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Initial–boundary-value problems for one-dimensional compressible Navier–Stokes equations with degenerate transport coefficients

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This paper is concerned with the construction of global, non-vacuum, strong, large amplitude solutions to initial–boundary-value problems for the one-dimensional compressible Navier–Stokes equations with degenerate transport coefficients. Our analysis derives the positive lower and upper bounds on the specific volume and the absolute temperature.

Keywords: one-dimensional compressible Navier–Stokes equations; initial–boundary-value problems; global solutions with large data; degenerate transport coefficients

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1. Introduction and main results

The dynamics of compressible, viscous, heat-conducting, one-dimensional flow can be described in Eulerian coordinates by the following one-dimensional compressible Navier–Stokes equations:

$$\left.\begin{aligned} \rho_\tau + (\rho u)_y &= 0,\\ (\rho u)_\tau + (\rho u^2 - \sigma)_y &= 0,\\ (\rho e + \tfrac{1}{2}\rho u^2)_\tau + (u(\rho e + \tfrac{1}{2}\rho u^2) - \sigma u + q)_y &= 0, \end{aligned}\right\} \tag{1.1}$$

where y and τ represent the space and time variables and ρ, u, σ, e and q denote the density, velocity, stress, internal energy and heat flux of the fluid, respectively.

Let

$$x = \int_{(0,0)}^{(\tau,y)} (\rho \, \mathrm{d}y - \rho u \, \mathrm{d}\tau)$$

be the Lagrangian space variable, let $t = \tau$ be the time variable and let $v = 1/\rho$ denote the specific volume. Then the system (1.1) can be rewritten as

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t - \sigma_x &= 0, \\ (e + \tfrac{1}{2}u^2)_t - (\sigma u - q)_x &= 0. \end{aligned}\right\} \tag{1.2}$$

For a Newtonian fluid, σ is given by

$$\sigma(v, \theta, u_x) = -p(v, \theta) + \frac{\mu(v, \theta)}{v} u_x,$$

and Fourier's law tells us that heat flux q satisfies

$$q(v, \theta, u_x) = -\frac{\kappa(v, \theta)}{v} \theta_x$$

with p and θ being the pressure and the absolute temperature, respectively.

The thermodynamic variables p, v, θ and e are related by the Gibbs equation $\mathrm{d}e = \theta \, \mathrm{d}s - p \, \mathrm{d}v$, with s being the specific entropy. $\kappa(v, \theta) > 0$ and $\mu(v, \theta) > 0$ denote the heat conductivity coefficient and viscosity coefficient, respectively.

In this paper we are concerned with the construction of global, non-vacuum, large amplitude, smooth solutions to the one-dimensional compressible Navier–Stokes equation (1.2) in the domain $\{(x, t) \mid x \in I = [0, 1],\ t \geqslant 0\}$ with the prescribed initial condition

$$(v(x, 0), u(x, 0), \theta(x, 0)) = (v_0(x), u_0(x), \theta_0(x)), \quad x \in [0, 1], \tag{1.3}$$

and one of the following three types of boundary condition:

$$\left.\begin{aligned} u(0, t) = u(1, t) = 0, \\ q(0, t) = q(1, t) = 0, \end{aligned}\right\} \tag{1.4}$$

$$\left.\begin{aligned} \sigma(0, t) = \sigma(1, t) = 0, \\ q(0, t) = q(1, t) = 0, \end{aligned}\right\} \tag{1.5}$$

and

$$\left.\begin{aligned} &\sigma(0, t) = \sigma(1, t) = -Q(t) < 0, \\ &q(0, t) = q(1, t) = 0. \end{aligned}\right\} \tag{1.6}$$

Here the outer pressure $Q(t) \in C^1(\mathbb{R}_+)$ is a given function.

Throughout this paper, we will focus on the ideal, polytropic gases:

$$e = C_v \theta = \frac{R\theta}{\gamma - 1}, \qquad p(v, \theta) = \frac{R\theta}{v} = A v^{-\gamma} \exp\left(-\frac{\gamma - 1}{R} s\right), \tag{1.7}$$

where $\gamma > 1$, R and A are suitable positive constants. Our main interest concerns the case when the transport coefficients μ and κ may depend on the specific volume and/or the absolute temperature, which are degenerate in the sense that κ and/or μ are not uniformly bounded from below or above by some positive constants for all $v > 0$ and $\theta > 0$. Note that if one derives the compressible Navier–Stokes equations from the Boltzmann equation through the Chapman–Enskog expansion, the corresponding thermodynamic variables p, v, θ and e satisfy the constitutive relations (1.7) and the transport coefficients are degenerate function of θ (see [3, 4, 7, 25]).

Compressible Navier–Stokes-type equations with density- and temperature-dependent transport coefficients arise in many applied sciences, such as in certain classes of solid-like materials [5, 6] and gases at very high temperatures [13, 26]. Such a dependence of μ and κ on v and θ will obviously influence the solutions of the field equations as well as the mathematical analysis, and establishing the corresponding well-posedness theory has been the subject of much recent research (see [5, 6, 8–10, 13, 16, 17, 21, 22], R.-H. Pan (personal communication, 2011) and the references therein). Such studies indicate that temperature dependence of the viscosity μ is especially challenging, but one can incorporate various forms of density dependence in μ and temperature dependence in κ. In this regard, Dafermos [5] and Dafermos and Hsiao [6] considered certain classes of solid-like materials in which the viscosity and/or the heat conductivity depend on density and in which the heat conductivity may depend on temperature. However, the latter is assumed to be bounded, as well as uniformly bounded away from zero. Kawohl [13] and Luo [17] considered a gas model that incorporates real-gas effects that occur in high-temperature regimes. In [13, 17] the viscosity depends only on density (or is constant) and it is uniformly bounded away from zero, while the thermal conductivity may depend on both density and temperature. For example, one of the assumptions in [13] is that there exist constants $\kappa_0 > 0$, $\kappa_1 >$ such that $\kappa(v, \theta)$ satisfies $\kappa_0(1 + \theta^q) \leqslant \kappa(v, \theta) \leqslant \kappa_1(1 + \theta^q)$, where $q \geqslant 2$. This type of temperature dependence is motivated by experimental results for gases at very high temperatures (see [26]). Jenssen and Karper [8] and R.-H. Pan (personal communication, 2011) studied the case when μ is a positive constant and $\kappa = \bar{k}\theta^b$ for some positive constant $\bar{k} > 0$. Such studies are motivated by the first level of approximation in kinetic theory, in which the viscosity μ and heat conductivity κ are power functions of the temperature alone.

We note, however, that, in all the above studies, although the viscosity coefficient μ may depend on v and the heat conductivity κ may depend on both v and θ, they impose the condition that either or both μ and κ are non-degenerate. This paper focuses on the case when μ is a function of v, κ depends on v and/or θ and both μ and κ are degenerate. To simplify our presentation, we will mainly concentrate on the case

$$\mu = v^{-a}, \qquad \kappa = \theta^b, \tag{1.8}$$

for some positive constants $a > 0$, $b > 0$, or on the $a = 0$ case, but κ is a general smooth function of v and θ satisfies $\kappa(v, \theta) > 0$ for $v > 0$, $\theta > 0$. For such a case, note that, for ideal polytropic gases, the assumptions imposed on μ in [13, 17] hold only when $a = 0$, i.e. the viscosity coefficient μ is a positive constant.

Now we state our main results. The first is concerned with the initial–boundary-value problem (IBVP) (1.2)–(1.4). In such a case, the transport coefficients μ and κ are assumed to satisfy one of the following two conditions.

(i) μ is a positive constant, $\kappa(v,\theta)$ is a smooth function of v and θ satisfying $\kappa(v,\theta)>0$ for $v>0$, $\theta>0$ and there exist positive constants μ_0 and $K(\tilde{v},\tilde{\theta})$ such that

$$\mu(v,\theta)=\mu_0>0, \qquad \min_{v\geqslant\tilde{v}>0,\ \theta\geqslant\tilde{\theta}>0}\kappa(v,\theta)=K(\tilde{v},\tilde{\theta})>0 \tag{1.9}$$

hold for each given positive constants $\tilde{v}>0$ and $\tilde{\theta}>0$.

(ii) μ and κ are given by (1.8) with the two positive constants a and b satisfying one of the following conditions:

$$\left.\begin{aligned} &\frac{1}{3}<a<\frac{1}{2}, \quad 1\leqslant b<\frac{2a}{1-a};\\ &\frac{1}{3}<a<\frac{1}{2}, \quad \frac{3-a+2a^2}{2+4a-4a^2}<b<1.\end{aligned}\right\} \tag{1.10}$$

Our result can be stated as follows.

Theorem 1.1. *Suppose that $(v_0(x),u_0(x),\theta_0(x))\in H^1(I)$. Let $\inf_{x\in I}v_0(x)>0$, $\inf_{x\in I}\theta_0(x)>0$ and assume that the initial data $(v_0(x),u_0(x),\theta_0(x))$ are compatible with the boundary condition (1.4). Then if the transport coefficients μ and κ are assumed to satisfy (1.9) or (1.8), (1.10), there exists a unique global solution $(v(x,t),u(x,t),\theta(x,t))$ to the IBVP (1.2)–(1.4) that satisfies*

$$\left.\begin{aligned} (v(x,t),u(x,t),\theta(x,t))&\in C^0(0,T;H^1(I)),\\ (u_x(x,t),\theta_x(x,t))&\in L^2(0,T;H^1(I)),\\ \underline{V}\leqslant v\leqslant\bar{V}, \quad \underline{\Theta}\leqslant\theta\leqslant\bar{\Theta} &\quad \forall(x,t)\in I\times[0,T].\end{aligned}\right\} \tag{1.11}$$

Here T is any given positive constant and $\underline{V}$, $\bar{V}$, $\underline{\Theta}$, $\bar{\Theta}$ are some positive constants that may depend on T and the initial data $(v_0(x),u_0(x),\theta_0(x))$.

Remark 1.2.

- The IBVP (1.2)–(1.4) has been studied in [13]. Since the argument developed by Kazhikhov and Shelukhin [14] is used in [13] to deduce the desired lower and upper bounds on the specific volume v, the assumption that μ is a positive constant should be imposed. See also [1, 2, 12, 27] for related studies. But, in theorem 1.1, if we focus on the ideal polytropic gas, then, on the one hand, we can deal with the case when μ and κ are given by (1.8) with a and b satisfying (1.10) (in such a case, both of them are degenerate) and, on the other hand, we only need to specify that the heat conductivity κ satisfies (1.9), which can be degenerate, even for the case when μ is a positive constant.

- Note that for the case when the transport coefficients μ and κ are given by (1.8), the assumptions imposed on a and b in theorem 1.1 exclude the case when $0<a\leqslant\frac{1}{3}$. We are convinced that the arguments used here can be modified to cover such a case.

For the IBVP (1.2), (1.3), (1.5), we have the following result.

THEOREM 1.3. *Suppose we have the following.*

(i) $(v_0(x), u_0(x), \theta_0(x)) \in H^1(I)$, $\inf_{x\in I} v_0(x) > 0$, $\inf_{x\in I} \theta_0(x) > 0$, *and the initial data* $(v_0(x), u_0(x), \theta_0(x))$ *are compatible with the boundary condition (1.5).*

(ii) *The transport coefficients* μ *and* κ *are assumed to satisfy one of the following two conditions:*

- μ *is a positive constant,* κ *satisfies* $\kappa(v,\theta) > 0$ *for* $v > 0$, $\theta > 0$ *and*

$$0 \leqslant \kappa(v,\theta) \leqslant C(V)(1+\theta^c), \quad 0 < V^{-1} \leqslant v \leqslant V, \tag{1.12}$$

holds for some positive constant $C(V) > 0$ *and* $\theta > 0$ *sufficiently large (here* $0 \leqslant c < 1$ *is a constant and* $V > 0$ *is any given positive constant);*

- μ *and* κ *are given by (1.8) with* a *and* b *satisfying*

$$0 \leqslant a < \tfrac{1}{5}, \qquad b \geqslant 2. \tag{1.13}$$

Then the initial–boundary-value problem (1.2), (1.3), (1.5) admits a unique global solution $(v(x,t), u(x,t), \theta(x,t))$ *such that (1.11) holds.*

REMARK 1.4. The IBVP (1.2), (1.3), (1.5) has also been studied in [13]. To deduce the desired lower and upper bound on the specific volume v, the viscosity coefficient $\mu(v)$ is assumed to satisfy

$$0 < \mu_0 \leqslant \mu(v) \leqslant \mu_1, \tag{1.14}$$

and the entropy $s(v,\theta)$ and the internal energy $e(v,\theta)$ are assumed to satisfy

$$s(v,\theta) \leqslant \left(\left|\int_1^v \frac{\mu(z)}{z}\,\mathrm{d}z\right|^r + 1\right) e(v,\theta) \tag{1.15}$$

in [13]. Here μ_0, μ_1 and $r < 2$ are some positive constants. For the ideal polytropic gas, if the transport coefficients μ and κ are assumed to satisfy (1.8), then (1.15) holds only if $a = 0$.

