Ergod. Th. & Dynam. Sys., (2022), **42**, 1898–1907
doi:10.1017/etds.2021.18

Continuity of Hausdorff dimension across generic dynamical Lagrange and Markov spectra II

ALINE CERQUEIRA†, CARLOS G. MOREIRA‡ and SERGIO ROMAÑA§

† *IMPA, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina, Jardim Botânico, Rio de Janeiro, RJ, CEP 22460-320, Brazil*
(e-mail: alineagc@gmail.com)

‡ *IMPA, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina, Jardim Botânico, Rio de Janeiro, RJ, CEP 22460-320, Brazil*
(e-mail: gugu@impa.br)

§ *UFRJ, Universidade Federal do Rio de Janeiro, Av. Athos da Silveira Ramos 149, Centro de Tecnologia (Bloco C), Cidade Universitária, Ilha do Fundão, Rio de Janeiro, RJ, CEP 21941-909, Brazil*
(e-mail: sergiori@im.ufrj.br)

(*Received* 6 *May* 2019 *and accepted in revised form* 15 *February* 2021)

Abstract. Let g_0 be a smooth pinched negatively curved Riemannian metric on a complete surface N, and let Λ_0 be a basic hyperbolic set of the geodesic flow of g_0 with Hausdorff dimension strictly smaller than two. Given a small smooth perturbation g of g_0 and a smooth real-valued function f on the unit tangent bundle to N with respect to g, let $L_{g,\Lambda,f}$ (respectively $M_{g,\Lambda,f}$) be the Lagrange (respectively Markov) spectrum of asymptotic highest (respectively highest) values of f along the geodesics in the hyperbolic continuation Λ of Λ_0. We prove that for generic choices of g and f, the Hausdorff dimensions of the sets $L_{g,\Lambda,f} \cap (-\infty, t)$ vary continuously with $t \in \mathbb{R}$ and, moreover, $M_{g,\Lambda,f} \cap (-\infty, t)$ has the same Hausdorff dimension as $L_{g,\Lambda,f} \cap (-\infty, t)$ for all $t \in \mathbb{R}$.

Key words: flows on 3-manifolds, Hausdorff dimension, horseshoes, Lagrange spectrum, Markov spectrum
2020 Mathematics Subject Classification: 37D40, 11J06 (Primary); 28A78 (Secondary)

1. *Introduction*

The first paper of this series [**CMM**] discussed the continuity properties of the Hausdorff dimension across dynamical Lagrange and Markov spectra of surface diffeomorphisms. In this article, our goal is to extend the results in [**CMM**] to the case of geodesic flows of negatively curved Riemannian surfaces.

1.1. *Dynamical Markov and Lagrange spectra.* Let M be a smooth manifold, $T = \mathbb{Z}$ or $\mathbb{R}$, and let $\phi = (\phi^t)_{t\in T}$ be a discrete-time ($T = \mathbb{Z}$) or continuous-time ($T = \mathbb{R}$) smooth dynamical system on M, that is, $\phi^t : M \to M$ are smooth diffeomorphisms, $\phi^0 = \mathrm{id}$, and $\phi^t \circ \phi^s = \phi^{t+s}$ for all $t, s \in T$.

Given a compact invariant subset $\Lambda \subset M$ and a function $f : M \to \mathbb{R}$, we define the *dynamical Markov (respectively Lagrange) spectrum* $M_{\phi,\Lambda,f}$ (respectively $L_{\phi,\Lambda)f}$ as

$$M_{\phi,\Lambda,f} = \{m_{\phi,f}(x) : x \in \Lambda\}, \quad \text{respectively} \quad L_{\phi,\Lambda,f} = \{\ell_{\phi,f}(x) : x \in \Lambda\},$$

where

$$m_{\phi,f}(x) := \sup_{t\in T} f(\phi^t(x)), \quad \text{respectively} \quad \ell_{\phi,f}(x) := \limsup_{t\to+\infty} f(\phi^t(x)).$$

Remark 1.1. An elementary compactness argument (cf. Remark in §3 of [**MoRo**]) shows that

$$\{\ell_{\phi,f}(x) : x \in A\} \subset \{m_{\phi,f}(x) : x \in A\} \subset f(A)$$

whenever $A \subset M$ is a compact ϕ-invariant subset.

1.2. *Statement of the main result.* In this paper, we study the fractal geometry of $M_{\phi,\Lambda,f} \cap (-\infty, t)$ and $L_{\phi,\Lambda,f} \cap (-\infty, t)$ as $t \in \mathbb{R}$ varies in the context of geodesic flows on negatively curved Riemannian surfaces.

