_o being a positive definite matrix for solving the Lyapunov equation $A_0^T P + PA_0 = -Q_o$, $f_o(e, Q_d, \theta, \eta, \epsilon) = [-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s(\hat{e}, Q_d, \theta)]$. By knowing the upper bound on the inertial parameter $\theta \in \Omega$ as well as the bound on the desired trajectories $Q_d \in \Omega_d$, we can simplify the modeling error term $f_o(e, Q_d, \theta, \eta, \epsilon) = [\psi(0, \eta, \epsilon) + \psi(e, 0, 0)]$ as $\|\psi(0, \eta, \epsilon)\| \leq \gamma_{sp}\|\eta\|$ and $\|\psi(e, 0, 0)\| \leq \gamma_s\|e\|$. Then, $\dot{V}_C$ can be written as $\dot{V}_C \leq -(1-d)[\alpha_1\|e\|^2 + \zeta_1\|e\|\|\eta\|] - d\frac{\alpha_2}{\epsilon}\|\eta\|^2 + 2d\|\eta\|\|P\|\gamma_{sp}\|\eta\| + 2d\|\eta\|\|P\|\gamma_s\|e\|$ with $\|B\| = I$. We now define $\Psi_1(e) = \|e\|$ and $\Psi_2(\eta) = \|\eta\|$. Then $\dot{V}_C$ can be further simplified as $\dot{V}_C \leq -(1-d)\alpha_1\Psi_1^2(e) - d\frac{\alpha_2}{\epsilon}\Psi_2^2(\eta) + (1-d)\zeta_1\Psi_1(e)\Psi_2(\eta) + d\zeta_2\Psi_1(e)\Psi_2(\eta) + d\gamma\Psi_2^2(\eta)$, where $\zeta_2 = 2\|P\|\gamma_s$ and $\gamma = 2\|P\|\gamma_{sp}$. Now, $\dot{V}_C$ can be expressed in compact matrix form as

$$\dot{V}_C \leq -\begin{bmatrix}\Psi_1(e)\\ \Psi_2(\eta)\end{bmatrix}^T U \begin{bmatrix}\Psi_1(e)\\ \Psi_2(\eta)\end{bmatrix}$$

with

$$U = \begin{bmatrix} (1-d)\alpha_1 & -\frac{1}{2}(1-d)\zeta_1 - \frac{1}{2}d\zeta_2 \\ -\frac{1}{2}(1-d)\zeta_1 - \frac{1}{2}d\zeta_2 & d\left(\left(\frac{\alpha_2}{\epsilon}\right) - \gamma\right) \end{bmatrix}.$$

This implies that $\dot{V}_C$ is negative definite if the matrix U is positive definite and satisfies the inequality $[d(1-d)\alpha_1(\frac{\alpha_2}{\epsilon} - \gamma)] > \frac{1}{4}[(1-d)\zeta_1 + d\zeta_2]$. This means that for a given $d \in (0, 1)$ the matrix U will be positive definite, and there exists a continuous interval $(0, \epsilon^*)$ such that $\forall\epsilon \in (0, \epsilon^*)$, where ϵ^* satisfies $\epsilon^*(d) = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}[(1-d)\zeta_1 + d\zeta_2]}$ for the maximum value of $d^* = \frac{\zeta_1}{\zeta_1 + \zeta_2}$, and one gets the bound on ϵ^* as $\epsilon^* = \frac{\alpha_1\alpha_2}{\zeta_1\zeta_2 + \alpha_1\gamma}$. Then $\dot{V}_c$ becomes, $\dot{V}_c \leq -\lambda_{\min}(U)\|\tilde{\eta}\|^2$, where $\tilde{\eta} = [\Psi_1^T(e),\quad \Psi_2^T(\eta)]^T$ and $\lambda_{\min}(U)$ is the minimum eigenvalue of the positive definite matrix U. This implies that all the signals in the singularly perturbed system (13)–(14) are bounded, and the trajectories converge close to zero within a finite time.

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4. Design and Implementation Process

To illustrate the design procedure of the proposed scheme, let us now consider a two-link robotic system.[9,17,20] The dynamic equation for this robot system can be defined as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad (25)$$

with $m_{11} = (\theta_1 + 2\theta_2 + 2\theta_2 \cos q_2)$, $m_{12} = (\theta_2 + \theta_2 \cos q_2)$, $m_{21} = (\theta_2 + \theta_2 \cos q_2)$, $m_{22} = \theta_2$, $c_{11} = -2\dot{q}_2\theta_2 \sin q_2$, $c_{12} = -\dot{q}_2\theta_2 \sin q_2$, $c_{21} = \dot{q}_1\theta_2 \sin q_2$, $c_{22} = 0$, $\theta_1 = m_1 l^2$, and $\theta_2 = m_2 l^2$; l is the link length and m_1 and m_2 are the masses of link 1 and link 2 respectively. The robot operates in the horizontal plane; so the gravitational force vector is zero, that is, $G = 0$. We consider that the parameters θ_1 and θ_2 of the above-mentioned robot dynamics are unknown but belong to comparatively large compact sets as $\theta \in \Omega = [-10, 10]$. We also consider that the initial states belong to the set $e(0) \in \Omega_{co} = [-2, 2]$. We now generate the reference trajectory $q_d(t)$ for the given robot model to follow, a square wave with a period of 8 s and an amplitude of ± 1 rad is prefiltered with a critically damped second-order linear filter using a bandwidth of $\omega_n = 2.0$ rad/s. Specifically, our main target is to use a desired trajectory that is usually used in industrial robotic systems. In practice, the step reference inputs are not preferred because such initial jump may reduce the lifetime of the bearing. If one uses the step reference trajectory, then very small step sizes are required. If we know the uncertain parameter θ of the given system, then the controller is a simple feedback-linearizing regulator as $\tau(e, Q_d, \theta) = M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + G(q) - \mathcal{K}S - K(\frac{S}{\phi})$ with positive constant K and ϕ. Then the origin of the closed-loop system under the above-given control law can be shown to be globally exponentially stable provided that the value of positive constant K and ϕ are chosen via well-known pole-placement technique. Now, if we consider that the parameter θ is unknown, then we can estimate the parameter $\hat{\theta}$ of θ. Then, the continuous SMC has the form $\tau(e, Q_d, \hat{\theta}) = \hat{M}(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{G}(q) - \mathcal{K}S - K sat(\frac{S}{\phi})$, where ϕ is a small positive constant and $sat(.)$ is a bounded saturation function that satisfies

$$sat(y) = \begin{cases} -1 & \text{if } y < -1, \\ y & \text{if } |y| \le 1, \\ 1 & \text{if } y > 1. \end{cases}$$

We first show how to design the values of K and ϕ for the given initial conditions of interest. The value of K and ϕ can be chosen to ensure that the closed-loop trajectories converge to an invariant set where $|S| \le \phi$. This requires to guarantee $K \ge |\triangle\beta|_{\max}$ with $\triangle G = 0$ as the given system operating in the horizontal plane. Our aim is now to show an exponential convergence property of the closed-loop trajectory that satisfies the Lyapunov-stability property as $\dot{V} + \sigma V \le 0$ with $\sigma = \frac{\lambda_{\min}\mathcal{K}}{\lambda_{\max}(M)}$. To show $\dot{V} + \sigma V \le 0$, the value of K and ϕ are required to satisfy the inequality as $\frac{K}{\phi} \ge 771.1321$ with $\Omega_{co} \subset \{S(0)^T M(0) S(0) \le 3357.2\}$. If $|S| > \phi$, then the control gain K requires satisfying the inequality as $K \ge 771.1321$. Note that for the given set, the designer can increase the value of ϕ and decrease the value of K to reduce the chattering phenomenon.