Finally, we consider the outer pressure problem (1.2), (1.3), (1.6). Under the assumption that the transport coefficients μ and κ satisfy (1.8) with

$$0 \leqslant a < \tfrac{1}{2}, \qquad b \geqslant \tfrac{1}{2}, \tag{1.16}$$

we have the following.

THEOREM 1.5. *Suppose that* $(v_0(x), u_0(x), \theta_0(x)) \in H^1(I)$. *Let* $\inf_{x\in I} v_0(x) > 0$, $\inf_{x\in I} \theta_0(x) > 0$, *and assume that the initial data* $(v_0(x), u_0(x), \theta_0(x))$ *are compatible with the boundary condition (1.6). Then, if the transport coefficients* μ *and* κ *are given by (1.8) with the two parameters* a *and* b *satisfying (1.16), then the IBVP (1.2), (1.3) and (1.6) has a unique global solution* $(v(x,t), u(x,t), \theta(x,t))$ *satisfying (1.11).*

Remark 1.6. In fact, the outer pressure problem (1.2), (1.3), (1.6) was studied in [17, 20] and the main aim of [17] was to remove the assumption (1.15) needed in [13] in the study of the IBVP (1.2), (1.3), (1.5). We note, however, that in [17] the assumption (1.14) is still imposed, together with the assumption that the heat conductivity coefficient $\kappa(v, \theta)$ is non-degenerate.

Remark 1.7. It is our pleasure to mention the result obtained by Mellet and Vasseur [19] for the barotropic case (when there is no temperature). The same type of condition $a < \frac{1}{2}$ is imposed in [19], and thus our result, in some sense, is reminiscent of the result obtained there.

Before concluding this section, we outline the main ideas used to deduce our main results. Our analysis is based on the continuation argument, and the main difficulty lies in controlling the possible growth of the solutions to the one-dimensional compressible Navier–Stokes equation (1.2) caused by the nonlinearities of the equations. If the initial data $(v_0(x), u_0(x), \theta_0(x))$ represent a small perturbation of the non-vacuum constant state $(v, u, \theta) = (\bar{v}, 0, \bar{\theta})$, with $\bar{v} > 0$ and $\bar{\theta} > 0$ being two given positive constants, then for the case when the transport coefficients μ and κ are general smooth functions of v and θ satisfying $\mu(v, \theta) > 0$ and $\kappa(v, \theta) > 0$ for $v > 0$, $\theta > 0$, the argument developed by Matsumura and Nishida in [18] can be used to deduce a satisfactory well-posedness theory in the class of functions that is a small perturbation of the constant state $(v, u, \theta) = (\bar{v}, 0, \bar{\theta})$.

However, for the construction of global, non-vacuum, large amplitude solutions to the one-dimensional compressible Navier–Stokes equation, the story is quite different, and the key aim is to deduce the positive lower and upper bounds on the specific volume v and the absolute temperature θ. Before we give the main ideas used to deduce our main results, we first outline the main idea used in [5,6,8,13,17] and R.-H. Pan (personal communication, 2011), i.e. to deduce the desired estimates on $v(x, t)$ and $\theta(x, t)$ separately. A key element of all proofs in [5, 6, 8, 13, 17] and R.-H. Pan (personal communication, 2011) is to first deduce the pointwise *a priori* estimates on the specific volume, which guarantee that no vacuum or concentration of mass occurs, and then, based on some sophisticated energy-type estimates, the upper bound on the absolute temperature can be obtained. The arguments used in [8, 13, 17] and R.-H. Pan (personal communication, 2011) to deduce the desired positive lower and upper bounds on the specific volume can be outlined as follows.

- For the IBVP (1.2)–(1.4), the viscosity coefficient μ is assumed to be a positive constant in [13], so the argument developed therein, together with the non-degenerate assumption on the heat conductivity coefficient κ, can indeed yield the desired lower and upper bounds on v (see [8, 13] and R.-H. Pan (personal communication, 2011)).

- For the IBVP (1.2), (1.3), (1.5), the viscosity coefficient μ, the entropy $s(v, \theta)$ and the internal energy $e(v, \theta)$ are assumed to satisfy (1.14) and (1.15), so an upper bound on the term $|\int_1^v \mu(z)/z \, \mathrm{d}z|$ was obtained in [13], from which the desired estimates on v follow immediately. A similar argument works for the outer pressure problem (1.2), (1.3), (1.5) (see [17]). In fact, as pointed out before, one of the main aims of [17] was to remove the assumption (1.15) needed in [13].

However, for the cases we consider in this paper, the gas is assumed to be ideal polytropic and the transport coefficients μ and κ are degenerate. Thus, the above argument cannot first be used to deduce the desired estimates on v. To overcome such a difficulty, our main idea is to estimate $v(x,t)$ and $\theta(x,t)$ simultaneously, and the key points in our analysis are outlined below.

(i) First, we must control the lower bound of the absolute temperature in terms of the lower bound of the specific volume (see the estimate (2.8) obtained in lemma 2.3).

(ii) Even for the case when the viscosity coefficient μ is a positive constant (as in one of the two cases considered in theorem 1.1), since the heat conductivity κ may be degenerate, we cannot hope to deduce the desired bounds on v and θ as in [8, 13] and R.-H. Pan (personal communication, 2011), i.e. deduce the lower and upper bounds on v first and then to bound θ. Instead, our trick, motivated by [22], is to first deduce the lower bound on v based on the explicit formula for v given in [14] for the case when both μ and κ are positive constants. We can thus deduce the lower bound on θ. With the lower bounds on v and θ in hand, we can then deduce an upper bound on v if the heat conductivity coefficient $\kappa(v,\theta)$ satisfies the assumption

$$\min_{v\geqslant\tilde{v}>0,\ \theta\geqslant\tilde{\theta}>0}\kappa(v,\theta)=K(\tilde{v},\tilde{\theta})>0$$

for any given positive constants $\tilde{v}>0$ and $\tilde{\theta}>0$. Having obtained these bounds, it remains to find the desired upper bound on θ, and the argument used here to deduce such a bound is similar to those in [5, 6, 8, 13, 17] and R.-H. Pan (personal communication, 2011).

(iii) When the transport coefficients μ and κ are given by (1.8) with $a>0$, $b>0$ as in the other case considered in theorem 1.1, we must estimate the lower and upper bounds on v and θ simultaneously. Our main idea is to first estimate the lower bound of θ in terms of the lower bound of v, (see lemma 2.3), and then, by employing Kanel′'s argument (see [11]), to control the lower and upper bounds of v in terms of $\|\theta^{1-b}\|_\infty$ as in (2.68) and (2.69). These estimates, together with the estimate on $\|\theta(t)\|_{L^\infty(I)}$ (see (2.72)), can yield the desired lower and upper bounds on v and θ provided that a and b satisfy certain relations stated in theorem 1.1.

(iv) The discussion on the IBVP (1.2), (1.3), (1.5) is more subtle due to the boundary condition (1.5). Our main trick here is to recover the $L^1([0,1])$-estimate on v, which is not obvious under the boundary condition (1.5).

The paper is organized as follows. The proofs of theorems 1.1, 1.3 and 1.5 are given in §§ 2–4, respectively.

Notation. Throughout the paper, $C>1$ is used to denote a generic constant, which may depend on $\inf_{x\in I}v_0(x)$, $\inf_{x\in I}\theta_0(x)$, T and $\|(v_0,u_0,\theta_0)\|_{H^1(I)}$. Here $T>0$ is some given constant. $C(\cdot)$ is used to denote some positive constant depending only on the arguments listed in the parentheses. Note that all these constants may vary in different places. $H^s(I)$ represents the usual Sobolev spaces on I with the standard norm $\|\cdot\|_{H^s(I)}$ and, for $1 \leqslant p \leqslant +\infty$, $L^p(I)$ denotes the usual L^p spaces equipped with the usual norm $\|\cdot\|_{L^p(I)}$. For simplicity, we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty(I \times [0,T])$.

2. Proof of theorem 1.1

This section is devoted to the proof of theorem 1.1, based on the continuation argument. Such an argument is a combination of the local existence result with certain *a priori* estimates on the local solutions constructed. Firstly, we state the local solvability result as follows.

THEOREM 2.1 (local existence result). *Under the assumptions in theorem 1.1, there exists a sufficiently small positive constant t_1 that depends on*

$$\inf_{x\in I} v_0(x), \quad \inf_{x\in I} \theta_0(x) \quad \text{and} \quad \|(v_0,u_0,\theta_0)\|_{H^1(I)},$$

such that the initial–boundary-value problem (1.2)–(1.4) admits a unique smooth solution $(v(x,t),u(x,t),\theta(x,t))$ defined on $I \times [0,t_1]$.

Moreover, $(v(x,t),u(x,t),\theta(x,t))$ satisfies

$$\left.\begin{aligned} (v(x,t),u(x,t),\theta(x,t)) &\in C^0(0,t_1;H^1(I)),\\ (u_x(x,t),\theta_x(x,t)) &\in L^2(0,t_1;H^1(I)),\\ \frac{1}{2}\inf_{x\in I} v_0(x) \leqslant v(x,t) \leqslant 2\sup_{x\in I} v_0(x) &\quad \forall (x,t)\in I\times[0,t_1],\\ \frac{1}{2}\inf_{x\in I} \theta_0(x) \leqslant \theta(x,t) \leqslant 2\inf_{x\in I} \theta_0(x) &\quad \forall (x,t)\in I\times[0,t_1], \end{aligned}\right\} \tag{2.1}$$

and

$$\sup_{[0,t_1]}(\|(v,u,\theta)(t)\|_{H^1(I)}) \leqslant 2\|(v_0,u_0,\theta_0)\|_{H^1(I)}. \tag{2.2}$$

Theorem 2.1 can be proved by using a similar approach to that in [14] or [23] in the three-dimensional case. We thus omit the details for brevity.

Suppose that the local solution $(v(x,t),u(x,t),\theta(x,t))$ constructed in theorem 2.1 has been extended to the time step $t = T \geqslant t_1$ and satisfies the following *a priori* assumption:

$$\underline{V}' \leqslant v(x,t) \leqslant \bar{V}', \quad \underline{\Theta}' \leqslant \theta(x,t) \leqslant \bar{\Theta}' \quad \forall (x,t)\in I\times[0,T]. \tag{H}$$

Here $\underline{V}'$, $\bar{V}'$, $\underline{\Theta}'$ and $\bar{\Theta}'$ are some positive constants. To extend this solution step by step to a global one, we only need to deduce certain *a priori* estimates on $(v(x,t),u(x,t),\theta(x,t))$ that are independent of $\underline{V}'$, $\bar{V}'$, $\underline{\Theta}'$ and $\bar{\Theta}'$ but may depend on T and the initial data $(v_0(x),u_0(x),\theta_0(x))$.

Using (1.7), we can rewrite (1.2) as

$$\left.\begin{aligned} v_t - u_x &= 0,\\ u_t + p_x &= \left(\frac{\mu(v)u_x}{v}\right)_x,\\ C_v\theta_t + pu_x &= \frac{\mu(v)u_x^2}{v} + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x. \end{aligned}\right\} \tag{2.3}$$

Set

$$\phi(x) = x - \ln x - 1. \tag{2.4}$$

Note that

$$\eta(v,u,\theta) = R\phi(v) + \tfrac{1}{2}u^2 + C_v\phi(\theta) \tag{2.5}$$

is a convex entropy of (2.3) and satisfies

$$\eta(v,u,\theta)_t + (pu)_x + \frac{\mu(v)u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} = \left(\frac{\mu(v)uu_x}{v} + p(1,1)u + \frac{\kappa(v,\theta)\theta_x(\theta-1)}{v\theta}\right)_x. \tag{2.6}$$

Then, by integrating (2.6) with respect to x and t over $I\times[0,T]$ and with the help of integrations by parts and the boundary condition (1.4), we can deduce the following lemma.

