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Symmetry Classification for Jackson Integrals Associated with the Root System BC_n

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Abstract. The Jackson integrals associated with the non-reduced root system are defined as multiple sums which are generalization of the Bailey's very-well-poised ${}_6\psi_6$ sum. They are classified by the number of their parameters when they can be expressed as a product of the Jacobi elliptic theta functions. The sums which appear in the classification list coincide with those investigated individually by Gustafson and van Diejen.

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Key words. Bailey's very-well-poised ${}_6\psi_6$ sum, Gustafson's C_n-type sum, Jackson integrals, non-reduced root system, van Diejen's BC_n-type sum.

1. Introduction

This paper is a sequel to the paper [Ito2]. In [Ito2], we introduced the Jackson integrals associated with irreducible reduced root systems, and discussed the condition if they can be expressed as a product of the Jacobi elliptic theta functions. In this paper, we study the Jackson integral for the non-reduced case. The Jackson integral associated with the non-reduced root system, i.e., the root system BC_n, is a multiple extension of Baily's very-well-poised ${}_6\psi_6$ sum (q-series).

The result is Theorem 3.3, which classifies the Jackson integrals associated with BC_n when they are expressed as a product of the Jacobi elliptic functions. Bailey's very-well-poised ${}_6\psi_6$ summation formula

$$\sum_{\nu=-\infty}^{\infty}\frac{(1-a^2q^{2\nu})(ab;q)_\nu(ac;q)_\nu(ad;q)_\nu(ae;q)_\nu}{(1-a^2)(qa/b;q)_\nu(qa/c;q)_\nu(qa/d;q)_\nu(qa/e;q)_\nu}\Big(\frac{q}{bcde}\Big)^\nu$$
$$=\frac{(q;q)_\infty(q/a^2;q)_\infty(qa^2;q)_\infty}{(q/ab;q)_\infty(q/ac;q)_\infty(q/ad;q)_\infty(q/ae;q)_\infty}\times$$
$$\times\frac{(q/bc;q)_\infty(q/bd;q)_\infty(q/be;q)_\infty(q/cd;q)_\infty(q/ce;q)_\infty(q/de;q)_\infty}{(qa/b;q)_\infty(qa/c;q)_\infty(qa/d;q)_\infty(qa/e;q)_\infty(q/bcde;q)_\infty}$$

can be regarded as the formula for the BC_1-type Jackson integral in Theorem 3.3. In [Gu1], Gustafson established multidimensional generalization of ${}_6\psi_6$ summation formula corresponding to semi-simple Lie algeblas. By using Gustafson's C_n-type sum,

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van Diejen [vD] proved a summation formula for his BC_n-type sum, which includes Aomoto's B_n and C_n-type sums as special cases. Gustafson's C_n-type sum and van Diejen's BC_n-type sum are included in the classification list in Theorem 3.3. The formulae appearing in Theorem 3.3 were essentially investigated by a series of Gustafson's works [Gu1, Gu2, Gu3, Gu4]. Although we were able to find a new summation formula [Ito4] for the root system F_4 in the irreducible reduced cases [Ito2], Theorem 3.3 assures, in our sense, his formulae are all those of BC_n-type. Note that Schlosser [S] also presented other multidimensional generalization of ${}_6\psi_6$ summation formula and his C_n-type sums are not included in our list.

Throughout this paper, we assume $0 < q < 1$ and use notation

$$(x;q)_\infty := \prod_{i=0}^{\infty}(1-xq^i) \quad \text{and} \quad (x;q)_\nu := (x;q)_\infty/(xq^\nu;q)_\infty.$$