We now show how to calculate the feedback controller gain $\mathcal{K}$ for the given initial conditions and bandwidth of the controller. This implies finding the value of the feedback controller gains $\mathcal{K}$ such that they ensure an acceptable transient performance of the closed-loop system. To show that, let us consider positive definite Lyapunov-like energy function as $V_r = \frac{1}{2} S^T M S$ with $S = [S_1^T, S_2^T]^T$, $M = [\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}]$ and $C = [\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}]$. First take the derivative and then use the control input $\tau(e_1, e_2) = -\mathcal{K}S$ and property (ii) to obtain $\dot{V}_r$ as $\dot{V}_r \le -\lambda_{\min.}(\mathcal{K})\|S\|^2 + \|S^T\|k_o$, where $\|[\lambda C(e_1 + q_d, e_1)\dot{q}_d - C(e_1 + q_d, \dot{q}_d)\dot{q}_d] + M(-\ddot{q}_d + \lambda e_2) + [\lambda C(e_1 + q_d, e_1) - C(e_1 + q_d, \dot{q}_d)]e_2\| \le k_o \quad \forall (e, Q_d) \in \Omega_{c_1} \times \Omega_d$ with $k_o > 0$. Now, applying $\|S\|^2 \ge \frac{V_r}{\lambda_{\max.}(\alpha_o)}$ and $\|S\| \le \sqrt{\frac{V_r}{\lambda_{\min.}(\alpha_o)}}$, $\dot{V}_r$ can be written as $\dot{V}_r \le -\psi_o V_r + \upsilon_o \sqrt{V_r}$ with $\alpha_o = 0.5[\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}]$, $\psi_o = \frac{\Xi}{\lambda_{\max}(\alpha_o)}$, $\Xi = \lambda_{\min}(\mathcal{K})$, and $\upsilon_o = \frac{k_o}{\sqrt{\lambda_{\min.}(\alpha_o)}}$. Then, we can find the bound on the error trajectory as $V_r(t) \le V_r(0)e^{-\gamma_o t} + \frac{\beta_o}{\gamma_o}(1 - e^{-\gamma_o t})$ for any $\alpha_o > 0$ with $\psi_o > \frac{\alpha_o}{2}$, $\gamma_o = (\psi_o - \frac{\alpha_o}{2})$, and $\beta_o = \frac{\upsilon_o^2}{2\alpha_o}$. Using $V_r \ge \frac{4k_o^2\lambda_{\max.}^2(\alpha_o)}{\lambda_{\min.}(\alpha_o)\Xi^2}$, the bound on $\dot{V}_r$ can be written as $\dot{V}_r \le -\kappa V_r$, where $\kappa = \frac{\psi_o}{2}$. Thus, the tracking error bound can be obtained as $V_r(t) \le V_r(0)e^{-\kappa t}$. This implies that the trajectory starting in a region where $V_r(0) > c_1$ will continue to decrease until the trajectory enters into the set Ω_{c_1}, where $\Omega_{c_1} = \{e \mid V_r \le c_1\}$. Now using $V_r = \frac{4k_o^2\lambda_{\max.}^2(\alpha_o)}{\lambda_{\min.}(\alpha_o)\Xi^2}$, we have $\dot{V}_r \le 0$, where $c_1 = \frac{2k_o^2\lambda_{\max.}^2(\alpha_o)}{\lambda_{\min.}(\alpha_o)\Xi^2}$, which implies that the trajectory starting in the set $\Omega_{c_1} = \{e \mid V_r \le c_1\}$ will remain there $\forall t \ge 0$, as $\dot{V}_r$ is negative on the boundary $V_r = c_1$. The boundary of the set Ω_{c_1} defines the maximum errors that one can expect from the controller $\tau = -\mathcal{K}S$. Notice from the relationship $c_1 = \frac{2k_o^2\lambda_{\max.}^2(\alpha_o)}{\lambda_{\min.}(\alpha_o)\Xi}$ that the set Ω_{c_1}, that is, the error bound, can be made arbitrarily small by increasing the minimal eigenvalue of the control gain $\mathcal{K}$. Then, we can say that for any given initial conditions of interest and λ, there exists a controller gain $\mathcal{K}$ such that the tracking error signals are bounded by a bound that can be made arbitrarily small close to the origin by increasing the minimal eigenvalue of the control gains $\mathcal{K}$.

Finally, we design observer gain matrices H_1 and H_2 such that the dynamics represented by A_o is Hurwitz. Then, for the given $e(0) \in \Omega_{co}$, $\theta(0) \in \Omega$, $\mathcal{K}$, H_1, H_2, and λ, we select the small value of ϵ that makes observer dynamics relatively faster than the closed-loop robot dynamics to guarantee robust reconstruction of the unknown velocity state. Note that for the given $e(0) \in \Omega_{co}$, $\theta(0) \in \Omega$, $\mathcal{K}$, H_1, H_2, and λ, one can calculate the minimum bound of ϵ *a priori* to ensure robust reconstruction of the unknown velocity signals. It is also important to note that the *a priori* calculated bound on ϵ may not be applied for the real-time application, as the value of ϵ depends on the sampling time, the output (position measurement obtained from encoders), and the input disturbances.

4.1. Implementation results

In this subsection, we follow the above-mentioned design steps to implement and evaluate the property of the proposed

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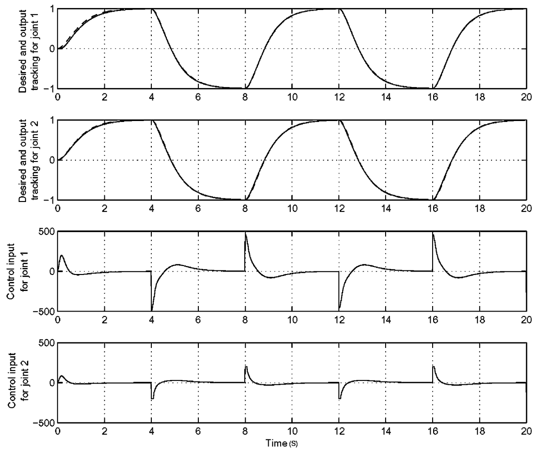

Fig. 1. The desired trajectory (dashed line), output tracking (in radians, solid lines), and control input (in newton meters) for joints 1 and 2 under state-feedback-based ASMC for $\tilde{\theta} = 10$ without using input and output disturbance noise.

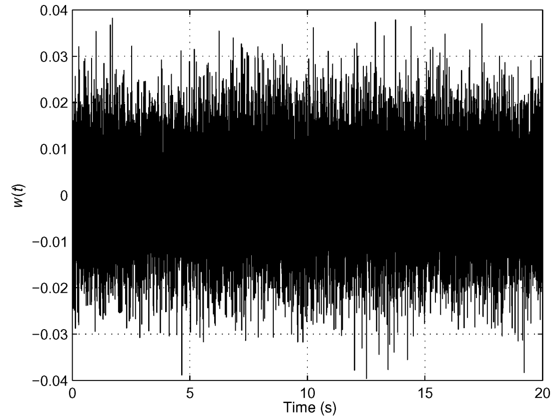

Fig. 2. The disturbance level $w(t)$ for the input $\tau(t)$ and the output $q(t)$.