LEMMA 2.2 (basic energy estimates). *Let the conditions in theorem 2.1 hold and suppose that the local solution* $(v(x,t),u(x,t),\theta(x,t))$ *constructed in theorem 2.1 satisfies the* a priori *assumption* (H). *Then, for* $0\leqslant t\leqslant T$ *we have that*

$$\int_0^1 \eta(v,u,\theta)\,\mathrm{d}x + \int_0^t\int_0^1 \left(\frac{\mu(v)u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\right)\mathrm{d}x\,\mathrm{d}s = \int_0^1 \eta(v_0,u_0,\theta_0)\,\mathrm{d}x. \tag{2.7}$$

The next lemma is concerned with estimating the lower bound of $\theta(x,t)$ in terms of the lower bound of $v(x,t)$.

LEMMA 2.3. *Under the conditions in lemma 2.2, we have*

$$\frac{1}{\theta(x,t)} \leqslant C + C\left\|\frac{1}{\mu(v)v}\right\|_\infty \quad \forall (x,t)\in I\times[0,T]. \tag{2.8}$$

Proof. First, $(2.3)_3$ implies

$$C_v\left(\frac{1}{\theta}\right)_t = -\frac{\mu(v)u_x^2}{v\theta^2} + \frac{Ru_x}{v\theta} - \frac{1}{\theta^2}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x. \tag{2.9}$$

From (2.9), for each $p>1$ we obtain that

$$\begin{aligned} &C_v\left[\left(\frac{1}{\theta}\right)^{2p}\right]_t + \frac{2p(2p+1)\kappa(v,\theta)\theta_x^2}{v\theta^{2p+2}}\\ &\quad = -2p\left(\frac{1}{\theta}\right)^{2p-1}\left[\frac{\mu(v)}{v}\left(\frac{u_x}{\theta} - \frac{R}{2\mu(v)}\right)^2 - \frac{R^2}{4\mu(v)v}\right] - \left(\frac{2p\kappa(v,\theta)\theta_x}{v\theta^{2p+1}}\right)_x. \end{aligned} \tag{2.10}$$

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Integrating (2.10) with respect to x over I, we have

$$C_v\left(\left\|\frac{1}{\theta}\right\|_{L^{2p}}^{2p}\right)_t \leqslant 2p\int_0^1 \frac{R^2}{4\mu(v)v}\left(\frac{1}{\theta}\right)^{2p-1}\mathrm{d}x \leqslant 2pC\left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}\left\|\frac{1}{\theta}\right\|_{L^{2p}}^{2p-1}, \tag{2.11}$$

which implies

$$\left\|\frac{1}{\theta}\right\|_{L^{2p}} \leqslant C\Big(\inf_{x\in I}\theta_0(x)\Big)^{-1} + C\int_0^t \left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}\mathrm{d}s. \tag{2.12}$$

Letting $p\to+\infty$ in (2.12), we can deduce (2.8) immediately. This completes the proof of lemma 2.3. □

To derive bounds on the specific volume v, we first define

$$g(v) := \int_1^v \frac{\mu(\xi)}{\xi}\,\mathrm{d}\xi. \tag{2.13}$$

Then we get

$$\left(\frac{\mu(v)u_x}{v}\right)_x = \left(\frac{\mu(v)v_t}{v}\right)_x = [g(v)]_{xt} \tag{2.14}$$

and $(2.3)_2$ can be rewritten as

$$u_t + p_x = [g(v)]_{xt}. \tag{2.15}$$

Integrating (2.15) over $[y,x]\times[0,t]$ yields

$$\begin{aligned} -g(v(x,t)) + \int_0^t p(x,s)\,\mathrm{d}s &= \int_y^x (u_0(z)-u(z,t))\,\mathrm{d}z - g(v(y,t)) \\ &\quad - g(v(x,0)) + g(v(y,0)) + \int_0^t p(y,s)\,\mathrm{d}s. \end{aligned} \tag{2.16}$$

For the case when the transport coefficients $\mu(v)$, $\kappa(v,\theta)$ satisfy (1.9), we have the following result.

Lemma 2.4. *Under the conditions in lemma 2.2 and assuming that the transport coefficients $\mu(v)$, $\kappa(v,\theta)$ satisfy (1.9), there exist positive constants $\underline{V}_1$, $\bar{V}_1$, and $\underline{\Theta}_1$ depending only on T and $(v_0(x),u_0(x),\theta_0(x))$ such that*

$$\underline{V}_1 \leqslant v(x,t) \leqslant \bar{V}_1, \quad \theta(x,t) \geqslant \underline{\Theta}_1 \quad \forall (x,t)\in I\times[0,T]. \tag{2.17}$$

Proof. Note that when the transport coefficient $\mu(v)\equiv\mu_0$ is a positive constant, we have

$$g(v) = \mu_0 \log v. \tag{2.18}$$

Without loss of generality, we assume $\int_0^1 v_0(x)\,\mathrm{d}x = 1$. Thus, integrating $(2.3)_1$ over $I\times[0,t]$ and using the boundary condition (1.4), we have

$$\int_0^1 v(x,t)\,\mathrm{d}x = 1. \tag{2.19}$$

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Hence, for each $t \in [0,T]$, there exists at least one number $a(t) \in [0,1]$ such that

$$v(a(t),t) = 1. \tag{2.20}$$

Set $y = a(t)$ in (2.16). Then, by (2.18) and (2.20) we can obtain

$$\begin{aligned} -\mu_0 \log v(x,t) &+ \int_0^t p(x,s)\,\mathrm{d}s \\ &= \int_{a(t)}^x (u_0(z) - u(z,t))\,\mathrm{d}z \\ &\qquad - \mu_0 \log v(x,0) + \mu_0 \log v(a(t),0) + \int_0^t p(a(t),s)\,\mathrm{d}s. \end{aligned} \tag{2.21}$$

Multiplying (2.21) by μ_0^{-1} and then taking the exponentials on the resulting identity, we arrive at

$$\frac{1}{v(x,t)} \exp\left\{\frac{1}{\mu_0}\int_0^t p(x,s)\,\mathrm{d}s\right\} = Y(t)B(x,t), \tag{2.22}$$

where

$$\left.\begin{aligned} Y(t) &= v_0(a(t)) \exp\left\{\frac{1}{\mu_0}\int_0^t p(a(t),s)\,\mathrm{d}s\right\}, \\ B(x,t) &= \frac{1}{v_0(x)} \exp\left\{\frac{1}{\mu_0}\int_{a(t)}^x (u_0(z) - u(z,t))\,\mathrm{d}z\right\}. \end{aligned}\right\} \tag{2.23}$$

For $Y(t)$, we can deduce immediately that

$$Y(t) \geqslant v_0(a(t)) \geqslant C^{-1} > 0 \quad \forall t \in [0,T], \tag{2.24}$$

and by (2.7) we have

$$C^{-1} \leqslant B(x,t) \leqslant C \quad \forall (x,t) \in I \times [0,T]. \tag{2.25}$$

Now, we estimate the upper bound on $Y(t)$. Using the argument in [14] and by (2.22) we have

$$v(x,t)Y(t) = B^{-1}(x,t)\left(1 + \frac{1}{\mu_0}\int_0^t p(x,s)v(x,s)Y(s)B(x,s)\,\mathrm{d}s\right). \tag{2.26}$$

Integrating (2.26) with respect to x over I and using (2.7), (2.19) and (2.25), we have

$$\begin{aligned} Y(t) &\leqslant C + C\int_0^t Y(s) \int_0^1 p(x,s)v(x,s)\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t Y(s) \int_0^1 \theta\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t Y(s)\,\mathrm{d}s. \end{aligned} \tag{2.27}$$

Then, by the Gronwall inequality, we get

$$Y(t) \leqslant C \quad \forall t \in [0,T]. \tag{2.28}$$

This, together with (2.26), yields the lower bound on v, i.e.

$$v(x,t) \geqslant \underline{V}_1 \quad \forall (x,t) \in I \times [0,T] \tag{2.29}$$

holds for some positive constant $\underline{V}_1$.

We can easily obtain the lower bound on $\theta(x,t)$ from (2.29) together with (2.8), i.e. there exists a positive constant $\bar{\Theta}_1$ depending on T and $(v_0(x), u_0(x), \theta_0(x))$ such that

$$\theta(x,t) \geqslant \underline{\Theta}_1 \quad \forall (x,t) \in I \times [0,T]. \tag{2.30}$$

Next, to finish the proof of lemma 2.4, we have to estimate the upper bound on $v(x,t)$. First, assumption (1.9), together with the estimates (2.29)–(2.30), implies that

$$\kappa(v,\theta) \geqslant K \tag{2.31}$$

holds for some positive number K depending on $\underline{\Theta}_1$ and $\underline{V}_1$ for all v and θ under consideration.

From (2.7) we have that for each t there exists at least one number $b(t) \in I$ such that $\theta(b(t),t) \leqslant C$. Then, we have

$$\begin{aligned} \theta(x,t) &\leqslant 4\theta(b(t),t) + 2(\sqrt{\theta(x,t)} - \sqrt{\theta(b(t),t)})^2 \\ &\leqslant 4\theta(b(t),t) + \frac{1}{2}\biggl(\int_{b(t)}^{x} \frac{\theta_y(y,s)}{\sqrt{\theta(y,s)}}\,\mathrm{d}y\biggr)^2 \\ &\leqslant C + C\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\,\mathrm{d}x \int_0^1 \frac{v\theta}{\kappa(v,\theta)}\,\mathrm{d}x \\ &\leqslant C + C\|v(t)\|_{L^\infty(I)} \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\,\mathrm{d}x. \end{aligned} \tag{2.32}$$

Using this result, together with (2.24)–(2.25) and (2.28), means we can deduce from (2.26) that

$$\begin{aligned} v(x,t) &\leqslant Y^{-1}(t)B^{-1}(x,t)\biggl(1 + \frac{R}{\mu_0}\int_0^t \theta(x,s)Y(s)B(x,s)\,\mathrm{d}s\biggr) \\ &\leqslant C + C\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s v(x,t) \\ &\leqslant Y^{-1}(t)B^{-1}(x,t)\biggl(1 + \frac{R}{\mu_0}\int_0^t \theta(x,s)Y(s)B(x,s)\,\mathrm{d}s\biggr) \\ &\leqslant C + C\int_0^t \|v(s)\|_{L^\infty(I)} \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{2.33}$$

Thus, with the aid of the Gronwall inequality and (2.7), we can get the upper bound on $v(x,t)$, which completes the proof of lemma 2.4. □

Now we deduce the upper bound on $\theta(x,t)$. The following is an immediate consequence of (2.32) and (2.17).

COROLLARY 2.5. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} \, \mathrm{d}s \leqslant C \tag{2.34}$$

and

$$\int_0^t \int_0^1 \theta^2(x,s) \, \mathrm{d}x \, \mathrm{d}s \leqslant C. \tag{2.35}$$

By (2.35), we can obtain the following.

LEMMA 2.6. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^1 u^2 \, \mathrm{d}x + \int_0^t \int_0^1 u_x^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant C. \tag{2.36}$$

Proof. Multiplying $(2.3)_2$ by u and integrating the resulting equation with respect to x and t over $I \times [0,t]$, one has

$$\int_0^1 \frac{u^2}{2} \, \mathrm{d}x + \int_0^t \int_0^1 \frac{\mu u_x^2}{v} \, \mathrm{d}x \, \mathrm{d}s \leqslant C\|u_0\|_{L^2}^2 + C \int_0^t \int_0^1 \frac{\theta^2}{\mu v} \, \mathrm{d}x \, \mathrm{d}s. \tag{2.37}$$

Thus, applying (2.17) and (2.35), we get (2.36). This proves lemma 2.6. □

To estimate the upper bound on θ, by employing the argument used in [22], we get the following.

