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INVOLUTIONS FIXING THE DISJOINT UNION OF ODD-DIMENSIONAL PROJECTIVE SPACES

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ABSTRACT. We show that any differentiable involution on a closed manifold whose fixed point set is a disjoint union of odd-dimensional real projective spaces must be a bounding involution.

1. **Introduction.** Suppose M^r is a closed manifold and $T: M \longrightarrow M$ is a differentiable involution on M. We let (M, T) denote the unoriented bordism class of this involution (see, for example, [2]). Let F denote the set of points fixed by T. In this paper we will prove the following:

THEOREM. *If F is any disjoint union of odd-dimensional real projective spaces, then (M, T) is a bounding involution.*

The issue of classifying (up to bordism) involutions fixing a single real projective space has been settled by Stong [5] in even dimensions and by Capobianco [1] in odd dimensions. Royster [4] gave a partial classification of involutions fixing two projective spaces; in particular, he showed that if the fixed set consists of the disjoint union of two odd-dimensional projective spaces then the involution bounds. In [6] it was shown that an involution will bound if F consists of an arbitrary disjoint union of real projective spaces of constant odd dimension. This paper generalizes these last two results.

2. **A word on Pascal's triangle.** In order to prove the theorem we will need to develop a few ideas about Pascal's triangle reduced modulo two. Let $n > 0$ be odd, and let A_n denote the $(\frac{n+1}{2}) \times (\frac{n+1}{2})$ matrix over $\mathbf{Z}_2$ formed from the upper corner of Pascal's triangle, as follows:

$$A_n = \begin{pmatrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \cdots & \binom{(n-1)/2}{(n-1)/2} \\ \binom{1}{0} & \binom{2}{1} & & & \\ \binom{2}{0} & & \ddots & & \\ \vdots & & & & \vdots \\ \binom{(n-1)/2}{0} & & \cdots & & \binom{n-1}{(n-1)/2} \end{pmatrix}$$

We adopt the unusual convention of labeling the rows and columns of A_n by the *odd* integers $1, 3, 5, \ldots, n$. The reason for this will be made apparent in the next section

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(essentially, column j of A_n represents a copy of $\mathbf{RP}^n$ in the fixed set F whose normal bundle has stable bordism class $(\mathbf{RP}^n, j\lambda)$, and we know [6] that j may take any odd value between 1 and n). Since $\binom{a}{b} = \binom{2a}{2b} \bmod 2$, the entry in row i and column j of A_n is then

$$A(i,j) = \binom{i+j-2}{j-1}.$$

Let $[\mathbf{RP}^n] \in H_n(\mathbf{RP}^n; \mathbf{Z}_2)$ denote the fundamental homology class, and let $\alpha \in H^1(\mathbf{RP}^n; \mathbf{Z}_2)$ denote the nonzero element.

LEMMA 1. *Let $2^N > n$. Then* $A(i,j) = \binom{2^N-j}{i-1} = \binom{2^N-j}{i} = \frac{\alpha^{n-i}}{(1+\alpha)^j}[\mathbf{RP}^n]$.

PROOF. Since 2^N is even and i and j are odd, $\binom{2^N-j}{i-1} = \binom{2^N-j}{i}$. Now,

$$\frac{\alpha^{n-i}}{(1+\alpha)^j}[\mathbf{RP}^n] = \frac{\alpha^{n-i}(1+\alpha)^{2^N-j}}{(1+\alpha)^{2^N}}[\mathbf{RP}^n] = \alpha^{n-i}(1+\alpha)^{2^N-j}[\mathbf{RP}^n] = \binom{2^N-j}{i}.$$

On the other hand, $\frac{\alpha^{n-i}}{(1+\alpha)^j}[\mathbf{RP}^n]$ is equal to the coefficient of α^i in $\frac{1}{(1+\alpha)^j}$, which is

$$\binom{i+j-1}{i} = \binom{i+j-1}{j-1} = \binom{i+j-2}{j-1} = A(i,j)$$

since i and j are odd. ∎

The next lemma illustrates the manner in which A_n contains copies of $A_{2^\lambda-1}$ and the zero matrix within itself:

$$A_n = \begin{array}{|cc|cc|cccc|cc}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\cline{1-4}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \cdots \\
\cline{1-8}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cline{1-4}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

