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GENERALIZED STOCHASTIC CONVEXITY AND STOCHASTIC ORDERINGS OF MIXTURES

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In this paper, a new concept called generalized stochastic convexity is introduced as an extension of the classic notion of stochastic convexity. It relies on the well-known concept of generalized convex functions and corresponds to a stochastic convexity with respect to some Tchebycheff system of functions. A special case discussed in detail is the notion of stochastic s-convexity ($s \in \mathbb{N}$), which is obtained when this system is the family of power functions $\{x^0, x^1, \ldots, x^{s-1}\}$. The analysis is made, first for totally positive families of distributions and then for families that do not enjoy that property. Further, integral stochastic orderings, said of Tchebycheff-type, are introduced that are induced by cones of generalized convex functions. For s-convex functions, they reduce to the s-convex stochastic orderings studied recently. These orderings are then used for comparing mixtures and compound sums, with some illustrations in epidemic theory and actuarial sciences.

1. INTRODUCTION

Let us consider a family of random variables $\{X_\theta, \theta \in \Theta\}$ valued in a subset $\mathcal{S}$ of the real line $\mathbb{R}$ and with law P_θ indexed by a single parameter $\theta \in \Theta \subseteq \mathbb{R}$.

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Now, given a function $\phi : \mathcal{S} \to \mathbb{R}$, let us construct the new function ϕ^* defined as

$$\phi^* : \Theta \to \mathbb{R} : \theta \mapsto \phi^*(\theta) \equiv E\phi(X_\theta), \tag{1.1}$$

provided that the expectation exists. A natural question is to which extent some properties of the function ϕ can be transmitted to the function ϕ^*. In other words, given the fact that ϕ belongs to some specific class of functions $\mathcal{F}_1^{\mathcal{S}}$, does it imply that ϕ^* belongs to some remarkable class of functions $\mathcal{F}_2^{\Theta}$?

Such a question is rather general and has been discussed for various problems in probability and statistics. Of special interest here is the work of Shaked and Shanthikumar [14] in which $\mathcal{F}_1^{\mathcal{S}}$ and $\mathcal{F}_2^{\Theta}$ are both classes of (increasing) convex functions. When the property of (increasingness) convexity is transmitted from ϕ to ϕ^*, then the family of laws $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$ is said to be stochastically (increasing) convex (Chap. VI, Shaked and Shanthikumar [15]).

It is well recognized, however, that the standard definition of convexity can be restrictive for various purposes in mathematics. In the field of probabilistic modeling, this arises, for instance, when one wants to compare certain statistics of practical importance. So, the more general concept of s-convex function ($s \in \mathbb{N}$) introduced by Popoviciu [12] is a very useful tool for comparing stochastic models in epidemic theory and actuarial sciences (see some recent works by Lefèvre and Utev [9], and Denuit and Lefèvre [1], Denuit, Lefèvre, and Shaked [2], and Denuit, Lefèvre, and Utev [3]).

The s-convexity itself is a special case of the classic concept of generalized convex functions. This convexity is defined with respect to an arbitrary Tchebycheff system of functions and reduces to the s-convexity when the system is the family of power functions $\{x^0, x^1, \ldots, x^{s-1}\}$. A study of Tchebycheff systems with applications in analysis and statistics is provided in Karlin and Studden [7] (see also Pečarić, Proschan, and Tong [11]).

In the present paper, we precisely investigate the possible transmission of the generalized convexity from a function ϕ to the function ϕ^* given by (1.1). Thus, $\mathcal{F}_1^{\mathcal{S}}$ being a class of generalized convex functions on $\mathcal{S}$, we would like to know whether $\mathcal{F}_2^{\Theta}$ is a class of generalized convex functions on Θ. If this is true, the family of distributions $\mathcal{P}(\Theta)$ is said to be stochastically generalized convex. We will present basic facts concerning the generalized convexity. We will then examine in detail the possible transmission of the more specific property of s-convexity.

The starting point of our approach is the analysis, in Section 3, of the generalized convexity for totally positive families of distributions. In this case, the generalized stochastic convexity for an appropriate Tchebycheff system is rather straightforward. Moreover, the property of stochastic s-convexity is then satisfied when, roughly speaking, each moment of order k in (1.1), $1 \le k \le s-1$, is a polynomial of θ of order k with positive highest coefficient. A few standard distributions will be given for illustration. For this part, we will make an extensive recourse to results given in Karlin [6].

Though the technique based on the total positivity seems to be very efficient, it does not allow us to cover all the desired examples. In Section 4, using direct arguments, we derive complementary results for the stochastic s-convexity which do not follow, surprisingly, from the previous general theory. These will be illustrated with some standard distributions $\mathcal{P}(\Theta)$ that are stochastically s-convex.

Next, we introduce a broad class of integral stochastic orderings, said of Tchebycheff-type, that are induced by cones of generalized convex functions. These orderings are considered implicitly in [7]. In the special case of s-convex functions, they correspond to the s-convex stochastic orderings studied and used recently in the four papers mentioned before. We briefly give some properties of these new orderings. We then focus on the s-convex comparison of mixtures and compound sums, with some illustrations in epidemic theory and actuarial sciences.

It is worth indicating that, for $s = 1$ or 2, further results can be obtained using the alternative concept of sample path convexity (see, e.g., [15], Chap. VI). Whether this approach can be generalized to any integer $s \geq 3$ is an interesting open problem.

2. MATHEMATICAL BACKGROUND

We start by recalling some standard definitions and basic results [6].

2.1. Tchebycheff Systems

A system of linearly independent real-valued functions $\Psi_s = \{\psi_0, \psi_1, \ldots, \psi_s\}$ defined on an ordered subset $\mathcal{S}$ of the real line $\mathbb{R}$ is called a Tchebycheff system (T-system in short) if for all $x_0 < x_1 < \cdots < x_s \in \mathcal{S}$,

$$D\begin{pmatrix} \psi_0, \psi_1, \ldots, \psi_s \\ x_0, x_1, \ldots, x_s \end{pmatrix} \equiv \begin{vmatrix} \psi_0(x_0) & \psi_0(x_1) & \ldots & \psi_0(x_s) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_s(x_0) & \psi_s(x_1) & \ldots & \psi_s(x_s) \end{vmatrix} > 0. \tag{2.1}$$

A system Ψ_s such that $\{\psi_0, \psi_1, \ldots, \psi_k\}$ is a T-system for every $k = 0, 1, \ldots, s$, is called a complete T-system (CT-system).