LEMMA 2.7. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\|\theta(t)\|_{L^\infty(I)} \leqslant C + C \int_0^t (\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2) \, \mathrm{d}s. \tag{2.38}$$

Proof. From $(2.3)_3$, we can easily deduce, for each $p > 1$, that

$$C_v(\theta^{2p})_t + 2p(2p-1)\theta^{2p-2}\frac{\kappa\theta_x^2}{v} = \left(2p\theta^{2p-1}\frac{\kappa\theta_x}{v}\right)_x + 2p\theta^{2p-1}\frac{\mu u_x^2}{v} - 2p\theta^{2p-1}\frac{R\theta u_x}{v}. \tag{2.39}$$

Integrating (2.39) with respect to x over I, one has

$$C_v(\|\theta(t)\|_{L^{2p}}^{2p})_t \leqslant 2p \int_0^1 \theta^{2p-1}\frac{\mu u_x^2}{v} \, \mathrm{d}x - 2pR \int_0^1 \theta^{2p-1}\frac{R\theta u_x}{v} \, \mathrm{d}x. \tag{2.40}$$

By exploiting the Hölder inequality and letting $p \to +\infty$, we get from (2.40) that

$$\|\theta(t)\|_{L^\infty(I)} \leqslant C\|\theta_0\|_{L^\infty(I)} + C \int_0^t \left(\left\| \frac{\mu u_x^2}{v} \right\|_{L^\infty(I)} + \left\| \frac{\theta u_x}{v} \right\|_{L^\infty(I)} \right) \mathrm{d}s. \tag{2.41}$$

Then, with the help of (2.29) and Cauchy's inequality, we can deduce (2.38), and the proof of lemma 2.7 is complete. □

To estimate

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^2 \, \mathrm{d}s,$$

we need the following result.

LEMMA 2.8. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}} \,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|\theta\|_\infty^r \quad \forall r \in (0,1). \tag{2.42}$$

Proof. Multiplying $(2.3)_3$ by θ^r and integrating the resulting equation with respect to x over I yields

$$\begin{aligned} C_v \int_0^1 \theta^{1+r}\,\mathrm{d}x &+ \int_0^t \int_0^1 \frac{r\kappa(v,\theta)\theta_x^2}{v\theta^{1-r}}\,\mathrm{d}x\,\mathrm{d}s \\ &= C_v \int_0^1 \theta_0^{1+r}\,\mathrm{d}x + \int_0^t\int_0^1 \frac{\mu_0\theta^r u_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s - R\int_0^t\int_0^1 \frac{\theta^{r+1}u_x}{v}\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C\|\theta_0\|_{L^\infty}^{1+r} + C\|\theta\|_\infty^r\biggl(\int_0^t\int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_0^1 u_x^2\,\mathrm{d}x\,\mathrm{d}s\biggr) \\ &\leqslant C + C\|\theta\|_\infty^r, \end{aligned} \tag{2.43}$$

where (2.35) and (2.36) are used. This is (2.42) and completes the proof. □

The following is a direct consequence of (2.42).

LEMMA 2.9. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^2\,\mathrm{d}s \leqslant C + C\|\theta\|_\infty^{1/2}. \tag{2.44}$$

Proof. Observe that (2.31) and (2.7) imply

$$\begin{aligned} \theta^2(x,t) &= \theta^2(b(t),t) + \int_{b(t)}^x 2\theta(y,t)\theta_y(y,t)\,\mathrm{d}y \\ &\leqslant C + C\|\theta(t)\|_{L^\infty(I)}^{1-r/2}\biggl(\int_0^1 \theta(x,t)\,\mathrm{d}x\biggr)^{1/2}\biggl(\int_0^1 \biggl(\frac{\theta_x^2}{\theta^{1-r}}\biggr)(x,t)\,\mathrm{d}x\biggr)^{1/2} \\ &\leqslant C + C\|\theta(t)\|_{L^\infty(I)}^{1-r/2}\biggl(\int_0^1 \biggl(\frac{\theta_x^2}{\theta^{1-r}}\biggr)(x,t)\,\mathrm{d}x\biggr)^{1/2}. \end{aligned}$$

From the above inequality together with the estimates (2.42) and (2.34), we can get that

$$\begin{aligned} \int_0^t &\|\theta(s)\|_{L^\infty(I)}^2\,\mathrm{d}s \\ &\leqslant C + C\int_0^t \biggl(\|\theta(s)\|_{L^\infty(I)}^{1-r/2}\biggl(\int_0^1\biggl(\frac{\theta_x^2}{\theta^{1-r}}\biggr)(x,s)\,\mathrm{d}x\biggr)^{1/2}\biggr)\mathrm{d}s \\ &\leqslant C + C\biggl(\int_0^t \|\theta(s)\|_{L^\infty(I)}^{2-r}\,\mathrm{d}s\biggr)^{1/2}\biggl(\int_0^t\int_0^1\biggl(\frac{\theta_x^2}{\theta^{1-r}}\biggr)(x,s)\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2} \\ &\leqslant C + C\|\theta\|_\infty^{(1-r)/2}\biggl(\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s\biggr)^{1/2}\biggl(\int_0^t\int_0^1\biggl(\frac{\theta_x^2}{\theta^{1-r}}\biggr)(x,s)\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2} \\ &\leqslant C + C\|\theta\|_\infty^{1/2}. \end{aligned} \tag{2.45}$$

This is exactly (2.44), and the proof of lemma 2.9 is complete. □

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We now estimate the term

$$\int_0^t \|u_x(s)\|^2_{L^\infty(I)}\,\mathrm{d}s$$

on the right-hand side of (2.38). To do so, we shall first estimate

$$\int_0^1 v_x^2\,\mathrm{d}x,$$

which is the main content of the following lemma.

LEMMA 2.10. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^1 v_x^2\,\mathrm{d}x + \int_0^t\int_0^1 \theta v_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|\theta\|^r_\infty \quad \forall r \in (0,1). \tag{2.46}$$

Proof. As in (2.14), we can rewrite $(2.3)_2$ as

$$u_t + \left(\frac{R\theta}{v}\right)_x = \left(\frac{\mu u_x}{v}\right)_x = \left(\frac{\mu v_t}{v}\right)_x = \left(\frac{\mu v_x}{v}\right)_t. \tag{2.47}$$

Multiplying the identity (2.47) by $\mu v_x/v$, we get that

$$\left(\frac{\mu^2 v_x^2}{2v^2}\right)_t = \left(\frac{\mu u v_x}{v}\right)_t - \left(\frac{\mu u u_x}{v}\right)_x + \frac{\mu u_x^2}{v} + \frac{R\mu v_x\theta_x}{v^2} - \frac{R\mu\theta v_x^2}{v^3}. \tag{2.48}$$

Integrating (2.48) with respect to x and t over $I \times [0,t]$, and with the aid of (2.7) and Cauchy's inequality, we get

$$\begin{aligned}
\int_0^1 v_x^2\,\mathrm{d}x &+ \int_0^t\int_0^1 \theta v_x^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C(\underline{V}_1, \bar{V}_1, \|v_{0x}\|_{L^2}) + C\int_0^t\int_0^1 \frac{u_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\|\theta\|^r_\infty \quad \forall r \in (0,1).
\end{aligned} \tag{2.49}$$

This is (2.46), and the proof of lemma 2.10 is complete. □

On the other hand, noting that

$$u_x^2(y,t) \leqslant \int_0^1 u_x^2(x,t)\,\mathrm{d}x + 2\int_0^1 |u_x(x,t)||u_{xx}(x,t)|\,\mathrm{d}x, \tag{2.50}$$

we obtain the following result by (2.36) and Hölder's inequality.

LEMMA 2.11. *Under the conditions in lemma 2.4, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^t \|u_x(s)\|^2_{L^\infty(I)}\,\mathrm{d}s \leqslant C + C\left(\int_0^t\int_0^1 u_{xx}(x,s)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}. \tag{2.51}$$

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Next, we need to estimate

$$\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s.$$

To this end, by differentiating $(2.3)_2$ with respect to x and multiplying the resulting equation by u_x, one has

$$\left(\frac{u_x^2}{2}\right)_t = \left[\left(\frac{\mu u_x}{v} - \frac{R\theta}{v}\right)_x u_x\right]_x - \left(\frac{\mu u_{xx}}{v} - \frac{\mu u_x v_x}{v^2} - \frac{R\theta_x}{v} + \frac{R\theta v_x}{v^2}\right) u_{xx}. \tag{2.52}$$

Note that the term

$$\left(\frac{\mu u_x}{v} - \frac{R\theta}{v}\right)_x \bigg|_{x=0,1} = u_t|_{x=0,1} = 0.$$

By integrating (2.52) with respect to x and t over $I \times [0,t]$, one has that

$$\begin{aligned}
&\int_0^1 u_x^2 \,\mathrm{d}x + \int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant C(\underline{V}_1, \bar{V}_1, \|u_{0x}\|_{L^2}) + C\int_0^t \int_0^1 (u_x^2 v_x^2 + \theta_x^2 + \theta^2 v_x^2) \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant C + C\int_0^t (\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2) \int_0^1 v_x^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \int_0^1 \theta_x^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant C(1 + \|\theta\|_\infty^{\max\{1/2+r,2r,1\}}) + \frac{1}{2}\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{2.53}$$

Here we use the fact that

$$\begin{aligned}
&\int_0^t (\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2) \int_0^1 v_x^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \leqslant C(1 + \|\theta\|_\infty^r)\left(1 + \|\theta\|_\infty^{1/2} + \int_0^t \|u_{xx}(s)\|_{L^2(I)} \,\mathrm{d}s\right) \\
&\qquad \leqslant C(1 + \|\theta\|_\infty^{\max\{1/2+r,2r\}}) + \frac{1}{2}\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{2.54}$$

and

$$\int_0^t \int_0^1 \theta_x^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \|\theta\|_\infty^{1-r} \int_0^t \int_0^1 \frac{\theta_x^2}{\theta^{1-r}} \,\mathrm{d}x\,\mathrm{d}s \leqslant C(1 + \|\theta\|_\infty). \tag{2.55}$$

Thus, we can immediately derive from (2.53) that

$$\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant C(1 + \|\theta\|_\infty^{\max\{1/2+r,2r,1\}}), \tag{2.56}$$

which combined with (2.51) implies

$$\int_0^t \|u_x(s)\|_{L^\infty(I)}^2 \,\mathrm{d}s \leqslant C(1 + \|\theta\|_\infty^{\max\{(1+2r)/4,r,1/2\}}). \tag{2.57}$$

Hence, together with (2.38), (2.44) and (2.57), we can obtain the upper bound on $\theta(x,t)$ immediately, since the parameter $r>0$ can be chosen sufficiently small.

Now, we deal with the case when the transport coefficients μ and κ satisfy (1.8) and (1.10). By (2.8), we have

$$\frac{1}{\theta(x,t)} \leqslant C + C\left\|\frac{1}{v}\right\|_\infty^{1-a} \quad \forall (x,t)\in I\times[0,t]. \tag{2.58}$$

For any number $\epsilon > -b$, since

$$\begin{aligned}\int_0^t \|\theta(s)\|_{L^\infty(I)}^{b+\epsilon}\,\mathrm{d}s &\leqslant C + C\int_0^t \left(\int_0^1 \theta^{(b+\epsilon)/2-1}|\theta_x|\,\mathrm{d}x\right)^2 \mathrm{d}s \\ &\leqslant C + C\int_0^t \left(\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\right)\left(\int_0^1 v\theta^\epsilon\,\mathrm{d}x\right)\mathrm{d}s,\end{aligned} \tag{2.59}$$

we can deduce that

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^{b+\epsilon}\,\mathrm{d}s \leqslant C + C\|\theta^\epsilon\|_\infty \tag{2.60}$$

or

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^{b+\epsilon}\,\mathrm{d}s \leqslant C + C\|v\|_\infty\|\theta^{\epsilon-1}\|_\infty \tag{2.61}$$

and

$$\int_0^t\int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s \leqslant C + C\|\theta^{1-b}\|_\infty, \tag{2.62}$$

where we have taken $\epsilon = 1-b$ in (2.60).