LEMMA 2. *Let $\lambda > 0$. If $\binom{i}{2^\lambda} = 1$ then*

$$A(i,j) = \begin{cases} A(i-2^\lambda, j) & \text{if } \binom{j}{2^\lambda} = 0 \\ 0 & \text{if } \binom{j}{2^\lambda} = 1. \end{cases}$$

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PROOF.

$$A(i,j) = \binom{2^N - j}{i} = \begin{cases} \binom{2^N-j}{i-2^\lambda} & \text{if } \binom{2^N-j}{2^\lambda} = 1 \\ 0 & \text{if } \binom{2^N-j}{2^\lambda} = 0 \end{cases}$$
$$= \begin{cases} A(i-2^\lambda, j) & \text{if } \binom{j}{2^\lambda} = 0 \\ 0 & \text{if } \binom{j}{2^\lambda} = 1. \end{cases}$$

The last equality follows from the fact that $\binom{2^N-j}{2^\lambda} = \binom{j}{2^\lambda} + 1 \bmod 2$, which is easily derived by an argument similar to that used in Lemma 1. ∎

LEMMA 3. *A_n is nonsingular for all (odd) n.*

PROOF.

$$A_n = \begin{pmatrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \binom{3}{3} & \cdots \\ \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & & \\ \binom{2}{0} & \binom{3}{1} & & & \\ \binom{3}{0} & & & & \\ \vdots & & & & \end{pmatrix}$$

Since $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$, if one replaces column 2 of A_n with (column 2 − column 1), column 3 with (column 3 − column 2), and so on, one obtains a matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \binom{1}{1} & \binom{2}{2} & \binom{3}{3} & & \\ 1 & \binom{2}{1} & \binom{3}{2} & & & \\ 1 & \binom{3}{1} & & & & \\ 1 & & & & & \\ \vdots & & & & & \end{pmatrix}$$

which has the same rank as A_n. One may continue in this manner (next time leaving the first two columns alone and replacing the third with the difference of the third and second), to obtain a lower triangular matrix with diagonal entries equal to one. Since this matrix has the same rank as A_n, A_n must be nonsingular. ∎

Now let $Q \subseteq \{1, 3, 5, \ldots, n\}$ be any nonempty subset (think of Q as being a subset of the columns of A_n). We define the integer $\nu(Q)$ to be the number labeling the first row of A_n such that the sum of the entries on that row from the columns labeled by the set Q is nonzero. That is

$$\nu(Q) = \min\Big\{ i \;\Big|\; \sum_{j \in Q} A(i,j) \neq 0 \bmod 2 \Big\}.$$

Since the columns of A_n are linearly independent, this definition makes sense. Clearly $\nu(Q)$ is odd with $1 \leq \nu(Q) \leq n$.

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LEMMA 4. *Let λ be a nonnegative integer with $2^\lambda < n$, and let $\emptyset \neq X \subseteq \{1,3,5,\ldots,n\}$ be such that $\binom{j}{2^\lambda} = 1$ for each $j \in X$. Let $Y \subseteq \{1,3,5,\ldots,n\}$ be such that $\binom{j}{2^\lambda} = 0$ for each $j \in Y$ (Y may be empty). Then*

$$\nu(X \cup Y) \leq \nu(X) + 2^\lambda.$$

PROOF. If $\lambda = 0$, then $Y = \emptyset$, and the result holds. So assume $\lambda > 0$. If λ is so large that $\nu(X) + 2^\lambda \geq n$, the result is also trivial, so assume $\nu(X) + 2^\lambda < n$.

Suppose first that $\sum_{j\in Y} A\big(\nu(X), j\big) = 0$. Since we have by the definition of $\nu(X)$ that $\sum_{j\in X} A\big(\nu(X), j\big) = 1$, and since $X \cap Y = \emptyset$, we have

$$\begin{aligned}\sum_{j\in X\cup Y} A\big(\nu(X), j\big) &= \sum_{j\in X} A\big(\nu(X), j\big) + \sum_{j\in Y} A\big(\nu(X), j\big)\\ &= 1 + 0\\ &= 1.\end{aligned}$$

Thus $\nu(X \cup Y) \leq \nu(X) < \nu(X) + 2^\lambda$ and the result holds.