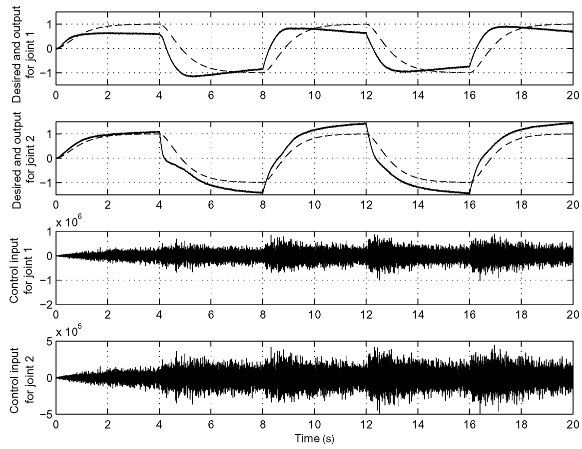

Fig. 3. The desired trajectory (dashed lines), output tracking (in radians, solid lines), and control input (in newton meters) for joints 1 and 2 under state-feedback-based ASMC for $\tilde{\theta} = 8$ with nonideal operating conditions.

method on the given system (25). In our evaluation, we will examine the tracking convergence property of the ASMC and the AOFBSMC approach with respect to estimation errors as $\tilde{\theta} = 10$ and $\tilde{\theta} = 8$. Because of the space limit, we have removed the evaluation of Theorem 1. The observer–controller design parameters are calculated along the line of the design steps given in previous section for the chosen initial conditions. For the given initial sets of interest, the design parameters for the ASMC approach are obtained as $\mathcal{K} = 125I_{2\times2}$, $K = 15I_{2\times2}$, $\Gamma = 10I_{2\times2}$, $\phi_1 = 0.7$, and $\phi_2 = 0.7$. With these control parameters, we first apply the ASMC design on the given system. The tested results are given in Fig. 1. Figure 1 shows the tracking performance with the chosen parameter estimator errors as $\tilde{\theta} = 10$. We now examine the tracking performance of ASMC algorithms under nonideal operating situations. For this purpose, we add a band-limited white noise $w(t)$ into the output $q(t)$ (as position measurement obtained from encoders is contaminated by disturbance noise[1,2]) and the input $\tau(t)$ to the system. For our evaluation, the level $w(t)$ for the output $q(t)$ and the input $\tau(t)$ are given in Fig. 2. Then, we apply ASMC on the given system by using the same design parameters as used in our last evaluation. The implemented results are given in Fig. 3. Figure 3 pictures the tracking performance under nonideal operating condition. The estimation errors for this test are set to $\tilde{\theta} = 8$. The design parameters are used for this experimentation as $\mathcal{K} = 125I_{2\times2}$, $K = 15I_{2\times2}$, $\Gamma = I_{2\times2}$, $\phi_1 = 0.7$, and $\phi_2 = 0.7$. Notice from Figs. 1 and 3 that the tracking errors increase with the increase of the disturbance noise associated with input and output measurement. More specifically, one can notice that the control effort become very large, which may not be realistic for the real-time operation, as available control inputs are restricted in most nonlinear control system designs. Such a large control effort is mainly due to the derivative action of the noisy position signals that are used in the ASMC based on position and velocity. Now our aim is to show that control chattering phenomenon under the ASMC design based on position and velocity can be reduced by using an AOFBSMC design. To explore that, we keep the same controller design parameters as applied for the evaluation of the ASMC algorithm based on position and velocity. Then, we define observer design constants as $H_1 = I_{2\times2}$, $H_2 = I_{2\times2}$, and $\epsilon = 0.05$ such that the observer dynamics is faster than the closed-loop robot dynamics. With these sets up, we now apply the AOFSMC algorithm on system (25). The implementation results are given in Fig. 4. By observing Figs. 1 and 4, we can see that the performance under the AOFBSMC design is similar to the performance under the ASMC design. Let us examine the robustness property of the AOFBSMC design in the presence of input and output disturbances noise. For this purpose, we add a band-limited white noise, $w(t)$, into the output measurement $q(t)$ and the input $\tau(t)$ of the system. For fair comparison of the performance of the AOFBSMC with the ASMC design, we use the same level of $w(t)$ as used for the evaluation of the ASMC design as depicted in Fig. 2. Then, using the observer–controller design parameters $\mathcal{K} = 125I_{2\times2}, \Gamma = 10I_{2\times2}, K = 15I_{2\times2}, \phi_1 = 0.7, \phi_2 = 0.7$, $\epsilon = 0.05$, $H_1 = I_{2\times2}$, and $H_2 = I_{2\times2}$, we implement the AOFBSMC on the system. The tested results are presented in Fig. 5 with $\tilde{\theta} = 10$. By comparing Figs. 3 (ASMC) and 5 (AOFBSMC), we can clearly observe that the tracking performance under the AOFBSMC is almost close to zero,

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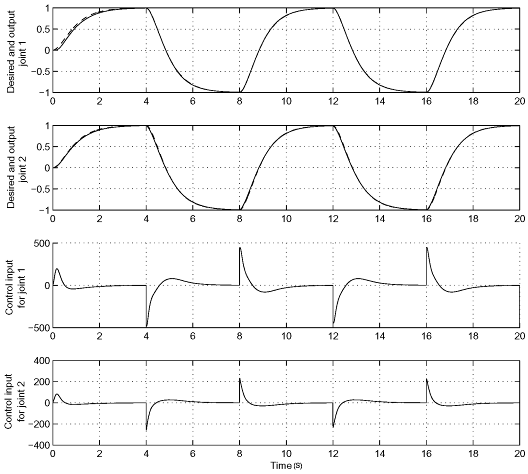

Fig. 4. The desired trajectory (dashed lines), output tracking (in radians, solid lines), and control input (in newton meters) for joints 1 and 2 under the AOFBSMC design with $\tilde{\theta} = 10$ and without using input and output disturbance noise $w(t)$.

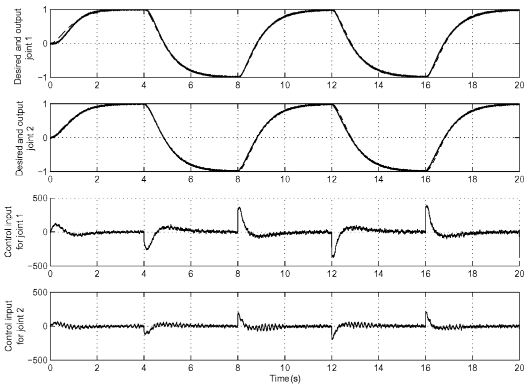

Fig. 5. The desired trajectory (dashed lines), output tracking (in radians, solid lines), and control input (in newton meters) for joints 1 and 2 under the AOFBSMC design using with disturbance noise $w(t)$ and the estimation error as $\tilde{\theta} = 10$.

while large tracking errors can be seen under the ASMC design even with smaller uncertainty. Notice also from these results that the required control effort under the AOFBSMC is much smaller than the control effort demanded under the ASMC design.

4.2. *Comparison with output feedback method*

In this subsection, we compare our approach with the method proposed in refs. [7, 14]. In comparison, the robot dynamics, uncertain model parameters, and nonlinear control inputs are not required in the proposed observer, while they should be known in ref. [7]. The method introduced in ref. [7] only provides local stability. In addition, we propose an ASMC approach, but the method[7] is not in adaptive control framework. For comparison, let us design nonlinear observer[7] for rigid robot manipulators as

$$\left.\begin{aligned} \dot{\hat{e}}_1 &= \hat{e}_2 + \frac{H_1}{\epsilon}\tilde{e}_1, \\ \dot{\hat{e}}_2 &= \frac{H_2}{\epsilon^2}\tilde{e}_1 - \ddot{q}_d + \phi_1(\hat{e}, q_d, \dot{q}_d) + \phi_2(\hat{e}_1, q_d) \\ &\quad \times \tau(\hat{e}, Q_d, \hat{\theta}). \end{aligned}\right\} \quad (26)$$

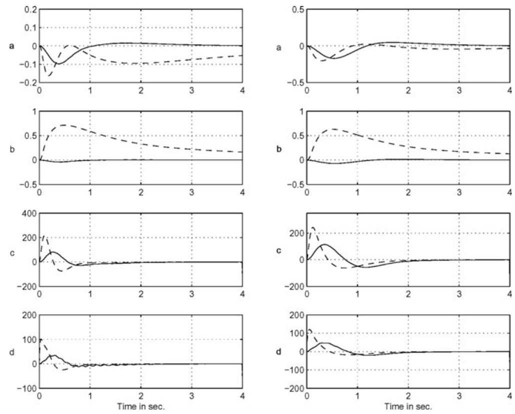

Fig. 6. The implementation results with the AOFBSMC based on observers (9) and (26) under ideal condition: the left column is for $\tilde{\theta}_{\text{observer}} = 4$, and the right column is for $\tilde{\theta}_{\text{observer}} = 0$; the dashed lines denote observer (26), and the solid lines denote observer (9). (a) Output tracking errors (in radians) for joint 1. (b) Output tracking errors (in radians) for joint 2. (c) Control input for joint 1. (d) Control input for joint 2.