More precisely, let N be a complete surface, let g_0 be a smooth (C^r, $r \geq 4$) pinched negatively curved Riemannian metric on N, that is, the curvature is bounded above and below by two negative constants. Let $\phi_{g_0} = (\phi^t_{g_0})_{t\in\mathbb{R}}$ be the geodesic flow on the unit tangent bundle $M = S_{g_0}N$ of N with respect to g_0. Consider a horseshoe Λ_0 of ϕ_{g_0} with Hausdorff dimension $\dim(\Lambda_0) < 2$ (throughout this paper, dim(A) denotes the Hausdorff dimension of set A). Denote by $\mathcal{U}$ a small (C^r, $r \geq 4$) neighborhood of g_0 such that Λ_0 admits a hyperbolic continuation Λ for all $g \in \mathcal{U}$.

THEOREM 1.2. *If $\mathcal{U}$ is sufficiently small, then there exists a Baire residual subset $\mathcal{U}^* \subset \mathcal{U}$ with the following property. For every $g \in \mathcal{U}^*$, there exists a Baire residual subset $\mathcal{H}_{\phi_g,\Lambda} \subset C^s(S_gN, \mathbb{R})$, $s \geq 4$, such that the function*

$$t \mapsto \dim(L_{\phi_g,\Lambda,f} \cap (-\infty, t))$$

is continuous and

$$\dim(L_{\phi_g,\Lambda,f} \cap (-\infty, t)) = \dim(M_{\phi_g,\Lambda,f} \cap (-\infty, t)) \quad \textit{for all } t \in \mathbb{R}$$

whenever $f \in \mathcal{H}_{\phi_g,\Lambda}$.

2. *Proof of the main result*

Morally speaking, our proof of Theorem 1.2 consists of a reduction to the context of the first paper of this series [**CMM**].

2.1. *Dimension reduction via Poincaré maps.* The notion of *good cross sections* was exploited in [MoRo] to describe the dynamics of ϕ_g on Λ (for $g \in \mathcal{U}$) in terms of Poincaré maps. More precisely, they constructed a finite number of disjoint smooth (C^r, $r \geq 3$) cross sections Σ_i, $1 \leq i \leq k$, of the flow ϕ such that the ϕ-orbit of any point of Λ intersects $\Theta := \bigsqcup_{i=1}^k \Sigma_i$, the subset $K := \Lambda \cap \Theta$ is disjoint from the boundary $\partial\Theta := \bigsqcup_{i=1}^k \partial\Sigma_i$, and K is a horseshoe of the Poincaré (first return) map $\mathcal{R} : D_{\mathcal{R}} \to \Theta$ from a neighborhood $D_{\mathcal{R}} \subset \Theta$ of K to Θ sending $y \in D_{\mathcal{R}}$ to the point $\mathcal{R}(y) = \phi^{t_+(y)}(y)$ where the forward ϕ-orbit of y first hits Θ.

The relation between the Hausdorff dimensions of K and Λ is described by the following lemma (compare with Lemma 14 in [MoRo]).

LEMMA 2.1. *In the previous setting, one has* $\dim(\Lambda) = \dim(K) + 1$.

Proof. We cover Λ with a finite number of tubular neighborhoods U_l, $1 \leq l \leq m$, of compact pieces of ϕ-orbits issued from points in Θ, say $U_l = \{\phi^t(y) : |t| < \gamma_l, y \in V_l\}$ where $V_l \subset \Theta - \partial\Theta$ is open and $\gamma_l \in \mathbb{R}$.

Since $\dim(\Lambda) = \max_{1\leq l\leq m}\{\dim(\Lambda \cap U_l)\}$ and $\dim(K) = \max_{1\leq l\leq m}\{\dim(K \cap V_l)\}$, we can select l_0 and l_1 such that $\dim(\Lambda) = \dim(\Lambda \cap U_{l_0})$ and $\dim(K) = \dim(K \cap V_{l_1})$. Because $\Lambda \cap V_l = K \cap V_l$ and U_l is a tubular neighborhood for each $1 \leq l \leq m$, we also have that $\Lambda \cap U_l$ is diffeomorphic to $(K \cap V_l) \times (-\gamma_l, \gamma_l)$.

It follows that

$$\dim(\Lambda) = \dim(\Lambda \cap U_{l_0}) = \dim(K \cap V_{l_0}) + 1 \leq \dim(K) + 1$$

and

$$\dim(K) + 1 = \dim(K \cap V_{l_1}) + 1 = \dim(\Lambda \cap U_{l_1}) \leq \dim(\Lambda).$$

This proves the lemma. □

The dynamical Lagrange and Markov spectra of Λ and K are related in the following way. Given a function $f \in C^s(S_g N, \mathbb{R})$, $s \geq 1$, let us denote by $F = \max_\phi f : D_{\mathcal{R}} \to \mathbb{R}$ the function

$$F(y) := \max_{0\leq t\leq t_+(y)} f(\varphi^t(y)).$$

Remark 2.2. $F = \max_\phi f$ might not be C^1 in general.