Suppose on the other hand that $\sum_{j\in Y} A\big(\nu(X), j\big) = 1$. By the definition of $\nu(X)$ we have $\sum_{j\in X} A\big(\nu(X), j\big) = 1$, so by Lemma 2 we have $\binom{\nu(X)}{2^\lambda} = 0$, and hence $\binom{\nu(X)+2^\lambda}{2^\lambda} = 1$. Thus

$$A\big(\nu(X) + 2^\lambda, j\big) = \begin{cases} A\big(\nu(X), j\big) & \text{if } j \in Y\\ 0 & \text{if } j \in X\end{cases}$$

again by Lemma 2. Now since $X \cap Y = \emptyset$,

$$\begin{aligned}\sum_{j\in X\cup Y} A\big(\nu(X) + 2^\lambda, j\big) &= \sum_{j\in X} A\big(\nu(X) + 2^\lambda, j\big) + \sum_{j\in Y} A\big(\nu(X) + 2^\lambda, j\big)\\ &= 0 + \sum_{j\in Y} A\big(\nu(X), j\big)\\ &= 0 + 1\\ &= 1.\end{aligned}$$

Thus $\nu(X \cup Y) \leq \nu(X) + 2^\lambda$. ∎

Now fix $\emptyset \neq Q \subseteq \{1,3,5,\ldots,n\}$. Let $Q_\lambda = \{j \in Q \mid \binom{j}{2^\lambda} = 1\}$.

LEMMA 5. *If $\lambda_1, \lambda_2, \ldots, \lambda_t$ are distinct nonnegative integers, and $Q_{\lambda_1} \cap Q_{\lambda_2} \cap \cdots \cap Q_{\lambda_t} \neq \emptyset$, then*

$$\nu(Q) \leq \nu(Q_{\lambda_1} \cap \cdots \cap Q_{\lambda_t}) + 2^{\lambda_1} + \cdots + 2^{\lambda_t}.$$

PROOF. Let $X_1 = Q_{\lambda_1}$ and $Y_1 = Q - Q_{\lambda_1}$, so $X_1 \cup Y_1 = Q$. For $i > 1$, let $X_i = Q_{\lambda_1} \cap \cdots \cap Q_{\lambda_i}$, and $Y_i = X_{i-1} - X_i$. Then $X_i \cup Y_i = X_{i-1}$. Now apply Lemma 4 to obtain

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$$\begin{aligned} v(Q) &\leq v(X_1) + 2^{\lambda_1} \\ &\leq v(X_2) + 2^{\lambda_1} + 2^{\lambda_2} \\ &\vdots \\ &\leq v(X_t) + 2^{\lambda_1} + \cdots + 2^{\lambda_t} \\ &= v(Q_{\lambda_1} \cap \cdots \cap Q_{\lambda_t}) + 2^{\lambda_1} + \cdots + 2^{\lambda_t}. \end{aligned}$$ ∎

Now let p be any nonnegative integer. If p is nonzero we write the dyadic expansion $p = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_t}$. The main result of this section is the following:

PROPOSITION 1. *If $u > n - v(Q) + p$ is any integer, then*

$$\sum_{j \in Q} \binom{j}{p} \frac{\alpha^u}{(1+\alpha)^j} [\mathbf{RP}^n] = 0.$$

PROOF. If $u > n$, then $\alpha^u = 0$ and we are done, so assume $u \leq n$. If u is even, then since j and n are odd,

$$\frac{\alpha^u}{(1+\alpha)^j} [\mathbf{RP}^n] = \frac{\alpha^{u+1}}{(1+\alpha)^j} [\mathbf{RP}^n]$$

so we may assume that u is even.