Then, we replace the unknown velocity signals by the output of the nonlinear observer (26) to formulate nonlinear-observer-based AOFBSMC. We use observer (26) in the ASMC that has been introduced in the current work to formulate the AOFBSMC. We examine the tracking performance under the ASMC approach based on observers (9) and (26). It is important to note that for fair comparison, we use adaptive controller instead of the nonadaptive SMC proposed in ref. [7]. As the observer is based on model parameters, we consider two different estimation error uncertainties for the observer error dynamics as $\tilde{\theta}_{\text{observer}} = (\theta - \hat{\theta}_{\text{observer}}) = 4$ and $\tilde{\theta}_{\text{observer}} = (\theta - \hat{\theta}_{\text{observer}}) = 0$; that is, the model parameters that are used in the observer dynamics are exactly known. Here, $\hat{\theta}_{\text{observer}}$ denotes the parameter estimates in nonlinear observer dynamics. Then we implement both observer-based (defined by (9) and (26)) AOFBSMC designs on the given robotic system. The results are given in Fig. 6 with the observer design constants $H_1 = 20I_{2\times 2}$, $H_2 = 20I_{2\times 2}$, and $\epsilon = 0.1$, but we keep the same set of controller design parameters that were applied for the evaluation of the ASMC. The left column of Fig. 6 presents the control performance when the estimation error for the nonlinear observer is set to $\tilde{\theta}_{\text{observer}} = 4$. The right column of Fig. 6 depicts the tracking convergence when the estimation error for the nonlinear observer is chosen to be $\tilde{\theta} = 0$. The dashed line of Fig. 6 is for the nonlinear observer-based AOFBSMC, while the solid line is for the linear observer-based AOFBSMC. By comparing the solid and dashed lines of Fig. 6, we can notice that the tracking error and control effort under the nonlinear observer-based AOFBSMC are larger than the tracking error and control effort under the model-free linear observer-based AOFBSMC. Let us now show the difficulties associated with the SMO-based output feedback design.[14] Like the method reported in refs. [4, 7], the SMO-basedoutput feedback SMC reported in ref. [14] only ensures local stability. The main assumption in SMO-based design is the requirement of the zero initial conditions $\tilde{e}_1 = 0$ and the growth condition on unknown velocity signals

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(see Eqs. (11) and (20) of ref. [14] for a certain class of system). Let us examine the SMO-based adaptive output feedback design for robotic systems. The following design and implementation for robotic system can be viewed as an extension of the work reported in ref. [14] for a class of nonlinear systems. To derive an SMO-based adaptive approach, let us construct the SMO for robotic systems (1):

$$\left.\begin{aligned}\dot{\hat{q}}_1 &= \hat{q}_2 - \Lambda_{1o}\tilde{e}_1 - \delta_{1o} sgn(\tilde{e}_1),\\ \dot{\hat{q}}_2 &= -\delta_{2o} sgn(\tilde{e}_1) - W(q, \dot{q}_r, \hat{\theta})\hat{s}\\ &\quad + W(q, \dot{q}_r, \hat{\theta})\delta_{1o} sgn(\tilde{e}_1) + v,\end{aligned}\right\} \tag{27}$$

where $\hat{q}_1$ is the observer estimate of the joint positions; $\hat{q}_2$ is the estimate of the joint velocities; and $\hat{s} = (\hat{q}_2 - \dot{q}_d) + \Lambda(q - q_d)$. The constants Λ_{1o}, δ_{1o}, and δ_{2o} are $n \times n$ positive definite diagonal design matrices. The design elements are $\Lambda_{1o} = diag(\gamma_{1o}, \gamma_{1o}, \ldots, \ \gamma_{1o})$, $\delta_{1o} = diag(\lambda_{1o}, \lambda_{1o}, \ldots, \ \lambda_{1o})$, $\delta_{2o} = diag(\lambda_{2o}, \lambda_{2o}, \ldots, \ \lambda_{2o})$. Notice from the SMO design (27) that the observer consists of undesirable high-frequency switching terms such as $sgn()$, $W(q, \dot{q}_r, \hat{\theta})$ and v. These switching terms are designed in such away that they meet the eventual Lyapunov-stability condition. To construct observer dynamics for system (25), we first define the matrix $W(q, \dot{q}_r, \hat{\theta})$ as $W(q, \dot{q}_r, \hat{\theta}) = [\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22}\end{smallmatrix}]$ and the diagonal matrix $\Lambda = [\Lambda_{s1} \ \ \Lambda_{s2}]$. The SMO for links 1 and 2 is given by Eq. (27) as $\dot{\hat{q}}_{11} = \hat{q}_{21} - \gamma_{1o}\tilde{e}_{11} - \lambda_{1o} sgn(\tilde{e}_{11})$, $\dot{\hat{q}}_{21} = -\lambda_2 sgn(\tilde{e}_{11}) - W_{11}\hat{s}_1 - W_{12}\hat{s}_2 + v_1$, $\dot{\hat{q}}_{12} = \hat{q}_{22} - \gamma_{1o}\tilde{e}_{12} - \lambda_{1o} sgn(\tilde{e}_{12})$, and $\dot{\hat{q}}_{22} = -\lambda_{2o} sgn(\tilde{e}_{12}) - W_{21}\hat{s}_1 - W_{22}\hat{s}_2 + v_2$, where $\tilde{e}_{11} = (\hat{q}_1 - q_1)$ is the known position estimation error for joint 1; $\tilde{e}_{12} = (\hat{q}_2 - q_2)$ is the known position estimation error for joint 2; $\hat{q}_{11}$ is the observer estimate of the position for joint 1; $\hat{q}_{21}$ is the estimate of the velocity for joint 1; $\hat{q}_{12}$ is the observer estimate of the position for joint 2; $\hat{q}_{22}$ is the estimate of the velocity for joint 2; and $W_{11} = [-2\cos(q_2)\hat{\theta}_2\Lambda_{s1} - 2\hat{\theta}_2\Lambda_{s1} - \hat{\theta}_1\Lambda_{s1} - \mathcal{K}_1]$, $W_{12} = [-\cos(q_2)\hat{\theta}_2\Lambda_{s2} - \hat{\theta}_2\Lambda_{s2} - 2\sin(q_2)\dot{q}_{r1}\hat{\theta}_2 - \sin(q_2)\dot{q}_{r2}\hat{\theta}_2]$, $W_{21} = [-\cos(q_2)\hat{\theta}_2\Lambda_{s1} - \hat{\theta}_2\,\Lambda_{s1} - \hat{\theta}_2\,\sin(q_2)\,\dot{q}_{r1}]$, and $W_{22} = [-\hat{\theta}_2\Lambda_{12} - \mathcal{K}_2]$ with $\dot{q}_{r1} = \dot{q}_{d1} - \Lambda_{s1}(q_1 - q_{d1})$ and $\dot{q}_{r2} = \dot{q}_{d2} - \Lambda_{s2}(q_2 - q_{d2})$. The nonlinear compensation laws v_1 and v_2 for joints 1 and 2 can be designed as follows

$$v_1 = \begin{cases} \psi(\hat{q}_2, \tau) sgn(\tilde{e}_{11}) & \text{if } |sgn(\tilde{e}_{11})| \neq 0,\\ 0 & \text{if } |sgn(\tilde{e}_{11})| = 0\end{cases}$$