By definition,

$$\limsup_{n\to+\infty} F(\mathcal{R}^n(x)) = \limsup_{t\to+\infty} f(\phi_g^t(x))$$

and

$$\sup_{n\in\mathbb{Z}} F(\mathcal{R}^n(x)) = \sup_{t\in\mathbb{R}} f(\phi_g^t(x))$$

for all $x \in K$. In particular,

$$L_{\phi_g,\Lambda,f} = L_{\mathcal{R},K,F} \quad \text{and} \quad M_{\phi_g,\Lambda,f} = M_{\mathcal{R},K,F}.$$

This reduces Theorem 1.2 to the following statement.

https://doi.org/10.1017/etds.2021.18 Published online by Cambridge University Press

THEOREM 2.3. *In the setting of Theorem 1.2, if $\mathcal{U}$ is sufficiently small, then there exists a Baire residual subset $\mathcal{U}^* \subset \mathcal{U}$ such that for each $g \in \mathcal{U}^*$, one can find a Baire residual subset $\mathcal{H}_{\phi_g,\Lambda} \subset C^s(S_g N, \mathbb{R})$, $s \geq 4$, so that the function*

$$t \mapsto \dim(L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t))$$

is continuous and

$$\dim(L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)) = \dim(M_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)) \quad \textit{for all } t \in \mathbb{R}$$

whenever $f \in \mathcal{H}_{\phi_g,\Lambda}$.

The proof of Theorem 2.3 starts as follows. Let $\{R_a\}_{a\in\mathcal{A}}$ be a Markov partition consisting of rectangles $R_a \simeq I_a^s \times I_a^u$ delimited by compact pieces I_a^s (respectively I_a^u) of stable (respectively unstable) manifolds of a finite collection of $\mathcal{R}$-periodic points of $K \subset \Theta$.

Recall that the stable and unstable manifolds of K can be extended to locally $\mathcal{R}$-invariant $C^{1+\varepsilon}$-foliations in $D_{\mathcal{R}}$ for some $\varepsilon > 0$. These foliations induce projections $\pi_a^u : R_a \to I_a^s \times \{i_a^u\}$ and $\pi_a^s : R_a \to \{i_a^s\} \times I_a^u$ of the rectangles into the connected components $I_a^s \times \{i_a^u\}$ and $\{i_a^s\} \times I_a^u$ of the stable and unstable boundaries of R_a, where $i_a^u \in \partial I_a^u$ and $i_a^s \in \partial I_a^s$ are fixed arbitrarily. In this way, we obtain stable and unstable Cantor sets

$$K^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(K \cap R_a) \quad \text{and} \quad K^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(K \cap R_a)$$

associated with K.

In the sequel, we will analyze the sets

$$K_t := \{y \in K : m_{\mathcal{R},\max_\phi f}(y) \leq t\},$$

$$K_t^s := \bigcup_{a\in\mathcal{A}} \pi_a^u(K_t \cap R_a) \quad \text{and} \quad K_t^u := \bigcup_{a\in\mathcal{A}} \pi_a^s(K_t \cap R_a).$$

2.2. *Upper semicontinuity.* Denote by $D_s(t)$ and $D_u(t)$ the upper box dimension of K_t^s and K_t^u. As was shown in [**CMM**, Proposition 2.6], an elementary compactness argument reveals the following.

PROPOSITION 2.4. *For any $g \in \mathcal{U}$ and $f \in C^0(S_g N, \mathbb{R})$, the functions $t \mapsto D_u(t)$ and $t \mapsto D_s(t)$ are upper semicontinuous.*

Therefore, it remains to study the lower semicontinuity of $D_s(t)$ and $D_u(t)$ and their relations with $L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)$ and $M_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)$. For this purpose, we introduce the Baire generic sets $\mathcal{U}^*$ and $\mathcal{H}_{\phi_g,\Lambda}$ in the statement of Theorem 2.3.

2.3. *Description of $\mathcal{U}^*$.* We say that $g \in \mathcal{U}^*$ whenever every subhorseshoe $\widetilde{K} \subset K_g$ satisfies the so-called *property (Hα)* of Moreira and Yoccoz [**MY**] and possesses a pair of periodic points whose logarithms of unstable eigenvalues are incommensurable, where K_g denotes the hyperbolic continuation of K.

https://doi.org/10.1017/etds.2021.18 Published online by Cambridge University Press

The set $\mathcal{U}^*$ was defined so that Moreira's dimension formula [**Mo**, Corollary 3] implies the following result.