Let $X = \{j \in Q \mid \binom{j}{p} = 1\}$. So if $p = 0$ then $X = Q$, and if $p \neq 0$ then $X = Q_{\lambda_1} \cap \cdots \cap Q_{\lambda_t}$. If $X = \emptyset$, then each summand is equal to zero (since each $\binom{j}{p} = 0$), and the result holds. If $X \neq \emptyset$ then

$$\begin{aligned} \sum_{j \in Q} \binom{j}{p} \frac{\alpha^u}{(1+\alpha)^j} [\mathbf{RP}^n] &= \sum_{j \in X} \frac{\alpha^u}{(1+\alpha)^j} [\mathbf{RP}^n] \\ &= \sum_{j \in X} A(n-u, j) \end{aligned}$$

by Lemma 1. But Lemma 5 implies that $v(Q) \leq v(X) + p$, so $n - u < v(Q) - p \leq v(X)$, and hence

$$\sum_{j \in X} A(n-u, j) = 0$$

and the result holds. ∎

3. **Proof of the Theorem.** Given any manifold N^n with vector bundle $\xi^c \longrightarrow N$, let $(N, \xi) \in \mathfrak{N}_n(\mathbf{BO})$ denote the stable bordism class determined by the map

$$N \longrightarrow \mathbf{BO}(c) \hookrightarrow \mathbf{BO}$$

classifying ξ. With n fixed, the classes $(\mathbf{RP}^n, \xi)$ as ξ ranges over all vector bundles form a $\mathbf{Z}_2$-subspace of $\mathfrak{N}_n(\mathbf{BO})$, where the sum $(\mathbf{RP}^n, \xi) + (\mathbf{RP}^n, \eta)$ represents the disjoint union. According to [6], if $\lambda \longrightarrow \mathbf{RP}^n$ denotes the canonical twisted line bundle, and if n is odd, a basis for this subspace is

$$\{(\mathbf{RP}^n, j\lambda) \mid j \text{ odd}, 1 \leq j \leq n\}.$$

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Now suppose (M^r, T) is a smooth involution on a closed manifold M^r, with fixed point set F a disjoint union of odd-dimensional real projective spaces. In [2], Conner and Floyd showed that if $\mathfrak{N}_r^{\mathbf{Z}_2}$ denotes the group of unoriented bordism classes of involutions on r-manifolds, the composition

$$\mathfrak{N}_r^{\mathbf{Z}_2} \longrightarrow \sum_{j=0}^{r} \mathfrak{N}_j\big(\mathbf{BO}(r-j)\big) \longrightarrow \sum_{j=0}^{r} \mathfrak{N}_j(\mathbf{BO})$$

is monic, where the first map assigns to each involution its fixed point data, and the second is induced from the standard inclusion $\mathbf{BO}(k) \hookrightarrow \mathbf{BO}$. Thus if $\nu \longrightarrow F$ denotes the normal bundle of F in M, then either $(F, \nu) = 0 \in \mathfrak{N}_*(\mathbf{BO})$ and (M, T) is a bounding involution, or we may write

$$(F, \nu) = \sum_{i=1}^{t}\Big(\sum_{j\in Q_i}(\mathbf{RP}^{n_i}, j\lambda)\Big) \tag{1}$$

where $n_1 > n_2 > \cdots > n_t$ are all odd, and Q_i is a nonempty subset of $\{1, 3, 5, \ldots, n_i\}$. We will show that the latter is impossible.

For suppose that we have an involution (M^r, T) with fixed data as given in equation 1. Without loss of generality we may assume that $r > n_1$, since $\mathbf{RP}^{n_1}$ is a boundary. Let $k_i = r - n_i$ be the codimension of $\mathbf{RP}^{n_i}$ in M^r.

LEMMA 6. *For each i, and each* $j \in Q_i$, $j \leq k_i$.

PROOF. Let $\nu_{ij}^{k_i} \longrightarrow \mathbf{RP}^{n_i}$ denote the normal bundle of the $\mathbf{RP}^{n_i}$ with fixed data $(\mathbf{RP}^{n_i}, j\lambda)$. Then the Stiefel-Whitney class $w_j(\nu_{ij}^{k_i}) = w_j(j\lambda) = \binom{j}{j}\alpha^j \neq 0$ since $j \leq n_i$, and hence $k_i \geq j$. ■

Now for each i let m_i denote the largest $j \in Q_i$, and let $v_i = v(Q_i)$ as defined in Section 2 after Lemma 3.