2.2. Convexity with Respect to a *T*-System

A function $\phi : \mathcal{S} \to \mathbb{R}$ is said to be convex with respect to the T-system Ψ_{s-1} if $\{\psi_0, \psi_1, \ldots, \psi_{s-1}, \phi\}$ is a *weak* T-system, that is, if for all $x_0 < x_1 < \cdots < x_s \in \mathcal{S}$,

$$D\begin{pmatrix} \psi_0, \psi_1, \ldots, \psi_{s-1}, \phi \\ x_0, x_1, \ldots, x_{s-1}, x_s \end{pmatrix} \geq 0. \tag{2.2}$$

The set of the convex functions with respect to Ψ_{s-1} is denoted by $\mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$. Obviously, this is a convex cone closed in the topology of pointwise convergence; it is usually referred to as a cone of generalized convex functions.

A special situation of interest is when the T-system Ψ_{s-1} is the CT-system of functions $\Pi_{s-1} = \{x^0, x, \ldots, x^{s-1}\}$. In this case, $\mathcal{C}^{\mathcal{S}}_{s-cx}(\Pi_{s-1})$ corresponds to the class of the *s-convex functions*; it is denoted by $\mathcal{U}^{\mathcal{S}}_{s-cx}$. If $\mathcal{S}$ is an interval and ϕ has a derivative of degree s (denoted by $\phi^{(s)}$), then

$$\phi \in \mathcal{U}^{\mathcal{S}}_{s-cx} \Leftrightarrow \phi^{(s)} \geq 0.$$

A function $\phi : \mathcal{S} \to \mathbb{R}$ is said to be increasing convex with respect to the *CT*-system Ψ_{s-1} if

$$\phi \in \bigcap_{k=1}^{s} \mathcal{C}^{\mathcal{S}}_{k-cx}(\Psi_{k-1}), \tag{2.3}$$

that is, if $\{\psi_0, \psi_1, \ldots, \psi_k, \phi\}$ is a weak T-system for $k = 0, 1, \ldots, s-1$. The set of such functions is denoted by $\mathcal{C}^{\mathcal{S}}_{s-icx}(\Psi_{s-1})$. In particular, $\mathcal{C}^{\mathcal{S}}_{s-icx}(\Pi_{s-1})$ represents the class of the s-increasing convex functions, denoted by $\mathcal{U}^{\mathcal{S}}_{s-icx}$.

2.3. Total Positivity

A real function (called a kernel) $K(.,.)$ of two variables ranging over linearly ordered subsets Θ and $\mathcal{S}$ of $\mathbb{R}$, respectively, is said to be totally positive of order s (TP_s in short) if for all $\theta_1 < \theta_2 < \cdots < \theta_k \in \Theta$ and $x_1 < x_2 < \cdots < x_k \in \mathcal{S}$, with $k = 1, 2, \ldots, s$,

$$K\begin{pmatrix} \theta_1, \theta_2, \ldots, \theta_k \\ x_1, x_2, \ldots, x_k \end{pmatrix} \equiv \begin{vmatrix} K(\theta_1, x_1) & K(\theta_1, x_2) & \ldots & K(\theta_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ K(\theta_k, x_1) & K(\theta_k, x_2) & \ldots & K(\theta_k, x_k) \end{vmatrix} \geq 0. \tag{2.4}$$

When the subscript "s" is omitted, then the property is understood for all values of s. Some classic examples of *TP* kernels are the exponential, power, triangular, Cauchy, and Gauss kernels.

A function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be a *Pólya function* of order s (PF_s in short) if $K(\theta, x) \equiv \phi(x - \theta)$ is TP_s when $\theta, x \in \mathbb{R}$. For $s = 2$, Pólya functions correspond to log-concave functions.

2.4. Totally Positive Family of Distributions

Let us assume that there exists a sigma-finite dominating measure μ for the family of distributions $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$, and let f_θ denote a density function for the distribution P_θ. The family $\mathcal{P}(\Theta)$ is said to be TP_s (resp. TP) if the kernel $K(\theta, x) \equiv f_\theta(x)$ is TP_s (resp. *TP*s for any s). A density function ϕ is PF_s if $K(\theta, x) \equiv \phi(x - \theta)$ is TP_s when $\theta, x \in \mathbb{R}$.

Every density is TP_1, and the TP_2 densities are those having a monotone likelihood ratio. Various standard distributions are *TP* (e.g., the one-parameter exponential families, the noncentral-χ^2_n and t densities). Moreover, several standard densities are PF_2 (e.g., the normal, gamma, and Weibull densities), but this is not true for the Cauchy law and many others (such as in Lemma 4.12).

In the sequel, we will apply several classic results from the theory of convexity and total positivity. For clarity and brevity reasons, we prefer to refer to them directly in the work and not to recall them explicitly in this section.

3. STOCHASTIC CONVEXITY FOR *TP* FAMILIES OF DISTRIBUTIONS

We are going to point out that the property of generalized stochastic convexity is satisfied by the families of distributions that are totally positive.

3.1. Generalized Stochastic Convexity

Let $\{X_\theta, \theta \in \Theta\}$ be a family of random variables valued in $\mathcal{S} \subseteq \mathbb{R}$, and let $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$ be the family of associated distributions. The question raised is whether the generalized convexity is transmitted from a function ϕ to the function ϕ^* defined in (1.1).

DEFINITION 3.1: *Consider an arbitrary function ϕ that belongs to $\mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$ for some T-system Ψ_{s-1}. If the function $\phi^*: \theta \mapsto \phi^*(\theta) = E\phi(X_\theta)$ belongs to $\mathcal{C}^{\Theta}_{s-cx}(\tilde{\Psi}_{s-1})$ for some T-system $\tilde{\Psi}_{s-1}$ related to ψ_{s-1}, then $\mathcal{P}(\Theta)$ is said to be stochastically convex in the pair $(\Psi_{s-1}, \tilde{\Psi}_{s-1})$. When for a CT-system ψ_{s-1}, $\phi \in \mathcal{C}^{\mathcal{S}}_{s-icx}(\Psi_{s-1})$ implies that $\phi^* \in \mathcal{C}^{\Theta}_{s-icx}(\tilde{\Psi}_{s-1})$ for some CT-system $\tilde{\Psi}_{s-1}$, $\mathcal{P}(\Theta)$ is said to be stochastically increasing convex in $(\Psi_{s-1}, \tilde{\Psi}_{s-1})$.*

From the classic composition formula [6, p. 284] we easily deduce that a sufficient condition for this property is that the family $\mathcal{P}(\Theta)$ is TP_s.

