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On the existence and 'blow-up' of solutions to a two-dimensional nonlinear boundary-value problem arising in corrosion modelling

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Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^2$. We prove that the boundary-value problem $\Delta v = 0$ in Ω, $\partial v/\partial \boldsymbol{n} = \lambda \sinh(v)$ on $\partial\Omega$, has infinitely many (classical) solutions for any given $\lambda > 0$. These solutions are constructed by means of a variational principle. We also investigate the limiting behaviour as $\lambda \to 0^+$; indeed, we prove that each of our solutions, as $\lambda \to 0^+$, after passing to a subsequence, develops a finite number of singularities located on $\partial\Omega$.

1. Introduction and main results

A very common boundary condition in corrosion modelling is associated with the names of Butler and Volmer. In its simplest form, it asserts that there is an exponential relationship between the boundary voltages and the boundary normal currents

$$\frac{\partial w}{\partial \boldsymbol{n}} = \lambda(\mathrm{e}^{\beta w} - \mathrm{e}^{-(1-\beta)w}) + 2g,$$

where the coefficient $0 < \beta < 1$ (frequently referred to as the transfer coefficient) is a 'constitutive constant'; it depends on the constituents of the electrochemical system, but only very mildly on their concentrations. The constant λ, on the other hand, is highly concentration dependent; it may take negative, as well as positive, values—values near zero corresponding to a transition between 'active' and 'passive' status of the boundary. The source term g represents externally imposed boundary currents. If we take $\beta = \frac{1}{2}$ and set $v = \frac{1}{2}w$, then the above boundary condition simplifies to

$$\frac{\partial v}{\partial \boldsymbol{n}} = \lambda \sinh(v) + g. \tag{1.1}$$

We shall closely examine this version in the two-dimensional setting. We shall assume that the domain in question, $\Omega \subset \mathbb{R}^2$, contains no sources or sinks and constitutively is modelled by a simple Laplace operator, so that the voltage potential v satisfies

$$\Delta v = 0 \quad \text{in } \Omega, \tag{1.2}$$

subject to the boundary condition (1.1) on all of $\partial\Omega$. The externally imposed current is equilibrated, that is,

$$\int_{\partial\Omega} g \, \mathrm{d}\sigma = 0.$$

For a detailed discussion of this and more elaborate models of corrosion, see, for instance, [6]. The boundary-value problem (1.2) and (1.1) is easily seen to have a unique solution for $\lambda \leqslant 0$ (modulo a constant for $\lambda = 0$). In [12], it is proven that there exists a positive value λ_* (depending on g) such that the unique solution that exists for $\lambda \leqslant 0$ may be 'continued' for values of λ in the interval $0 < \lambda < \lambda_*$. Numerical experiments presented in that same paper strongly suggest that solutions exist for even larger values of λ; the same experiments also indicate that the solution will not be unique for $\lambda > 0$.

The special case of $\Omega = D =$ the unit disc is studied in detail in [4], the goal being to examine the multiplicity of solutions and their 'blow-up' behaviour. For that purpose, it is assumed that there are no externally imposed boundary currents, so that the boundary condition reads

$$\frac{\partial v}{\partial \boldsymbol{n}} = \lambda \sinh(v) \quad \text{on } \partial\Omega. \tag{1.3}$$

The function $v = 0$ is now a solution for all values of λ. For $\lambda < 0$, it is the only solution, but for $\lambda > 0$, the picture is radically different. At each non-negative integer value of λ, a non-zero family of solutions branches off the zero solution. The fact that the bifurcation happens at non-negative integer values of λ owes to the fact that these are the (Steklov) eigenvalues for the boundary-value problem, where the nonlinear boundary condition (1.3) is replaced by its linearized counterpart $\partial v/\partial \boldsymbol{n} = \lambda v$. For $\lambda = 0$, the new family of solutions is trivial, it simply consists of the constants, but the families corresponding to $\lambda = 1, 2, \dots$ are much more interesting. These solutions continue to exist for parameter values all the way down to $\lambda = 0^+$, and, as $\lambda \to 0^+$, they exhibit very interesting blow-up behaviour. Quite surprisingly, all these solutions are given by explicit formulae. Let k be any positive integer, let $K(x, y)$ denote the function $K(x, y) = \log(x^2 + y^2)$, let p_j, $0 \leqslant j \leqslant 2k-1$, denote the $2k$ points on the unit circle, which in 'complex notation' are given by $p_j = \mathrm{e}^{\mathrm{i}j\pi/k}$, and define

$$\mu_k(\lambda) := \left(\frac{k+\lambda}{k-\lambda}\right)^{1/2k}.$$

Then the functions

$$v_{2k,\lambda}(x, y) := \sum_{j=0}^{2k-1} (-1)^j K((x, y) - \mu_k(\lambda) p_j)$$

are indeed solutions to the boundary-value problem (1.2), (1.3) for $0 < \lambda < k$. We may thus conclude that

(i) given any $\lambda > 0$, there exist infinitely many non-trivial solutions to the boundary-value problem (1.2), (1.3), namely $v_{2k,\lambda}$, $k \geqslant [\lambda]+1$, where $[\lambda]$ denotes the integer part of λ.

These solutions have the properties that

(i) as $\lambda \to 0^+$, $v_{2k,\lambda}$ develops $2k$ singularities located at the points p_j, with $0 \leqslant j \leqslant 2k-1$;

(ii) as $\lambda \to 0^+$, $\partial v_{2k,\lambda}/\partial \boldsymbol{n}$ converges to $2\pi \sum_{j=0}^{2k-1}(-1)^{j+1}\delta_{p_j}$ in the sense of measures; and

(iii) $\|\nabla v_{2k,\lambda}\|^2_{L^2(D)} = 8k\pi \log(1/\lambda) + O(1)$ as $\lambda \to 0^+$.

If we introduce the energy

$$E(v) = \tfrac{1}{2}\|\nabla v\|^2_{L^2(D)} - \lambda \int_{\partial D} [\cosh(v(\sigma)) - 1]\,\mathrm{d}\sigma,$$

then all these solutions also have energies that are of the order $\log(1/\lambda)$ as $\lambda \to 0^+$.

The goal of this paper is to show that the problem (1.2), (1.3) qualitatively has a very similar 'solution structure' for an arbitrary two-dimensional bounded $C^{2,\alpha}$ domain Ω.

1.1. Existence

Concerning existence, we prove the following.

Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\alpha}$ domain. The boundary-value problem (1.2) and (1.3) has infinitely many classical solutions for any fixed $\lambda > 0$. To be quite precise, one may construct countably many families of solutions $\{v_{k,\lambda}\}_{k=2}^{\infty}$. The family $v_{k,\lambda}$ is defined for $0 < \lambda < \mu_k$, where μ_k is the kth (Steklov) eigenvalue associated with the linear boundary-value problem

$$\Delta\varphi = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial \boldsymbol{n}} = \mu\varphi \quad \text{on } \partial\Omega.$$

Each of these families of solutions satisfies the following estimates as $\lambda \to 0^+$:

$$c_*(k)\log\left(\frac{1}{\lambda}\right) \leqslant \|\nabla v_{k,\lambda}\|^2_{L^2(\Omega)} \leqslant C_*(k)\log\left(\frac{1}{\lambda}\right), \tag{1.4}$$

$$c_*(k)\log\left(\frac{1}{\lambda}\right) \leqslant E(v_{k,\lambda}) \leqslant C_*(k)\log\left(\frac{1}{\lambda}\right). \tag{1.5}$$

Here, $c_*(k)$ and $C_*(k)$ are two positive constants, and $E(v)$ denotes the energy expression

$$E(v) = \tfrac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} [\cosh(v(\sigma)) - 1]\,\mathrm{d}\sigma.$$

It may happen that the Steklov eigenvalues μ_k have multiplicities greater than one—in this case, it is not impossible that the sequence of solutions we construct will contain elements that are repeated a finite number of times. We do suspect, however, that such multiple eigenvalues give rise to additional solutions. Consider, for example, the case of the unit disc. In this case,

$$\mu_1 = 0 \quad \text{and} \quad \mu_{2k} = \mu_{2k+1} = k \quad \text{for } k \geqslant 1.$$

Earlier, we gave an explicit formula for the solution $v_{2k,\lambda}$; additional solutions arise by arbitrary rotation of this solution. The eigenvalue $\mu_{2k} = \mu_{2k+1} = k$, of multiplicity two, is thus associated with a set of solutions to (1.2) and (1.3) of genus two, much as one might have expected.

1.2. Blow-up

Let $\{v_n\}_n$ be a sequence of continuous functions defined on $\bar{\Omega}$. We introduce the following notion of blow-up points for $\{v_n\}$.

DEFINITION 1.1. A point $x \in \bar{\Omega}$ is said to be a blow-up point for the sequence $\{v_n\}$ if and only if there exists a sequence of points $x_n \to x$, $x_n \in \bar{\Omega}$, such that $|v_n(x_n)| \to \infty$ as $n \to \infty$.

Concerning the blow-up behaviour of solutions, we prove the following.

Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\alpha}$ domain, and let v_{λ_n}, $\lambda_n \to 0^+$, be a sequence of solutions to the boundary-value problem (1.2) and (1.3), satisfying (1.4) and (1.5), and set

$$v^0_{\lambda_n} = v_{\lambda_n} - |\partial\Omega|^{-1} \int_{\partial\Omega} v_{\lambda_n} \, \mathrm{d}\sigma.$$

Then there exists a subsequence, for simplicity also called λ_n, a regular finite Borel measure μ, supported on $\partial\Omega$, and a finite non-empty set of points $S \subset \partial\Omega$ such that[1]

$$\frac{\partial v_{\lambda_n}}{\partial \boldsymbol{n}} \to \mu \quad \textit{in the sense of measures, as } \lambda_n \to 0^+,$$

and

S is the set of blow-up points for the sequence $\{v^0_{\lambda_n}\}$;
the set S is also exactly the set of points $\sigma \in \partial\Omega$ for which $\mu(\{\sigma\}) \neq 0$.

The main technique used to classify solutions to the boundary-value problem (1.2) and (1.3) is one that is frequently associated with the names of Lyusternik and Schnirelman, or Palais and Smale. This technique characterizes critical values of the energy as appropriate inf sup involving sets of a fixed genus $\geqslant k$. The associated critical points are solutions to (1.2) and (1.3). Estimates for the critical values lead to corresponding estimates for the energy and the L^2-norm of the gradient.