From (2.37), we have

$$\int_0^1 u^2\,\mathrm{d}x + \int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\int_0^t\int_0^1 \frac{\theta^2}{v^{1-a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\left\|\frac{1}{v}\right\|_\infty^{1-a}\|\theta^{1-b}\|_\infty. \tag{2.63}$$

On the other hand, integrating (2.48) over $I\times[0,T]$, we get

$$\begin{aligned}&\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x + \int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leqslant C + C\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \frac{\theta_x^2}{v^{1+a}\theta}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leqslant C + C\left\|\frac{1}{v}\right\|_\infty^{1-a}\|\theta^{1-b}\|_\infty + C\left\|\frac{1}{v}\right\|_\infty^{a}\|\theta^{1-b}\|_\infty\int_0^t\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leqslant C + C\left(\left\|\frac{1}{v}\right\|_\infty^{1-a} + \left\|\frac{1}{v}\right\|_\infty^{a}\right)\|\theta^{1-b}\|_\infty.\end{aligned} \tag{2.64}$$

Setting

$$\Phi(v) = \int_1^v \frac{\sqrt{\phi(z)}}{z^{1+a}}\,\mathrm{d}z, \tag{2.65}$$

it is easy to see that there exist two positive constants, C_1 and C_2, such that

$$|\Phi(v)| \geqslant C_1(v^{-a} + v^{1/2-a}) - C_2. \tag{2.66}$$

Since

$$\begin{aligned} |\Phi(v)| &= \left| \int_0^x \Phi(v(y,t))_y \,\mathrm{d}y \right| \\ &\leqslant \int_0^1 \left| \frac{\sqrt{\phi(v)}}{v^{1+a}} v_x \right| \mathrm{d}x \\ &\leqslant \left(\int_0^1 \phi(v)\,\mathrm{d}x \right)^{1/2} \left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x \right)^{1/2} \\ &\leqslant C + C\left(\left\| \frac{1}{v} \right\|_\infty^{(1-a)/2} + \left\| \frac{1}{v} \right\|_\infty^{1/2} a \right) \|\theta^{1-b}\|_\infty^{1/2}, \end{aligned} \tag{2.67}$$

by combining (2.67) with (2.66) and making use of Young's inequality, we have from the assumption $\frac{1}{3} < a < \frac{1}{2}$ that

$$\frac{1}{v(x,t)} \leqslant C + C\|\theta^{1-b}\|_\infty^{1/(3a-1)} \tag{2.68}$$

and

$$v(x,t) \leqslant C + C\|\theta^{1-b}\|_\infty^{2a/(3a-1)(1-2a)}. \tag{2.69}$$

With (2.68) and (2.69) in hand, (2.63) and (2.64) can be rewritten as

$$\int_0^1 u^2\,\mathrm{d}x + \int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|\theta^{1-b}\|_\infty^{2a/(3a-1)} \tag{2.70}$$

and

$$\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x + \int_0^t \int_0^1 \frac{\theta v_x^2}{v^{3+a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|\theta^{1-b}\|_\infty^{2a/(3a-1)}. \tag{2.71}$$

From (2.41), we get

$$\|\theta(t)\|_{L^\infty(I)} \leqslant C + C\int_0^t \left(\left\| \frac{u_x^2}{v^{1+a}} \right\|_{L^\infty(I)} + \left\| \frac{\theta^2}{v^{1-a}} \right\|_{L^\infty(I)} \right) \mathrm{d}s. \tag{2.72}$$

Thus, to deduce a nice bound on $\|\theta(t)\|_{L^\infty(I)}$, we need to estimate

$$\int_0^t \left\| \frac{u_x^2}{v^{1+a}} \right\|_{L^\infty(I)} \mathrm{d}s \quad \text{and} \quad \int_0^t \left\| \frac{\theta^2}{v^{1-a}} \right\|_{L^\infty(I)} \mathrm{d}s.$$

The next lemma is concerned with the first term.

LEMMA 2.12. *Under the conditions in lemma 2.2 and assuming that the transport coefficients* μ *and* κ *satisfy (1.8) and (1.10), for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^1 u_x^2 \,\mathrm{d}x + \int_0^t \int_0^1 \frac{u_{xx}^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \leqslant C + C\|\theta^{1-b}\|_\infty^{(2+4a-4a^2)/(3a-1)(1-2a)} + C\|\theta^{1-b}\|_\infty^{(2a-2a^2)/(3a-1)(1-2a)} \|\theta^{1-b-\delta}\|_\infty \|\theta\|_\infty^\delta. \tag{2.73}$$

Here $\delta > 0$ *is a positive constant that can be chosen as small as necessary.*

Proof. Integrating (2.52) with respect to x and t over $I \times [0, t]$, we have

$$\int_0^1 u_x^2 \,\mathrm{d}x + \int_0^t \int_0^1 \frac{u_{xx}^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \leqslant C + C \int_0^t \int_0^1 \left(\frac{u_x^2 v_x^2}{v^{3+a}} + \frac{\theta^2 v_x^2}{v^{3+a}} + \frac{\theta_x^2}{v^{1-a}} \right) \mathrm{d}x \,\mathrm{d}s, \tag{2.74}$$

and the terms on the right-hand side of (2.74) can be estimated term by term as in the following.

Firstly, (2.68)–(2.70) together with (2.50) imply that

$$\begin{aligned} \int_0^t \|u_x(s)\|_{L^\infty(I)}^2 \,\mathrm{d}s &\leqslant C \int_0^t \int_0^1 u_x^2 \,\mathrm{d}x \,\mathrm{d}s + C \left(\int_0^t \int_0^1 u_x^2 \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2} \left(\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2} \\ &\leqslant C\|v\|_\infty^{1+a} \int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \\ &\quad + C\|v\|_\infty^{1+a} \left(\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2} \left(\int_0^t \int_0^1 \frac{u_{xx}^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2} \\ &\leqslant C + C\|\theta^{1-b}\|_\infty^{(4a-2a^2)/(3a-1)(1-2a)} \\ &\quad + C\|\theta^{1-b}\|_\infty^{3a/(3a-1)(1-2a)} \left(\int_0^t \int_0^1 \frac{u_{xx}^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2}. \end{aligned} \tag{2.75}$$

Then, by (2.71), the first term on the right-hand side of (2.74) can be controlled by

$$\begin{aligned} \int_0^t \int_0^1 \frac{u_x^2 v_x^2}{v^{3+a}} \,\mathrm{d}x \,\mathrm{d}s &\leqslant \left\| \frac{1}{v} \right\|_\infty^{1-a} \int_0^t \|u_x(s)\|_{L^\infty(I)}^2 \left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}} \,\mathrm{d}x \right) \mathrm{d}s \\ &\leqslant C + C\|\theta^{1-b}\|_\infty^{(2+4a-4a^2)/(3a-1)(1-2a)} + \frac{1}{2} \int_0^t \int_0^1 \frac{u_{xx}^2}{v^{1+a}} \,\mathrm{d}x \,\mathrm{d}s. \end{aligned} \tag{2.76}$$

Secondly, taking $\epsilon = 2 - b$ in (2.61), we have

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^2 \,\mathrm{d}s \leqslant C + C\|v\|_\infty \|\theta^{1-b}\|_\infty \leqslant C + C\|\theta^{1-b}\|_\infty^{(7a-6a^2-1)/(3a-1)(1-2a)}, \tag{2.77}$$

and the second term on the right-hand side of (2.74) can be estimated as

$$\begin{aligned}\int_0^t\int_0^1 \frac{\theta^2 v_x^2}{v^{3+a}}\,\mathrm{d}x\,\mathrm{d}s &\leqslant \left\|\frac{1}{v}\right\|_\infty^{1-a}\int_0^t \|\theta(s)\|^2_{L^\infty(I)}\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x\right)\mathrm{d}s\\ &\leqslant C + C\|\theta^{1-b}\|_\infty^{(6a-8a^2)/(3a-1)(1-2a)}.\end{aligned} \tag{2.78}$$

To bound the third term on the right-hand side of (2.74), by multiplying $(2.3)_3$ by θ^δ, with δ an arbitrary positive number, and integrating the resulting equation with respect to x and t over $I\times[0,t]$ we have that

$$\begin{aligned}&\int_0^1 \theta^{1+\delta}\,\mathrm{d}x + \int_0^t\int_0^1 \frac{\theta^{b-1+\delta}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leqslant C + C\int_0^t\int_0^1 \frac{u_x^2\theta^\delta}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \frac{\theta^{2+\delta}}{v^{1-a}}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leqslant C + C\|\theta^{1-b}\|_\infty^{2a/(3a-1)}\|\theta\|_\infty^\delta.\end{aligned} \tag{2.79}$$

From which we can deduce that

$$\begin{aligned}\int_0^t\int_0^1 \frac{\theta_x^2}{v^{1-a}}\,\mathrm{d}x\,\mathrm{d}s &\leqslant \|v\|_\infty^a\|\theta^{1-b-\delta}\|_\infty\int_0^t\int_0^1 \frac{\theta^{b-1+\delta}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\|\theta^{1-b}\|_\infty^{(2a-2a^2)/(3a-1)(1-2a)}\|\theta^{1-b-\delta}\|_\infty\|\theta\|_\infty^\delta.\end{aligned} \tag{2.80}$$

Thus, (2.74), together with (2.76), (2.78) and (2.80), implies (2.73) and the proof of the lemma is complete. □

Plugging (2.73) into (2.75) and using (2.68) gives the following.

Lemma 2.13. *Under the conditions in lemma 2.12, for $0\leqslant t\leqslant T$ we have that*

$$\begin{aligned}\int_0^t \left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty(I)}\mathrm{d}s &\leqslant C + C\|\theta^{1-b}\|_\infty^{(2+4a-4a^2)/(3a-1)(1-2a)}\\ &\quad + C\|\theta^{1-b}\|_\infty^{(1+3a-3a^2)/(3a-1)(1-2a)}\|\theta^{1-b-\delta}\|_\infty^{1/2}\|\theta\|_\infty^{\delta/2}.\end{aligned} \tag{2.81}$$

Here $\delta>0$ is a sufficiently small positive constant.

Now, we estimate

$$\int_0^t \left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}\mathrm{d}s.$$

Thus, for $0\leqslant t\leqslant T$ we have the following result.

Lemma 2.14. *Under the conditions in lemma 2.12, for $0\leqslant t\leqslant T$ we have that*

$$\int_0^t \left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}\mathrm{d}s \leqslant C + C\|\theta^{1-b}\|_\infty^{(4a-4a^2)/(3a-1)(1-2a)}. \tag{2.82}$$

Proof. Taking $\epsilon = 1$ in (2.61), one has

$$\begin{aligned}\int_0^t \left\| \frac{\theta^2}{v^{1-a}} \right\|_{L^\infty(I)} \mathrm{d}s &\leqslant \left\| \frac{1}{v} \right\|_\infty^{1-a} \int_0^t \|\theta^2(s)\|_{L^\infty(I)} \,\mathrm{d}s \\ &\leqslant C(1 + \|\theta^{1-b}\|_\infty^{(1-a)/(3a-1)})\|\theta^{1-b}\|_\infty \int_0^t \|\theta^{1+b}(s)\|_{L^\infty(I)} \,\mathrm{d}s \\ &\leqslant C(1 + \|\theta^{1-b}\|_\infty^{2a/(3a-1)})(1 + \|v\|_\infty). \end{aligned} \tag{2.83}$$

Thus, with the aid of (2.69), we get (2.82), which completes the proof of lemma 2.14. □

Putting (2.81) and (2.82) together, we derive from (2.72) that, for all $\delta > 0$,

$$\begin{aligned}\|\theta(t)\|_{L^\infty(I)} \leqslant C &+ C\|\theta^{1-b}\|_\infty^{(2+4a-4a^2)/(3a-1)(1-2a)} \\ &+ C\|\theta^{1-b}\|_\infty^{(1+3a-3a^2)/(3a-1)(1-2a)}\|\theta^{1-b-\delta}\|_\infty^{1/2}\|\theta\|_\infty^{\delta/2}. \end{aligned} \tag{2.84}$$

With the above preparations in hand, we now deduce the desired lower and upper bounds on v and θ for the case when the transport coefficients μ and θ are given by (1.8). In fact, we have the following result.