PROPERTY 3.2: *Let Ψ_{s-1} be a T-system (resp. a CT-system). If the family $\mathcal{P}(\Theta)$ is TP_s, then $\mathcal{P}(\Theta)$ is stochastically convex (resp. increasing convex) in the pair $(\Psi_{s-1}, \tilde{\Psi}_{s-1})$, where the functions $\tilde{\psi}_k$, $k = 0,1,\ldots,s-1$, constituting $\tilde{\Psi}_{s-1}$ are given by*

$$\tilde{\psi}_k(\theta) \equiv E\psi_k(X_\theta) = \int_{x\in\mathcal{S}} \psi_k(x) f_\theta(x)\, d\mu(x), \qquad \theta \in \Theta. \tag{3.1}$$

Now, let us examine the three cases indicated below where the parametrization in $\mathcal{P}(\Theta)$ is achieved by shift or convolution of random variables with Pólya density functions. From the definition of PF_s and the closure-type property of PF_s densities by convolution [6, p. 286], we know that $\mathcal{P}(\Theta)$ is TP_s under each of these transformations. Therefore, applying Property 3.2 yields directly the following result.

PROPERTY 3.3: *Let $\{Y, Y_n : n \geq 1\}$ be a sequence of i.i.d. real-valued random variables with PF_s density function. Consider a family of random variables $\{X_\theta, \theta \in \Theta\}$ that is defined by one of the three following transformations:*

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i. $X_\theta = Y + \theta$, *with* $\Theta = \mathbb{R}$;

ii. $X_\theta = \sum_{n=1}^{\theta} Y_n$, *with* $\Theta = \mathbb{N}$ *and under the additional assumption that* $Y \geq 0$ *a.s.*;

iii. $X_\theta = \sum_{n=1}^{N_\theta} Y_n$, *under the assumption that* $Y \geq 0$ *a.s. and where* $\{N_\theta, \theta \in \Theta\}$ *is a family of non-negative integer-valued random variables, independent of the* Y_n*'s, and with* TP_s *distributions.*

Then, the associated family of distributions $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$ *is stochastically (increasing) convex in any pair* $(\Psi_{s-1}, \tilde{\Psi}_{s-1})$.

3.2. Stochastic *s*-Convexity

A particular situation met in many applications is when the T-systems Ψ_{s-1} and $\tilde{\Psi}_{s-1}$ correspond to the family $\Pi_{s-1} = \{1, x, \ldots, x^{s-1}\}$. If the s-convexity is transmitted from ϕ to ϕ^*, that is, if $\phi^* \in \mathcal{U}^{\Theta}_{s-cx}$ (resp. $\mathcal{U}^{\Theta}_{s-icx}$) whenever $\phi \in \mathcal{U}^{\mathcal{S}}_{s-cx}$ (resp. $\mathcal{U}^{\mathcal{S}}_{s-icx}$), then $\{P_\theta, \theta \in \Theta\}$ is said to be stochastically s-convex (resp. s-increasing convex). For $s = 1$ or 2, $\mathcal{P}(\Theta)$ is stochastically increasing or convex in the usual sense [14,15].

The result below follows from Karlin [6, p. 24] and gives a sufficient condition that guarantees this property. It corresponds to a special case of Property 3.2 and will be illustrated with several standard distributions.

PROPERTY 3.4: *Let* $\mathcal{P}(\Theta)$ *be a* TP_s *family such that for* $k = 1, 2, \ldots, s-1$,

$$EQ_k(X_\theta) \equiv \int_{x \in \mathcal{S}} f_\theta(x) Q_k(x)\, d\mu(x) = \tilde{Q}_k(\theta), \qquad \theta \in \Theta, \tag{3.2}$$

where Q_k *is an arbitrary real polynomial of exact degree* k *with positive highest coefficient and* $\tilde{Q}_k$ *is an associated real polynomial of the same type. Then,* $\mathcal{P}(\Theta)$ *is stochastically* s*-(increasing) convex.*

Example 3.5: The following families of distributions are stochastically s-(increasing) convex (for any $s \in \mathbb{N}_0$):

i. the family of the Poisson distributions with mean $\theta \in \mathbb{R}_0^+$;

ii. the family of the continuous uniform distributions on $[0, \theta]$, with parameter $\theta \in \mathbb{R}_0^+$;

iii. the family of the negative binomial distributions with fixed exponent n and with mean $n\theta$, $\theta \in \mathbb{R}_0^+$;

iv. the family of the negative exponential distributions parameterized by its mean $\theta \in \mathbb{R}_0^+$.

PROOF: It is well known that each of these families is *TP*. Thus, it suffices to check that condition (3.2) is satisfied. For the Poisson distribution, we have

$$\sum_{x=0}^{+\infty} e^{-\theta} \frac{\theta^x}{x!} x^k = \sum_{j=0}^{k} S(k, j) \theta^j, \qquad k \in \mathbb{N},$$

where the $S(k,j)$'s are Stirling numbers of the second kind, which are positive, so that (3.2) holds true. This is also the case for the uniform distribution, since

$$\frac{1}{\theta}\int_{x=0}^{\theta} x^k\,dx = \frac{\theta^k}{k!}, \qquad k \in \mathbb{N},$$

for the negative binomial random variable X_θ, since

$$E\binom{X_\theta}{k} = \binom{n+k-1}{k}\theta^k, \qquad k \in \mathbb{N},$$

and for the exponential random variable X_θ, since

$$EX_\theta^k = k!\,\theta^k, \qquad k \in \mathbb{N}. \qquad \blacksquare$$

Example 3.6: Let $\{X_\theta = (\xi_\theta)^a, \theta \in \Theta\}$ where ξ_θ is an exponential random variable with mean θ and a is some positive non-integer real number. This family is *TP*, but it does not transfer moments into polynomials, since

$$EX_\theta^k = \Gamma(ka+1)\theta^{ka}, \qquad k \in \mathbb{N}.$$

By Property 3.2, however, we see that $\mathcal{P}(\Theta)$ is stochastically convex in the pair $(\Pi_{s-1}, \tilde{\Psi}_{s-1})$, with $\tilde{\psi}_k : \theta \mapsto \theta^{ka}$, $k = 0,1,\ldots,s-1$.

4. STOCHASTIC CONVEXITY FOR NON-*TP* FAMILIES OF DISTRIBUTIONS

In this section, we derive complementary results for families of distributions that are not necessarily totally positive.