[1]Here, 'convergence in the sense of measures' refers to convergence in the weak* topology on the dual of $C^0(\partial\Omega)$; in other words,

$$\int_{\partial\Omega} \frac{\partial v_{\lambda_n}}{\partial \boldsymbol{n}} \varphi \, \mathrm{d}\sigma \to \int_{\partial\Omega} \varphi \, \mathrm{d}\mu$$

for all $\varphi \in C^0(\partial\Omega)$.

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The basis for our analysis of the limiting blow-up behaviour is a uniform $L^1(\partial\Omega)$ estimate for $\partial v_\lambda/\partial \boldsymbol{n}$ (which follows directly from the aforementioned energy estimates) combined with an adaptation of techniques developed by Brezis and Merle [3]. In the case of the unit disc, the measure μ consists entirely of a sum of Dirac delta functions supported on the set S. For more general domains, we can currently not exclude that μ has a non-zero regular part as well. An interesting open question concerns the specific form of this regular part (if, indeed, it exists), as well as the exact location of the points of the set S.

We want to point out that our analysis does not claim to capture all solutions to (1.2) and (1.3), or even all possible kinds of asymptotic behaviour. For example, for certain (non-simply connected) domains it is not hard to construct a family of solutions whose normal flux blows up everywhere on $\partial\Omega$ – this family fails to satisfy the upper energy bounds in (1.4) and (1.5).

This paper is organized as follows. In § 2 we gather some preliminary results and prove the existence of infinitely many solutions via an appropriate variational approach. In § 3 we establish lower and upper bounds for the energy of the solutions constructed in the previous section. In this section we also establish a uniform L^1 bound for the normal derivatives. Finally, in § 4 we examine the limiting behaviour of these solutions and their normal derivatives as $\lambda \to 0^+$.

2. Existence of infinitely many solutions

Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\alpha}$ domain, and let λ denote a positive number. We wish to study the existence, as well as the asymptotic behaviour, of non-trivial (finite-energy) solutions to the boundary-value problem

$$\left.\begin{aligned} \Delta v &= 0 && \text{in } \Omega,\\ \frac{\partial v}{\partial \boldsymbol{n}} &= \lambda \sinh(v) && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1}$$

Denoting by $\|\cdot\|$ the norm on $L^2(\Omega)$, we consider the functional

$$E(v) := \tfrac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} [\cosh(v(\sigma)) - 1]\,\mathrm{d}\sigma \tag{2.2}$$

on $H^1(\Omega)$; as one can easily deduce from lemma 2.1 below, E is an even C^∞ functional on $H^1(\Omega)$ and the critical points of E in this space are (finite-energy) solutions to (2.1). Due to the regularity of $\partial\Omega$, standard elliptic regularity theory implies that these critical points are indeed classical ($C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$) solutions to (2.1). In the following, the space $H^1(\Omega)$ is equipped with the scalar product

$$(u|v)_1 := \int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} u(\sigma)v(\sigma)\,\mathrm{d}\sigma; \tag{2.3}$$

the associated norm is denoted $\|\cdot\|_1$.

We recall the two following results, which are both consequences of the fact that, for any $a, b \in \mathbb{R}$, the Sobolev space $H^{1/2}(a,b)$ is continuously embedded in the Orlicz space associated with the convex function $\Phi(t) := \mathrm{e}^{t^2} - 1$ (see [1, 8, 12]). For the convenience of the reader, we provide outlines of proofs that derive these results from the more classical two-dimensional Moser–Trudinger inequality.

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LEMMA 2.1. *For any $\alpha \in \mathbb{R}$, there exist two constants, $C > 0$ and $\beta > 0$, such that*

$$\int_{\partial\Omega} \cosh(\alpha v(\sigma))\, d\sigma \leqslant C \exp(\beta \|v\|_1^2) \quad \forall v \in H^1(\Omega).$$

Proof. It is a simple consequence of a well-known inequality that

$$\int_{\Omega} e^{\alpha v}\, dx \leqslant C e^{\beta \|v\|_1^2} \tag{2.4}$$

(see [1,11]). Indeed, theorem 7.15 of [11, p. 162] asserts that

$$\int_{\Omega} e^{(v/c_1 \|\nabla v\|_{L^2(\Omega)})^2} \leqslant c_2 |\Omega| \quad \forall v \in H_0^1(\Omega),$$

so

$$\int_{\Omega} e^{\alpha v}\, dx \leqslant \int_{\Omega} e^{(\alpha c_1 \|\nabla v\|_{L^2(\Omega)})^2} e^{(v/c_1 \|\nabla v\|_{L^2(\Omega)})^2}\, dx \leqslant C' e^{\beta' \|\nabla v\|_{L^2(\Omega)}^2} \quad \forall v \in H_0^1(\Omega).$$

Let $\Omega \subset\subset \tilde{\Omega}$, and, given any $v \in H^1(\Omega)$, let $\tilde{v} \in H_0^1(\tilde{\Omega})$ be such that

$$\tilde{v} = v \quad \text{on } \Omega \quad \text{and} \quad \|\nabla \tilde{v}\|_{L^2(\tilde{\Omega})} \leqslant C \|v\|_{H^1(\Omega)}.$$

Then

$$\int_{\Omega} e^{\alpha v}\, dx \leqslant \int_{\tilde{\Omega}} e^{\alpha \tilde{v}}\, dx \leqslant \tilde{C} e^{\tilde{\beta} \|\nabla \tilde{v}\|_{L^2(\tilde{\Omega})}^2} \leqslant C e^{\beta \|v\|_{H^1(\Omega)}^2} \quad \forall v \in H^1(\Omega).$$

This verifies (2.4). It is also well known that

$$\int_{\partial\Omega} |w|^p\, d\sigma \leqslant C_p \biggl(\int_{\Omega} |\nabla w|^p\, dx + \int_{\Omega} |w|^p\, dx \biggr)$$

for any $p > 1$. We now pick a fixed $1 < p < 2$ and estimate

$$\int_{\partial\Omega} e^{\alpha v}\, d\sigma = \int_{\partial\Omega} |e^{\alpha v/p}|^p\, d\sigma \leqslant C \biggl(\int_{\Omega} |\nabla (e^{\alpha v/p})|^p\, dx + \int_{\Omega} e^{\alpha v}\, dx \biggr). \tag{2.5}$$

By differentiation and use of Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |\nabla (e^{\alpha v/p})|^p\, dx &= \int_{\Omega} \biggl| \frac{\alpha}{p} e^{\alpha v/p} \nabla v \biggr|^p dx \\ &= \biggl(\frac{\alpha}{p} \biggr)^p \int_{\Omega} |\nabla v|^p e^{\alpha v}\, dx \\ &\leqslant C \biggl(\int_{\Omega} |\nabla v|^2\, dx \biggr)^{p/2} \biggl(\int_{\Omega} (e^{\alpha v})^{2/(2-p)}\, dx \biggr)^{(2-p)/2} \\ &\leqslant C \|v\|_1^p e^{\beta' \|v\|_1^2}. \end{aligned} \tag{2.6}$$

For the last inequality, we used the estimate (2.4). Insertion of (2.6) into (2.5), and use of (2.4), now yields

$$\int_{\partial\Omega} e^{\alpha v}\, d\sigma \leqslant C(\|v\|_1^p + 1) e^{\beta' \|v\|_1^2} \leqslant C e^{\beta \|v\|_1^2},$$

which immediately leads to the estimate of this lemma. □

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LEMMA 2.2. *If $\{v_n\}_n$ is a sequence in $H^1(\Omega)$ converging weakly to $v \in H^1(\Omega)$, then*

$$\sinh(\alpha v_n) \to \sinh(\alpha v) \quad \textit{and} \quad \cosh(\alpha v_n) \to \cosh(\alpha v)$$

strongly in $L^2(\partial\Omega)$ for any fixed $\alpha \in \mathbb{R}$.

Proof. Consider the first convergence result. The sequence $\{v_n\}_n$ is bounded in $H^1(\Omega)$ and, from Sobolev's imbedding theorem and the compactness of the imbedding $H^{1/2}(\partial\Omega) \subset L^p(\partial\Omega)$, $p < \infty$, we get that $v_n \to v$ in $L^4(\partial\Omega)$. By the mean-value theorem,

$$\sinh(\alpha v_n(x)) - \sinh(\alpha v(x)) = \alpha(v_n - v)(x)\cosh(\alpha\xi_n(x)),$$

with $\min\{v_n(x), v(x)\} \leqslant \xi_n(x) \leqslant \max\{v_n(x), v(x)\}$. Therefore,

$$|\sinh(\alpha v_n(x)) - \sinh(\alpha v(x))|^2 \leqslant |\alpha|^2|(v_n - v)(x)|^2\cosh(|\alpha|(|v_n(x)| + |v(x)|))^2,$$

and so

$$\begin{aligned}
&\|\sinh(\alpha v_n) - \sinh(\alpha v)\|_{L^2(\partial\Omega)} \\
&\qquad \leqslant |\alpha|\|v_n - v\|_{L^4(\partial\Omega)}\|\cosh(|\alpha v_n| + |\alpha v|)\|_{L^4(\partial\Omega)} \\
&\qquad \leqslant 2|\alpha|\|v_n - v\|_{L^4(\partial\Omega)}\|\cosh(|\alpha v_n|)\cosh(|\alpha v|)\|_{L^4(\partial\Omega)} \\
&\qquad \leqslant C\|v_n - v\|_{L^4(\partial\Omega)}\|\cosh(8\alpha v_n)\|_{L^1(\partial\Omega)}^{1/8}\|\cosh(8\alpha v)\|_{L^1(\partial\Omega)}^{1/8} \\
&\qquad \leqslant C\|v_n - v\|_{L^4(\partial\Omega)}.
\end{aligned}$$

Here we have used the inequality $\cosh(x+y) \leqslant 2\cosh(x)\cosh(y)$ in the second line, and lemma 2.1 in the last line. This establishes the first convergence result. The proof of the second result follows along the exact same lines. □

In order to construct a family of critical points for E, and thus a family of solutions to (2.1), we have two possibilities. The first is to construct critical points for the *undefined functional* E on $H^1(\Omega)$, where *undefined* refers to the fact that E is neither bounded from below nor from above. The second possibility is to introduce a functional J, which is bounded from one side, and which is such that critical points for J on some submanifold Σ yield critical points for E in $H^1(\Omega)$ (via some simple transform). In what follows, we adopt the second approach.

As suggested in the works of Pohožaev [9] and Bahri [2] (see, for instance, [7, ch. 4 and 5]), we define

$$J(u) := \sup_{t>0} E(tu), \quad u \neq 0.$$

Note that $J(u) \geqslant 0$, and

(i) $J(u) = \infty$ if and only if $u \in H_0^1(\Omega) = H^1(\Omega) \cap \{u : u|_{\partial\Omega} = 0\}$;

(ii) $J(u) = 0$ if and only if $E(tu) \leqslant 0\ \forall t > 0$.