LEMMA 7. *For each i*, $v_i \leq k_i$.

PROOF. Consider the matrix A_{n_i}. The upper left corner of A_{n_i} is a copy of A_{m_i}: the first $\frac{m_i+1}{2}$ rows intersected with the first $\frac{m_i+1}{2}$ columns. Since A_{m_i} is nonsingular, and since v_i denotes the smallest row number of A_{n_i} with $\sum_{j\in Q_i} A(v_i, j) \neq 0$, $v_i \leq m_i$. But $m_i \leq k_i$ by Lemma 6. ■

Now reindex the $n_1, n_2, \ldots, n_t$ if necessary so that

$$n_1 - v_1 = n_2 - v_2 = \cdots = n_s - v_s > n_{s+1} - v_{s+1} \geq \cdots \geq n_t - v_t \text{ and}$$
$$n_1 > \cdots > n_s. \tag{2}$$

We then have $k_1 < \cdots < k_s$.

LEMMA 8. $2k_i > k_s$ *for* $1 \leq i \leq s$.

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PROOF.
$$k_s - k_i = n_i - n_s = v_i - v_s < v_i \le k_i$$
by Lemma 7. ∎

According to Kosniowski and Stong [3], if $f(x_1, x_2, \ldots, x_r)$ is any symmetric polynomial over $\mathbf{Z}_2$ in r variables of degree at most r, then

$$f(x_1, \ldots, x_r)[M^r] = \sum_{i=1}^{t} \sum_{j \in Q_i} \frac{f(1+y_1, 1+y_2, \ldots, 1+y_{k_i}, z_1, z_2, \ldots, z_{n_i})}{(1+y_1)(1+y_2)\cdots(1+y_{k_i})}[\mathbf{RP}^{n_i}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_q(x)$, $\sigma_q(y)$, and $\sigma_q(z)$ by the Stiefel-Whitney classes $w_q(M)$, $w_q(j\lambda)$, and $w_q(\mathbf{RP}^{n_i})$ respectively, and evaluating the resulting cohomology class on the fundamental homology class $[M]$ or $[\mathbf{RP}^{n_i}]$. In our case, since each $j \in Q_i$ satisfies $j \le k_i$, we have

$$f(x_1, \ldots, x_r)[M^r] = \sum_{i=1}^{t} \sum_{j \in Q_i} \frac{f(1+y_1, \ldots, 1+y_{k_i}, z_1, \ldots, z_{n_i})}{(1+\alpha)^j}[\mathbf{RP}^{n_i}].$$

Now let $g(x_1, \ldots, x_r) = \sigma_1(x_1, \ldots, x_r) + (r+1)$. Then for each $1 \le i \le t$ and $j \in Q_i$,

$$\begin{aligned} g(1+y_1, \ldots, 1+y_{k_i}, z_1, \ldots, z_{n_i}) &= [k_i + w_1(j\lambda) + w_1(\mathbf{RP}^{n_i})] + (r+1) \\ &= [k_i + \alpha + 0] + (n_i + k_i + 1) \\ &= \alpha \end{aligned}$$

since j and n_i are odd. Now for any d with $0 \le d < k_s/2$, let

$$f_d(x_1, \ldots, x_r) = \sigma_d(x_1, \ldots, x_r)^2 g(x_1, \ldots, x_r)^{n_s - v_s}.$$

Then the degree of f is $2d + n_s - v_s < k_s + n_s - v_s < k_s + n_s = r$, so

$$f_d(x_1, \ldots, x_r)[M^r] = 0.$$

On the other hand, Kosniowski and Stong [3] showed that for each i, $1 \le i \le t$, and each $j \in Q_i$,

$$\sigma_d(1+y_1, \ldots, 1+y_{k_i}, z_1, \ldots, z_{n_i}) = \sum_{p+q \le d} \binom{k_i - p}{d-p-q}\binom{j}{p}\binom{n_i+1}{q}\alpha^{p+q},$$

so since $\binom{a}{b} = \binom{a}{b}^2 \bmod 2$,

$$f_d(1+y, z) = \sum_{p+q \le d} \binom{k_i - p}{d-p-q}\binom{j}{p}\binom{n_i+1}{q}\alpha^{n_s - v_s + 2p + 2q}.$$