4.1. Generalized Stochastic Convexity

Hereafter Ψ_{s-1} is assumed to have the following rather general representation [6, p. 276]. Let $\mathcal{S} = [a,b]$, $a,b \in \mathbb{R}$, b possibly infinite, and let $\omega_0, \omega_1, \ldots, \omega_{s-1}$ be positive functions on $\mathcal{S}$ such that $\omega_k \in \mathcal{C}^s(\mathcal{S})$ (i.e., ω_k has a continuous sth derivative in the interior of $\mathcal{S}$). Then, $\psi_0(x) = \omega_0(x)$ and for $k = 1,2,\ldots,s-1$,

$$\psi_k(x) = \omega_0(x)\int_{\xi_1=a}^{x}\omega_1(\xi_1)\int_{\xi_2=a}^{\xi_1}\omega_2(\xi_2)\ldots\int_{\xi_k=a}^{\xi_{k-1}}\omega_k(\xi_k)\,d\xi_k\ldots d\xi_2\,d\xi_1. \tag{4.1}$$

Such a representation does hold true when the ψ_k's possess certain smoothness properties. In this case, $\phi \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$, $s \geq 2$, implies that $\phi \in \mathcal{C}^{s-2}(\mathcal{S})$; for $s = 1$, it implies that ϕ/ψ_0 is nondecreasing on $\mathcal{S}$. Moreover, $\mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1}) \cap \mathcal{C}^s(\mathcal{S})$ is weakly dense in $\mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$. Now, let $\varphi_{0,t}(x)$ be the function $\omega_0(x)$ for $x \geq t$ and 0 otherwise, and when $s \geq 2$, define

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$$\varphi_{s-1,t}(x) = \omega_0(x)\int_{\xi_1=t}^{x}\omega_1(\xi_1)\int_{\xi_2=t}^{\xi_1}\omega_2(\xi_2)\ldots\int_{\xi_{s-1}=t}^{\xi_{s-2}}\omega_{s-1}(\xi_{s-1})\,d\xi_{s-1}\ldots d\xi_2\,d\xi_1 \tag{4.2}$$

for $t \le x$. It is well known that for any t, $\varphi_{s-1,t} \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$, and that any $\phi \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$ can be expanded in terms of the $\varphi_{k,t}$'s, $k = 0,1,\ldots,s-1$ and $t \in \mathcal{S}$ [7, p. 387]. This implies the next result.

Property 4.1: *The family $\mathcal{P}(\Theta)$ is stochastically generalized increasing convex in the pair $(\Psi_{s-1},\tilde{\Psi}_{s-1})$ if and only if $\phi^* \in \mathcal{C}^{\Theta}_{s-icx}(\tilde{\Psi}_{s-1})$ for all $\phi \in \mathcal{C}^{\mathcal{S}}_{s-icx}(\Psi_{s-1}) \cap \mathcal{C}^s(\mathcal{S})$, or, equivalently, for all $\phi \in \{\varphi_{0,a},\varphi_{1,a},\ldots,\varphi_{s-1,a};\varphi_{s-1,t}, t \in \mathcal{S}\}$. $\mathcal{P}(\Theta)$ is stochastically generalized convex in the pair $(\Psi_{s-1},\tilde{\Psi}_{s-1})$ if and only if $\phi^* \in \mathcal{C}^{\Theta}_{s-cx}(\tilde{\Psi}_{s-1})$ for all $\phi \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1}) \cap \mathcal{C}^s(\mathcal{S})$, or, equivalently, for all $\phi \in \{\pm\varphi_{0,a},\pm\varphi_{1,a},\ldots,\pm\varphi_{s-1,a};\varphi_{s-1,t}, t \in \mathcal{S}\}$.*

4.2. Stochastic *s*-Convexity

Property 4.1 leads to the following characterization of the stochastic *s*-(increasing) convexity, which generalizes Theorem 6.A.6 in Shaked and Shanthikumar [15].

Property 4.2: *For $\mathcal{S} = [a,b]$, $a,b \in \mathbb{R}$, b possibly infinite, the family $\mathcal{P}(\Theta)$ is stochastically s-increasing convex if and only if $\phi^* \in \mathcal{U}^{\Theta}_{s-icx}$ for all $\phi \in \mathcal{U}^{[a,b]}_{s-icx} \cap \mathcal{C}^s([a,b]) = \{\phi:[a,b]\to\mathbb{R} \mid \phi^{(k)}(x) \ge 0$ for $k = 1,2,\ldots,s$, $\forall x \in [a,b]\}$, or, equivalently, for all $\phi \in \{(x-a)^k, k = 1,2,\ldots,s-1; (x-t)_+^{s-1}, t \in \mathcal{S}\}$. $\mathcal{P}(\Theta)$ is stochastically s-convex if and only if $\phi^* \in \mathcal{U}^{\Theta}_{s-cx}$ for all $\phi \in \mathcal{U}^{[a,b]}_{s-cx} \cap \mathcal{C}^s([a,b]) = \{\phi:[a,b]\to\mathbb{R} \mid \phi^{(s)}(x) \ge 0\ \forall x \in [a,b]\}$, or, equivalently, for all $\phi \in \{\pm(x-a)^k$, $k = 1,2,\ldots,s-1; (x-t)_+^{s-1}, t \in \mathcal{S}\}$.*

With the composition of two *s*-increasing convex functions giving an *s*-increasing convex function, we directly obtain the following property that points out the preservation of the stochastic *s*-increasing convexity by some transformations.

Property 4.3:

i. *Let $\varphi:\mathcal{S}\to\mathbb{R}$ be a measurable function that belongs to $\mathcal{U}^{\mathcal{S}}_{s-icx}$. If $\{X_\theta,\theta \in \Theta\}$ is stochastically s-increasing convex, then $\{\varphi(X_\theta),\theta \in \Theta\}$ also is.*
ii. *Let $\vartheta:\Theta\to\Theta$ be a function belonging to $\mathcal{U}^{\Theta}_{s-icx}$. If $\{X_\theta,\theta \in \Theta\}$ is stochastically s-increasing convex, then $\{X_{\vartheta(\theta)},\theta \in \Theta\}$ also is.*

The three results below are concerned with cases where the parametrization in $\mathcal{P}(\Theta)$ corresponds to changes of scale or origin. Parts (i) and (iii) extend Examples 6.A.5 and 6.A.12 in [15]. We note that for these parts, the convexity result directly follows from a property (here stronger) of sample path *s*-convexity.

PROPERTY 4.4:

i. *Let Y be a nonnegative random variable, and define* $X_\theta = \theta Y$. *Then,* $\{X_\theta, \theta \in \mathbb{R}\}$ *is stochastically s-(increasing) convex, for any* $s \in \mathbb{N}_0$.
ii. *Let Y be a real-valued random variable with a symmetric density at* 0, *and define* $X_\theta = \theta Y$. *Then,* $\{X_\theta, \theta \in \mathbb{R}^+\}$ *is stochastically* 2*s-(increasing) convex, for any* $s \in \mathbb{N}_0$.
iii. *Let Y be a real-valued random variable, and define* $X_\theta = Y + \theta$. *Then,* $\{X_\theta, \theta \in \mathbb{R}\}$ *is stochastically s-(increasing) convex, for any* $s \in \mathbb{N}_0$.