Corollary 2.15. *Under the conditions in lemma 2.12, if we further assume that* $\frac{1}{3} < a < \frac{1}{2}$ *and that* b *satisfies either*

$$1 \leqslant b < \frac{2a}{1-a} \quad \text{or} \quad \frac{3-a+2a^2}{2+4a-4a^2} < b < 1,$$

then there exist positive constants $\underline{V}_2$, $\bar{V}_2$, $\underline{\Theta}_2$ *and* $\bar{\Theta}_2$, *such that*

$$\underline{V}_2 \leqslant v(x,t) \leqslant \bar{V}_2, \quad \underline{\Theta}_2 \leqslant \theta(x,t) \leqslant \bar{\Theta}_2 \quad \forall (x,t) \in I \times [0,t]. \tag{2.85}$$

Proof. We first consider the $b \geqslant 1$ case. In such a case, as a direct consequence of (2.58) and (2.68), we have

$$\frac{1}{\theta(x,t)} \leqslant C + C\|\theta^{1-b}\|_\infty^{(1-a)/(3a-1)} \leqslant C + C\left\| \frac{1}{\theta} \right\|_\infty^{(1-a)(b-1)/3a-1}, \tag{2.86}$$

which implies, under the assumption $1 < b < 2a/(1-a)$, that there exists one positive constant $\underline{\Theta}_2$ such that

$$\theta(x,t) \geqslant \underline{\Theta}_2 \quad \forall (x,t) \in I \times [0,t]. \tag{2.87}$$

Then (2.68) and (2.69), together with the fact that $b \geqslant 1$ and (2.87) holds, imply that there exist two positive constants, $\underline{V}_2$ and $\bar{V}_2$, such that

$$\underline{V}_2 \leqslant v(x,t) \leqslant \bar{V}_2 \quad \forall (x,t) \in I \times [0,t]. \tag{2.88}$$

On the other hand, note that we can choose δ small enough in (2.84) that the upper bound on $\theta(x,t)$ can be obtained by Young's inequality.

When $b < 1$, by choosing some δ belonging to $(0, \frac{1}{2}(1-b)]$, we have from (2.84) that

$$\begin{aligned}\|\theta(t)\|_{L^\infty(I)} &\leqslant C + C\|\theta\|_\infty^{(2+4a-4a^2)(1-b)/(3a-1)(1-2a)} \\ &\quad + C\|\theta\|_\infty^{(1+3a-3a^2)(1-b)/(3a-1)(1-2a)}\|\theta\|_\infty^{(1-b-\delta)/2}\|\theta\|_\infty^{\delta/2} \\ &\leqslant C + C\|\theta\|_\infty^{(2+4a-4a^2)(1-b)/(3a-1)(1-2a)}. \end{aligned} \tag{2.89}$$

Hence, under the assumption

$$\frac{3-a+2a^2}{2+4a-4a^2} < b < 1,$$

we deduce the upper bound on $\theta(x,t)$ from (2.89).

With this, the lower and upper bound on $v(x,t)$ can be obtained from (2.68), (2.69), and (2.58) implies that we can deduce the lower bound on $\theta(x,t)$ immediately. This completes the proof. □

With the above results in hand, theorem 1.1 follows immediately from the continuation argument and we omit the details for brevity.

3. Proof of theorem 1.3

The main aim of this section is to prove theorem 1.3 by the continuation argument. Since the local solvability of the IBVP (1.2), (1.3), (1.5) is well established (see [13, 24]), if we suppose that the local solution $(v(x,t), u(x,t), \theta(x,t))$ to the IBVP (1.2), (1.3), (1.5) has been extended to the time step $t = T > 0$ for some $T > 0$, then to extend such a solution $(v(x,t), u(x,t), \theta(x,t))$ step by step to a global one we only need to deduce certain *a priori* estimates on $(v(x,t), u(x,t), \theta(x,t))$ based on the *a priori* assumption (H) given in § 2. Note that, as in § 2, among these *a priori* estimates, it suffices to deduce the lower and upper bounds on the specific volume and the absolute temperature, which are independent of $\underline{V}'$, $\bar{V}'$, $\underline{\Theta}'$ and $\bar{\Theta}'$, but may depend on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

Before deriving the desired *a priori* estimates, we point out that, due to the change in boundary condition, some estimates valid in § 2 may no longer hold, and we need to pay particular attention to the boundary terms that appear when performing energy-type estimates.

Our first result is concerned with the estimate on the total energy. For this purpose, we obtain the following lemma from $(1.2)_3$ and (1.7).

Lemma 3.1 (estimate on the total energy). *Let the conditions in theorem 1.3 hold and suppose that $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the initial–boundary-value problem (1.2), (1.3), (1.5) defined on $I \times [0,T]$ for some $T > 0$. If we assume further that $(v(x,t), u(x,t), \theta(x,t))$ satisfies the* a priori *assumption (H), then for $0 \leqslant t \leqslant T$ we have that*

$$\int_0^1 \left(C_v\theta + \frac{u^2}{2}\right) \mathrm{d}x = \int_0^1 \left(C_v\theta_0 + \frac{u_0^2}{2}\right) \mathrm{d}x. \tag{3.1}$$

First, we consider the case when the transport coefficients μ and κ satisfy (1.9) and (1.12).

LEMMA 3.2. *Under the conditions in lemma 3.1 and assuming that the transport coefficients μ and κ satisfy (1.9) and (1.12), there exist positive constants $\underline{V}_3$, $\bar{V}_3$ and $\underline{\Theta}_3$ depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$ such that*

$$\underline{V}_3 \leqslant v(x,t) \leqslant \bar{V}_3 \quad \forall (x,t) \in I \times [0,T] \tag{3.2}$$

and

$$\theta(x,t) \geqslant \underline{\Theta}_3 \quad \forall (x,t) \in I \times [0,T]. \tag{3.3}$$

Proof. Set $y = 0$ in (2.16). Then, with the boundary condition (1.5), we have

$$-\mu_0 \log v(x,t) + \int_0^t p(x,s)\,\mathrm{d}s = \int_0^x (u_0(z) - u(z,t))\,\mathrm{d}z - \mu_0 \log v_0(x). \tag{3.4}$$

By (3.4), together with the fact that $p(x,t) > 0$ and the estimate (3.1) holds, we can easily obtain the lower bound of $v(x,t)$, and the lower bound on $\theta(x,t)$ can be obtained by combining the lower bound estimate on $v(x,t)$ with (2.8), i.e.

$$v(x,t) \geqslant \underline{V}_3, \quad \theta(x,t) \geqslant \underline{\Theta}_3 \quad \forall (x,t) \in I \times [0,T]. \tag{3.5}$$

Consequently, (2.31) holds for some positive constant K for all v and θ under consideration. Here, K depends on $\underline{V}_3$ and $\underline{\Theta}_3$.

To deduce an upper bound on $v(x,t)$ by exploiting the argument in lemma 2.4, we need only to recover the dissipative estimates

$$\int_0^t \int_0^1 \left(\frac{\mu u_x^2}{v\theta} + \frac{\kappa \theta_x^2}{v\theta^2} \right) \mathrm{d}x\,\mathrm{d}s.$$

To do this, by multiplying $(2.3)_3$ by θ^{-1} and integrating the resulting identity with respect to x and t over $I \times [0,t]$, one obtains

$$\begin{aligned}
\int_0^t \int_0^1 \frac{\mu_0 u_x^2}{v\theta}\,\mathrm{d}x\,\mathrm{d}s &+ \int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\,\mathrm{d}x\,\mathrm{d}s \\
&= C_v \int_0^1 \log\theta\,\mathrm{d}x - C_v \int_0^1 \log\theta_0\,\mathrm{d}x + R\int_0^1 \log v\,\mathrm{d}x - R\int_0^1 \log v_0\,\mathrm{d}x \\
&\leqslant C + R\int_0^1 \log v\,\mathrm{d}x,
\end{aligned} \tag{3.6}$$

where (3.1) and (3.5) are used.

As for the last term on the right-hand side of (3.6), by integrating (3.4) with respect to x over $[0,1]$ we have that

$$\int_0^1 \mu_0 \log v\,\mathrm{d}x \leqslant C + \int_0^t \int_0^1 p(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant C, \tag{3.7}$$

which, together with (3.6), implies that

$$\int_0^t \int_0^1 \frac{\mu_0 u_x^2}{v\theta}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{3.8}$$

Having obtained (3.8), we can deduce the upper bound on $v(x,t)$ by repeating the argument in lemma 2.4. This completes the proof of lemma 3.2. □

Now, we deduce the upper bound on $\theta(x,t)$ for the case when the transport coefficients μ and κ satisfy (1.9) and (1.12).

First, note that once we have obtained lemma 3.2, since the analysis in § 2 leading to corollary 2.5 and lemmas 2.6–2.9 and 2.11 involves only the boundary condition $\sigma(0,t)=\sigma(1,t)=0$, we can deduce that the estimates (2.34)–(2.36), (2.38), (2.42), (2.44) and (2.51) obtained therein hold. Now, we estimate $\|v_x(t)\|_{L^2(I)}$, which is the main content of lemma 2.10. To this end, by multiplying the identity (2.47) by $\mu v_x/v$, we get

$$\left(\frac{\mu^2 v_x^2}{2v^2}\right)_t = \left(\frac{\mu u v_x}{v}\right)_t - (u\sigma)_x + \frac{\mu u_x^2}{v} - (up)_x + \frac{\mu p_x v_x}{v}. \tag{3.9}$$

Integrating (3.9) with respect to x and t over $I\times[0,t]$, with the help of (3.1), Cauchy's inequality and the fact $\sigma(0,t)=\sigma(1,t)=0$, yields

$$\begin{aligned}\int_0^1 v_x^2\,\mathrm{d}x + \int_0^t\int_0^1 \theta v_x^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant C + C\int_0^t\int_0^1 \left(u_x^2 + u^2\theta + \frac{\theta_x^2}{\theta} + \theta^2\right)\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}}\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{3.10}$$

where (2.35) and (2.36) are used. Then, by (2.42), we can easily obtain (2.46).

By employing the arguments in [6, 13, 15], we can control

$$\int_0^t\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}s,$$

as in the following lemma.

Lemma 3.3. *Under the conditions in lemma 3.2, for $0\leqslant t\leqslant T$ we have that*

$$\int_0^t\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+\|\theta\|_\infty^2). \tag{3.11}$$

Proof. Set

$$U(x,t) = \int_0^x u(y,t)\,\mathrm{d}y. \tag{3.12}$$

Under the boundary condition

$$\sigma(0,t)=\sigma(1,t)=0, \tag{3.13}$$

by integrating $(2.3)_2$ over $(0,x)$ and by using (3.13) we can obtain that

$$\left.\begin{aligned} U_t - \frac{\mu}{v}U_{xx} &= -p(x,t),\\ U(x,0) &= \int_0^x u_0(y)\,\mathrm{d}y,\\ U(0,t) &= 0,\\ U(1,t) &= \int_0^1 u_0(x)\,\mathrm{d}x.\end{aligned}\right\} \tag{3.14}$$

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Hence, the standard L^p-estimates for solutions to the linear problem (3.14) (see [15]) yield

$$\int_0^t\int_0^1 U_{xx}^4\,\mathrm{d}x\,\mathrm{d}s \leqslant C(\|u_0\|_{L^2(I)}) + C\int_0^t\int_0^1 p^4\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\int_0^t\int_0^1 \theta^4\,\mathrm{d}x\,\mathrm{d}s. \tag{3.15}$$

Thus, by (2.35), we get (3.11), and the proof of lemma 3.3 is complete. □

For the estimate on

$$\int_0^t \|u_{xx}(s)\|_{L^2(I)}^2\,\mathrm{d}s,$$

we have the following.