In the following three lemmas, we shall carefully examine the regularity properties of J. First we show that, whenever $0 < J(u) < \infty$, there exists a (unique) real parameter $t(u) > 0$ such that $J(u) = E(t(u)u)$; indeed, $t(u)$ is the solution to

$$t\|\nabla u\|_{L^2(\Omega)}^2 - \lambda\int_{\partial\Omega}\sinh(tu(\sigma))u(\sigma)\,\mathrm{d}\sigma = 0.$$

We furthermore show that $t(u)$ is a smooth function of u on the open set $\{0 < J < \infty\}$. This immediately implies that J is a C^∞ functional on the open set $\{0 < J < \infty\}$. A slight extension of this argument yields that J is continuous from $H^1(\Omega)\backslash\{0\}$ onto $[0,\infty]$ (it is very simple to see that J may not be continuously extended to all of $H^1(\Omega)$). Finally, we also demonstrate how critical points for J on the unit sphere yield critical points for E in $H^1(\Omega)$, by the simple transformation $u \to t(u)u$.

Lemma 2.3. *For fixed $u \in H^1(\Omega) \setminus \{0\}$, define*

$$f(t) := E(tu) = \tfrac{1}{2}t^2\|\nabla u\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} [\cosh(tu(\sigma)) - 1]\,\mathrm{d}\sigma, \quad t \in \mathbb{R}.$$

Then f is an even function and $f \in C^\infty(\mathbb{R})$. Suppose u is such that $\sup_{t>0} f(t)$ is finite and positive. Then there exists a unique positive real number $t(u)$, satisfying $f(t(u)) = \sup_{t>0} f(t) = J(u)$. The map $w \mapsto t(w)$ is well defined in an $H^1(\Omega)$ neighbourhood of u, and it is of class C^∞.

Proof. It is clear that f is C^∞, even on $\mathbb{R}$, with $f(0) = 0$. Assuming that $\sup_{t>0} f(t)$ is finite and positive, we also easily conclude that u does not identically vanish on $\partial\Omega$, and thus that $f(t) \to -\infty$ as $t \to +\infty$. As a consequence, there exists $t(u) > 0$ such that $f(t(u)) = \max_{t>0} f(t)$. A straightforward calculation shows that

$$\begin{aligned} f'(t) &= t\|\nabla u\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} \sinh(tu(\sigma))u(\sigma)\,\mathrm{d}\sigma, \\ f''(t) &= \|\nabla u\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} \cosh(tu(\sigma))u^2(\sigma)\,\mathrm{d}\sigma, \\ f'''(t) &= -\lambda \int_{\partial\Omega} \sinh(tu(\sigma))u^3(\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

From the expression for f''', it follows that f' is strictly concave on $]0,+\infty[$ (remember that $u \notin H^1_0(\Omega)$ and $\lambda > 0$). We also have that $f'(0) = 0$ and $f'(t) \to -\infty$ as $t \to +\infty$. Since $\sup_{t>0} f(t) > 0 = f(0)$, the derivative f' must take some positive values on $]0,+\infty[$. A combination of these facts gives that there exists a unique $t_* > 0$ such that $f'(t_*) = 0$; moreover, $f(t)$, as well as $f'(t)$, are positive for $0 < t < t_*$. Using the fact that t_* is defined by the relation

$$\|\nabla u\|^2_{L^2(\Omega)} = t_*^{-1}\lambda \int_{\partial\Omega} \sinh(t_*u(\sigma))u(\sigma)\,\mathrm{d}\sigma,$$

we get

$$f''(t_*) = \lambda \int_{\partial\Omega} \left[\frac{\tanh(t_*u)}{t_*u} - 1\right] \cosh(t_*u)u^2\,\mathrm{d}\sigma < 0,$$

since $s^{-1}\tanh(s) < 1$ for all $s \in \mathbb{R} \setminus \{0\}$. We conclude that there exists a unique $t(u) := t_* > 0$ such that $f(t(u)) = \max_{t>0} f(t)$, and from the last inequality and the implicit functions theorem it follows that the mapping $w \mapsto t(w)$ is well defined and C^∞ in a neighbourhood of u. □

Lemma 2.4. *The functional $J : H^1(\Omega) \setminus \{0\} \to [0,\infty]$ is even and continuous. J is finite on $H^1(\Omega) \setminus H^1_0(\Omega)$.*

Proof. That J is even is obvious, and so is the fact that it is finite on $H^1(\Omega)\backslash H_0^1(\Omega)$. In order to prove continuity, we consider $u_n \in H^1(\Omega) \setminus \{0\}$, with $u_n \to u_0 \in H^1(\Omega) \setminus \{0\}$, as $n \to \infty$, and seek to establish that $J(u_n) \to J(u_0)$. Three different cases occur: (i) $J(u_0) = \infty$; (ii) $J(u_0) = 0$; and (iii) $0 < J(u_0) < \infty$.

Case i. For any fixed $t > 0$, we have

$$J(u_n) = \sup_{s>0} E(su_n) \geqslant E(tu_n) \quad \forall n$$

and

$$E(tu_n) \to E(tu_0) \quad \text{as } n \to \infty.$$

Therefore,

$$\liminf_{n\to\infty} J(u_n) \geqslant \lim_{n\to\infty} E(tu_n) = E(tu_0) \quad \forall t > 0,$$

and so

$$\liminf_{n\to\infty} J(u_n) \geqslant \sup_{t>0} E(tu_0) = J(u_0) = \infty.$$

This immediately implies that $J(u_n) \to \infty = J(u_0)$ as $n \to \infty$.

Case ii. The set $\{u \in H^1(\Omega) \setminus \{0\} : J(u) < \infty\}$ is easily seen to be open, and, as a consequence, we have that $J(u_n) < \infty$ for n sufficiently large. By taking away those u_n for which $J(u_n) = 0$ (and for which convergence to $J(u_0)$ is no problem), it remains to consider a sequence with $0 < J(u_n) < \infty$, for which $u_n \to u_0$, $J(u_0) = 0$. We shall prove that the corresponding parameters $t(u_n)$ converge to zero as $n \to \infty$; if not, there exists a subsequence, $t(u_{n_k})$, and a constant $c > 0$, such that $0 < c < t(u_{n_k})$ for all k. Since $E(tu_{n_k}) \geqslant 0$ for all $0 \leqslant t \leqslant t(u_{n_k})$ (see the proof of lemma 2.3), it follows immediately that

$$E(tu_0) = \lim_{k\to\infty} E(tu_{n_k}) \geqslant 0 \quad \text{for all } 0 \leqslant t \leqslant c.$$

This implies that $E(\cdot u_0)$ vanishes on the interval $[0, c]$, and after differentiation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3 E(tu_0) = 0 \quad \text{for all } 0 \leqslant t \leqslant c. \tag{2.7}$$

Due to the formula

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3 E(tu_0) = -\lambda \int_{\partial\Omega} \sinh(tu_0(\sigma))u_0(\sigma)^3 \,\mathrm{d}\sigma,$$

the identity (2.7) represents a contradiction to the facts that $\lambda > 0$ and that u_0 does not vanish identically on $\partial\Omega$. We conclude that $t(u_n) \to 0$ as $n \to \infty$, and thus $J(u_n) = E(t(u_n)u_n) \to E(0) = 0 = J(u_0)$ as $n \to \infty$.

Case iii. The set $\{u \in H^1(\Omega) \setminus \{0\} : 0 < J(u) < \infty\}$ is easily seen to be open, and, as a consequence, we have that $0 < J(u_n) < \infty$ for n sufficiently large. The fact that the mapping $u \to t(u)$ is smooth (C^∞) now implies that $t(u_n) \to t(u_0)$, and so $J(u_n) = E(t(u_n)u_n) \to E(t(u_0)u_0) = J(u_0)$ as $n \to \infty$. □

Next we observe that critical points for J yield critical points for E.

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LEMMA 2.5. *For $u \in H^1(\Omega) \setminus \{0\}$, with $\sup_{t\in\mathbb{R}} E(tu) > 0$, let $t(u)$ be as defined in lemma 2.3. The functional J is even and of class C^∞ on the open even set $\{0 < J < \infty\}$. For $u \in \{0 < J < \infty\}$, we have $J'(u) = t(u)E'(t(u)u)$. Moreover, if $u \in H^1(\Omega)$ is a critical point for J on the sphere*

$$\Sigma := \left\{u \in H^1(\Omega); \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x + \int_{\partial\Omega} |u(\sigma)|^2 \, \mathrm{d}\sigma = 1\right\} \tag{2.8}$$

for which the critical value $c := J(u)$ is positive (and finite), then $v := t(u)u$ is a critical point for E in $H^1(\Omega)$ corresponding to the same critical value c.

Proof. It is clear that J is even and C^∞ on the set $\{0 < J < \infty\}$. Now, as J is constant on the rays $\{su;\ s > 0\}$, the Lagrange multiplier corresponding to the critical point, u, of J on Σ, must be zero. Indeed, if we define $F(w) := (w|w)_1$, $w \in H^1(\Omega)$, and if $u \in \Sigma$ and $\mu \in \mathbb{R}$ are such that

$$J'(u) = \mu F'(u) \quad \text{in } (H^1(\Omega))',$$

then it follows that

$$2\mu = 2\mu\left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_{\partial\Omega} |u(\sigma)|^2 \, \mathrm{d}\sigma\right) = \mu\langle F'(u), u\rangle = \langle J'(u), u\rangle = 0,$$

since, for all $s \in \mathbb{R}$, we have $J(su) = J(u)$, and thus, for all $s \in \mathbb{R}$ (and, in particular, for $s = 1$),

$$\frac{\mathrm{d}}{\mathrm{d}s} J(su) = \langle J'(su), u\rangle = 0.$$

Therefore, the positive critical values for J on Σ are also critical values in $H^1(\Omega)$, while a critical point for J on Σ yields a ray of critical points for J in $H^1(\Omega)$. It is also easy to see that a critical point, u, for J in $H^1(\Omega)$, with a corresponding positive (finite) critical value, gives rise to a critical point, $u/\|u\|_1$, for J on Σ, with the same critical value. For $u \in \{0 < J < \infty\}$ and for any $w \in H^1(\Omega)$, we have

$$\langle J'(u), w\rangle = t(u)\langle E'(t(u)u), w\rangle + \langle E'(t(u)u), u\rangle\langle t'(u), w\rangle.$$