and

$$v_2 = \begin{cases} \psi(\hat{q}_2, \tau) sgn(\tilde{e}_{12}) & \text{if } \left|sgn(\tilde{e}_{12})\right| \neq 0,\\ 0 & \text{if } |sgn(\tilde{e}_{12})| = 0,\end{cases}$$

where $\psi(\hat{q}_2, \tau)$ is given as $\psi(\hat{q}_2, \tau) = \alpha_1|\hat{q}_2|^2 + 2\alpha_1|\hat{q}_2|\lambda_{1o} + \lambda_{1o}^2 + \alpha_2 + \alpha_3|\tau|$. Then, the adaptive controller with SMO observer can be designed as

$$\begin{aligned}\tau &= \hat{M}(q)\ddot{\hat{q}}_r + \hat{C}(q, \hat{q}_2)\dot{q}_r + \hat{G}(q) - \mathcal{K}\hat{s}\\ &= Y^{\tau}(q, \hat{q}_2, \dot{q}_r, \ddot{\hat{q}}_r)\hat{\theta} - \mathcal{K}\hat{s}\end{aligned} \tag{28}$$

with the positive definite matrix $\mathcal{K} \in \Re^{n\times n}$ and the regressor model as $Y^{\tau}(q, \hat{q}_2, \dot{q}_r, \ddot{\hat{q}}_r) = [\begin{smallmatrix} Y_{11}^{\tau} & Y_{12}^{\tau} \\ Y_{21}^{\tau} & Y_{22}^{\tau}\end{smallmatrix}]$, where $Y_{11}^{\tau} = \ddot{\hat{q}}_{r1}$, $Y_{12}^{\tau} = 2\ddot{\hat{q}}_{r1} + 2\cos(q_2)\ddot{\hat{q}}_{r1} + \ddot{\hat{q}}_{r2} + \cos(q_2)\ddot{\hat{q}}_{r2} - 2\hat{q}_{22}\sin(q_2)\dot{q}_{r1} - \hat{q}_{22}\sin(q_2)\dot{q}_{r2}$, $Y_{21}^{\tau} = 0$ and $Y_{22}^{\tau} = \ddot{\hat{q}}_{r1} + \cos(q_2)\ddot{\hat{q}}_{r1} + \ddot{\hat{q}}_{r2} + \sin(q_2)\dot{q}_{r1}\hat{q}_{21}$ in which $\ddot{\hat{q}}_{r1} = \ddot{q}_{d1} - \Lambda_{s1}(\hat{q}_{21} - \dot{q}_{d1})$ and $\ddot{\hat{q}}_{r2} = \ddot{q}_{d2} - \Lambda_{s2}(\hat{q}_{22} - \dot{q}_{d2})$ with $q_{d1} = q_{d2} = q_d$. Let us now formulate projection-free parameter adaptation law. For this purpose, we define the adaptation gain matrix as $\Gamma^{-1} = diag[\eta_1 \ \ \eta_2]$, and the standard adaptation law takes the following form: $\dot{\hat{\theta}} = -\Gamma^{-1}Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)s$, where $Y_{11} = \ddot{q}_{r1}$, $Y_{12} = 2\ddot{q}_{r1} + 2\cos(q_2)\ddot{q}_{r1} + \ddot{q}_{r2} + \cos(q_2)\ddot{q}_{r2} - 2\dot{q}_2\sin(q_2)\dot{q}_{r1} - \dot{q}_2\sin(q_2)\dot{q}_{r2}$, $Y_{21} = 0$, and $Y_{22} = \ddot{q}_{r1} + \cos(q_2)\ddot{q}_{r1} + \ddot{q}_{r2} + \sin(q_2)\dot{q}_{r1}\dot{q}_1$. The parameter adaptation law can be written as $\dot{\hat{\theta}}_1 = -\eta_1[Y_{11}s_1 + Y_{21}s_2]$ and $\dot{\hat{\theta}}_2 = -\eta_2[Y_{12}s_1 + Y_{22}s_2$. The unknown terms in adaptation mechanism can be defined as follows: $s_1 = \hat{s}_1 - \lambda_{1o} sgn(\tilde{e}_{11})$, $s_2 = \hat{s}_2 - \lambda_{1o} sgn(\tilde{e}_{12})$, $\ddot{q}_{r1} = \ddot{\hat{q}}_{r1} + \lambda_{1o}\Lambda_{s1} sgn(\tilde{e}_{11})$, and $\ddot{q}_{r2} = \ddot{\hat{q}}_{r2} + \lambda_{1o}\Lambda_{s2} sgn(\tilde{e}_{12})$. The observer design starts by choosing a particular observer estimation error sliding regime such that $\lambda_{1o} \geq |\tilde{e}_2|$ (the bound on the velocity error signals; see Eqs. (11) and (20) of ref. [14]). Now, the question is how such *a priori* bound on unknown velocity signals should be chosen. Another important assumption in SMO design is the requirement of strict condition of $\tilde{e}_{11} = 0$ and $\tilde{e}_{12} = 0$. For our evaluation, we choose $\tilde{e}_{11} = 0$ and $\tilde{e}_{12} = 0$ and find an appropriate value of λ_{1o} via trial-and-error search technique without injecting the discontinuous control term. We notice that once we inject the discontinuous control term in (28), the closed-loop control system immediately exhibits finite escape time phenomenon as the observer–controller system generates excessive chattering from four different control switching operation. The design parameters α_1 to α_3, γ_{1o}, λ_{1o}, λ_{2o}, Λ_{s1}, Λ_{s2}, η_1, and η_2 are determined by trial and error. The observer estimation error sliding regime is arbitrarily selected as $\lambda_{1o} = 0.1$. One then uses a trial-and-error procedure to find appropriate parameters such that $|\tilde{q}_2| \leq 0.1$. The controller and observer parameters are selected as follows: $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 1$, $\mathcal{K}_1 = 25$, $\mathcal{K}_2 = 30$, $\lambda_{1o} = 0.1$, $\lambda_{2o} = 5$, $\gamma_{1o} = 2$, $\Lambda_{s1} = 0.5$, $\Lambda_{s2} = 0.5$, and $\eta_1 = \eta_2 = 3$. The true values of the parameters are set at $\theta_1 = 1$ and $\theta_2 = 2$. Then, the parameter estimation are initialized to $\tilde{\theta}_1 = 1$ and $\tilde{\theta}_2 = 2$. The sampling period is chosen as 1 ms for all the results reported in the current paper. With these sets up, we now implement SMO-based output feedback design (27)–(28) on the given system (25). The results are depicted in Fig. 7. Figure 7 shows the observer estimation error in the position of joint 1. The switching behavior is clearly visible. It is clear from Fig. 7 that the assumption in SMO design, i.e., the estimation error of link 1 position, is not zero, $\tilde{e}_{11} \neq 0$. Consequently, Fig. 9 demonstrates the assumption on the derivative of the observer estimation error of joint 1, $\dot{\tilde{e}}_{11}$, is also not zero. Figure 8 also illustrates the observer velocity estimation error. In view of Figs. 7 and 9, we notice that the assumptions that are used in SMO design, i.e., the zero initial conditions on $\tilde{e}_{11}$ and $\dot{\tilde{e}}_{11}$, are not met even when the control system is operating under ideal conditions with small estimation error uncertainty. Figure 10 depicts the effect of

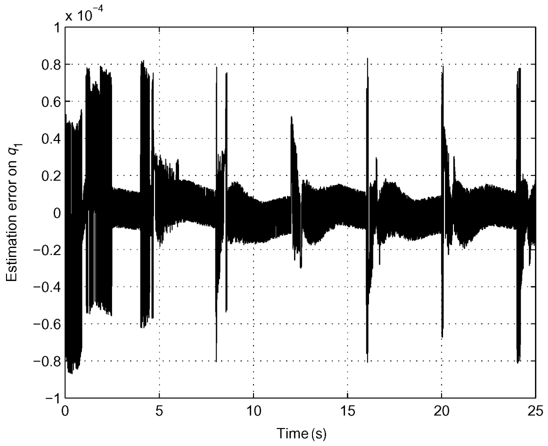

Fig. 7. Observer estimation error $\tilde{e}_{11}$ for joint 1.