PROPOSITION 2.5. *Suppose that $g \in \mathcal{U}^*$. Then, given any subhorseshoe $\widetilde{K} \subset K_g$ and any C^1 function $H : D_{\mathcal{R}} \to \mathbb{R}$ whose gradient is transverse to the stable and unstable directions of $\mathcal{R}$ at some point of $\widetilde{K}$, one has*

$$\dim(H(\widetilde{K})) = \min\{\dim(\widetilde{K}), 1\}.$$

For later use, we observe that $\mathcal{U}^*$ is a topologically large subset of $\mathcal{U}$.

LEMMA 2.6. *$\mathcal{U}^*$ is a Baire generic subset of $\mathcal{U}$.*

Proof. By the results in §§4.3 and 9 of [**MY**], every subhorseshoe $\widetilde{K} \subset K$ satisfies the property (Hα) whenever the so-called *Birkhoff invariant* (cf. [**MY**, Appendix A]) of all periodic points of $\mathcal{R}$ in K are non-zero. As it turns out, the non-vanishing of the Birkhoff invariant is an open, dense and conjugation-invariant condition on the third jet of a germ of an area-preserving automorphism of $(\mathbb{R}^2, 0)$ (compare with Lemma 32 in [**MoRo**]). It follows from Klingenberg and Takens' theorem [**KT**, Theorem 1] that the subset $\mathcal{V}$ of $g \in \mathcal{U}$ such that every subhorseshoe $\widetilde{K} \subset K_g$ satisfies the property (Hα) is C^r-Baire generic (for all $r \geq 4$).

On the other hand, given any pair p and q of distinct periodic orbits in K, if we denote by γ_p and γ_q the corresponding g-geodesics on N, then we can select a piece $l \subset \gamma_p$ disjoint from γ_q (because distinct geodesics intersect transversely), and we can apply Klingenberg and Takens' theorem [**KT**, Theorem 2] to (the first jet of the Poincaré map along) l to ensure that the logarithms of the unstable eigenvalues of p and q are incommensurable for a C^r-Baire generic subset $\mathcal{W}_{p,q}$ of $\mathcal{U}$ (for all $r \geq 2$).

It follows that the subset

$$\mathcal{U}^{**} = \mathcal{V} \cap \bigcap_{\substack{p,q \in \mathrm{Per}(\mathcal{R}) \cap K \\ p \neq q}} \mathcal{W}_{p,q}$$

is a countable intersection of C^r-Baire generic subsets (for all $r \geq 4$) such that $\mathcal{U}^{**} \subset \mathcal{U}^*$. This proves the lemma. □

2.4. *Description of $\mathcal{H}_{\phi_g,\Lambda}$.* Let $\mathcal{H}_{\phi_g,\Lambda}$ be the set of functions $f \in C^s(S_g N, \mathbb{R})$, $s \geq 4$, such that there exists a finite collection J of C^1-curves in Θ so that, for each $n \in \mathbb{N}$, the complement V_n of the $1/n$-neighborhood of J in Θ contains a finite collection L_n of C^1-curves with the property that $F = \max_\phi f$ is C^1 on $V_n \setminus L_n$ and the gradient of $F|_{V_n \setminus L_n}$ is transverse to the stable and unstable directions of $\mathcal{R}$ at all points of $K \cap (V_n \setminus L_n)$.

We want to show the following.

LEMMA 2.7. *$\mathcal{H}_{\phi_g,\Lambda}$ is Baire residual.*

For this, we need two auxiliary sets, $\mathcal{M}_{\phi_g,\Lambda} \subset \mathcal{N}_{\phi_g,\Lambda} \subset C^s(S_g N, \mathbb{R})$, $s \geq 4$, defined as follows.

https://doi.org/10.1017/etds.2021.18 Published online by Cambridge University Press

Once again we cover Λ with a finite number of open tubular neighborhoods U_l, $1 \le l \le m$, containing the good cross-sections $\Theta = \bigsqcup_{i=1}^{k} \Sigma_i$ mentioned above. For each l, let us fix coordinates $(x_1(l), x_2(l), x_3(l))$ on U_l such that $x_3(l)$ is the flow direction and $U_l \cap \Theta = \{x_3(l) = 0\} \cup \{x_3(l) = 1\}$. We may assume that the boundaries of U_l are $\{x_3(l) = -\varepsilon\} \cup \{x_3(l) = 1 + \varepsilon\}$, for some small $\varepsilon > 0$.