Now, by the very definition of $t(u)$, we have $\langle E'(t(u)u), u\rangle = f'(t(u)) = 0$, and therefore

$$\langle J'(u), w\rangle = t(u)\langle E'(t(u)u), w\rangle,$$

i.e. $J'(u) = t(u)E'(t(u)u)$. Consequently, we see that if $u \in \Sigma$ is such that $0 < J(u) < \infty$, and if u is a critical point for J on Σ, then $v := t(u)u$ is a critical point for E in $H^1(\Omega)$. □

REMARK 2.6. It is important to observe that if $v \in H^1(\Omega)$ is a non-trivial solution to (2.1), then $E(v) > 0$, and $v \notin H^1_0(\Omega)$, so $0 < J(v) < \infty$. The fact that $v \notin H^1_0(\Omega)$ is obvious. Furthermore, multiplying (2.1) by v and integrating by parts, we obtain

$$\|\nabla v\|^2_{L^2(\Omega)} = \lambda \int_{\partial\Omega} v(\sigma)\sinh(v(\sigma)) \, \mathrm{d}\sigma,$$

and so

$$E(v) = \lambda \int_{\partial\Omega} [\tfrac{1}{2} v(\sigma)\sinh(v(\sigma)) - \cosh(v(\sigma)) + 1] \, \mathrm{d}\sigma > 0.$$

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The strict positivity follows from the fact that, for all $\theta \in \mathbb{R} \setminus \{0\}$,

$$\tfrac{1}{2}\theta\sinh(\theta) - \cosh(\theta) + 1 = \sum_{n\geqslant 1} \frac{1}{(2n-1)!}\left[\frac{1}{2} - \frac{1}{2n}\right]\theta^{2n} = \frac{1}{2}\sum_{n\geqslant 2}\frac{n-1}{(2n-1)!n}\theta^{2n} > 0.$$

Thus it is quite natural to seek critical points for E only on the set $\{0 < J < \infty\}$.

In order to see that (2.1) has infinitely many solutions, it suffices to prove that J has an unbounded sequence of critical values on Σ. In constructing such a sequence, we rely on what is frequently known as the Lyusternik–Schnirelman technique. We recall that, given a closed even subset A of Σ, the genus $\gamma(A)$ is the smallest integer k for which there exists a continuous odd mapping of A into $\mathbb{R}^k \setminus \{0\}$ (an even set is one for which $A = -A$). For integer $k \geqslant 1$, we now define the sets

$$\mathfrak{A}_k := \{A \subset \Sigma;\ A \text{ is closed},\ A = -A \text{ and } \gamma(A) \geqslant k\} \tag{2.9}$$

and the numbers

$$c_k := c_k(\lambda) := \inf_{A\in\mathfrak{A}_k} \sup_{u\in A} J(u). \tag{2.10}$$

It is easy to see that, given any $k \geqslant 1$, there exists a compact even subset $A \subset \Sigma \setminus H_0^1(\Omega)$, with $\gamma(A) \geqslant k$. Simply take

$$A = \left\{\sum_{j=1}^{k}\alpha_j F_j(x) : \int_\Omega \left(\sum_{j=1}^{k}\alpha_j \nabla F_j(x)\right)^2 \mathrm{d}x + \int_{\partial\Omega}\left(\sum_{j=1}^{k}\alpha_j F_j(\sigma)\right)^2 \mathrm{d}\sigma = 1\right\},$$

where the functions $\{F_j\}_{j=1}^k \subset H^1(\Omega)$ are selected so that their boundary traces $f_j = F_j|_{\partial\Omega}$ are linearly independent. This, in combination with the continuity of J (see lemma 2.4), implies that $c_k < \infty$ for any $k \geqslant 1$. The goal is to prove that the values c_k form a non-decreasing set of critical values for J, converging to $+\infty$.

LEMMA 2.7. *Let $\{\mu_k\}_{k\geqslant 1}$, $\{\varphi_k\}_{k\geqslant 1}$ be the sequence of Steklov eigenvalues and (normalized) eigenfunctions defined by*

$$\Delta\varphi_k = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi_k}{\partial \boldsymbol{n}} = \mu_k\varphi_k \quad \text{on } \partial\Omega,$$
$$(\varphi_j|\varphi_k)_1 = \int_\Omega \nabla\varphi_k\nabla\varphi_j\,\mathrm{d}x + \int_{\partial\Omega}\varphi_k\varphi_j\,\mathrm{d}\sigma = \delta_k^j.$$

Suppose that λ is fixed, with $\mu_{k_0} \leqslant \lambda < \mu_{k_0+1}$, $k_0 \geqslant 1$. Let H_0 denote the span of $\varphi_1, \dots, \varphi_{k_0}$, and let $H_0^\perp$ denote the orthogonal supplement of H_0 in $H^1(\Omega)$ with respect to the scalar product $(\cdot|\cdot)_1$. Then there exist two constants $R > 0$ and $a > 0$ such that

$$v \in H_0^\perp \quad \text{and} \quad \|v\|_1 = R \quad \Rightarrow \quad E(v) \geqslant a. \tag{2.11}$$

Proof. Indeed, for $v \in H_0^\perp$, we have

$$\|\nabla v\|^2_{L^2(\Omega)} \geqslant \mu_{k_0+1}\int_{\partial\Omega} v^2(\sigma)\,\mathrm{d}\sigma$$

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and

$$E(v) = \tfrac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \tfrac{1}{2}\lambda \int_{\partial\Omega} v^2(\sigma)\,\mathrm{d}\sigma - \lambda \int_{\partial\Omega} [\cosh(v(\sigma)) - 1 - \tfrac{1}{2}v^2(\sigma)]\,\mathrm{d}\sigma$$
$$\geqslant \frac{1}{2}\biggl(1 - \frac{\lambda}{\mu_{k_0+1}}\biggr)\|\nabla v\|^2_{L^2(\Omega)} - C\|\nabla v\|^4_{L^2(\Omega)}\mathrm{e}^{\beta\|v\|_1^2}. \tag{2.12}$$

Here we use the fact that (due to lemma 2.1) there exist constants $C > 0$ and $\beta > 0$ such that

$$\begin{aligned}\int_{\partial\Omega} [\cosh(v(\sigma)) - 1 - \tfrac{1}{2}v^2(\sigma)] &\leqslant C \int_{\partial\Omega} v^4 \cosh(v)\,\mathrm{d}\sigma \\ &\leqslant C\biggl(\int_{\partial\Omega} v^8\,\mathrm{d}\sigma\biggr)^{1/2}\biggl(\int_{\partial\Omega} \cosh(2v)\,\mathrm{d}\sigma\biggr)^{1/2} \\ &\leqslant C\|v\|_1^4 \exp(\beta\|v\|_1^2) \\ &\leqslant C\|\nabla v\|^4_{L^2(\Omega)} \exp(\beta\|\nabla v\|^2_{L^2(\Omega)})\end{aligned}$$

for all $v \in H^1(\Omega)$ with $\int_{\partial\Omega} v\,\mathrm{d}\sigma = 0$. We also use the fact that $v \in H_0^{\perp}$ implies that $\int_{\partial\Omega} v\,\mathrm{d}\sigma = 0$, since $\varphi_1 = 1/\sqrt{|\partial\Omega|}$ (and $\mu_1 = 0$). From (2.12), we conclude that if $R > 0$ is chosen sufficiently small, then, for some $\delta > 0$ and any $v \in H_0^{\perp}$ with $\|v\|_1 = R$, we have $E(v) \geqslant \delta R^2 =: a$. □

We now show that for k sufficiently large (exactly how large depends on λ), we have that $c_k > 0$.

LEMMA 2.8. *Let $\lambda > 0$ be fixed, with $\mu_{k_0} \leqslant \lambda < \mu_{k_0+1}$, $k_0 \geqslant 1$. Then $0 < c_{k_0+1}$, and therefore $0 < c_{k_0+1} \leqslant c_k < \infty$ for $k \geqslant k_0 + 1$.*

Proof. Let $k \geqslant k_0 + 1$ and let $A \in \mathfrak{A}_k$ be given. Since the genus of A is strictly larger than k_0, we may infer that there exists $u_* \in A \cap H_0^{\perp}$. If $A \cap H_0^{\perp} = \emptyset$, then the orthogonal projection onto H_0 would yield a continuous odd mapping of A into $H_0 \setminus \{0\}$; by taking the coordinates (relative to any basis of H_0), we would now get a continuous odd mapping of A into $\mathbb{R}^{k_0} \setminus \{0\}$, and this would imply $\gamma(A) \leqslant k_0$ – a contradiction. Let $a > 0$ and $R > 0$ be as in lemma 2.7. Then $J(u_*) \geqslant E(Ru_*) \geqslant a > 0$, and therefore

$$\sup_{u\in A} J(u) \geqslant J(u_*) \geqslant a$$

for any $A \in \mathfrak{A}_k$. We conclude that $0 < a \leqslant c_{k_0+1} \leqslant c_k < \infty$ for all $k \geqslant k_0 + 1$. □

Before being in a position to conclude that each $0 < c_k < \infty$ is a critical value for J on Σ, we need to show that J satisfies the following Palais–Smale condition (as before, $F(u) := (u|u)_1 := \|\nabla u\|^2_{L^2(\Omega)} + \int_{\partial\Omega} |u(\sigma)|^2\,\mathrm{d}\sigma$).

LEMMA 2.9. *Let c be a given positive finite value. Assume that $\{u_n, \alpha_n\}_{n\geqslant 1}$ is a sequence in $\Sigma \times \mathbb{R}$ such that*

$$J(u_n) \to c \quad \text{and} \quad \varepsilon_n := J'(u_n) - \alpha_n F'(u_n) \to 0 \quad \text{in } (H^1(\Omega))'$$

as $n \to +\infty$. Then $\alpha_n \to 0$, and there exists $u \in \Sigma$ and a subsequence $\{u_{n_j}\}_{j\geqslant 1}$ such that $u_{n_j} \to u$ in $H^1(\Omega)$.