2. Jackson Integral Associated with the Root System BC_n

Let $\{\varepsilon_1,\dots,\varepsilon_n\}$ be the standard basis of $E=\mathbf{R}^n$ satisfying $\langle\varepsilon_i,\varepsilon_j\rangle=\delta_{ij}$ for all $i,j=1,\dots,n$, where δ_{ij} is the Kronecker delta. Let R be the root system BC_n provided by $R=R_1\cup R_2\cup 2R_1$ where

$$R_1 := \{\pm\varepsilon_i;\ 1\leqslant i\leqslant n\}, \qquad R_2 := \{\pm\varepsilon_i\pm\varepsilon_j;\ 1\leqslant i<j\leqslant n\}$$

and

$$2R_1 := \{\pm 2\varepsilon_i;\ 1\leqslant i\leqslant n\}.$$

We set

$$R_1^+ := \{\varepsilon_i;\ 1\leqslant i\leqslant n\}, \qquad R_2^+ := \{\varepsilon_i\pm\varepsilon_j;\ 1\leqslant i<j\leqslant n\}$$

and

$$2R_1^+ := \{2\varepsilon_i;\ 1\leqslant i\leqslant n\}.$$

For each $\alpha\in R$, let $\alpha^\vee=2\alpha/\ \langle\alpha,\alpha\rangle$. The root systems B_n and C_n are the sets $R_1\cup R_2$ and $R_2\cup 2R_1$, respectively. Let $\{\alpha_1,\dots,\alpha_n\}$ be a basis of the root system C_n given by

$$\alpha_1=\varepsilon_1-\varepsilon_2,\ \ \alpha_2=\varepsilon_2-\varepsilon_3,\dots,\alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\ \ \alpha_n=2\varepsilon_n.$$

Let $\{\chi_1,\dots,\chi_n\}$ be the set of the fundamental coweights of C_n given by

$$\chi_1=\varepsilon_1,\ \ \chi_2=\varepsilon_1+\varepsilon_2,\dots,\chi_{n-1}=\varepsilon_1+\cdots+\varepsilon_{n-1},\ \ \chi_n=(\varepsilon_1+\cdots+\varepsilon_n)/2,$$

which satisfy $\langle\alpha_i,\chi_j\rangle=\delta_{ij}$ for all $i,j=1,\dots,n$. We denote by P the coweight lattice of C_n defined by $P:=\mathbf{Z}\chi_1+\cdots+\mathbf{Z}\chi_n$, and Q be the coroot lattice of C_n defined by

$$Q:=\mathbf{Z}\alpha_1^\vee+\cdots+\mathbf{Z}\alpha_n^\vee=\mathbf{Z}\varepsilon_1+\cdots+\mathbf{Z}\varepsilon_n\subset P.$$

Let L be any sublattice of P of rank n. We assume L is W-stable, i.e., $L=wL$ for $w\in W$. The scalar product $\langle\cdot,\cdot\rangle$ is uniquely extended linearly to $E_{\mathbf{C}}=E\otimes_{\mathbf{R}}\mathbf{C}\simeq\mathbf{C}^n$. For $x\in E_{\mathbf{C}}$, we define a function $\Phi_{B_n}(\{b_i\}_{i=1}^s,\{c_j\}_{j=1}^l;x)$ as

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$$\begin{aligned}
&\Phi_{B_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; x) \\
&:= \prod_{i=1}^s \prod_{\alpha\in R_1^+} q^{(1/2-b_i)\langle\alpha,x\rangle} \frac{(q^{1-b_i+\langle\alpha,x\rangle};q)_\infty}{(q^{b_i+\langle\alpha,x\rangle};q)_\infty} \prod_{j=1}^l \prod_{\alpha\in R_2^+} q^{(1/2-c_j)\langle\alpha,x\rangle} \frac{(q^{1-c_j+\langle\alpha,x\rangle};q)_\infty}{(q^{c_j+\langle\alpha,x\rangle};q)_\infty} \\
&= \prod_{i=1}^s \left(\prod_{k=1}^n q^{(1/2-b_i)\langle\varepsilon_k,x\rangle} \frac{(q^{1-b_i+\langle\varepsilon_k,x\rangle};q)_\infty}{(q^{b_i+\langle\varepsilon_k,x\rangle};q)_\infty} \right)\times \\
&\quad \times \prod_{j=1}^l \left(\prod_{1\leqslant\mu<\nu\leqslant n} q^{(1-2c_j)\langle\varepsilon_\mu,x\rangle} \frac{(q^{1-c_j+\langle\varepsilon_\mu-\varepsilon_\nu,x\rangle};q)_\infty}{(q^{c_j+\langle\varepsilon_\mu-\varepsilon_\nu,x\rangle};q)_\infty} \frac{(q^{1-c_j+\langle\varepsilon_\mu+\varepsilon_\nu,x\rangle};q)_\infty}{(q^{c_j+\langle\varepsilon_\mu+\varepsilon_\nu,x\rangle};q)_\infty} \right),
\end{aligned}$$