Definition 2.8. We say that $f \in \mathcal{N}_{\phi_g,\Lambda}$, whenever:

(i) 0 is a regular value of the restriction of $\partial f/\partial x_3(l)$ to $U_l \cap \Theta$;

(ii) 0 is a regular value of $\partial^3 f/\partial x_3(l)^3$;

(iii) 0 is a regular value of the functions $\partial^2 f/\partial x_3(l)^2$ and $\partial^2 f/\partial x_3(l)^2|_{\{\partial^3 f/\partial x_3(l)^3=0\}}$;

(iv) 0 is a regular value of the functions $\partial f/\partial x_3(l)|_{\{\partial^2 f/\partial x_3(l)^2=0\}}$ and $\partial f/\partial x_3(l)|_{\{\partial^3 f/\partial x_3(l)^3=0\}\cap\{\partial^2 f/\partial x_3(l)^2=0\}}$

for each $1 \le l \le m$.

LEMMA 2.9. *$\mathcal{N}_{\phi_g,\Lambda}$ is Baire residual.*

Proof. Given a function $f \in C^s(S_g N, \mathbb{R})$, $s \ge 4$, let us consider the three-parameter family

$$f_{a,b,c}(x_1, x_2, x_3) = f(x_1, x_2, x_3) - cx_3^3/6 - bx_3^2/2 - ax_3$$

where $a, b, c \in \mathbb{R}$.

By Sard's theorem, we can fix first a very small regular value $c \approx 0$ of $\partial^3 f/\partial x_3^3$, then a very small regular value $b \approx 0$ of both $(\partial^2 f/\partial x_3^2) - cx_3$ and its restriction to $\{\partial^3 f/\partial x_3(l)^3 = c\}$, and finally a very small regular value $a \approx 0$ of $((\partial f/\partial x_3) - cx_3^2/2 - bx_3)|_{\{(\partial^2 f/\partial x_3^2)-cx_3=b\}}$, $((\partial f/\partial x_3) - cx_3^2/2 - bx_3)|_{\{\partial^3 f/\partial x_3^3=c\}\cap\{(\partial^2 f/\partial x_3^2)-cx_3=b\}}$ and $((\partial f/\partial x_3) - cx_3^2/2 - bx_3)|_{\{x_3=0\}\cup\{x_3=1\}}$.

For a choice of parameters (a, b, c) as above, we have that $f_{a,b,c}$ satisfies the transversality conditions (i), (ii), (iii) and (iv) on all points of U_l; indeed, this happens because $\partial^3 f_{a,b,c}/\partial x_3^3 = (\partial^3 f/\partial x_3^3) - c$, $\partial^2 f_{a,b,c}/\partial x_3^2 = (\partial^2 f/\partial x_3^2) - cx_3$ and $\partial f_{a,b,c}/\partial x_3 = (\partial f/\partial x_3) - cx_3^2/2 - bx_3$. Notice that $f_{a,b,c}$ is arbitrarily close to f.

$$\tilde{f}_{a,b,c}(x_1, x_2, x_3) = f(x_1, x_2, x_3) - \xi_l^{(n)}(x_1, x_2, x_3)(cx_3^3/6 - bx_3^2/2 - ax_3)$$

can be naturally extended as f outside U_l and coincide with $f_{a,b,c}$ in $U_l^{(n)}$. Thus the set of smooth functions f which satisfy the transversality conditions (i), (ii), (iii) and (iv) on all points of $U_l^{(n)}$ is dense (by the above argument) and open (by compactness of $U_l^{(n)}$). Their intersection (and, after that, the intersection of these sets for $1 \le l \le m$) is a Baire residual set, and any map in their intersection belongs to $\mathcal{N}_{\phi_g,\Lambda}$. This concludes the proof of the lemma. □

By Definition 2.8, if $f \in \mathcal{N}_{\phi_g,\Lambda}$, then $\mu_l := \{\partial f/\partial x_3(l) = 0\} \cap U_l$ is a curve (owing to (i)), and $J_l := \{\partial f/\partial x_3(l) = 0\} \cap \{\partial^2 f/\partial x_3(l)^2 = 0\}$ is a curve intersecting the surface $\{\partial^3 f/\partial x_3(l)^3 = 0\}$ at a finite set Π_l of points (owing to (ii), (iii) and (iv)).

Note that if $(x_1, x_2, 0), (x_1, x_2, 1) \notin \mu_l$ and the piece of orbit (x_1, x_2, z), $0 \le z \le 1$, does not intersect J_l, then there is a neighborhood V of $(x_1, x_2, 0) \in U_l \cap \Theta$ and a finite collection of disjoint graphs $\{(x, y, \psi_j(x, y)) : (x, y, 0) \in V\}$, $1 \le j \le n$, such that

https://doi.org/10.1017/etds.2021.18 Published online by Cambridge University Press

if $F(x_1', x_2') = \max_\phi f(x_1', x_2') = f(x_1', x_2', t')$ with $(x_1', x_2', 0) \in V$, then $t' = \psi_j(x_1', x_2')$ for some j.