We now use Proposition 1: If either $i > s$ or $p+q > 0$, then $n_s - v_s + 2p + 2q > n_i - v_i + p$, so the hypothesis of Proposition 1 is satisfied, and

$$\sum_{j \in Q_i} \binom{j}{p} \frac{\alpha^{n_s - v_s + 2p + 2q}}{(1+\alpha)^j}[\mathbf{RP}^{n_i}] = 0.$$

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Thus for each d

$$\begin{aligned}
0 &= f_d(x_1,\ldots,x_r)[M]\\
&= \sum_{i=1}^{t}\sum_{j\in Q_i}\frac{f_d(1+y,z)}{(1+\alpha)^j}[\mathbf{RP}^{n_i}]\\
&= \sum_{i=1}^{t}\sum_{p+q\le d}\binom{k_i-p}{d-p-q}\binom{n_i+1}{q}\Big[\sum_{j\in Q_i}\binom{j}{p}\frac{\alpha^{n_s-v_s+2p+2q}}{(1+\alpha)^j}[\mathbf{RP}^{n_i}]\Big]\\
&= \sum_{i=1}^{s}\binom{k_i}{d}\Big[\sum_{j\in Q_i}\frac{\alpha^{n_s-v_s}}{(1+\alpha)^j}[\mathbf{RP}^{n_i}]\Big]\\
&= \sum_{i=1}^{s}\binom{k_i}{d}\Big[\sum_{j\in Q_i}A_{n_i}\big(n_i-(n_s-v_s),j\big)\Big]\\
&= \sum_{i-1}^{s}\binom{k_i}{d}\Big[\sum_{j\in Q_i}A_{n_i}(v_i,j)\Big]\\
&= \sum_{i=1}^{s}\binom{k_i}{d} \qquad (3)
\end{aligned}$$

by Lemma 1, equation 2, and the definition of v_i.

But this is impossible. Recall that $0\le d<k_s/2$, and k_s may be odd. Let c be the largest integer which is strictly less than $k_s/2$. Since $k_1<k_2<\cdots<k_s$, and each $k_i>k_s/2$ (Lemma 8), and $c+1\ge k_s/2$, we have for each i

$$k_i+c+1>k_s\Longrightarrow k_i+c\ge k_s\Longrightarrow k_i\ge k_s-c.$$

Let B be the matrix of binomial coefficients reduced modulo 2 given as follows:

$$B=\begin{pmatrix}
\binom{k_s-c}{0} & \binom{k_s-c+1}{0} & \binom{k_s-c+2}{0} & \cdots & \binom{k_s}{0}\\
\binom{k_s-c}{1} & \binom{k_s-c+1}{1} & & & \\
\binom{k_s-c}{2} & & \ddots & & \vdots\\
\vdots & & & & \\
\binom{k_s-c}{c} & & \cdots & & \binom{k_s}{c}
\end{pmatrix}$$

To show that equation 3 cannot hold for all d with $0\le d<k_s/2$, it suffices to show that B is nonsingular. But this is clear: Obtain a matrix of the same rank as B by replacing column 2 of B with (column 2 − column 1), column 3 with (column 3 − column 2), *etc.* Since $\binom{a}{b}+\binom{a}{b+1}=\binom{a+1}{b+1}$, one obtains the matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0\\
* & \binom{k_s-c}{0} & \binom{k_s-c+1}{0} & \binom{k_s-c+2}{0} & \cdots & \binom{k_s-1}{0}\\
* & \binom{k_s-c}{1} & \binom{k_s-c+1}{1} & & & \\
* & \binom{k_s-c}{2} & & \ddots & & \vdots\\
\vdots & & & & & \\
* & \binom{k_s-c}{c-1} & & \cdots & & \binom{k_s-1}{c-1}
\end{pmatrix}$$

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and B is nonsingular by induction.

Thus the assumption that $(F, \nu) \neq 0 \in \mathfrak{N}_*(\mathbf{BO})$ leads to a contradiction, and we conclude that (M, T) is a bounding involution. ∎

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