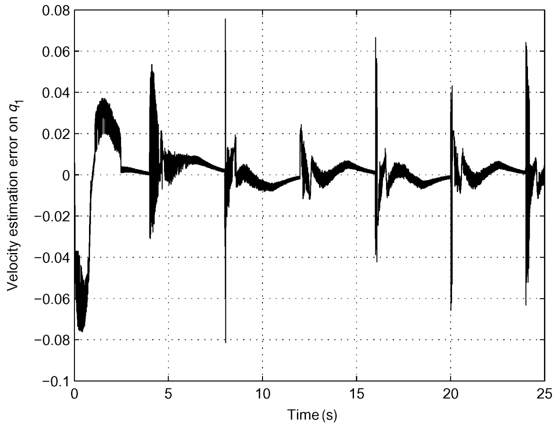

Fig. 8. Observer velocity estimation error $\tilde{q}_{21}$ for joint 1.

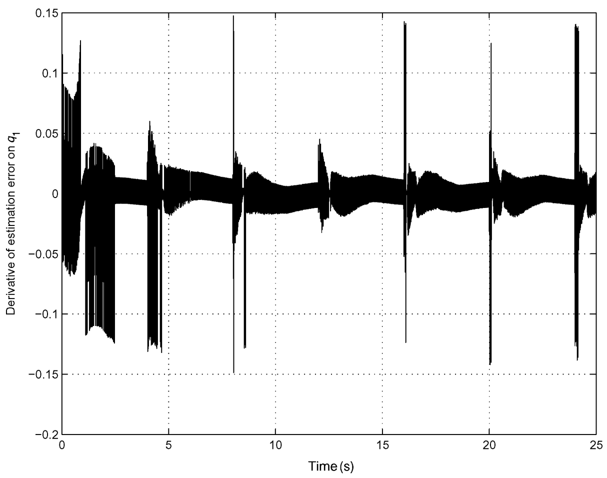

Fig. 9. Derivative of observer estimation error $\dot{\tilde{e}}_{11}$ for joint 1.

the switching term v in SMO dynamics for joint 1. Finally, we show the drifting effect of the projection-free adaptation mechanism in Fig. 11. Notice from Fig. 11 that the parameter estimates of θ_1 approach the true value of $\theta_1 = 1.0$ and then drift away from the correct value. For brevity, we remove the discontinuous property of the estimates θ_2.

Remark 3: It is assumed in our evaluation that with the chosen design parameters, the sliding regime is established,

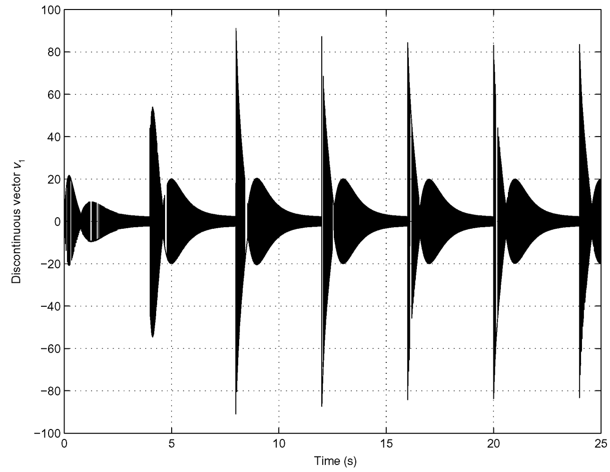

Fig. 10. Discontinuous nonlinear vector v_1 for joint 1.

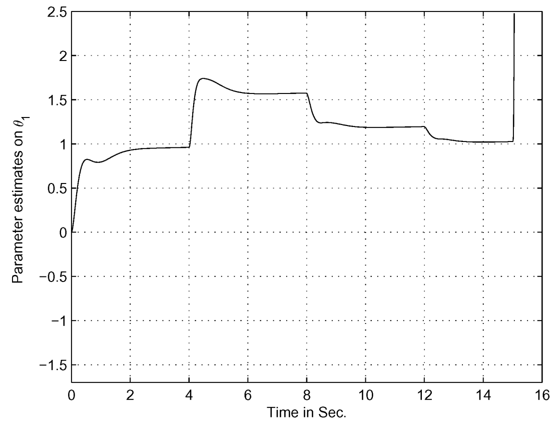

Fig. 11. The parameter estimate $\hat{\theta}_1$.

which satisfies the bounded inequality $\lambda_{1o} \geq |\tilde{e}_2|$. This condition, however, makes the SMO design unrealistic, as it is very hard for the designer to select the bound that provides the required sliding regime. On the other hand, because of the presence of control switching terms in SMO design, it is also very difficult for this method to be robust with respect to large parametric uncertainty and disturbance noise with $q(t)$ and $\tau(t)$. Therefore, in the face of large-scale uncertainty, the SMO-based output feedback SMC design cannot achieve the desired tracking objectives, as the designer cannot increase the observer–controller gain because of excessive control chattering phenomenon.

5. Conclusion

In the current paper, an output feedback sliding mode robot control method has been proposed to deal with the problem associated with the velocity-based SMC design. The Lyapunov method has been utilized to establish the stability condition of all the closed-loop signals. This property has been shown using a parameter projection and control saturation mechanism. The evaluation on a two-degree-of-freedom robotic system validates the effectiveness of the proposed SMC approach and the Lyapunov-stability arguments of the proposed method. Furthermore, the introduced method is compared with the exiting results,

https://doi.org/10.1017/S0263574709990816 Published online by Cambridge University Press

which confirms the superiority of the proposed design. The implementation of the proposed theorems on an industrial robotic system will be focused on in our future work. Future work will also be focused on developing a multiple-model-based ASMC along the lines of the design introduced in ref. [17].

Appendix: Proof of Theorem 1

The proof of the Theorem 1 consists of two parts. In the first part, we show that there exists a short transient period $T_1(\epsilon) \in [0, T_2]$ during which the fast variable η approaches a function $O(\epsilon)$, while the slow variables $(e, \hat{\theta})$ remain in a subset of the domain of attraction. In the second part, we establish the boundedness of the signal $e(t)$ for all $t \in [T_1(\epsilon), T_3]$, where $T_1(\epsilon) \in (0, \frac{T_2}{2}]$ and $T_3 \geq T_2$ is the first time $(e(t), \hat{\theta}(t))$ exists from the set Ω_c. This part shows that the state variables $(e(t), \hat{\theta}(t))$ remain bounded for $t \geq 0$. This proof makes use of the fact that the fast variable η is $O(\epsilon)$ such that $W(\eta(t)) \leq \epsilon^2 \beta$ for all $t \in [T_1(\epsilon), T_3]$.