where $s, l \in \mathbf{Z}_{\geqslant 0}$ and $b_i, c_j \in \mathbf{C}$. We denote by $\Delta_{C_n}(x)$ the Weyl denominator of C_n as

$$\begin{aligned}
\Delta_{C_n}(x) &:= \prod_{\alpha\in R_2^+} (q^{\langle\alpha,x\rangle/2} - q^{-\langle\alpha,x\rangle/2}) \prod_{\alpha\in 2R_1^+} (q^{\langle\alpha,x\rangle/2} - q^{-\langle\alpha,x\rangle/2}) \\
&= \prod_{\alpha\in R_1^+} (q^{\langle\alpha,x\rangle} - q^{-\langle\alpha,x\rangle}) \prod_{\alpha\in R_2^+} (q^{\langle\alpha,x\rangle/2} - q^{-\langle\alpha,x\rangle/2}) \\
&= \prod_{k=1}^n (q^{\langle\varepsilon_k,x\rangle} - q^{-\langle\varepsilon_k,x\rangle}) \prod_{1\leqslant\mu<\nu\leqslant n} (q^{\langle\varepsilon_\mu-\varepsilon_\nu,x\rangle/2} - q^{-\langle\varepsilon_\mu-\varepsilon_\nu,x\rangle/2})(q^{\langle\varepsilon_\mu+\varepsilon_\nu,x\rangle/2} - q^{-\langle\varepsilon_\mu+\varepsilon_\nu,x\rangle/2}).
\end{aligned}$$

Let W be the *Weyl group* generated by orthogonal reflections with respect to the hyperplane perpendicular to $\alpha \in R$. For $w \in W$, we define $wF(x) := F(w^{-1}x)$ for a function $F(x)$ of $x \in E_{\mathbf{C}}$. For $w \in W$, we denote $U_w(x)$ by a function defined by

$$U_w(x) := \prod_{i=1}^s \prod_{\substack{\alpha\in R_1^+ \\ -w^{-1}\alpha\in R_1^+}} q^{(2b_i-1)\langle\alpha,x\rangle} \frac{\theta(q^{b_i+\langle\alpha,x\rangle};q)}{\theta(q^{1-b_i+\langle\alpha,x\rangle};q)} \prod_{j=1}^l \prod_{\substack{\alpha\in R_2^+ \\ -w^{-1}\alpha\in R_2^+}} q^{(2c_j-1)\langle\alpha,x\rangle} \frac{\theta(q^{c_j+\langle\alpha,x\rangle};q)}{\theta(q^{1-c_j+\langle\alpha,x\rangle};q)},$$

where $\theta(\xi;q) := (\xi;q)_\infty(q/\xi;q)_\infty$. The function $\theta(\xi;q)(q;q)_\infty$ is called the *Jacobi elliptic theta function*. Since $\theta(\xi;q)$ has a property $\theta(q\xi;q) = -\theta(\xi;q)/\xi$, we see $U_w(x)$ is an invariant under the shift $x \to x+\chi$ for $\chi \in P$. Under the action of $w \in W$, the function $:= \Phi_R(\{b_i\},\{c_j\};x)$ changes as follows:

$$w\Phi_{B_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; x) = U_w(x)\, \Phi_{B_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; x), \quad w \in W. \tag{1}$$