PROOF: We will only establish (ii). Given $\phi \in \mathcal{U}^{\mathbb{R}}_{2s-icx} \cap \mathcal{C}^{2s}(\mathbb{R})$, we have that for $k = 1, 2, \ldots, s$,

$$\frac{d^{2k}}{d\theta^{2k}} \phi^*(\theta) = E[Y^{2k}\phi^{(2k)}(X_\theta)] \geq 0,$$

and

$$\frac{d^{2k-1}}{d\theta^{2k-1}} \phi^*(\theta) = E\{Y^{2k-1}[\phi^{(2k-1)}(X_\theta) - \phi^{(2k-1)}(0)]\} + \phi^{(2k-1)}(0)EY^{2k-1} \geq 0,$$

since $EY^{2k-1} = 0$ (Y being symmetric about 0) and $y_{2k-1}[\phi^{(2k-1)}(\theta y) - \phi^{(2k-1)}(0)] \geq 0$ for all $y \in \mathbb{R}$ (because $\phi^{(2k)} \geq 0$ and $\theta \in \mathbb{R}^+$). Thus, by Property 4.2 we deduce (ii). ∎

Example 4.5: The family of the normal distributions with mean 0 and with standard deviation $\sigma \in \mathbb{R}^+$ is stochastically 2*s*-(increasing) convex, while the family of the normal distributions with mean $\mu \in \mathbb{R}$ and with fixed standard deviation σ is stochastically *s*-(increasing) convex.

We underline that in Property 4.4 (iii) it is not assumed, as it is in Property 3.3 (i), that Y has a PF_s density function. In other words, working with the stochastic *s*-(increasing) convexity leads to a stronger result which does not follow from the previous general theory. This will be supported by Lemma 4.12 (i).

In the next property (which extends Example 6.A.3 in Shaked and Shanthikumar [15]), θ is assumed to take values in $\mathbb{N}$. We recall [3] that $\mathcal{U}^{\mathbb{N}}_{s-cx}$ can be defined equivalently as

$$\mathcal{U}^{\mathbb{N}}_{s-cx} = \{\phi : \mathbb{N} \to \mathbb{R} | \Delta^s \phi(i) \geq 0 \quad \text{for all } i \in \mathbb{N}\}, \tag{4.3}$$

where Δ^s is the *s*th iterated of the forward difference operator Δ, defined for a function $\phi : \mathbb{N} \to \mathbb{R}$ by $\Delta\phi(i) = \phi(i+1) - \phi(i)$.

PROPERTY 4.6: *Let* $\{Y_n, n \geq 1\}$ *be a sequence of i.i.d. random variables valued in* $\mathcal{S} = \mathbb{N}$ *or* $\mathbb{R}^+$, *and define* $X_\theta = \sum_{n=1}^{\theta} Y_n$. *Then,* $\{X_\theta, \theta \in \mathbb{N}\}$ *is stochastically s-(increasing) convex.*

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PROOF: The argument is inspired from Rolski [13, Ex. 2.6]. Let us consider, for instance, the s-convex case when $\mathcal{S}=\mathbb{R}^+$. Let F be the distribution function of Y_1 and let $F^{*(\theta)}$ be the θth convolution of F. We have to show that the function ϕ^* given by

$$\phi^*:\mathbb{N}\to\mathbb{R}:\theta\mapsto\phi^*(\theta)\equiv E\phi\left(\sum_{n=1}^{\theta}Y_n\right)=\int_{x\in\mathbb{R}^+}\phi(x)\,dF^{*(\theta)}(x)$$

belongs to $\mathcal{U}_{s-cx}^{\mathbb{N}}$ for all $\phi\in\mathcal{U}_{s-cx}^{\mathbb{R}^+}$. Define a function φ on $\mathbb{R}^+$ by

$$\varphi(t)=\int_{x\in\mathbb{R}^+}\phi(x+t)\,dF^{*(\theta)}(x).$$

Since $\phi\in\mathcal{U}_{s-cx}^{\mathbb{R}^+}$, we directly see that $\varphi\in\mathcal{U}_{s-cx}^{\mathbb{R}^+}$. Denote by Δ_h the forward difference operator with increment h. Using a well-known result (see, e.g., [11, Formula (1.4.1)]), we then get that for all $h_1,h_2,\dots,h_s\in\mathbb{R}^+$,

$$(\Delta_{h_1}\Delta_{h_2}\dots\Delta_{h_s})\varphi(0)=\int_{\xi_1=0}^{h_1}\int_{\xi_2=\xi_1}^{\xi_1+h_2}\cdots\int_{\xi_s=\xi_{s-1}}^{\xi_{s-1}+h_s}\varphi^{(s)}(\xi_s)\,d\xi_1\,d\xi_2\dots d\xi_s\geq 0. \tag{4.4}$$

Now, we notice that $\phi^*(\theta)=\varphi(0)$, and it is easily checked that $\Delta^s\phi^*(\theta)$ can be expressed as

$$\Delta^s\phi^*(\theta)=\int_{h_1\in\mathbb{R}^+}\int_{h_2\in\mathbb{R}^+}\cdots\int_{h_s\in\mathbb{R}^+}(\Delta_{h_1}\Delta_{h_2}\dots\Delta_{h_s})\varphi(0)\,dF(h_1)\,dF(h_2)\dots dF(h_s). \tag{4.5}$$

Therefore, from Eqs. (4.4) and (4.5), we deduce that $\Delta^s\phi^*(\theta)\geq 0$ as required. ■

Example 4.7: The following families of distributions are stochastically s-(increasing) convex:

i. the family of the chi-square distributions with parameter $n\in\mathbb{N}$;
ii. the family of the binomial distributions with parameter $n\in\mathbb{N}$ and with mean np;
iii. the family of the negative binomial distributions with parameter $n\in\mathbb{N}$ and with mean $n\theta$.

Here too, we indicate that Property 4.6 does not rely, as does Property 3.3 (ii), on the assumption that the Y_n's have a PF_s density function. See Lemma 4.12 (ii) below for a supporting example.

Property 4.6 together with Property 4.3 (i) allow us to state the following result, which generalizes Theorem 6.6.6 in Shaked and Shanthikumar [15].

PROPERTY 4.8: *Let $\{Y_n,n\geq 1\}$ be a sequence of i.i.d. random variables valued in $\mathcal{S}=\mathbb{N}$ or $\mathbb{R}^+$. Let $\{N_\theta,\theta\in\Theta\}$ be a family of non-negative integer-valued random*

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variables, independent of the Y_n's, and define $X_\theta = \sum_{n=1}^{N_\theta} Y_n$. If $\{N_\theta, \theta \in \Theta\}$ is stochastically s-increasing convex, then $\{X_\theta, \theta \in \Theta\}$ also is.