Part 1: We first show that there exists a finite time T_2, independent of ϵ, such that for all $t \in [0, T_2]$ the slow variable remains bounded in the set Ω_c. To show that, let us define the following positive definite Lyapunov-like function candidate:

$$V = \frac{1}{2} S^T M S. \tag{29}$$

We consider that all initial conditions are bounded. Thus, we choose $e(0) \in \Omega_{co} \subseteq \Omega_c$, which includes $e_1(0) \in \Omega_{co_1}$ and $e_2(0) \in \Omega_{co_2}, \theta(0) \in \Omega$, where Ω_c is the domain of attraction and Ω_{co} is the compact set chosen to cover any bounded initial condition. Then, for the given initial set of e_1 and e_2, we have $c_4 = max_{e_1 \in \Omega_{co1}, e_2 \in \Omega_{co2}} \frac{1}{2} S^T M S$, where $c_4 > 0$. Then define the compact set as $e(t) \in \Omega_c$ with $c > c_4$. Now, our aim is to prove that the energy function V remains bounded by a constant c. To verify that, we first take the derivative of the Lyapunov function (29). Then we use property (ii) and the bounded output feedback law on the set Ω_{cr} as $\tau^s(\hat{e}, Q_d, \hat{\theta}) = [\tau^s(e, Q_d, \hat{\theta}) - \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta})]$ with $\tau(e, Q_d, \hat{\theta}) = \tau^s(e, Q_d, \hat{\theta})$ to simplify the derivative of V as

$$\dot{V} = -S^T \mathcal{K} S - \sum_{i=1}^{n} \left(S_i \left[K_i sat\left(\frac{|S_i|}{\phi_i}\right) - \triangle\hat{\beta}_i \right] \right) - S^T \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta}), \tag{30}$$

where $\hat{\triangle}\beta$ is defined as

$$\triangle\hat{\beta} = \triangle\beta + \left[(\hat{M} - M)\lambda\eta_2 + (\hat{C} - C)\lambda\epsilon\eta_1 \right] \tag{31}$$

with $\hat{S} = S - (\eta_2 + \lambda\epsilon\eta_1)$, $\ddot{\hat{q}}_r = \ddot{q}_r + \lambda\eta_2$, and $\dot{\hat{q}}_r = \dot{q}_r + \lambda\epsilon\eta_1$. Since ϵ is a small positive constant and the fast variable η enters into the slow dynamics via the bounded function $\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})$, $\forall e \in \Omega_c$, $\forall Q_d \in \Omega_d$, $\forall\theta \in \Omega$, and $\forall\hat{\theta} \in \Omega$, in view of the dynamical properties of (i) and (iii), the second part of (31) satisfies the following inequality:

$$\|(\hat{M} - M)\lambda\eta_2 + (\hat{C} - C)\lambda\epsilon\eta_1\| \leq k_{sm} \tag{32}$$

for $k_{sm} > 0$. We then simplify the derivative of (30) as

$$\dot{V} \leq -\sum_{i=1}^{n} \left(S_i \left[K_i sat\left(\frac{|S_i|}{\phi_i}\right) - \triangle\beta_i - k_{sm_i} \right] \right) - S^T \mathcal{K} S - S^T \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta}). \tag{33}$$

Now, for $|K_i| \geq |\triangle\beta_i|$, we have $[K_i sat(\frac{|S_i|}{\phi}) - \triangle\beta_i] \geq 0$. Then Eq. (33) can be written in a simplified form as

$$\dot{V} \leq -S^T \mathcal{K} S + \|S\| k_{sm} - S^T \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta}). \tag{34}$$

As η enters into slow subsystem via bounded function, the above equation can be simplified as

$$\dot{V} \leq -\varpi_o V + \alpha_o, \tag{35}$$

$\forall e \in \Omega_c$, $\forall\hat{\theta} \in \Omega$, and $\forall\theta \in \Omega$, where $\alpha_o = \gamma_s k_{sm} + \gamma_s \alpha_1$, $\varpi_o = \frac{\lambda_{\min}(\mathcal{K})}{\lambda_{\max}(Q_{sm})}$, and γ_s and α_1 are the bound for $\|S\|$ and $\|\eta\|$ over the set Ω_c, respectively. Then, the solution of the differential equation (35) can be derived as follows:

$$V(t) \leq V(0) e^{-\varpi_o t} + \frac{\alpha_o}{\varpi_o} \left(1 - e^{-\varpi_o t}\right). \tag{36}$$

As $V(0) \leq c_4 < c$, we conclude that there always exists a finite time T_2, independent of ϵ, such that for all $t \in [0, T_2]$, $V(t) \leq c$. Note that during $t \in [0, T_2]$, we can alternatively show the ultimate boundedness property of the closed-loop trajectories as follows: $\dot{V} = -S^T \mathcal{K} S - \sum_{i=1}^{n} (S_i [K_i sat(\frac{|S_i|}{\phi_i}) - \triangle\beta_i]) - S^T \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta})$. Using $[K_i sat(\frac{|S_i|}{\phi_i}) - \triangle\beta_i] \geq 0$, we have $\dot{V} \leq -S^T \mathcal{K} S + S^T \tau^s(\zeta(\epsilon)\eta, Q_d, \hat{\theta})$. Since ϵ is a bounded by a positive constant and the fast variable η enters into the slow dynamics via the bounded function $\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})$, $\forall e \in \Omega_c$, $Q_d \in \Omega_d$, $\forall\theta \in \Omega$, and $\forall\hat{\theta} \in \Omega$, we have $\dot{V} \leq -\varpi_o V + \alpha_o$, $\forall e \in \Omega_c$, $\forall\hat{\theta} \in \Omega$, and $\forall\theta \in \Omega$, where $\alpha_o = \gamma_s \alpha_1$, and γ_s and α_1 are the bound for $\|S\|$ and $\|\eta\|$ over the set Ω_c, respectively.

We now prove that over the time interval $[0, T_2]$ the fast variable η converge to a very small value. On the basis of the choice of the observer design parameter ϵ, one can make the bound on η very small. To show that, let us consider the following Lyapunov function candidate for the fast observer error model (14):

$$W(\eta) = \eta^T P \eta, \tag{37}$$

where $P = P^T > 0$ is the solution of the Lyapunov equation $PA_0 + A_0^T P = -I$. Applying $PA_0 + A_0^T P = -I$, one can simplify the derivative of (37) along the trajectory (14),

$$\dot{W}(\eta) \leq -\frac{1}{\epsilon}\|\eta\|^2 + 2\eta^T P B[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})]. \tag{38}$$

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Now, for any given $e(0) \in \Omega_{co}$, $\hat{e}(0) \in \Omega_{co}$, $\theta(0) \in \Omega$, and $\hat{\theta}(0) \in \Omega$, the inequality $\|[-\ddot{q}_d + \phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s((e - \zeta(\epsilon)\eta), Q_d, \hat{\theta})]\| \leq k_1$ holds. Substituting this inequality, we obtain $\dot{W}(\eta)$ as

$$\dot{W}(\eta) \leq -\frac{1}{\epsilon}\|\eta\|^2 + 2k_1\|P\|\|\eta\| \tag{39}$$

with $\|B\| = I$. Using the Lyapunov equation (37), Eq. (39) can be simplified as

$$\dot{W}(\eta) \leq -\frac{W(\eta)}{\epsilon\lambda_{\max.}(P)} + 2k_1\|P\|\sqrt{\frac{W(\eta)}{\lambda_{\min.}(P)}}. \tag{40}$$

We now investigate the property of η for three different situations as $W(\eta) \geq \epsilon^2\beta$, $W(\eta) = \epsilon^2\beta$, and $W(\eta) \leq \epsilon^2\beta$. If $W(\eta) \geq \epsilon^2\beta$, then we can write (40) in the following form:

$$\dot{W}(\eta) \leq -\frac{\gamma}{\epsilon}W(\eta) \tag{41}$$

with $W(\eta) \geq \epsilon^2\beta$, where $\beta = 16\|P\|^2k_1^2\lambda_{\max.}(P) = 16\|P\|^3k_1^2$ and $\|P\| = \lambda_{\max.}(P)$. Since $\{V \leq c\}$ and $\{W \geq \epsilon^2\beta\}$, we have the solution of the differential equation (41) as

$$W(\eta(t)) \leq W(0)\exp^{(-\frac{\gamma t}{\epsilon})}. \tag{42}$$

As we know that initial conditions are bounded for the estimates $\hat{e}(0) \in \Omega_{co}$, the corresponding scaled initial-state estimation error are also bounded by $\eta(0) = \frac{e(0)-\hat{e}(0)}{\epsilon} \leq \frac{k}{\epsilon}$ for some positive constant k that depends on the initial set Ω_{co}. Then, Eq. (42) can be further simplified as follows:

$$W(\eta(t)) \leq W(0)\exp^{(-\frac{\gamma t}{\epsilon})} \leq \frac{k_o}{\epsilon^2}\exp^{(-\frac{\gamma t}{\epsilon})}, \tag{43}$$

where $k_o = k^2\lambda_{\max.}(P) = \frac{k^2}{2\gamma}$ with $\gamma = \frac{1}{2\lambda_{\max.}(P)}$. We now calculate the transient peaking time $T_1(\epsilon)$ when $W(\eta(t)) = \epsilon^2\beta$. Let $\epsilon_1^* > 0$ be small enough so that for all $0 < \epsilon < \epsilon_1^*$, the time $T_1(\epsilon)$ is calculated when $W(\eta(t)) = \epsilon^2\beta$ as

$$T_1(\epsilon) = \frac{\epsilon}{\gamma}\ln\left(\frac{k_o}{\beta\epsilon^4}\right) \leq \frac{1}{2}T_2. \tag{44}$$