The Weyl denominator $\Delta_{C_n}(x)$ changes by the action of W as

$$w\Delta_{C_n}(x) = \mathrm{sgn} w\ \Delta_{C_n}(x). \tag{2}$$

For $z \in E_{\mathbf{C}}$, we now define the *Jackson integral associated with the root system BC_n* as

$$J_{BC_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; L; z) := \sum_{\chi\in L} \Phi_{B_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; z+\chi)\, \Delta_{C_n}(z+\chi). \tag{3}$$

For simplicity, we abbreviate $J_{BC_n}(\{b_i\}_{i=1}^s, \{c_j\}_{j=1}^l; L; z)$ by $J_{BC_n}(\{b_i\},\{c_j\};L;z)$. By definition, the Jackson integral $J_{BC_n}(\{b_i\},\{c_j\};L;z)$ is obviously invariant under the shift $z \to z+\chi$ for $\chi \in L$:

$$J_{BC_n}(\{b_i\},\{c_j\};L;z+\chi) = J_{BC_n}(\{b_i\},\{c_j\};L;z). \tag{4}$$

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From (1) and (2), for $w \in W$, we have the following property of $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$:

$$wJ_{BC_n}(\{b_i\}, \{c_j\}; L; z) = \operatorname{sgn} w\, U_w(z)\, J_{BC_n}(\{b_i\}, \{c_j\}; L; z), \quad w \in W.$$

3. Product Formula

In this section, we discuss the sum $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ which can be expressed as a product of the *Jacobi elliptic theta function* $\theta(\xi; q)(q; q)_\infty$. The function $\theta(\xi; q)$ has a quasi-periodic property such as $\theta(q\xi; q) = -\theta(\xi; q)/\xi$. By using this property, for $\chi \in L$ and $\theta(q^{c+\langle\alpha,z\rangle}; q)$, we have

$$\theta(q^{c+\langle\alpha,z+\chi\rangle}; q) = (-1)^{\langle\alpha,\chi\rangle} q^{(1/2-c)\langle\alpha,\chi\rangle-\langle\alpha,\chi\rangle^2/2-\langle\alpha,z\rangle\langle\alpha,\chi\rangle}\theta(q^{c+\langle\alpha,z\rangle}; q), \tag{5}$$

which is used in the subsequent discussion.

LEMMA 3.1. *Assume that* $L = P$ *or* Q. *For* $\alpha \in R_2^+ \cup 2R_1^+$, *if* $\langle\alpha, z\rangle = 0$, *then* $J_{BC_n}(\{b_i\}, \{c_j\}; L; z + \chi) = 0$ *for all* $\chi \in P$.

Proof. See [Ito2, p. 332 Lemma 4.2]. □

PROPOSITION 3.2. *For* $L = P$ *or* Q, *the sum* $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ *is expressed as*

$$f(z) \prod_{\alpha\in R_1^+} \frac{q^{\left(\frac{s}{2}-1-\sum_{i=1}^s b_i\right)\langle\alpha,z\rangle}\theta(q^{\langle2\alpha,z\rangle}; q)}{\prod_{i=1}^s \theta(q^{b_i+\langle\alpha,z\rangle}; q)} \prod_{\alpha\in R_2^+} \frac{q^{\left(\frac{l-1}{2}-\sum_{j=1}^l c_j\right)\langle\alpha,z\rangle}\theta(q^{\langle\alpha,z\rangle}; q)}{\prod_{j=1}^l \theta(q^{c_j+\langle\alpha,z\rangle}; q)}$$

where $f(z)$ *is a holomorphic function of* $z \in E_{\mathbf{C}}$.