Definition 2.10. We say that $f \in \mathcal{M}_{\phi_g,\Lambda}$ if $f \in \mathcal{N}_{\phi_g,\Lambda}$ and there exists a finite collection J of C^1-curves in Θ so that, for each $n \in \mathbb{N}$, the complement V_n of the $1/n$-neighborhood of J in Θ contains a finite collection L_n of C^1-curves with the property that for each $y \in V_n \setminus L_n$, there is an unique $0 \leq t(y) \leq t_+(y)$ with $F(y) = f(\phi^{t(y)}(y))$, and, moreover, the function $y \mapsto \phi^{t(y)}(y)$ is C^1 on $V_n \setminus L_n$.

LEMMA 2.11. *$\mathcal{M}_{\phi_g,\Lambda}$ is Baire residual.*

Proof. Consider $f \in \mathcal{N}_{\phi_g,\Lambda}$ as above. Our discussion so far says that the curves μ_l and the projections of the curves J_l in the flow direction (x_3-coordinate) are a finite union J of C^1 curves contained in Θ such that, for each $y \in D_{\mathcal{R}} \setminus J$, the value $F(z)$ for z near y is described by the values of f at a finite collection of graphs transverse to the flow direction.

In other terms, using the notation in the paragraph before Definition 2.10, our task is reduced to perturbing f in such a way that $F(x_1', x_2', t')$ are given by the values of f on an *unique* graph $(x_1', x_2', \psi(x_1', x_2'))$.

In this direction, we employ the argument from Lemma 19 in [**MoRo**]. More precisely, given $N \in \mathbb{N}$, the value of F at any point $(x, y) \in V_N$ is described by finitely many disjoint graphs ψ_j, $1 \leq j \leq n$ (where n depends on N). As is explained in Lemma 19 in [**MoRo**], we can perform small perturbations of f on V_N in such a way that 0 is a simultaneous regular value of the functions $(x_1, x_2) \mapsto g_{ji}(x_1, x_2) := f(x_1, x_2, \psi_j(x_1, x_2)) - f(x_1, x_2, \psi_i(x_1, x_2))$ for all choices of $1 \leq j < i \leq n$. In this situation, $L_n = \bigcup_{1 \leq j < i \leq n} g_{ji}^{-1}(0)$ is a finite collection of C^1-curves such that, for each $y \in V_n \setminus L_n$, the values of F near y are described by the values of f on an unique graph. Hence, for each $y \in V_n \setminus L_n$, one has that $F(y) = f(\phi^{t(y)}(y))$ for an unique $0 \leq t(y) \leq t_+(y)$ depending in a C^1 way on y.

This shows the lemma. □

At this point, we are ready to establish that $\mathcal{H}_{\phi_g,\Lambda}$ is Baire residual.

Proof of Lemma 2.7. Given a function $f \in C^s(S_g N, \mathbb{R})$, we apply Lemma 2.11 in order to perform a preliminary perturbation so that $f \in \mathcal{M}_{\phi_g,\Lambda}$. In this context, our task is simply to prove that some appropriate perturbations of f render the gradient of $F = \max_\phi f$ transverse to the stable and unstable directions at all points of $K \setminus (\bigcup_{n \in \mathbb{N}} L_n \cup J)$.

For this purpose, we fix $n \in \mathbb{N}$ and consider a point $x \in K \cap (V_n \setminus L_n)$. Recall that in a small neighborhood of x, the values of $F = \max_\phi f$ are given by the values of f on a graph $(x_1, x_2, \psi(x_1, x_2))$. Since the Hausdorff dimension of K is strictly smaller than one (cf. Lemma 2.1), we can employ the argument in Proposition 2.7 in [**CMM**] to find arbitrarily small vectors $v = (v_1, v_2) \in \mathbb{R}^2$ such that the functions $f_v(x_1, x_2, t) := f(x_1, x_2, t) - v_1 x_1 - v_2 x_2$ near the graph $(x_1, x_2, \psi(x_1, x_2))$ (and coinciding with f elsewhere) have the property that the gradient of $F_v := \max_\phi f_v$ is transverse to the stable and unstable directions of any point of K close to (x_1, x_2). Because $n \in \mathbb{N}$ and $x \in K \cap (V_n \setminus L_n)$ were arbitrary, the proof of the lemma is complete. □

2.5. *Lower semicontinuity.* The first step towards the lower semicontinuity $D_u(t)$ and $D_s(t)$ is the following analog of Proposition 2.10 in [**CMM**].