Proof. We begin by showing that the sequence $v_n := t(u_n)u_n$ is bounded in $H^1(\Omega)$, or, equivalently, that the sequence $\{t(u_n)\}_n$ is bounded in $\mathbb{R}$. From the definition of $t(u)$,

$$t(u)^2\|\nabla u\|^2 = \lambda \int_{\partial\Omega} t(u)u \sinh(t(u)u)\,\mathrm{d}\sigma \tag{2.13}$$

for any $u \in \Sigma$ such that $0 < J(u) < \infty$. We also note that there exists a constant $b > 0$ such that $\cosh(\theta) - 1 \leqslant \frac{1}{4}\theta\sinh(\theta) + b$ for all $\theta \in \mathbb{R}$. Therefore, as

$$J(u) = \tfrac{1}{2}t(u)^2\|\nabla u\|^2 - \lambda \int_{\partial\Omega} [\cosh(t(u)u(\sigma)) - 1]\,\mathrm{d}\sigma,$$

we have

$$\begin{aligned} J(u) &\geqslant \tfrac{1}{2}t(u)^2\|\nabla u\|^2 - \tfrac{1}{4}\lambda \int_{\partial\Omega} t(u)u\sinh(t(u)u)\,\mathrm{d}\sigma - \lambda b|\partial\Omega| \\ &= \tfrac{1}{4}t(u)^2\|\nabla u\|^2 - \lambda b|\partial\Omega|. \end{aligned}$$

Since $\{J(u_n)\}_n$ is bounded, it follows that so is the sequence $\{\|\nabla v_n\|\}_n = \{t(u_n)\|\nabla u_n\|\}_n$. Using (2.13), we now also get that $\{v_n \sinh(v_n)\}_n$ is bounded in $L^1(\partial\Omega)$, and thus that the sequence $\{v_n\}_n$ is bounded in $L^2(\partial\Omega)$. We may finally conclude that $\{v_n\}_n$ is bounded in $H^1(\Omega)$.

As $\{u_n\}_n$ is a sequence in the unit ball of $H^1(\Omega)$, and as $0 < J(u_n) < \infty$ for n sufficiently large,

$$-2\alpha_n = \langle J'(u_n), u_n\rangle - \alpha_n\langle F'(u_n), u_n\rangle = \langle \varepsilon_n, u_n\rangle \to 0.$$

Here we use the facts that $\langle J'(u_n), u_n\rangle = 0$ and $\langle F'(u_n), u_n\rangle = 2F(u_n) = 2$. Thus the sequence $\{\alpha_n\}_n$ converges to zero as $n \to +\infty$.

Using the relation between $E'(v_n)$ and $J'(u_n)$ stated in lemma 2.5, we observe that v_n satisfies the variational identity

$$\begin{aligned} t(u_n)\biggl[\int_{\Omega} \nabla v_n(x)\cdot\nabla w(x)\,\mathrm{d}x - \lambda\int_{\partial\Omega} \sinh(v_n(\sigma))w(\sigma)\,\mathrm{d}\sigma\biggr] &= \langle J'(u_n), w\rangle \\ &= \langle \varepsilon_n, w\rangle + 2\alpha_n(u_n|w)_1 \end{aligned} \tag{2.14}$$

for all $w \in H^1(\Omega)$. The sequence $\{v_n\}_n$ being bounded, we may extract a subsequence $\{v_{n_j}\}_{j\geqslant 1}$ such that $v_{n_j} \rightharpoonup v$ weakly in $H^1(\Omega)$, $v_{n_j} \to v$ strongly in $L^2(\Omega)$. Due to lemma 2.2, it follows that $\sinh(v_{n_j}) \to \sinh(v)$ strongly in $L^2(\partial\Omega)$. By further extraction of a subsequence, if necessary, we may also obtain that $t(u_{n_j}) \to t \geqslant 0$. The limit t must indeed satisfy $t > 0$; because, if $t(u_{n_j}) \to 0$, then we would have that $v_{n_j} \to 0$ in $H^1(\Omega)$ and thus $E(v_{n_j}) \to 0$, but this would contradict the fact that $E(v_{n_j}) = J(u_{n_j}) \to c > 0$. Consider the linear forms $R_{n_j} \in (H^1(\Omega))'$ as being defined by

$$R_{n_j}(w) := \int_{\Omega} v_{n_j}(x)w(x)\,\mathrm{d}x + \lambda\int_{\partial\Omega}\sinh(v_{n_j})w\,\mathrm{d}\sigma + \frac{1}{t(u_{n_j})}[\langle\varepsilon_{n_j}, w\rangle + 2\alpha_{n_j}(u_{n_j}|w)_1]$$

for $w \in H^1(\Omega)$. Rearranging (2.14), we get

$$\int_{\Omega} \nabla v_{n_j}(x)\cdot\nabla w(x)\,\mathrm{d}x + \int_{\Omega} v_{n_j}(x)w(x)\,\mathrm{d}x = R_{n_j}(w). \tag{2.15}$$

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From the discussion above, it follows that the linear forms R_{n_j} converge to R, defined by

$$R(w) := \int_\Omega vw\,\mathrm{d}x + \lambda\int_{\partial\Omega}\sinh(v)w\,\mathrm{d}\sigma$$

in $(H^1(\Omega))'$. We therefore conclude that $v_{n_j} \to v_*$ in $H^1(\Omega)$, where v_* solves

$$\int_\Omega \nabla v_*\cdot\nabla w\,\mathrm{d}x + \int_\Omega v_* w\,\mathrm{d}x = R(w) = \int_\Omega vw\,\mathrm{d}x + \lambda\int_{\partial\Omega}\sinh(v)w\,\mathrm{d}\sigma.$$

As $v_{n_j} \rightharpoonup v$ in $H^1(\Omega)$, by the uniqueness of the limit, it follows that $v_* = v$, that is, $v_{n_j} \to v$ in $H^1(\Omega)$, where v solves

$$\Delta v = 0 \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \boldsymbol{n}} = \lambda\sinh(v) \quad \text{on } \partial\Omega.$$

□

We can now formulate and prove our main result concerning the existence of infinitely many solutions to the problem (2.1).

THEOREM 2.10. *Let $\lambda > 0$ be fixed, with $\mu_{k_0} \leqslant \lambda < \mu_{k_0+1}$, $k_0 \geqslant 1$. Then $\{c_k\}_{k\geqslant k_0+1}$ is a non-decreasing sequence of finite positive critical values for J, and $c_k \to +\infty$ as $k \to +\infty$. In particular, for any fixed $\lambda > 0$, the boundary-value problem (2.1) has infinitely many solutions $\{v_k\}_k$ such that $E(v_k) \to +\infty$, and $\|v_k\|_1 \to +\infty$, as $k \to +\infty$.*

Proof. This is a classical result in the theory of critical points. Any even non-constant C^1 function J (with a lower bound), which satisfies a Palais–Smale condition, such as the one given in lemma 2.9, possesses an unbounded non-decreasing sequence of critical values on the sphere Σ, constructed exactly as we have defined the sequence $\{c_k\}$ (see, for example, [7, théorème 5.5, ch. 4, pp. 212–213] or [10]). Lemmas 2.4 and 2.8 ensure that we may disregard the 'set of non-differentiability' $\{J(u) = 0\} \cup \{J(u) = \infty\}$. The fact that $E(v_k) \to \infty$ follows from the unboundedness of the critical values; to see that $\|v_k\|_1 \to \infty$, simply note that $\frac{1}{2}\|v_k\|_1^2 \geqslant E(v_k)$. □

3. Auxiliary results and *a priori* estimates for the variational solutions

In this section, we establish some results concerning lower and upper bounds for the solutions constructed in the previous section. We suppose that

$$0 = \mu_1 < \lambda < \mu_2.$$

Briefly speaking, we establish two main results. Lemmas 3.2 and 3.5 show that a branch of solutions corresponding to any of the critical values $c_k(\lambda)$, $k \geqslant 2$, blows up (in energy) as $\lambda \to 0^+$, the energy being of the order $\log(1/\lambda)$. Corollary 3.7 shows that the normal currents $\partial v_k/\partial\boldsymbol{n}$ stay bounded in $L^1(\partial\Omega)$ as $\lambda \to 0^+$.

Our first task is to establish a lower bound for the energy of solutions. Consider a finite-energy solution, $v \neq 0$, to

$$\left.\begin{aligned} -\Delta v &= 0 && \text{in } \Omega,\\ \frac{\partial v}{\partial \boldsymbol{n}} &= \lambda\sinh(v) && \text{on } \partial\Omega.\end{aligned}\right\} \tag{3.1}$$

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We wish to prove that the energy, $E(v)$, as well as the expression $\|\nabla v\|^2_{L^2(\Omega)}$, may be bounded from below by $a\log(1/\lambda)$ (as $\lambda \to 0^+$). We first observe that, by Green's formula,

$$0 = \int_{\partial\Omega} \frac{\partial v(\sigma)}{\partial \boldsymbol{n}}\, \mathrm{d}\sigma = \lambda \int_{\partial\Omega} \sinh(v(\sigma))\, \mathrm{d}\sigma.$$

Therefore, the solution v may be written in the form $v = v^0 + s$, with $v^0 \in H^1(\Omega)$, and $s \in \mathbb{R}$, satisfying

$$\left.\begin{aligned} -\Delta v^0 &= 0 && \text{in } \Omega, \\ \frac{\partial v^0}{\partial \boldsymbol{n}} &= \lambda \sinh(v^0 + s) && \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.2}$$

and

$$\int_{\partial\Omega} v^0(\sigma)\, \mathrm{d}\sigma = 0, \qquad \int_{\partial\Omega} \sinh(v^0(\sigma) + s)\, \mathrm{d}\sigma = 0. \tag{3.3}$$

It is easy to find a formula for s in terms of v^0. Indeed, due to the relation

$$\sinh(\alpha + \beta) = \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta),$$

the second identity in (3.3) yields

$$\tanh(s) = \frac{-\int_{\partial\Omega} \sinh(v^0(\sigma))\, \mathrm{d}\sigma}{\int_{\partial\Omega} \cosh(v^0(\sigma))\, \mathrm{d}\sigma},$$

or, equivalently (since $\tanh(s) = (\mathrm{e}^{2s} - 1)/(\mathrm{e}^{2s} + 1)$),

$$s = s(v^0) := \tfrac{1}{2} \log \frac{\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\, \mathrm{d}\sigma}{\int_{\partial\Omega} \mathrm{e}^{v^0(\sigma)}\, \mathrm{d}\sigma}. \tag{3.4}$$

We need the following estimate for s in terms of $\|\nabla v^0\|^2_{L^2(\Omega)}$.