Proof. Since $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ has poles lying in the set

$$\left\{ z \in E_{\mathbf{C}}\ ;\ \prod_{i=1}^s \prod_{\alpha\in R_1^+} \theta(q^{b_i+\langle\alpha,z\rangle}; q) \prod_{j=1}^l \prod_{\alpha\in R_2^+} \theta(q^{c_j+\langle\alpha,z\rangle}; q) = 0 \right\},$$

the sum $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ is written as

$$J_{BC_n}(\{b_i\}, \{c_j\}; L; z) = g(z)\frac{\prod_{\alpha\in R_1^+} q^{\left(\frac{s}{2}-1-\sum_{i=1}^s b_i\right)\langle\alpha,z\rangle} \prod_{\alpha\in R_2^+} q^{\left(\frac{l-1}{2}-\sum_{j=1}^l c_j\right)\langle\alpha,z\rangle}}{\prod_{i=1}^s \prod_{\alpha\in R_1^+} \theta(q^{b_i+\langle\alpha,z\rangle}; q) \prod_{j=1}^l \prod_{\alpha\in R_2^+} \theta(q^{c_j+\langle\alpha,z\rangle}; q)},$$

where $g(z)$ is a holomorphic function of $z \in E_{\mathbf{C}}$. By Lemma 3.1, the function $g(z)$ is divided out by the product $\prod_{\alpha\in R_1^+} \theta(q^{\langle2\alpha,z\rangle}; q) \prod_{\alpha\in R_2^+} \theta(q^{\langle\alpha,z\rangle}; q)$. □

Now we consider the case where the holomorphic function $f(z)$ in Proposition 3.2 is a constant not depending on z. As we see in the next theorem, its classification list includes not only two cases for arbitrary n, but three exceptional cases.

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THEOREM 3.3. *For $L = P$ or Q, the sum $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ is expressed as*

$$C_{BC_n}(\{b_i\}, \{c_j\}; L) \prod_{k=1}^{n} \frac{q^{\left(\frac{s}{2}-1-\sum_{i=1}^{s} b_i\right)\langle\varepsilon_k, z\rangle}\theta\left(q^{\langle 2\varepsilon_k, z\rangle}; q\right)}{\prod_{i=1}^{s} \theta(q^{b_i+\langle\varepsilon_k, z\rangle}; q)} \times$$

$$\times \prod_{1 \leqslant \mu < \nu \leqslant n} \frac{q^{\left(l-1-2\sum_{j=1}^{l} c_j\right)\langle\varepsilon_\mu, z\rangle}\theta(q^{\langle\varepsilon_\mu-\varepsilon_\nu, z\rangle}; q)\theta(q^{\langle\varepsilon_\mu+\varepsilon_\nu, z\rangle}; q)}{\prod_{j=1}^{l} \theta(q^{c_j+\langle\varepsilon_\mu-\varepsilon_\nu, z\rangle}; q)\theta(q^{c_j+\langle\varepsilon_\mu+\varepsilon_\nu, z\rangle}; q)},$$

where $C_{BC_n}(\{b_i\}, \{c_j\}; L)$ is a constant not depending on $z \in E_{\mathbf{C}}$, if and only if (s, l) satisfies the following:

n	(s,l)
any	(4,1)
any	$(2n+2,0)$

or

n	(s,l)
2	(0,3)
2	(2,2)
3	(0,2)

.

Remark 3.3.1. In particular, if $n = 1$ in Theorem 3.3, we have $s = 4$ and it is nothing but Bailey's very-well-poised ${}_6\psi_6$ summation formula (see [vD, p. 484]).