Example 4.9: The family of the compound Poisson sums with parameter $\theta \in \mathbb{R}^+$, for random variables valued in $\mathbb{N}$ or $\mathbb{R}^+$, is stochastically s-increasing convex.

By comparison with Property 3.3 (iii), we observe again that Property 4.8 does not ask that the Y_n's have a PF_s density function. This is important, in particular, for the compound Poisson sums; see Lemma 4.12 (iii) below.

Sometimes it is simpler to check the stochastic convexity by a direct argument rather than by a general technique. Here are two illustrations.

Example 4.10: The family of the binomial distributions with fixed exponent n and with mean np, $p \in [0,1]$, is stochastically s-(increasing) convex. Indeed, it is easily shown that

$$\frac{d^k}{dp^k}\,\phi^*(p) = \frac{n!}{(n-k)!}\,E\Delta^k\phi(X_{n-k}), \qquad 1 \le k \le n,$$

so that $\phi \in \mathcal{U}_{s-(i)cx}^{\mathcal{D}_n}$ obviously implies that $\phi^* \in \mathcal{U}_{s-(i)cx}^{[0,1]}$.

Example 4.11: The family of the discrete uniform distributions on $\{0,1,\ldots,n\}$, with parameter $n \in \mathbb{N}_0$, is stochastically s-(increasing) convex. This is a consequence of the following formula, which can be proved by induction,

$$(n+s+1)\binom{n+s}{s}\Delta^s\phi^*(n) = \sum_{k=0}^{n}\binom{k+s}{s}\Delta^s\phi(k).$$

The following lemma illustrates that when dealing with the stochastic s-(increasing) convexity, rather than with the generalized stochastic convexity, it is possible to relax the *TP* hypothesis for certain results (as stated before).

LEMMA 4.12: *Let $\{Y, Y_n : n \ge 1\}$ be a sequence of i.i.d. real-valued random variables with a density function $f(x) = 0.5$ when $x \in [0,1] \cup [2,3]$. Consider a family of random variables $\{X_\theta, \theta \in \Theta\}$ that is defined by one of the three following transformations*:

i. $X_\theta = Y + \theta$, *with* $\Theta = \mathbb{R}$;
ii. $X_\theta = \sum_{n=1}^{\theta} Y_n$, *with* $\Theta = \mathbb{N}$;
iii. $X_\theta = \sum_{n=1}^{N_\theta} Y_n$ *with* $\Theta = \mathbb{R}^+$ *and where* $\{N_\theta, \theta \in \mathbb{R}^+\}$ *is a family of Poisson random variables with parameter* θ, *independent of the* Y_n*'s.*

Then, the associated family of distributions $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$ is stochastically s-(increasing) convex but not TP_2.

PROOF:

Case (i): By Property 4.4 (iii), $\{X_\theta, \theta \in \mathbb{R}\}$ is stochastically s-(increasing) convex. Let us show that the kernel $K(\theta, x) = f(x-\theta)$ is not TP_2, so that f is not *PF*. First, take $0 < \theta_1 < x_1 < 1$ and $2 < \theta_2 < x_2 < 3$, yielding $x_1 - \theta_2 < 0$ and thus

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$$K\begin{pmatrix}\theta_1, & \theta_2\\ x_1, & x_2\end{pmatrix} = f(x_1-\theta_1)f(x_2-\theta_2) = \frac{1}{4} > 0.$$

Now, for $\theta_1 < x_1 - 1$, $1 < \theta_2 < x_1 < 2$, $2 + \theta_1 < x_2 < 3$, we have $1 < x_1 - \theta_1 < 2$ and thus

$$K\begin{pmatrix}\theta_1, & \theta_2\\ x_1, & x_2\end{pmatrix} = -f(x_2-\theta_1)f(x_1-\theta_2) = -\frac{1}{4} < 0.$$

Case (ii): By Property 4.6, $\{X_\theta, \theta \in \mathbb{N}\}$ is stochastically s-(increasing) convex. We now prove that the kernel

$$K(\theta,x) = f^{*(\theta)}(x), \qquad \theta \in \mathbb{N}, x \in \mathbb{R}^+,$$

is not TP_2. Indeed, for $x_1 < x_2$,

$$K\begin{pmatrix}1, & 2\\ x_1, & x_2\end{pmatrix} = f(x_1)f^{*(2)}(x_2) - f(x_2)f^{*(2)}(x_1), \tag{4.6}$$

and we observe that $f^{*(2)}(t) > 0$ for all $t \in [0,6]$, $t \neq 0,2,4,6$. Thus, for $0 < x_1 < 1$ and $1 < x_2 < 2$, (4.6) is equal to $f(x_1)f^{*(2)}(x_2) > 0$, while for $1 < x_1 < 2$ and $2 < x_2 < 3$, (4.6) is equal to $-f(x_2)f^{*(2)}(x_1) < 0$.

Case (iii): By Property 4.8, $\{X_\theta, \theta \in \mathbb{R}^+\}$ is stochastically s-(increasing) convex. The corresponding kernel is given by

$$K(\theta,x) = \sum_{n=1}^{+\infty} P(N_\theta = n) f^{*(n)}(x), \qquad \theta \in \mathbb{R}^+, x \in \mathbb{R}^+,$$

and we now check that it is not TP_2. Clearly, we have for θ, h, x_1 and $x_2 > 0$,

$$\begin{aligned}\frac{1}{h}K\begin{pmatrix}\theta, & \theta+h\\ x_1, & x_2\end{pmatrix} &= K(\theta,x_1)\,\frac{K(\theta+h,x_2)-K(\theta,x_2)}{h}\\ &\quad - K(\theta,x_2)\,\frac{K(\theta+h,x_1)-K(\theta,x_1)}{h}.\end{aligned} \tag{4.7}$$

Putting

$$T(\theta,x_1,x_2) = K(\theta,x_1)\,\frac{\partial K(\theta,x_2)}{\partial\theta} - K(\theta,x_2)\,\frac{\partial K(\theta,x_1)}{\partial\theta}, \tag{4.8}$$

we obtain that

$$\begin{aligned}T(\theta,x_1,x_2) &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} f^{*(n)}(x_1)f^{*(m)}(x_2)\\ &\quad \times \left\{P(N_\theta=n)\,\frac{d}{d\theta}\,P(N_\theta=m) - P(N_\theta=m)\,\frac{d}{d\theta}\,P(N_\theta=n)\right\}\\ &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} f^{*(n)}(x_1)f^{*(m)}(x_2)P(N_\theta=n)P(N_\theta=m)\,\frac{m-n}{\theta},\end{aligned}$$

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which yields

$$\frac{1}{\theta^2} T(\theta, x_1, x_2) = \frac{e^{-2\theta}}{2} [f(x_1) f^{*(2)}(x_2) - f(x_2) f^{*(2)}(x_1)] + o(\theta). \quad \textbf{(4.9)}$$

Now, by (4.7) and (4.8), $K(\theta, x)$ TP_2 would imply that $T(\theta, x_1, x_2)$ is always nonnegative, but this does not hold true from (4.9) and (ii) above. ■

5. STOCHASTIC ORDERINGS OF MIXTURES

We start by introducing integral stochastic orderings that are induced by cones of generalized convex functions.