Lemma 3.1. *Let $s(v^0)$ be as defined in (3.4) for $v^0 \in H^1(\Omega)$, with $\int_{\partial\Omega} v^0(\sigma)\, \mathrm{d}\sigma = 0$. There exist two positive constants, C_1 and C_2, depending only on Ω, such that*

$$|s(v^0)| \leqslant C_1 + C_2 \|\nabla v^0\|^2_{L^2(\Omega)}.$$

Proof. By Hölder's inequality, we have

$$\left(\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\, \mathrm{d}\sigma\right)\left(\int_{\partial\Omega} \mathrm{e}^{v^0(\sigma)}\, \mathrm{d}\sigma\right) \geqslant |\partial\Omega|^2.$$

Since $s(-v^0) = -s(v^0)$, we may assume, without loss of generality, that $s(v^0) \geqslant 0$, that is,

$$\int_{\partial\Omega} \mathrm{e}^{v^0(\sigma)}\, \mathrm{d}\sigma \leqslant \int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\, \mathrm{d}\sigma$$

(if not, we simply prove the estimate of this lemma for $-v^0$ in place of v^0). Therefore,

$$\frac{\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\, \mathrm{d}\sigma}{\int_{\partial\Omega} \mathrm{e}^{v^0(\sigma)}\, \mathrm{d}\sigma} \leqslant \frac{1}{|\partial\Omega|^2}\left(\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\, \mathrm{d}\sigma\right)^2,$$

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and, as $|s(v^0)| = s(v^0)$,

$$|s(v^0)| \leqslant -\log|\partial\Omega| + \log\biggl(\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\,\mathrm{d}\sigma\biggr). \tag{3.5}$$

Lemma 2.1 yields that

$$\int_{\partial\Omega} \mathrm{e}^{-v^0(\sigma)}\,\mathrm{d}\sigma \leqslant \int_{\partial\Omega} 2\cosh(v^0)\,\mathrm{d}\sigma \leqslant C\exp(\beta\|v^0\|_1^2) \leqslant C\exp(\beta\|\nabla v^0\|_{L^2(\Omega)}^2).$$

This inequality, together with (3.5), immediately leads to the desired estimate. □

We can now state and prove our result concerning the lower estimate for the energy of solutions.

LEMMA 3.2. *Assume that* $0 < \lambda < \mu_2$ *and that* $v \in H^1(\Omega)$, $v \neq 0$, *is a solution to (3.1). There exists two constants* $a, b > 0$, *independent of* λ *and* v, *such that*

$$E(v) \geqslant a\log\biggl(\frac{1}{\lambda}\biggr) - b, \qquad \|\nabla v\|_{L^2(\Omega)}^2 \geqslant a\log\biggl(\frac{1}{\lambda}\biggr) - b.$$

Proof. Multiplying equation (3.1) by v and integrating by parts, we obtain

$$\|\nabla v\|_{L^2(\Omega)}^2 = \lambda\int_{\partial\Omega} v(\sigma)\sinh(v(\sigma))\,\mathrm{d}\sigma. \tag{3.6}$$

For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$0 \leqslant \cosh(\theta) - 1 \leqslant C_\varepsilon + \varepsilon\theta\sinh(\theta) \quad \forall\theta\in\mathbb{R}.$$

It follows, due to (3.6), that

$$\begin{aligned}
\tfrac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2 &\geqslant E(v) \\
&= \tfrac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2 - \lambda\int_{\partial\Omega}[\cosh(v(\sigma)) - 1]\,\mathrm{d}\sigma \\
&\geqslant \tfrac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2 - \lambda\varepsilon\int_{\partial\Omega} v(\sigma)\sinh(v(\sigma))\,\mathrm{d}\sigma - \lambda C_\varepsilon \\
&\geqslant (\tfrac{1}{2} - \varepsilon)\|\nabla v\|_{L^2(\Omega)}^2 - \lambda C_\varepsilon.
\end{aligned}$$

This shows that finding a lower bound for $\|\nabla v\|_{L^2(\Omega)}^2$ is equivalent to finding a lower bound for $E(v)$ (for λ in a bounded interval). Let $v = v^0 + s(v^0)$ be the splitting introduced earlier. By the mean-value theorem,

$$\sinh(v^0(\sigma) + s(v^0)) - \sinh(s(v^0)) = v^0(\sigma)\cosh(s(v^0) + \theta v^0(\sigma))$$

for some $\theta \in [0, 1]$, and thus, noting that $\int_{\partial\Omega} v^0(\sigma)\sinh(s(v^0))\,\mathrm{d}\sigma = 0$, we calculate

$$\begin{aligned}
\|\nabla v^0\|_{L^2(\Omega)}^2 &= \lambda\int_{\partial\Omega} v^0(\sigma)[\sinh(v^0(\sigma) + s(v^0)) - \sinh(s(v^0))]\,\mathrm{d}\sigma \\
&= \lambda\int_{\partial\Omega} |v^0(\sigma)|^2\cosh(s(v^0) + \theta v^0(\sigma))\,\mathrm{d}\sigma
\end{aligned}$$

$$\leqslant \lambda e^{|s(v^0)|} \int_{\partial\Omega} |v^0(\sigma)|^2 e^{|v^0(\sigma)|} \,d\sigma$$
$$\leqslant \lambda e^{|s(v^0)|} \|v^0\|^2_{L^4(\partial\Omega)} \|e^{|v^0|}\|_{L^2(\partial\Omega)}$$
$$\leqslant \lambda\sqrt{2} e^{|s(v^0)|} \|v^0\|^2_{L^4(\partial\Omega)} \|\cosh(2v^0)\|\|^{1/2}_{L^1(\partial\Omega)}.$$

Here we have also used the fact that

$$\cosh(s(v^0) + \theta v^0(\sigma)) \leqslant \exp(|s(v^0)| + |v^0(\sigma)|).$$

As $\|v^0\|^2_{L^4(\partial\Omega)} \leqslant C\|\nabla v^0\|^2_{L^2(\Omega)}$, the above estimate and lemma 2.1 yield

$$\|\nabla v^0\|^2_{L^2(\Omega)} \leqslant \lambda C e^{|s(v^0)|} \|\nabla v^0\|^2_{L^2(\Omega)} e^{\beta\|\nabla v^0\|^2_{L^2(\Omega)}}. \tag{3.7}$$

Since v is a non-zero solution, it is also non-constant. This implies $\nabla v^0 \neq 0$ and, from (3.7), we now get that

$$\lambda e^{|s(v^0)|} e^{\beta\|\nabla v^0\|^2_{L^2(\Omega)}} \geqslant c$$

for some constant $c > 0$, independent of λ and v. This estimate, in combination with lemma 3.1, implies that $\|\nabla v^0\|^2_{L^2(\Omega)} \geqslant a\log(\lambda^{-1}) - b$ for some positive constants a and b, independent of λ and v. □

Remark 3.3. One may easily check that the pair v^0, s is a solution to (3.2)–(3.3) if and only if $s = s(v^0)$ (as given by (3.4)) and v^0 is a critical point of the functional

$$E_0(w) := \tfrac{1}{2}\|\nabla w\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} [\cosh(w(\sigma) + s(w)) - 1] \,d\sigma$$

on the space

$$\tilde{H}_0 := \biggl\{ w \in H^1(\Omega);\ \int_{\partial\Omega} w(\sigma)\,d\sigma = 0 \biggr\}.$$

Also note that $E(v) = E(v^0 + s(v^0)) = E_0(v^0)$. Furthermore, the v^0 corresponding to any non trivial solution to (3.2)–(3.3) belongs to the manifold

$$\tilde{\Sigma} := \biggl\{ w \in \tilde{H}_0;\ w \neq 0,\ \int_{\Omega} |\nabla w|^2 \,dx = \lambda \int_{\partial\Omega} \sinh(w(\sigma) + s(w)) w(\sigma)\,d\sigma \biggr\},$$

and using the same kind of arguments as in § 2, one can show that, for $0 < \lambda < \mu_2$, there exists such a $v^0 \in \tilde{\Sigma}$ that additionally minimizes the energy E_0 on $\tilde{\Sigma}$. In this way, one sees that, for this range of λ, equation (2.1) possesses a *ground state*, that is, a non-trivial solution that has minimal energy among all possible solutions. We are not going into the details of this argument in the present paper.

In order to be able to analyse the blow-up, as $\lambda \to 0^+$, of the variational solutions we obtained in the previous section, we need to estimate the critical values $c_k(\lambda)$ for a fixed $k \geqslant 2$, assuming $0 = \mu_1 < \lambda < \mu_2$. As mentioned in § 1, the special case when Ω is a disc was considered in [4], and an entire set of explicit solutions was constructed using as building blocks appropriately modified versions of the fundamental solution $G(x) := -\log(|x|^2)/4\pi$. The main idea in the following proof

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of upper estimates for $c_k(\lambda)$ (on an arbitrary domain) is to introduce sets $A \in \mathfrak{A}_k$, constructed from similarly modified versions of G.

To be more specific, we choose k distinct points $\sigma_1, \dots, \sigma_k$ on the boundary $\partial\Omega$ and, for $\varepsilon > 0$, $R > 0$, we introduce the functions

$$G_j(x) := -\log(\varepsilon^2 + |x - \sigma_j|^2)$$

and the set

$$A_{\varepsilon,R} := \biggl\{\sum_{j=1}^k \alpha_j G_j;\ \int_\Omega \biggl(\sum_{j=1}^k \alpha_j \nabla G_j(x)\biggr)^2 \,\mathrm{d}x + \int_{\partial\Omega} \biggl(\sum_{j=1}^k \alpha_j G_j(\sigma)\biggr)^2 \,\mathrm{d}\sigma = R^2\biggr\}. \tag{3.8}$$

LEMMA 3.4. *Let $k \geqslant 2$ and the points $\sigma_j \in \partial\Omega$, $1 \leqslant j \leqslant k$, be fixed, and let the set $A_{\varepsilon,R}$, $0 < \varepsilon$, $0 < R$, be defined as above. There exists $\lambda_* > 0$, only depending on k, the points σ_j, and Ω, such that, given any $0 < \lambda < \lambda_*$, one may find $\varepsilon > 0$ (of the order λ) and $R > 0$ (of the order $\sqrt{(\log 1/\lambda)}$) for which one has*

$$\|\nabla v\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} v(\sigma) \sinh(v(\sigma)) \,\mathrm{d}\sigma \leqslant 0 \quad \forall v \in A_{\varepsilon,R}.$$

Proof. A simple calculation shows that $\nabla G_j(x) = -2(x - \sigma_j)(\varepsilon^2 + |x - \sigma_j|^2)^{-1}$, so that

$$\|\nabla G_j\|^2_{L^2(\Omega)} = C_j \log\biggl(\frac{1}{\varepsilon}\biggr) + O(1)$$

and

$$\begin{aligned} c\sum_{j=1}^k \alpha_j^2 \log\biggl(\frac{1}{\varepsilon}\biggr) &\leqslant \int_\Omega \biggl(\sum_{j=1}^k \alpha_j \nabla G_j(x)\biggr)^2 \,\mathrm{d}x + \int_{\partial\Omega} \biggl(\sum_{j=1}^k \alpha_j G_j(\sigma)\biggr)^2 \,\mathrm{d}\sigma \\ &\leqslant C \sum_{j=1}^k \alpha_j^2 \log\biggl(\frac{1}{\varepsilon}\biggr), \end{aligned} \tag{3.9}$$

say, for $\varepsilon < \frac{1}{2}$. The positive constants C_j, C and c depend only on the points σ_j and on Ω.