Remark 3.3.2. The cases $(s, l) = (4, 1)$ and $(2n + 2, 0)$ were investigated by Gustafson [Gu1, Gu2, Gu3] and van Diejen [vD]. The explicit forms of the constants $C_{BC_n}(\{b_i\}, \{c_j\}; Q)$ are the following:

$$C_{BC_n}(b_1, b_2, b_3, b_4, c_1; Q)$$
$$= (q; q)_\infty^n \prod_{j=1}^{n} \frac{(q^{1-jc_1}; q)_\infty}{(q^{1-c_1}; q)_\infty} \frac{\prod_{1 \leqslant \mu<\nu \leqslant 4}(q^{1-(j-1)c_1-b_\mu-b_\nu}; q)_\infty}{(q^{1-(n+j-2)c_1-b_1-b_2-b_3-b_4}; q)_\infty},$$

$$C_{BC_n}(b_1, \dots, b_{2n+2}; Q) = (q; q)_\infty^n \frac{\prod_{1 \leqslant \mu<\nu \leqslant 2n+2}(q^{1-b_\mu-b_\nu}; q)_\infty}{(q^{1-b_1-\cdots-b_{2n+2}}; q)_\infty}.$$

For C_n-type, the sum $J_{C_n}(\{b_i\}, \{c_j\}; L; z)$ of $(s, l) = (1, 1)$ and $(0, (n + 1)/2)$, n: odd, in [Ito2, p. 334 Theorem 4.5] are the special cases of the cases above.

Remark 3.3.3. In [Gu4], Gustafson studied the *generalized Nasrallah–Rahman integral*. There exist the generalized Nasrallah–Rahman integrals corresponding to the Jackson integrals $J_{BC_n}(\{b_i\}, \{c_j\}; Q; z)$ in the list of Theorem 3.3 (see [Gu4, pp. 447–449]). The explicit forms of the constants $C_{BC_2}(c_1, c_2, c_3; Q)$, $C_{BC_2}(b_1, b_2, c_1, c_2; Q)$ and $C_{BC_3}(c_1, c_2; Q)$ are the following:

$$C_{BC_2}(c_1, c_2, c_3; Q)$$
$$= \frac{(q; q)_\infty^2 (q^{1-c_1-c_2}; q)_\infty (q^{1-c_1-c_3}; q)_\infty (q^{1-c_2-c_3}; q)_\infty}{(q^{-c_1-c_2-c_3}; q)_\infty} \prod_{i=1}^{3} \frac{(q^{1-2c_j}; q)_\infty}{(q^{1-c_j}; q)_\infty},$$

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$$\begin{aligned}
&C_{BC_2}(b_1, b_2, c_1, c_2; Q)\\
&\quad = (q;q)_\infty^2 (q^{1-b_1-b_2};q)_\infty (q^{1-b_1-b_2-c_1};q)_\infty (q^{1-b_1-b_2-c_2};q)_\infty \times\\
&\qquad \times (q^{1-c_1-c_2};q)_\infty (q^{1-2b_1-c_1-c_2};q)_\infty (q^{1-2b_2-c_1-c_2};q)_\infty \times\\
&\qquad \times \frac{(q^{1-b_1-b_2-c_1-c_2};q)_\infty}{(q^{1-2b_1-2b_2-2c_1-2c_2};q)_\infty} \prod_{i=1}^{2} \frac{(q^{1-2c_j};q)_\infty}{(q^{1-c_j};q)_\infty},
\end{aligned}$$

$$\begin{aligned}
&C_{BC_3}(c_1, c_2; Q)\\
&\quad = (q;q)_\infty^3 (q^{1-c_1-2c_2};q)_\infty (q^{1-2c_1-c_2};q)_\infty \times\\
&\qquad \times \frac{(q^{1-c_1-c_2};q)_\infty (q^{1-2c_1-2c_2};q)_\infty}{(q^{-3c_1-3c_2};q)_\infty} \prod_{i=1}^{2} \frac{(q^{1-2c_j};q)_\infty}{(q^{1-c_j};q)_\infty} \frac{(q^{1-3c_j};q)_\infty}{(q^{1-c_j};q)_\infty}.
\end{aligned}$$