Lemma 3.4. *Under the conditions in lemma 3.2, for $0 \leqslant t \leqslant T$ we have that*

$$\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|\theta\|_\infty^{\max\{2r,1,c+1\}}. \tag{3.16}$$

Proof. By differentiating $(2.3)_2$ with respect to x and multiplying the resulting equation by $u_x - R\theta/\mu_0$, we have

$$\left(\frac{u_x^2}{2} - \frac{R\theta u_x}{\mu_0}\right)_t = -u_x\left(\frac{R\theta}{\mu_0}\right)_t + \left(\frac{v\sigma\sigma_x}{\mu_0}\right)_x - \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x. \tag{3.17}$$

Integrating (3.17) with respect to x and t over $[0,1]\times[0,t]$, one has

$$\int_0^1\left(\frac{u_x^2}{2} - \frac{R\theta u_x}{\mu_0}\right)\mathrm{d}x \leqslant C - \frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_t\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_0^1 \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x\mathrm{d}x\,\mathrm{d}s. \tag{3.18}$$

Since by (1.12), (2.36), (2.42), (2.44), (2.46), (2.51) and (3.11) we have

$$\begin{aligned}
&-\int_0^t\int_0^1 \sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant -\frac{\underline{V}_3}{\mu_0}\int_0^t\int_0^1 \sigma_x^2\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\mu_0}\int_0^t\int_0^1 \sigma\sigma_x v_x\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant -\frac{\underline{V}_3}{2\mu_0}\int_0^t\int_0^1 \sigma_x^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \sigma^2 v_x^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant -\frac{\mu_0\underline{V}_3}{4\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 (u_x^2+\theta^2)v_x^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \theta_x^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant -\frac{\mu_0\underline{V}_3}{4\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + C\int_0^t (\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2)\|v_x(s)\|_{L^2(I)}^2\,\mathrm{d}s \\
&\qquad + C\|\theta\|_\infty^{1-r}\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant -\frac{\mu_0\underline{V}_3}{8\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s + C\|\theta\|_\infty^{\max\{2r,r+1/2,1\}}
\end{aligned} \tag{3.19}$$

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and

$$
\begin{aligned}
&-\frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=-\frac{R}{\mu_0C_v}\int_0^t\int_0^1 u_x\left[\left(\frac{\kappa\theta_x}{v}\right)_x+\frac{\mu_0u_x^2}{v}-\frac{R\theta u_x}{v}\right]\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant\frac{\mu_0\underline{V}_3}{16\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s+C\int_0^t\int_0^1(\kappa^2(v,\theta)\theta_x^2+u_x^3+\theta u_x^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant\frac{\mu_0\underline{V}_3}{16\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+C\left(\int_0^t\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^1(u_x^2+\theta^2)\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\\
&\qquad+C\|\theta\|_\infty^{c+1-r}\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant\frac{\mu_0\underline{V}_3}{16\bar{V}_3^2}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s+C\|\theta\|_\infty^{\max\{1,c+1\}}.
\end{aligned}\tag{3.20}
$$

Combing the two estimates above together with (3.1), (3.18) and Cauchy's inequality yields (3.16). This completes the proof of lemma 3.4. □

Having obtained (2.38), (2.44), (2.51) and (3.16), we can obtain the upper bound on $\theta(x,t)$ if the parameter c is chosen such that $c<1$. Here we have used the fact that $r>0$ can be chosen as small as required.

Now, we consider the case when the transport coefficients μ and κ satisfy (1.8) with $0\leqslant a<\frac{1}{5}$ and $b\geqslant 2$. For such a case, (3.4) should be replaced by

$$
-g(v(x,t))+\int_0^t p(x,s)\,\mathrm{d}s=\int_0^x(u_0(z)-u(z,t))\,\mathrm{d}z+g(v_0(x))\tag{3.21}
$$

with

$$
g(v)=\begin{cases}\dfrac{1-v^{-a}}{a}, & a>0,\\[2mm] \ln v, & a=0.\end{cases}
$$

With (3.21) in hand, by repeating the argument used in the proof of lemma 3.2, in particular the way to deduce (3.5), (3.6), we can deduce that there exist some positive constants $\underline{V}_3>0$ and $\underline{\Theta}_3>0$ such that

$$
v(x,t)\geqslant\underline{V}_3,\qquad \theta(x,t)\geqslant\underline{\Theta}_3
$$

hold for all $(x,t)\in I\times[0,T]$. But, since the boundary condition (1.5) does not yield any L^p-estimate on v, we can deduce from the fact that $|\ln v|\leqslant\|v\|_\infty^\varepsilon$ for any $\varepsilon>0$ that

$$
\int_0^t\int_0^1\frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s\leqslant C+C\|v\|_\infty^\varepsilon.\tag{3.22}
$$

To deduce an upper bound on $v(x,t)$, we aim to recover the L^1-estimate on $v(x,t)$, which plays an important role in deriving the upper bound on $v(x,t)$ for the case

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when the transport coefficients μ and κ satisfy (1.8). To do this, integrating $(2.3)_1$ with respect to x and t over $I \times [0,t]$, we get

$$\begin{aligned}\int_0^1 v\,\mathrm{d}x &\leqslant \int_0^1 v_0\,\mathrm{d}x + \int_0^t\int_0^1 u_x\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\|v\|_\infty^{1/2}a\biggl(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}\biggl(\int_0^t\int_0^1 v\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}\\ &\leqslant C + C\|v\|_\infty^{a}\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_0^1 v\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}\tag{3.23}$$

Then, by the Gronwall inequality, we can easily deduce that

$$\int_0^1 v\,\mathrm{d}x \leqslant C + C\|v\|_\infty^{a}\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.24}$$

Since $b \geqslant 2$, we have

$$\begin{aligned}\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s &\leqslant C\int_0^t \|\theta(s)\|_{L^\infty(I)}^{1/2}b\,\mathrm{d}s + C\\ &\leqslant C + C\int_0^t\int_0^1 \theta^{b/2-1}|\theta_x|\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\biggl(\int_0^t\int_0^1 v\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}\biggl(\int_0^t\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}\\ &\leqslant C + C\|v\|_\infty^{(\varepsilon+a)/2}\biggl(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}, \end{aligned}\tag{3.25}$$

which implies that

$$\int_0^t\int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|v\|_\infty^{(\varepsilon+a)/2}\biggl(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}. \tag{3.26}$$

Thus, with the help of (2.37), we have

$$\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|v\|_\infty^{(\varepsilon+a)/2}\biggl(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s\biggr)^{1/2}. \tag{3.27}$$

Then, by Cauchy's inequality and (3.24)–(3.27), we can easily obtain the following results.

LEMMA 3.5. *Under the conditions in lemma 3.2, for* $0 \leqslant t \leqslant T$ *we have that*

$$\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|v\|_\infty^{\varepsilon+a}, \tag{3.28}$$

$$\int_0^1 v\,\mathrm{d}x \leqslant C + C\|v\|_\infty^{\varepsilon+2a}, \tag{3.29}$$

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s \leqslant C + C\|v\|_\infty^{\varepsilon+a}, \tag{3.30}$$

and

$$\int_0^t\int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\|v\|_\infty^{\varepsilon+a}. \tag{3.31}$$

To estimate $\|v_x(t)\|_{L^2(I)}$, by integrating (3.9) with respect to x and t over $I\times[0,t]$ and with the help of (3.1) and Cauchy's inequality, we have that

$$\begin{aligned}\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x + \int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}\,\mathrm{d}x\,\mathrm{d}s &\leqslant C + C\int_0^t\int_0^1 \left(\frac{u_x^2}{v^{1+a}} + \frac{u^2\theta}{v^{1-a}} + \frac{\theta_x^2}{v^{1+a}\theta} + \frac{\theta^2}{v^{1-a}}\right)\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\|v\|_\infty^{\varepsilon+a} + \int_0^t\int_0^1 \frac{\theta_x^2}{v\theta}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.32}$$

To control

$$\int_0^t\int_0^1 \frac{\theta_x^2}{v\theta}\,\mathrm{d}x\,\mathrm{d}s,$$

by multiplying $(2.3)_3$ by θ^{-b} and integrating the resulting identity over $I\times[0,t]$, we have that

$$\begin{aligned}\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}\theta^b}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_0^1 \frac{\theta_x^2}{v\theta}\,\mathrm{d}x\,\mathrm{d}s &\leqslant C + C\int_0^t\int_0^1 \frac{|u_x|}{v}\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\|v\|_\infty^{(\varepsilon+a)/2}, \end{aligned} \tag{3.33}$$

and the above estimate, together with (3.32), implies

$$\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x \leqslant C + C\|v\|_\infty^{\varepsilon+a}. \tag{3.34}$$

Since

$$\begin{aligned} v(y,t) &\leqslant \int_0^1 v(x,t)\,\mathrm{d}x + \int_0^1 |v_x|\,\mathrm{d}x \\ &\leqslant C + C\|v\|_\infty^{\varepsilon+2a} + C\|v\|_\infty^{1/2+a}\left(\int_0^1 v\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\,\mathrm{d}x\right)^{1/2} \\ &\leqslant C + C\|v\|_\infty^{\varepsilon+1/2+5a/2}, \end{aligned} \tag{3.35}$$

and the assumption $0 \leqslant a < \frac{1}{5}$, we can deduce that

$$v(x,t) \leqslant \bar{V}_3 \quad \forall (x,t) \in I\times[0,T] \tag{3.36}$$

holds for some positive constant $\bar{V}_3$ which depends only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$. As a by-product of the estimate (3.36), we can deduce that the terms on the right-hand side of the inequalities in lemma 3.5 and (3.34) can all be bounded by some constant C depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

Now, we turn to derive the upper bound on $\theta(x,t)$. To do this, we multiply $(2.3)_3$ by $\theta^{-\gamma}$ for some $\gamma \in (0,1)$ and integrate the resulting identity over $I \times [0,t]$ to obtain

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}\theta^\gamma} \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_0^1 \frac{\theta^{b-1-\gamma}\theta_x^2}{v} \,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{3.37}$$

Then, by (3.15), we have

$$\begin{aligned}
\int_0^t \int_0^1 u_x^4 \,\mathrm{d}x\,\mathrm{d}s &\leqslant C + C\int_0^t \int_0^1 \theta^4 \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\int_0^t \|\theta(s)\|_{L^\infty(I)}^3 \,\mathrm{d}s \\
&\leqslant C + C\int_0^t \left(\int_0^1 \sqrt{\theta}|\theta_x| \,\mathrm{d}x\right)^2 \mathrm{d}s \\
&\leqslant C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}} \int_0^t \int_0^1 \theta^{b-1-\gamma}\theta_x^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}}.
\end{aligned} \tag{3.38}$$

Now, we set

$$X := \int_0^t \int_0^1 \theta^b \theta_t^2 \,\mathrm{d}x\,\mathrm{d}s, \qquad Y := \max_t \int_0^1 \theta^{2b}\theta_x^2 \,\mathrm{d}x, \qquad Z := \max_t \int_0^1 u_{xx}^2 \,\mathrm{d}x. \tag{3.39}$$

Observe that

$$\begin{aligned}
\theta^{2b+2} &\leqslant C + C\int_0^1 \theta^{2b+1}|\theta_x| \,\mathrm{d}x \\
&\leqslant C + C\|\theta\|_{L^\infty(I)}^{b+1/2} \left(\int_0^1 \theta \,\mathrm{d}x\right)^{1/2} \left(\int_0^1 \theta^{2b}\theta_x^2 \,\mathrm{d}x\right)^{1/2} \\
&\leqslant C + C\|\theta\|_{L^\infty(I)}^{b+1/2} Y^{1/2},
\end{aligned} \tag{3.40}$$

which implies

$$\|\theta\|_{L^\infty(I)} \leqslant C + CY^{1/(2b+3)}. \tag{3.41}$$

Combining (2.50) with the inequality

$$\int_0^1 u_x^2 \,\mathrm{d}x \leqslant C\int_0^1 u^2 \,\mathrm{d}x + C\left(\int_0^1 u^2 \,\mathrm{d}x\right)^{1/2} \left(\int_0^1 u_{xx}^2 \,\mathrm{d}x\right)^{1/2}, \tag{3.42}$$

and (3.1), we have

$$\max_t \int_0^1 u_x^2 \,\mathrm{d}x \leqslant C + CZ^{1/2} \tag{3.43}$$

and

$$\|u_x\|_{L^\infty(I)} \leqslant C + CZ^{3/8}. \tag{3.44}$$

Our next result will show that X and Y can be controlled by Z.