This implies that at the time $T_1(\epsilon)$, $W(\eta(t)) = \epsilon^2\beta$ and $\dot{W}(\eta) \leq -\gamma\epsilon\beta$. Hence, $W(\eta)$ will continue to decrease, and for the time $t > T_1(\epsilon)$ the inequality $W(\eta(t)) \leq \epsilon^2\beta$ holds. We now investigate the property of η for the case in which $W(\eta(t)) \leq \epsilon^2\beta$. One can see from (44) that $T_1(\epsilon)$ tends to zero as $\epsilon \to 0$. This means that the right-hand side of the above-mentioned inequality tends to zero as $\epsilon \to 0$. This implies that we can choose $T_1(\epsilon)$ small enough such that $T_1(\epsilon) \in (0, \frac{1}{2}T_2]$. That is, for $t \in [0, T_2]$, there always exists a short transient period $T_1(\epsilon) \in (0, \frac{1}{2}T_2]$ such that for all $t \in [T_1(\epsilon), T_3]$, $W(\eta(t)) \leq \epsilon^2\beta$. The time $T_3 \geq T_2$ is the first time $(e(t), \hat{\theta}(t))$ exists from the set Ω_c, and the time T_3 may be equal to infinity. Since the time $T_1(\epsilon)$ in (44) is a function of ϵ, one can make $T_1(\epsilon) \to 0$ as $\epsilon \to 0$. In view of the Lyapunov equation (37), we can conclude that as $W(\eta(t)) \leq \epsilon^2\beta$ $\forall t \in [T_1(\epsilon), T_3]$, $\|\eta\| \leq K_o\epsilon$, $K_o > 0$. This means that η is $O(\epsilon)$ over the time interval $\forall t \in [T_1(\epsilon), T_3]$.

Part 2: Let us now study the slow subsystems (13) over the time interval $[T_1(\epsilon), T_3]$, that is, study the property of the slow variable $(e, \hat{\theta})$ when $\|\eta\|$ converges close to the origin. For the time interval $[T_1(\epsilon), T_3]$, we choose $e(0) \in \Omega_{co}$ and $\hat{\theta}(0) \in \Omega$. Then we can write the tracking error model as

$$\dot{e}_2 = \left[\phi_1(e, q_d, \dot{q}_d) + \phi_2(e_1, q_d)\tau^s(e, Q_d, \hat{\theta}) - \ddot{q}_d\right] + \left[\phi_2(e_1, q_d)\tau^s(\hat{e}, Q_d, \hat{\theta}) - \phi_2(e_1, q_d)\tau^s(e, Q_d, \hat{\theta})\right]. \tag{45}$$

This can be viewed as a perturbed closed-loop model under state feedback over the time interval $[T_1(\epsilon), T_3]$. From part 1, we already know that the perturbation term $\|\eta\|$ decays exponentially fast to a level at which $\|\eta\|$ is $O(\epsilon)$ and $W(\eta(t)) \leq \epsilon^2\beta$. Now $\forall (e, \hat{\theta}, \eta) \in \{e \in \Omega_c\} \times \{\hat{\theta} \in \Omega\} \times \{\eta \in \Omega_\epsilon\}$ with $\Omega_\epsilon = \{\eta \mid W(\eta(t)) \leq \epsilon^2\beta\}$ for all $t \in [T_1(\epsilon), T_3]$, and we have

$$\begin{aligned} &\|\tau^s(\hat{e}, Q_d, \hat{\theta}) - \tau^s(e, Q_d, \hat{\theta})\| \leq k_{\epsilon 1}\|\eta\|, \\ &\|(\hat{M} - M)\lambda\eta_2 + (\hat{C} - C)\lambda\epsilon\eta_1\| \leq k_{sm_1}\|\eta\| \end{aligned} \tag{46}$$

with $k_{\epsilon 1} > 0$ and $k_{sm_1} > 0$. Now using perturbed error model (45) as well as inequality (46), the derivative of the lyapunov function (29) has the following simplified form:

$$\dot{V} \leq -S^T\mathcal{K}S + \|S\|k_{sm_1}\|\eta\| + k_{\epsilon 1}\|\eta\|\|S\|. \tag{47}$$

As we have shown in part 1 that η is bounded by a constant $K_o\epsilon$ with $K_o > 0$ and $\epsilon > 0$, the term $\|\eta\|$ is also bounded. Then, Eq. (47) can be simplified as

$$\dot{V} \leq -\lambda_{\min}(\mathcal{K})\|S\|^2 + K_o\epsilon k_{\epsilon 1}\|S\| + \|S\|k_{sm_1}K_o\epsilon. \tag{48}$$

Now, $\forall e \in \Omega_c$, $\forall\hat{\theta} \in \Omega$, and $\forall\theta \in \Omega$; we can further simplify $\dot{V}$ as

$$\dot{V} \leq -\varpi_o V + \chi\epsilon, \tag{49}$$

$\forall e \in \Omega_c$, $\forall\hat{\theta} \in \Omega$, and $\forall\theta \in \Omega$, where $\chi = K_o\gamma_p k_{\epsilon 1} + \gamma_p k_{sm_1}K_o$, $\varpi_o = \frac{\lambda_{\min.}(\mathcal{K})}{\lambda_{\max.}(Q_{sm})}$, and γ_p is the bound for the error state $\|S\|$ over the set Ω_c. From (49), we can see that on the boundary $V = c$, $\dot{V} < 0$ when $c > \frac{\chi\epsilon}{\varpi_o}$. Since ϵ is a small bounded positive constant and c is chosen strictly greater than c_4 and $\frac{\chi\epsilon}{\varpi_o}$, we can conclude that the set $\Omega_c \times \Omega \times \Omega_\epsilon$ is a positively invariant set, and inside this set, $e \in \Omega_c$. This implies that the state trajectory $(e, \hat{\theta}, \eta)$ is confined inside the set $\Omega_c \times \Omega \times \Omega_\epsilon$. By making ϵ small, one can make this invariant set very small close to the origin. Hence, we conclude that $T_3 = \infty$, and all the signals in (13) and (14) are bounded $\forall t \geq 0$ and η is $O(\epsilon)$ $\forall t \in [T_1, T_3]$. Now integrating (49) from $t = 0$ to $t = T$ yields $V(t) - V(0) \leq -\int_0^T \alpha_1\|e\|^2 dt + T\chi\epsilon$. Since $V(0) \geq 0$, and furthermore from (29) we can write the above equation as $\int_0^T \alpha_1\|e\|^2 dt \leq \frac{1}{2}S(0)^T MS(0) + k_6\epsilon$ with $k_6 = T\chi$. This implies that if $\epsilon \to 0$, then the output feedback SMC (Lyapunov-stability

https://doi.org/10.1017/S0263574709990816 Published online by Cambridge University Press

condition defined by (49)) can recover the performance (the bound on error trajectories) achieved by the state-feedback-based SMC controller (Lyapunov-stability condition defined by (7)).

Acknowledgement
This work is partially supported by the Natural Science and Engineering Research Council (NSERC) of Canada and the Canada Research Chairs Program.

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