Proof of Theorem 3.3. By the q-periodicity (4) of $J_{BC_n}(\{b_i\}, \{c_j\}; L; z)$ and (5), for $\chi \in L$, the function $f(z)$ in Proposition 3.2 satisfies

$$f(z+\chi) = V_\chi(z) f(z), \tag{6}$$

where

$$V_\chi(z) = \prod_{\alpha \in R_1^+} (-1)^{s\langle\alpha,\chi\rangle} q^{(s-4)(-\langle\alpha,\chi\rangle^2/2 - \langle\alpha,z\rangle\langle\alpha,\chi\rangle)} \prod_{\alpha \in R_2^+} (-1)^{(l-1)\langle\alpha,\chi\rangle} q^{(l-1)(-\langle\alpha,\chi\rangle^2/2 - \langle\alpha,z\rangle\langle\alpha,\chi\rangle)}. \tag{7}$$

If $f(x)$ is a constant, $V_\chi(z) = 1$. From (7), it suffices to find (s, l) satisfying

$$s \sum_{\alpha \in R_1^+} \langle\alpha, \chi_i'\rangle + (l-1) \sum_{\alpha \in R_2^+} \langle\alpha, \chi_i'\rangle \equiv 0 \pmod 2 \tag{8}$$

for $i = 1, \dots, n$, and

$$(s-4) \sum_{\alpha \in R_1^+} \langle\alpha, \chi_i'\rangle\langle\alpha, \chi_j'\rangle + (l-1) \sum_{\alpha \in R_2^+} \langle\alpha, \chi_i'\rangle\langle\alpha, \chi_j'\rangle = 0 \tag{9}$$

for $i, j = 1, \dots, n$, where $\{\chi_1', \dots, \chi_n'\}$ is a basis of L. We define positive definite integral symmetric matrices $A_L = (a_{ij})_{i,j=1}^n$ and $B_L = (b_{ij})_{i,j=1}^n$ as

$$a_{ij} = \sum_{\alpha \in R_1^+} \langle\alpha, \chi_i'\rangle\langle\alpha, \chi_j'\rangle, \qquad b_{ij} = \sum_{\alpha \in R_2^+} \langle\alpha, \chi_i'\rangle\langle\alpha, \chi_j'\rangle,$$

Since we see $B_L = (2n-2)A_L$ by [Ito2, p.335 Lemma 4.6], Equation (9) implies that

$$s - 4 + (2n-2)(l-1) = 0. \tag{10}$$

From (10), we have (s, l) as in Theorem 3.3. Equation (8) is valid for such (s, l). □

THEOREM 3.4. *If (s, l) satisfies the condition in Theorem 3.3, then the following relation holds for $L = P$ or Q:*

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$$J_{BC_n}(\{b_i\},\{c_j\};P;z) = 2\ J_{BC_n}(\{b_i\},\{c_j\};Q;z),$$

in particular,

$$C_{BC_n}(\{b_i\},\{c_j\};P) = 2\ C_{BC_n}(\{b_i\},\{c_j\};Q).$$

Proof. We set $\chi_n + Q := \{\chi_n + \lambda;\ \lambda \in Q\}$. Since the lattice P is described as $P = Q \cup (\chi_n + Q)$, by the definition (3) of $J_{BC_n}(\{b_i\},\{c_j\};L;z)$, we have

$$J_{BC_n}(\{b_i\},\{c_j\};P;z) = J_{BC_n}(\{b_i\},\{c_j\};Q;z) + J_{BC_n}(\{b_i\},\{c_j\};Q;z+\chi_n)$$

From the theta product expression of $J_{BC_n}(\{b_i\},\{c_j\};Q;z)$ in Theorem 3.3, it follows that

$$J_{BC_n}(\{b_i\},\{c_j\};Q;z+\chi_n) = J_{BC_n}(\{b_i\},\{c_j\};Q;z).$$

Thus we have $J_{BC_n}(\{b_i\},\{c_j\};P;z) = 2J_{BC_n}(\{b_i\},\{c_j\};Q;z)$. This concludes the proof. □

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