Using the notation $v = \sum_{j=1}^k \alpha_j G_j \in A_{\varepsilon,R}$, we also have

$$\int_{\partial\Omega} v \sinh(v) \,\mathrm{d}\sigma = \int_{\partial\Omega} |v| \sinh(|v|) \,\mathrm{d}\sigma = \frac{1}{2} \int_{\partial\Omega} |v(\sigma)| \exp(|v(\sigma)|) \,\mathrm{d}\sigma + O(R). \tag{3.10}$$

Let j_0 denote an index with $|\alpha_{j_0}| := \max(|\alpha_1|, \dots, |\alpha_k|)$, and choose $\rho > 0$ sufficiently small so that

$$\sum_{j \neq j_0} |\log|\sigma - \sigma_j|^2| < \tfrac{1}{2} |\log|\sigma - \sigma_{j_0}|^2| \quad \text{on } \partial\Omega \cap B_\rho(\sigma_{j_0}).$$

For ε sufficiently small (exactly how small depends only on the points σ_j, $1 \leqslant j \leqslant k$), we then have

$$\biggl|\sum_{j \neq j_0} \alpha_j G_j(\sigma)\biggr| \leqslant \tfrac{1}{2} |\alpha_{j_0} G_{j_0}(\sigma)| \quad \text{on } \partial\Omega \cap B_\rho(\sigma_{j_0}),$$

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and therefore $|v(\sigma)| \geqslant \frac{1}{2}|\alpha_{j_0} G_{j_0}(\sigma)|$ on $\partial\Omega \cap B_\rho(\sigma_{j_0})$, so that

$$|v| \exp(|v(\sigma)|) \geqslant \tfrac{1}{2}|\alpha_{j_0} G_{j_0}(\sigma)| \exp(\tfrac{1}{2}|\alpha_{j_0} G_{j_0}(\sigma)|) \quad \text{on } \partial\Omega \cap B_\rho(\sigma_{j_0}). \tag{3.11}$$

By elementary calculations, one easily sees that there exist constants $a_0, b_0 > 0$ and a_1, b_1, depending continuously on $\alpha \geqslant 1$, such that

$$\int_0^{1/\varepsilon} (1+s^2)^{-\alpha} \, \mathrm{d}s = a_0(\alpha) + a_1(\alpha)\varepsilon^{2\alpha-1} + o(\varepsilon^{2\alpha}),$$

$$\int_0^{1/\varepsilon} \log(1+s^2)(1+s^2)^{-\alpha} \, \mathrm{d}s = b_0(\alpha) + b_1(\alpha)\log(\varepsilon)\varepsilon^{2\alpha-1} + o(\log(\varepsilon)\varepsilon^{2\alpha-1}),$$

and therefore

$$\begin{aligned}\int_0^d |\log(\varepsilon^2+t^2)|(\varepsilon^2+t^2)^{-\alpha} \, \mathrm{d}t \\ &= \varepsilon^{-2\alpha+1} \int_0^{d/\varepsilon} |\log(\varepsilon^2) + \log(1+s^2)|(1+s^2)^{-\alpha} \, \mathrm{d}t \\ &= 2a_0(\alpha)\varepsilon^{-2\alpha+1} \log\left(\frac{1}{\varepsilon}\right) + o(\varepsilon^{-2\alpha+1}\log\varepsilon),\end{aligned} \tag{3.12}$$

where the remainder term depends on the constant d, but is 'little o' of $\varepsilon^{-2\alpha+1}\log\varepsilon$ independently of α and ε. Using the estimate (3.12), together with the inequality (3.10) and (3.11), we get (provided $|\alpha_{j_0}| \geqslant 2$)

$$\begin{aligned}\int_{\partial\Omega} v \sinh(v) \, \mathrm{d}\sigma &\geqslant \frac{1}{2} \int_{\partial\Omega \cap B_\rho(\sigma_{j_0})} |\alpha_{j_0} G_{j_0}(\sigma)| \exp(\tfrac{1}{2}|\alpha_{j_0} G_{j_0}(\sigma)|) \, \mathrm{d}\sigma + O(R) \\ &\geqslant c\tilde{a}_0(|\alpha_{j_0}|) \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{1-|\alpha_{j_0}|} + |\alpha_{j_0}| o(\varepsilon^{1-|\alpha_{j_0}|}\log(\varepsilon)) + O(R),\end{aligned}$$

where $\tilde{a}_0(|\alpha_{j_0}|) := a_0(\frac{1}{2}|\alpha_{j_0}|)|\alpha_{j_0}|$ and $c > 0$ is a constant, independent of ε, α_{j_0} and R. Using this lower estimate we have, for $v = \sum_{j=1}^k \alpha_j G_j \in A_{\varepsilon,R}$,

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} v \sinh(v) \, \mathrm{d}\sigma \\ &\leqslant R^2 - \lambda\left[c\tilde{a}_0(|\alpha_{j_0}|) \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{1-|\alpha_{j_0}|} + |\alpha_{j_0}| o(\varepsilon^{1-|\alpha_{j_0}|}\log(\varepsilon)) + O(R)\right] \\ &\leqslant R^2 - \lambda c\tilde{a}_0(|\alpha_{j_0}|) \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{1-|\alpha_{j_0}|} + \lambda|\alpha_{j_0}| o(\varepsilon^{1-|\alpha_{j_0}|}\log(\varepsilon)) + \lambda O(R),\end{aligned} \tag{3.13}$$

provided that $|\alpha_{j_0}| \geqslant 2$. According to (3.9), for $v = \sum_{j=1}^k \alpha_j G_j \in A_{\varepsilon,R}$, we have that

$$\frac{R}{\sqrt{kC\log(1/\varepsilon)}} \leqslant |\alpha_{j_0}| \leqslant \frac{R}{\sqrt{c\log(1/\varepsilon)}}. \tag{3.14}$$

If we now select

$$R = 2\sqrt{kC\log(1/\varepsilon)}, \tag{3.15}$$

with C being the same constant as in equation (3.14), then it follows that $2 \leqslant |\alpha_{j_0}| \leqslant 2\sqrt{kC/c}$, and thus $\tilde{a}_0(|\alpha_{j_0}|) > 0$ is uniformly bounded and bounded away from zero. From a combination of this and inequality (3.13), we now get, for some constants C_1 and $c_1 > 0$,

$$\begin{aligned}\|\nabla v\|^2 - \lambda \int_{\partial\Omega} v \sinh(v)\,\mathrm{d}\sigma \\ &\leqslant 4kC \log\left(\frac{1}{\varepsilon}\right) - \lambda c \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{1-|\alpha_{j_0}|} + \lambda o(\varepsilon^{1-|\alpha_{j_0}|}\log(\varepsilon)) \\ &\leqslant C_1 \log\left(\frac{1}{\varepsilon}\right) - \lambda c_1 \log\left(\frac{1}{\varepsilon}\right)\varepsilon^{-1} \qquad (3.16)\end{aligned}$$

for ε sufficiently small (how small depends only on the points σ_j, k and Ω). By choosing $\varepsilon = c_1\lambda/C_1$, the estimate (3.16) yields

$$\|\nabla v\|^2_{L^2(\Omega)} - \lambda \int_{\partial\Omega} v \sinh(v)\,\mathrm{d}\sigma \leqslant 0$$

for all $v \in A_{\varepsilon,R}$ and for all $\lambda < \lambda_*$ (to guarantee that ε is sufficiently small). With the above choice for ε, R is given by

$$R^2 = C \log\left(\frac{1}{\lambda}\right) + D \qquad (3.17)$$

for some constants C and D, C being positive. This concludes the proof of the lemma. □

We may now obtain the desired upper estimate on $c_k(\lambda)$.

LEMMA 3.5. *For any $k \geqslant 2$, there exist positive constants $C_*(k)$ and λ_* such that*

$$c_k(\lambda) \leqslant C_*(k) \log\left(\frac{1}{\lambda}\right), \quad 0 < \lambda < \lambda_*.$$

Proof. Let ε, R and $A_{\varepsilon,R}$ be as described in lemma 3.4. For λ sufficiently small (and thus ε sufficiently small), the set $A_{\varepsilon,1}$ (defined in (3.8)) is in $\mathfrak{A}_k$, and $A_{\varepsilon,1} \cap H^1_0(\Omega)$ is empty. Therefore,

$$0 < c_k(\lambda) = \inf_{A\in\mathfrak{A}_k} \sup_{v\in A} J(v) \leqslant \max_{v\in A_{\varepsilon,1}} J(v) < \infty.$$

Due to the facts that $A_{\varepsilon,R} = RA_{\varepsilon,1}$ and that $J(Rv) = J(v)$ for $v \in A_{\varepsilon,1}$, we conclude that

$$0 < c_k(\lambda) \leqslant \max_{v\in A_{\varepsilon,R}} J(v) = J(v_*) < \infty$$

for some $v_* \in A_{\varepsilon,R}$. Recall that in lemma 2.3 we defined $f(t) := E(tv_*)$, and recall that $J(v_*) = f(t(v_*)v_*)$. Using the estimate of R given in lemma 3.4 (see (3.17)), we immediately get

$$\begin{aligned} J(v_*) &= \tfrac{1}{2}t(v_*)^2\|\nabla v_*\|^2_{L^2(\Omega)} - \lambda\int_{\partial\Omega}[\cosh(t(v_*)v_*) - 1]\,\mathrm{d}\sigma \\ &\leqslant \tfrac{1}{2}t(v_*)^2\|\nabla v_*\|^2_{L^2(\Omega)} \\ &\leqslant \tfrac{1}{2}t(v_*)^2R^2 \\ &\leqslant C_*(k)t(v_*)^2\log\biggl(\frac{1}{\lambda}\biggr) \end{aligned}$$

for some constant $C_*(k)$ and for λ sufficiently small, say, $0 < \lambda < \lambda_*$. As we have seen in the proof of lemma 2.3, f achieves its maximum at a unique point $t(v_*) > 0$. We also know that f' is strictly concave on $(0,\infty)$ (see the proof of lemma 2.3), while $f'(0) = f'(t(v_*)) = 0$, and therefore $f'(t) > 0$ for $0 < t < t(v_*)$. By lemma 3.4, we have

$$f'(1) = \|\nabla v_*\|^2_{L^2(\Omega)} - \lambda\int_{\partial\Omega} v_*\sinh(v_*)\,\mathrm{d}\sigma \leqslant 0,$$

which now, in turn, implies that $t(v_*) \leqslant 1$. This finally yields the estimate

$$0 < c_k(\lambda) \leqslant C_*(k)t(v_*)^2\log\biggl(\frac{1}{\lambda}\biggr) \leqslant C_*(k)\log\biggl(\frac{1}{\lambda}\biggr), \quad 0 < \lambda < \lambda_*,$$

for some constants $C_*(k)$ and λ_*, depending only on k and Ω. □

Next we state an elementary lemma from integration theory.