PROPOSITION 2.12. *Suppose that $g \in \mathcal{U}$ and $f \in \mathcal{H}_{\phi_g,\Lambda}$. Given $t \in \mathbb{R}$ such that $D_u(t) > 0$, respectively $D_s(t) > 0$, and $0 < \eta < 1$, there exist $\delta > 0$ and a (complete) subhorseshoe $K' \subset K_{t-\delta}$ such that*

$$\dim((K')^u) > (1-\eta)D_u(t) \quad \textit{and} \quad \dim((K')^s) > (1-\eta)D_u(t),$$

respectively

$$\dim((K')^u) > (1-\eta)D_s(t) \quad \textit{and} \quad \dim((K')^s) > (1-\eta)D_s(t).$$

In particular, $D_u(t) = D_s(t) = d_u(t) = d_s(t)$ for all $t \in \mathbb{R}$.

Proof. By symmetry (that is, replacing the flow by its inverse), it suffices to prove the statement when $D_u(t) > 0$.

We consider the construction of K in terms of its Markov partition R_a, $a \in \mathcal{A}$, introduced above. Given an admissible† word $\alpha = (a_0, \ldots, a_k)$ on the alphabet $\mathcal{A}$, denote by $I^u(\alpha) = \pi^u_{a_0}(\{x \in R_{a_0} : \mathcal{R}^i(x) \in R_{a_i}$ for all $i = 1, \ldots, k\})$. In this setting, the *unstable scale* $r^u(\alpha)$ is $\lfloor \log(1/(\text{length of } I^u(\alpha))) \rfloor$.

For each $r \in \mathbb{N}$, define

$$P^u_r := \{\alpha = (a_0, \ldots, a_k) \text{ admissible word} : r^u(\alpha) \geq r \text{ and } r^u(a_0, \ldots, a_{k-1}) < r\},$$

$$C^u(t, r) := \{\alpha \in P^u_r : I^u(\alpha) \cap K^u_t \neq \emptyset\}$$

and $N_u(t, r) := \#C^u(t, r)$.

Of course, we have similar notions for $I^s(\beta)$, etc.

Denote $\tau = \eta/100$. By the definition of limit capacity, we can fix r_0 sufficiently large such that

$$\left| \frac{\log N_u(t, r)}{r} - D_u(t) \right| < \frac{\tau}{6} D_u(t)$$

for all $r \geq r_0$.

Recall the fact that $f \in \mathcal{H}_{\phi_g,\Lambda}$ is associated with a finite collection J of C^1-curves in Θ so that, for each $n \in \mathbb{N}$, the complement V_n of the $1/n$-neighborhood of J in Θ contains a finite collection L_n of C^1-curves with the property that $F = \max_\phi f$ is C^1 on $V_n \setminus L_n$ and the gradient of $F|_{V_n \setminus L_n}$ is transverse to the stable and unstable directions of $\mathcal{R}$ at all points of $K \cap (V_n \setminus L_n)$.

As is explained in Lemma 18 in [**MoRo**], it is possible to select a subset $B^u(r_0) \subset C^u(t, r_0)$ such that

$$\frac{\log \#B^u(r_0)}{r_0} \geq \frac{\log N_u(t, r_0)}{r_0} - \frac{\tau}{6} D_u(t)$$

and the subhorseshoe $K^{(r_0)} \subset K$ associated with the admissible words in B^u is disjoint from J.

† That is, there is a point $x \in K$ such that $\mathcal{R}^i(x) \in R_{a_i}$ for all $i = 0, \ldots, k$.

https://doi.org/10.1017/etds.2021.18 Published online by Cambridge University Press

By selecting $n_0 \in \mathbb{N}$ large so that $K^{(r_0)} \subset V_{n_0}$ and by applying again the arguments in Lemma 18 in [**MoRo**], we can find a subset $B^u \subset B^u(r_0)$ such that

$$\frac{\log \# B^u}{r_0} \geq \frac{\log B^u(r_0)}{r_0} - \frac{\tau}{6} D_u(t)$$

and the subhorseshoe $K'' \subset K$ associated with the admissible words in B^u is contained in $V_n \setminus L_n$.

In summary, we have obtained a subset $B^u \subset C^u(t, r_0)$ with

$$\left| \frac{\log \# B_u}{r} - D_u(t) \right| < \frac{\tau}{2} D_u(t)$$

such that the subhorseshoe $K'' \subset K$ associated with B^u is contained in $V_{n_0} \setminus L_{n_0}$ and, *a fortiori*, the gradient of $F = \max_\phi f$ is transverse to the stable and unstable directions at all points of K''.

In this scenario, we can use the arguments from Proposition 2.10 in [**CMM**] in order to locate a subhorseshoe $K' \subset K''$ with the desired features. □

At this stage, we are ready to show the lower semicontinuity of $D_u(t)$ and $D_s(t)$.