DEFINITION 5.1: *Consider two random variables X and Y valued in* $\mathcal{S} \subseteq \mathbb{R}$. *Given a T-system* Ψ_{s-1} *of the form* (4.1), *X is said to be smaller than Y in the* Ψ_{s-1}*-Tchebycheff ordering, denoted by* $X \preceq^{\mathcal{S}}_{\Psi_{s-1}-cx} Y$, *when* $E\phi(X) \le E\phi(Y)$ *for all* $\phi \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\psi_{s-1})$ *for which the expectations exist. Given a CT-system* Ψ_{s-1} *of the form* (4.1), *X is said to be smaller than Y in the* Ψ_{s-1}*-increasing Tchebycheff ordering, denoted by* $X \preceq^{\mathcal{S}}_{\Psi_{s-1}-icx} Y$, *when* $E\phi(X) \le E\phi(Y)$ *for all* $\phi \in \mathcal{C}^{\mathcal{S}}_{s-icx}(\Psi_{s-1})$ *for which the expectations exist.*

These orderings have been considered implicitly in [7, Chap. IV, Sect. 5]. In the particular case, where Ψ_{s-1} is the family Π_{s-1}, they correspond to the stochastic s-convex and s-increasing convex orderings, denoted by $\preceq^{\mathcal{S}}_{s-cx}$ and $\preceq^{\mathcal{S}}_{s-icx}$, respectively.

Note that since $\pm\psi_k \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1})$ for $k = 0,1,\dots,s-1$, we have

$$X \preceq^{\mathcal{S}}_{\Psi_{s-1}-cx} Y \Rightarrow E\psi_k(X) = E\psi_k(Y) \quad \text{for } k = 0,1,\dots,s-1. \quad \textbf{(5.1)}$$

In other words, the ordering $\preceq^{\mathcal{S}}_{\Psi_{s-1}-cx}$ can only be used to compare random variables such that the expectations of their transformations by $\psi_0, \psi_1, \dots, \psi_{s-1}$ are identical. For the ordering $\preceq^{\mathcal{S}}_{s-cx}$, the constraint is that the random variables have the same $s-1$ first moments.

This observation highlights natural reasons for generalizing $\preceq^{\mathcal{S}}_{s-cx}$ into $\preceq^{\mathcal{S}}_{\Psi_{s-1}-cx}$. Indeed, for some random variable Z, the moments EZ^k may not exist but, for instance, all the expectations $E[Z^k \exp(-Z)]$ do exist. Thus, $\preceq^{\mathcal{S}}_{s-cx}$ is not applicable but one could consider $\preceq^{\mathcal{S}}_{\Psi_{s-1}-cx}$, where $\psi_k(x) = x^k \exp(-x)$, $k = 0,1,\dots,s-1$. Furthermore, in probabilistic modeling, the comparison of models often leads us to fix certain expected values. These quantities, however, are not always the moments of some random variable Z, but can represent the expectation of certain functions, more complex, of Z.

As with Property 4.1, a density argument yields the following characterization of $\preceq^{\mathcal{S}}_{\Psi_{s-1}-cx}$.

PROPERTY 5.2: *For two random variables X and Y valued in* $\mathcal{S}$, $X \preceq^{\mathcal{S}}_{\Psi_{s-1}-icx} Y$ *if and only if* $E\phi(X) \le E\phi(Y)$ *for all* $\phi \in \mathcal{C}^{\mathcal{S}}_{s-icx}(\Psi_{s-1}) \cap \mathcal{C}^s(\mathcal{S})$ *for which the expectations exist, or, equivalently,*

$$X \preceq^{\mathcal{S}}_{\Psi_{s-1}-icx} Y \Leftrightarrow \begin{cases} E\varphi_{k,a}(X) \le E\varphi_{k,a}(Y) & \text{for } k = 0,1,\ldots,s-1, \\ E\varphi_{s-1,t}(X) \le E\varphi_{s-1,t}(Y) & \text{for all } t \in \mathcal{S}. \end{cases}$$

$X \preceq^{\mathcal{S}}_{\Psi_{s-1}-cx} Y$ *if and only if* $E\phi(X) \le E\phi(Y)$ *for all* $\phi \in \mathcal{C}^{\mathcal{S}}_{s-cx}(\Psi_{s-1}) \cap \mathcal{C}^s(\mathcal{S})$ *for which the expectations exist*, *or*, *equivalently*,

$$X \preceq^{\mathcal{S}}_{\psi_{s-1}-cx} Y \Leftrightarrow \begin{cases} E\varphi_{k,a}(X) = E\varphi_{k,a}(Y) & \text{for } k = 0,1,\ldots,s-1, \\ E\varphi_{s-1,t}(X) \le E\varphi_{s-1,t}(Y) & \text{for all } t \in \mathcal{S}. \end{cases}$$

To check the possible existence of a $\preceq^{\mathcal{S}}_{\Psi_{s-1}-cx}$ ordering, it is convenient to have a condition of crossing-type between the distributions. Let $S^-(\phi)$ denote the number of sign changes of the function ϕ on its domain. The same argument as in [2] for the s-convex ordering yields the sufficient condition below (see also [7, p. 407]).

Property 5.3: *For two random variables* X *and* Y *valued in* $\mathcal{S}$, $X \preceq^{\mathcal{S}}_{\Psi_{s-1}-cx} Y$ *if* $E\psi_k(X) = E\psi_k(Y)$ *for* $k = 0,1,\ldots,s-1$, *and if either the distribution functions satisfy* $S^-(F_X - F_Y) = s-1$ *and* $F_X \ge F_Y$ *near* ∞, *or the density functions satisfy* $S^-(f_X - f_Y) = s$ *and* $f_Y \ge f_X$ *near* ∞.

Now, these concepts of generalized stochastic convexity and stochastic orderings of Tchebycheff-type allow us to deduce directly a rather general result for the comparison of mixtures.