LEMMA 3.6. *Under the conditions in lemma 3.2, we have*

$$X+Y \leqslant C + CZ^{3/4}. \tag{3.45}$$

Proof. Multiplying $(2.3)_3$ by $\theta^b\theta_t$ and integrating the resulting identity over $I\times[0,t]$, one has

$$X+Y \leqslant C + C\int_0^t\int_0^1 (\theta^{b+1}|u_x||\theta_t| + \theta^b u_x^2|\theta_t| + \theta^{2b}|u_x|\theta_x^2)\,\mathrm{d}x\,\mathrm{d}s, \tag{3.46}$$

since by Cauchy's inequality and (3.36)–(3.38), we can get from (3.41) and (3.44) that

$$\begin{aligned}\int_0^t\int_0^1 \theta^{b+1}|u_x||\theta_t|\,\mathrm{d}x\,\mathrm{d}s &\leqslant \tfrac{1}{4}X + C\|\theta\|_\infty^{b+2}\int_0^t\int_0^1 u_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \tfrac{1}{4}X + CY^{(b+2)/(2b+3)},\end{aligned} \tag{3.47}$$

$$\begin{aligned}\int_0^t\int_0^1 \theta^b u_x^2|\theta_t|\,\mathrm{d}x\,\mathrm{d}s &\leqslant \tfrac{1}{4}X + C\|\theta\|_\infty^{b}\int_0^t\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \tfrac{1}{4}X + CY^{\max\{b,2+\gamma\}/(2b+3)},\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\int_0^t\int_0^1 \theta^{2b}|u_x|\theta_x^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant \|u_x\|_\infty\|\theta\|_\infty^{b+1+\gamma}\int_0^t\int_0^1 \theta^{b-1-\gamma}\theta_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant CY^{(b+1+\gamma)/(2b+3)}(1+Z^{3/8}).\end{aligned} \tag{3.49}$$

Based on the above three estimates and (3.46) and by employing the Cauchy inequality, we can get (3.45) immediately if we choose $\gamma \in (0, \frac{1}{2})$. This completes the proof of lemma 3.6. □

Our last result in this section shows that Z can be bounded by X and Y.

LEMMA 3.7. *Under the conditions in lemma 3.2, we have*

$$Z \leqslant C + CY^{(2+\gamma)/(2b+3)} + CX + CZ^{3/4} \quad \text{for some } \gamma \in (0,1). \tag{3.50}$$

Proof. Using $(2.3)_2$, we can easily get the following identity:

$$u_{xx} = v^{1+a}\left(u_t + p_x + \frac{(1+a)v_x u_x}{v^{2+a}}\right). \tag{3.51}$$

Integrating (3.51) with respect to x and t over $I\times[0,t]$ yields

$$\begin{aligned}\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant C\int_0^t\int_0^1 (u_t^2 + \theta_x^2 + \theta^2 v_x^2 + v_x^2 u_x^2)\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C\int_0^t\int_0^1 u_t^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \frac{\theta^{b-1-\gamma}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + C(\|\theta\|_\infty^2 + \|u_x\|_\infty^2)\int_0^t\int_0^1 v_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\int_0^t\int_0^1 u_t^2\,\mathrm{d}x\,\mathrm{d}s + CY^{2/(2b+3)} + CZ^{3/4}.\end{aligned} \tag{3.52}$$

Next, we need to estimate

$$\int_0^t \int_0^1 u_t^2 \, \mathrm{d}x \, \mathrm{d}s$$

to complete the proof of this lemma. To this end, by differentiating $(2.3)_2$ with respect to t and multiplying the resulting identity by u_t we have that

$$\left(\frac{u_t^2}{2}\right)_t + \frac{u_{xt}^2}{v^{1+a}} = (\sigma_t u_t)_x + \frac{(1+a)u_x^2 u_{xt}}{v^{2+a}} + \frac{R\theta_t u_{xt}}{v} - \frac{R\theta u_x u_{xt}}{v^2}. \tag{3.53}$$

Integrating (3.53) with respect to x and t over $I\times[0,t]$ and with the help of Cauchy's inequality, one has

$$\begin{aligned}\int_0^1 u_t^2 \, \mathrm{d}x + \int_0^t \int_0^1 u_{xt}^2 \, \mathrm{d}x \, \mathrm{d}s &\leqslant C + C\int_0^t \int_0^1 (u_x^4 + \theta_t^2 + \theta^2 u_x^2) \, \mathrm{d}x \, \mathrm{d}s \\ &\leqslant C + CY^{(2+\gamma)/(2b+3)} + CX, \end{aligned}\tag{3.54}$$

which, together with (3.52) implies (3.50) and the proof of lemma 3.7. □

Combining (3.45) and (3.50), we can obtain $Y \leqslant C$. Then we derive the upper bounds on $\theta(x,t)$ from (3.41).

In summary, we have obtained the desired lower and upper bounds on v and θ, provided that the transport coefficients μ and κ satisfy the conditions in theorem 1.3. Theorem 1.3 can then be proved by employing the continuation argument.

Remark 3.8. For the case when $\mu(v)$ is a smooth function of v satisfying $\mu(v) > 0$ for $v > 0$ and $\kappa(\theta) = \theta^b$, if the specific volume v is bounded both from below and from above and the absolute temperature θ is bounded from below, i.e. there exist some positive constants $\underline{V}_3 > 0$, $\bar{V}_3 > 0$ and $\underline{\Theta}_3 > 0$ such that

$$\underline{V}_3 \leqslant v(x,t) \leqslant \bar{V}_3, \qquad \theta(x,t) \geqslant \underline{\Theta}_3 > 0$$

hold for $(x,t) \in I \times [0,T]$, then the argument used above can be employed to derive the upper bound on $\theta(x,t)$, provided that $b \geqslant 0$.

4. Proof of theorem 1.5

The local solvability for the outer pressure problem (1.2), (1.3), (1.6) is simpler than that of the IBVP (1.2), (1.3), (1.5), due to the fact that $0 < Q(t) \in C^1(\mathbb{R}_+)$. Thus, to prove theorem 1.5 by the continuation argument, it remains to show that if $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the outer pressure problem (1.2), (1.3), (1.6) defined on $I \times [0,T]$ for some $T > 0$ and satisfies the *a priori* assumption (H), then $v(x,t)$ and $\theta(x,t)$ are bounded, both from below and above, by some positive constants depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

To this end, we first derive the following from $(1.2)_3$.

Lemma 4.1 (estimate on the total energy). *Let the conditions in theorem 1.5 hold and suppose that $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the outer pressure problem (1.2), (1.3), (1.6) defined on $I \times [0,T]$ for some $T > 0$, and it satisfies the* a priori *assumption (H). Then we have*

$$\int_0^1 \left(\theta + \frac{u^2}{2} + v\right) \mathrm{d}x \leqslant C. \tag{4.1}$$

Proof. Integrating $(1.2)_3$ with respect to x and t over $I \times [0,t]$ and making use of the boundary condition (1.6) yields

$$\int_0^1 \left(C_v\theta + \frac{u^2}{2}\right) \mathrm{d}x + Q(t)\int_0^1 v\,\mathrm{d}x = \int_0^1 \left(C_v\theta_0 + \frac{u_0^2}{2}\right) \mathrm{d}x + \int_0^t Q'(s)\int_0^1 v\,\mathrm{d}x\,\mathrm{d}s. \tag{4.2}$$

Then by the Gronwall inequality and the assumption on $Q(t)$, we get (4.1). This proves lemma 4.1. □

To derive the desired lower bound estimate on v, we integrate (2.15) over $[0,x] \times [0,t]$ to get that

$$-g(v) + \int_0^t p(x,s)\,\mathrm{d}s = \int_0^x (u_0(z) - u(z,t))\,\mathrm{d}z + \int_0^t Q(s)\,\mathrm{d}s + g(v_0(x)), \tag{4.3}$$

where

$$g(v) = \begin{cases} \dfrac{1-v^{-a}}{a}, & a > 0, \\ \ln v, & a = 0. \end{cases} \tag{4.4}$$

Thus, we can easily deduce the upper bound for $-g(v)$, from which, together with the fact that $a \geqslant 0$, one can immediately obtain the lower bound on $v(x,t)$. Having obtained the lower bound for $v(x,t)$, we can deduce the lower bound on $\theta(x,t)$ from (2.8).

A direct consequence of (3.6) and (4.1) is

$$\int_0^t \int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{4.5}$$

To derive the upper bound on $v(x,t)$, we shall use the following estimates.

Lemma 4.2. *Under the conditions in lemma 4.1, for $0 \leqslant t \leqslant T$ we have that*

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}\,\mathrm{d}s \leqslant C \tag{4.6}$$

and

$$\int_0^t \int_0^1 \theta^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{4.7}$$

Proof. By (4.1) and (4.5), we have

$$\begin{aligned} \int_0^t \|\theta(s)\|_{L^\infty(I)}^b\,\mathrm{d}s &\leqslant C + C\int_0^t \left(\int_0^1 \theta^{b/2-1}|\theta_x|\,\mathrm{d}x\right)^2 \mathrm{d}s \\ &\leqslant C + C\int_0^t \left(\int_0^1 v\,\mathrm{d}x\right)\left(\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}\,\mathrm{d}x\right)\mathrm{d}s \\ &\leqslant C. \end{aligned} \tag{4.8}$$

If $b \geqslant 1$, we get (4.6) immediately.

Now we deal with the $\frac{1}{2} \leqslant b < 1$ case. By (4.8), we have

$$\int_0^t \int_0^1 \theta^{b+1}\, \mathrm{d}x\, \mathrm{d}s \leqslant C. \tag{4.9}$$

Multiplying $(2.3)_3$ by θ^{-s} for some $s > 0$ to be determined, and integrating the resulting identity with respect to x and t over $I \times [0,t]$, one has

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}\theta^s}\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_0^1 \frac{\theta^{b-1-s}\theta_x^2}{v}\, \mathrm{d}x\, \mathrm{d}s \leqslant C + C\int_0^t \int_0^1 \theta^{2-s}\, \mathrm{d}x\, \mathrm{d}s. \tag{4.10}$$

Hence, by (4.9) we get

$$\int_0^t \int_0^1 \frac{\theta^{b-1-s}\theta_x^2}{v}\, \mathrm{d}x\, \mathrm{d}s \leqslant C \quad \forall s \geqslant 1-b > 0. \tag{4.11}$$

Setting $s = b$ in (4.11), this reduces to

$$\int_0^t \int_0^1 \frac{\theta_x^2}{v\theta}\, \mathrm{d}x\, \mathrm{d}s \leqslant C. \tag{4.12}$$

Then

$$\begin{aligned}\int_0^t \|\theta(s)\|_{L^\infty(I)}\, \mathrm{d}s &\leqslant C + C\int_0^t \left(\int_0^1 \frac{|\theta_x|}{\sqrt{\theta}}\, \mathrm{d}x\right)^2 \mathrm{d}s \\ &\leqslant C + C\int_0^t \left(\int_0^1 v\, \mathrm{d}x\right)\left(\int_0^1 \frac{\theta_x^2}{v\theta}\, \mathrm{d}x\right) \mathrm{d}s \\ &\leqslant C, \end{aligned} \tag{4.13}$$

which implies that (4.6) holds for all $b \geqslant \frac{1}{2}$, and (4.7) can be obtained directly. This completes the proof of lemma 4.2. □

Formula (2.37), together with (4.7), implies

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}}\, \mathrm{d}x\, \mathrm{d}s \leqslant C + C\int_0^t \int_0^1 \frac{\theta^2}{v^{1-a}}\, \mathrm{d}x\, \mathrm{d}s \leqslant C. \tag{4.14}$$

Integrating (3.9) with respect to x and t over $I \times [0,t]$, and with the help of (3.1) and Cauchy's inequality, we have

$$\begin{aligned}&\int_0^1 \frac{v_x^2}{v^{2(1+a)}}\, \mathrm{d}x + \int_0^t \int_0^1 \frac{\theta v_x^2}{v^{3+a}}\, \mathrm{d}x\, \mathrm{d}s \\ &\quad \leqslant C + C\int_0^t \int_0^1 \left(\frac{u_x^2}{v^{1+a}} + \frac{\theta^2}{v^{1-a}} + \frac{u^2\theta}{v^{1-a}} + \frac{\theta_x^2}{v^{1+a}\theta}\right) \mathrm{d}x\, \mathrm{d}s \\ &\quad \leqslant C, \end{aligned} \tag{4.15}$$

where (4.6), (4.7), (4.12) and (4.14) are used.

Hence, as in (3.35), we get the upper bound on $v(x,t)$.

Note that from (4.10) and (4.14) we have (3.37) with $\gamma \in (0,1)$. On the other hand, as in (3.15) and with the aid of $Q(t) \in C^1(\mathbb{R}_+)$, we can obtain the inequality (3.38). Thus, as pointed out in remark 3.8, the upper bound on $\theta(x,t)$ can be obtained by employing the argument in § 3. This completes the proof of theorem 1.5.

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