PROPOSITION 2.13. *For $g \in \mathcal{U}^*$ and $f \in \mathcal{H}_{\phi_g,\Lambda}$, the functions $t \mapsto D_u(t)$ and $t \mapsto D_s(t)$ are lower semicontinuous, and*

$$D_s(t) + D_u(t) = 2D_u(t) = \dim(L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)) = \dim(M_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)).$$

Proof. Consider $t \in \mathbb{R}$ with $D_u(t) > 0$ and fix $\eta > 0$. By Proposition 2.12, we can find $\delta > 0$ and a subhorseshoe $K' \subset K_{t-\delta}$ such that

$$(1 - \eta)(D_u(t) + D_s(t)) = 2(1 - \eta) D_u(t) \leq \dim(K').$$

Since the gradient of $F = \max_\phi f$ is transverse to the stable and unstable directions of K' (cf. the proof of Proposition 2.12 above), we can use Proposition 2.16 in [**CMM**] to show that for each $\varepsilon > 0$, there exists a subhorseshoe $K'_\varepsilon \subset K'$ with $\dim(K'_\varepsilon) \geq \dim(K') - \varepsilon$, and a C^1 height function H_ε whose gradient is transverse to the stable and unstable directions of K'_ε such that

$$H_\varepsilon(K'_\varepsilon) \subset \ell_{\mathcal{R},\max_\phi f}(K').$$

By Proposition 2.5, it follows that

$$\dim(K') - \varepsilon \leq \dim(K'_\varepsilon) = \dim(H_\varepsilon(K'_\varepsilon)) \leq \dim(\ell_{\mathcal{R},\max_\phi f}(K'))$$

for all $\varepsilon > 0$. In particular, $\dim(K') \leq \dim(\ell_{\mathcal{R},\max_\phi f}(K'))$.

Because $K' \subset K_{t-\delta}$, one has $\ell_{\mathcal{R},\max_\phi f}(K') \subset L_{\phi_g,\Lambda,f} \cap (-\infty, t - \delta)$. Thus, our discussion so far can be summarized by the following estimates:

$$\begin{aligned} 2(1 - \eta) D_u(t) &\leq \dim(K') \leq \dim(\ell_{\mathcal{R},\max_\phi f}(K')) \\ &\leq \dim(L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t - \delta)) \leq \dim(M_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t - \delta)) \\ &\leq \dim(\max_{\phi_g} f(K_{t-\delta})) \leq 2D_u(t - \delta). \end{aligned}$$

This proves the proposition. □

2.6. *End of proof of Theorem 2.3.* Let $g \in \mathcal{U}^*$ and $f \in \mathcal{H}_{\phi_g,\Lambda}$. Note that $\mathcal{U}^*$ is a Baire residual subset of $\mathcal{U}$ owing to Lemma 2.6, and $\mathcal{H}_{\phi_g,\Lambda}$ is Baire residual in $C^s(S_g N, \mathbb{R})$ for $s \geq 4$ owing to Lemma 2.7.

By Propositions 2.4 and 2.13, the function

$$t \mapsto D_s(t) = D_u(t) = \tfrac{1}{2}\dim(L_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t)) = \tfrac{1}{2}\dim(M_{\mathcal{R},K,\max_{\phi_g} f} \cap (-\infty, t))$$

is continuous.

This completes the proof of Theorem 2.3 (and, *a fortiori*, Theorem 1.2).

Acknowledgements. We thank Carlos Matheus for very helpful discussions and suggestions during the preparation of this paper. We also would like to thank the anonymous referee for his valuable corrections and suggestions, which improved the writing of this paper. A.C. was partially supported by CNPq-Brazil. Also, she acknowledges the hospitality of Collège de France and IMPA-Brazil during the preparation of this article. C.G.M. was partially supported by CNPq-Brazil.

References

[CMM] A. Cerqueira, C. Matheus and C. G. Moreira. Continuity of Hausdorff dimension across generic dynamical Lagrange and Markov spectra. *J. Mod. Dyn.* **12** (2018), 151–174.

[KT] W. Klingenberg and F. Takens. Generic properties of geodesic flows. *Math. Ann.* **197** (1972), 323–334.

[Mo] C. G. Moreira. Geometric properties of images of Cartesian products of regular Cantor sets by differentiable real maps. *Preprint*, 2016, arXiv:1611.00933.

[MoRo] C. G. Moreira and S. Romaña. On the Lagrange and Markov dynamical spectra for geodesic flows in surfaces with negative curvature. *Preprint*, 2015, arXiv:1505.05178.

[MY] C. G. Moreira and J.-C. Yoccoz. Tangences homoclines stables pour des ensembles hyperboliques de grande dimension fractale. *Ann. Sci. Éc. Norm. Supér. (4)* **43** (2010), 1–68.