Property 5.4: *Let* $\{X_\theta, \theta \in \Theta\}$ *be a family of random variables valued in* $\mathcal{S}$ *and with law* P_θ. *Let* X_Λ *denote a random variable distributed as a mixture of these* X_θ*'s with mixing law* Λ, *that is*,

$$P(X_\Lambda \le x) = \int_{\theta\in\Theta} P(X_\theta \le x)\, dP(\Lambda \le \theta), \qquad x \in \mathcal{S}.$$

If the family $\mathcal{P}(\Theta) = \{P_\theta, \theta \in \Theta\}$ *is stochastically* (*increasing*) *convex in the pair* $(\Psi_{s-1}, \tilde{\Psi}_{s-1})$, *then*

$$\Lambda_1 \preceq^{\Theta}_{\tilde{\Psi}_{s-1}-(i)cx} \Lambda_2 \Rightarrow X_{\Lambda_1} \preceq^{\mathcal{S}}_{\Psi_{s-1}-(i)cx} X_{\Lambda_2}. \tag{5.2}$$

In particular, *if* $\mathcal{P}(\Theta)$ *is stochastically* s-(*increasing*) *convex*, *then*

$$\Lambda_1 \preceq^{\Theta}_{s-(i)cx} \Lambda_2 \Rightarrow X_{\Lambda_1} \preceq^{\mathcal{S}}_{s-(i)cx} X_{\Lambda_2}. \tag{5.3}$$

Example 5.5: Let $X_\Lambda = (\xi_\Lambda)^a$ where ξ_Λ is a mixed exponential random variable with random mean Λ as mixing parameter and a is some positive non-integer real number. From Property 5.4 and Example 3.6, we deduce that

$$\Lambda_1 \preceq^{\mathbb{R}^+}_{\tilde{\Psi}_{s-1}-(i)cx} \Lambda_2 \Rightarrow X_{\Lambda_1} \preceq^{\mathbb{R}^+}_{s-(i)cx} X_{\Lambda_2}, \tag{5.4}$$

where $\tilde{\Psi}_{s-1} = \{1, x^a, x^{2a}, \ldots, x^{(s-1)a}\}$.

Example 5.6: Let X_Λ be a mixed Poisson random variable with random mean Λ as mixing parameter. From Property 5.4 and Example 3.5 (i), we get that (5.3) holds true. A similar s-(increasing) convex ordering is valid for the mixed random variables built from the parametric distributions discussed in Examples 3.5 (ii)–(iv), 4.5, 4.7 (i)–(iii), 4.9, 4.10, and 4.11.

Combining Properties 4.6 and 5.4 yields the following comparison result for compound sums.

PROPERTY 5.7: *Let $\{Y_n, n \geq 1\}$ be a sequence of i.i.d. random variables valued in $\mathcal{S} = \mathbb{N}$ or $\mathbb{R}^+$, and let N be some non-negative integer-valued random variable independent of the Y_n's. Then,*

$$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2 \Rightarrow \sum_{n=1}^{N_1} Y_n \preceq^{\mathcal{S}}_{s-(i)cx} \sum_{n=1}^{N_2} Y_n.$$

Moreover, let $\{Z_n, n \geq 1\}$ be another sequence of i.i.d. random variables of the same type and such that $Y_n \preceq^{\mathcal{S}}_{s-(i)cx} Z_n$ for all $n \geq 1$. Then,

$$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2 \Rightarrow \sum_{n=1}^{N_1} Y_n \preceq^{\mathcal{S}}_{s-(i)cx} \sum_{n=1}^{N_2} Z_n.$$

Extrema with respect to the s-convex orderings have been derived for discrete random variables [1] and for continuous random variables [2]. Using Property 5.4, the latter extrema can allow us to approximate mixed distributions when only the moments $E\Lambda^k$, $k = 1,2,\ldots,s-1$, of the mixing parameter are known. By Property 5.7, the former extrema can provide approximations to compound sums when only the moments EN^k, $k = 1,2,\ldots,s-1$, of the number of terms are fixed. For modelling problems (as in the two illustrations below), such approximations are useful when only partial information on some components of the model are available.

Illustration 5.8 (Carrier-borne epidemic model): Let us consider a closed community subdivided initially into n susceptibles and m carriers. Each carrier j, say, is infectious during a random period of time of length T_j. During that period, the carrier can contact any given susceptible according to a Poisson process with rate β. All the infectious periods and contact processes are independent. A susceptible, if ever contacted, is immediately detected and removed from the population. Let $S(t)$ denote the number of susceptibles still present at time t, $t \geq 0$. We easily see that the probability $Q_j(t)$ that any given susceptible escapes contacts with carrier j until time t is given by $Q_j(t) = \exp[-\beta \min(t, T_j)]$. Therefore, $S(t)$ has a mixed binomial distribution with fixed exponent n and with random parameter $\Pi_{j=1}^m Q_j(t)$ as mixing parameter.

Now, let us assume that the initial number of susceptibles, for instance, is not known with precision, which is rather frequent in practice. Therefore, we replace the constant n above by some random variable N, say. The dependence on N is marked by writing $S_N(t)$. From Examples 4.7 and 5.6, we then deduce that

$$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2 \Rightarrow S_{N_1}(t) \preceq^{\mathbb{N}}_{s-(i)cx} S_{N_2}(t). \tag{5.5}$$

This result extends a property given in Malice and Lefèvre [10]. We mention that the effects of heterogeneity in carrier-borne epidemics have been investigated by Lefèvre and Malice [8] and Tong [16].

Illustration 5.9 (Collective risk model in actuarial sciences): Let us consider an insurer who manages a homogeneous portfolio of n risks over a given period of time. During that period, each policyholder i, say, can have a claim with probability p, which is represented by the indicator variable $I_i(p)$. If claim i occurs, its amount is of random level Y_i (valued in $\mathbb{R}^+$). The indicators $I_i(p)$ are independent, and the amounts Y_i are i.i.d. and independent of the $I_i(p)$'s. Then, the total claim amount, denoted by X_p, is given by

$$X_p = \sum_{i=1}^{n} Y_i I_i(p). \tag{5.6}$$

Now, let us assume that the probability p is no longer a constant but corresponds to some random variable π, say, valued in $[0,1]$. This can be used to translate a possible variability effect in the claim occurrences of the portfolio. Let X_π denote the associated total claim amount. Clearly, X_π can be expressed as

$$X_\pi = \sum_{i=1}^{N} Y_i, \tag{5.7}$$

where the random variable N has a mixed binomial distribution with fixed exponent n and with random parameter π as mixing parameter. From Examples 4.10 and 5.6 and Property 5.7, we then obtain that

$$\pi_1 \preceq_{s-(i)cx}^{[0,1]} \pi_2 \Rightarrow X_{\pi_1} \preceq_{s-(i)cx}^{\mathbb{R}^+} X_{\pi_2}. \tag{5.8}$$

For a treatment of actuarial models and methods with various comparison problems, see De Vylder [4] and Goovaerts, Kaas, Van Heerwaarden, and Bauwelinckx [5].

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