LEMMA 3.6. *Let $(X, \mathrm{d}\mu)$ be a non-negative measure space, let a, b be two given positive constants, and let w be a measurable function on X with the property that, for a certain $\lambda \in (0,1)$, one has*

$$\int_X |w|\mathrm{e}^{|w|}\,\mathrm{d}\mu \leqslant \frac{a}{\lambda}\log\biggl(\frac{1}{\lambda}\biggr) + b. \tag{3.18}$$

Then there exists two positive constants C_1, C_2, depending only on a, b and $\mu(X)$, such that

$$\int_X \mathrm{e}^{|w|}\,\mathrm{d}\mu \leqslant \frac{C_1}{\lambda} + C_2.$$

Proof. For any $\varepsilon \in (0,1)$, there exists a constant $C(\varepsilon)$ such that

$$\mathrm{e}^s \leqslant \varepsilon s\mathrm{e}^s + C(\varepsilon)$$

for any $s \geqslant 0$. In fact, $C(\varepsilon) := \max_{s\geqslant 0}(1-\varepsilon s)\mathrm{e}^s = (1-\varepsilon\theta)\mathrm{e}^\theta$, where θ is given by $\theta := (1-\varepsilon)/\varepsilon$, and so $C(\varepsilon) = \varepsilon\exp((1-\varepsilon)/\varepsilon)$. It follows that

$$\begin{aligned} \int_X \mathrm{e}^{|w|}\,\mathrm{d}\mu &\leqslant C(\varepsilon)\mu(X) + \varepsilon\int_X |w|\mathrm{e}^{|w|}\,\mathrm{d}\mu \\ &\leqslant C(\varepsilon)\mu(X) + \frac{a\varepsilon}{\lambda}\log\biggl(\frac{1}{\lambda}\biggr) + \varepsilon b \\ &\leqslant \varepsilon\biggl(\mu(X)\exp\biggl(\frac{1-\varepsilon}{\varepsilon}\biggr) + \frac{a}{\lambda}\log\biggl(\frac{1}{\lambda}\biggr) + b\biggr). \end{aligned} \tag{3.19}$$

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Now choose ε to be given by

$$\frac{1}{\varepsilon} = 1 + \log\left(b + 1 + \frac{a}{\lambda}\log\frac{1}{\lambda}\right).$$

This ensures that $c(1+\log(1/\lambda))^{-1} \leqslant \varepsilon \leqslant C(1+\log(1/\lambda))^{-1}$, with positive constants c and C only depending on a and b. It also ensures that

$$\exp\left(\frac{1-\varepsilon}{\varepsilon}\right) = b + 1 + \frac{a}{\lambda}\log\frac{1}{\lambda}.$$

A combination of these two facts with the estimate (3.19) yields the result of the lemma. □

Corollary 3.7. *For $k \geqslant 2$ let $\lambda_* < \mu_2$ be as in lemma 3.5 and assume that $0 < \lambda < \lambda_*$. There exists a constant $D_*(k)$, depending only on k and Ω, such that, if v_k (equal to $v_{k,\lambda}$) is a variational solution obtained in theorem 2.10, with $E(v_k) = c_k(\lambda)$, then*

$$\int_{\partial\Omega}\left|\frac{\partial v_k(\sigma)}{\partial \boldsymbol{n}}\right| \mathrm{d}\sigma = \lambda \int_{\partial\Omega} |\sinh(v_k(\sigma))| \,\mathrm{d}\sigma \leqslant D_*(k).$$

Proof. From the beginning of the proof of lemma 3.2, it follows (by taking $\varepsilon = \frac{1}{4}$) that

$$\|\nabla v_k\|^2_{L^2(\Omega)} \leqslant 4E(v_k) + \lambda C.$$

Upon combination with lemma 3.5, we now arrive at

$$\begin{aligned}\lambda \int_{\partial\Omega} v_k(\sigma)\sinh(v_k(\sigma))\,\mathrm{d}\sigma &= \|\nabla v_k\|^2_{L^2(\Omega)} \\ &\leqslant 4E(v_k) + \lambda C \\ &\leqslant C\log(\lambda^{-1}) + D\end{aligned}$$

for some positive constants C and D, depending only on k and Ω. Using the fact that $|\theta|\mathrm{e}^{|\theta|} \leqslant 2\theta\sinh(\theta) + \mathrm{e}^{-1}$ for all $\theta \in \mathbb{R}$, we conclude that there exist two positive constants a and b such that

$$\int_{\partial\Omega} |v_k(\sigma)|\mathrm{e}^{|v_k(\sigma)|}\,\mathrm{d}\sigma \leqslant \frac{a}{\lambda}\log\left(\frac{1}{\lambda}\right) + b.$$

Now, lemma 3.6 shows that $\int_{\partial\Omega} \mathrm{e}^{|v_k(\sigma)|}\,\mathrm{d}\sigma \leqslant C_1\lambda^{-1} + C_2$, and this establishes the claim of the corollary. □

4. Blow-up and limit of solutions

Let $0 < \lambda_n < \lambda_*$ be a sequence tending to 0 and, for a fixed $k \geqslant 2$, let $v_{\lambda_n} = v_{k,\lambda_n}$ be solutions of

$$\left.\begin{aligned}\Delta v_{\lambda_n} &= 0 && \text{in } \Omega,\\ \frac{\partial v_{\lambda_n}}{\partial \boldsymbol{n}} &= \lambda_n \sinh(v_{\lambda_n}) && \text{on } \partial\Omega,\end{aligned}\right\} \tag{4.1}$$

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corresponding to the kth critical value (as constructed earlier in this paper). We have already established in lemmas 3.2 and 3.5 (see also the proof of corollary 3.7) that, for some positive constants a, b, C and D,

$$E(v_{\lambda_n}) \geqslant a \log\left(\frac{1}{\lambda_n}\right) - b, \tag{4.2}$$

$$\|\nabla v_{\lambda_n}\|^2 \leqslant C \log\left(\frac{1}{\lambda_n}\right) + D. \tag{4.3}$$

The fact that

$$\lambda_n \int_{\partial\Omega} |\sinh(v_{\lambda_n})| \, \mathrm{d}\sigma \leqslant C \tag{4.4}$$

is a direct consequence of lemma 3.6 and the estimate (4.3) (as we have already seen in the proof of corollary 3.7). We shall also use the decomposition $v_{\lambda_n} = v^0_{\lambda_n} + s_{\lambda_n}$ introduced earlier. In this context, we already have an estimate for the constant s_{λ_n} in terms of $v^0_{\lambda_n}$. However, here it will be more convenient to use an estimate based on the relation

$$|s_{\lambda_n}| = \left|\frac{1}{|\partial\Omega|}\int_{\partial\Omega} v_{\lambda_n} \, \mathrm{d}\sigma\right| \leqslant \int_{\partial\Omega} |v_{\lambda_n}| \frac{\mathrm{d}\sigma}{|\partial\Omega|}$$

and (4.4). We obtain

$$\begin{aligned}
|s_{\lambda_n}| &\leqslant \log\left(\exp\left(\int_{\partial\Omega} |v_{\lambda_n}| \frac{\mathrm{d}\sigma}{|\partial\Omega|}\right)\right) \\
&\leqslant \log\left(\int_{\partial\Omega} \exp(|v_{\lambda_n}|) \frac{\mathrm{d}\sigma}{|\partial\Omega|}\right) \\
&\leqslant \log\left(\int_{\partial\Omega} \frac{2}{|\partial\Omega|} |\sinh(v_{\lambda_n})| \, \mathrm{d}\sigma + 1\right) \\
&\leqslant \log\left(\frac{C}{\lambda_n} + 1\right) \\
&\leqslant \log\frac{1}{\lambda_n} + D.
\end{aligned} \tag{4.5}$$

Here we have used Jensen's inequality for the exponential function, and the inequality $\exp|t| \leqslant 2|\sinh t| + 1$. An essential aspect of the above estimate is that the constant in front of $\log(1/\lambda_n)$ is 1. In this section, we establish the following result concerning the behaviour of v_{λ_n} as $\lambda_n \to 0^+$.

THEOREM 4.1. *Let $v_{\lambda_n} \in H^1(\Omega)$, $\lambda_n \to 0^+$, be a sequence of solutions to (4.1), which additionally satisfy (4.2) and (4.3), and define*

$$v^0_{\lambda_n} = v_{\lambda_n} - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v_{\lambda_n} \, \mathrm{d}\sigma.$$

There exists a subsequence, also referred to as v_{λ_n}, a regular finite Borel measure μ (on $\partial\Omega$) and a finite set of points $\{x^{(i)}\}_{i=1}^N \subset \partial\Omega$, $N \geqslant 1$, such that

$$\lambda_n \sinh(v_{\lambda_n})|_{\partial\Omega} \to \mu,$$

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in the sense of measures, i.e.

$$\int_{\partial\Omega} \lambda_n \sinh(v_{\lambda_n})\varphi \,\mathrm{d}\sigma \to \int_{\partial\Omega} \varphi \,\mathrm{d}\mu \quad \textit{for all } \varphi \in C^0(\partial\Omega).$$

The points $x^{(i)}$, $i = 1, \dots, N$, are exactly the points at which μ has point masses, i.e. at which $\mu(\{x\}) \neq 0$. The same points $x^{(i)}$, $i = 1, \dots, N$, also represent the blow-up points for the sequence $v^0_{\lambda_n}$, in the sense that

$$\{x^{(i)}\}_{i=1}^N = \{x \in \bar{\Omega} : \exists x_n \to x,\ x_n \in \bar{\Omega},\ \textit{with } |v^0_{\lambda_n}(x_n)| \to \infty\}.$$

The proof of this theorem consists of an adaptation of the proof of the somewhat similar results for solutions to the boundary-value problem

$$\begin{aligned} \Delta v &= U(x)\mathrm{e}^v \quad \text{in } \Omega, \\ v &= 0 \qquad\quad\ \ \text{on } \partial\Omega \end{aligned}$$

found in [3] (see also [5] for a related result). For the convenience of the reader, we provide the details of this adaptation. We begin with two lemmas.

Lemma 4.2. *Let w be a classical solution to $\Delta w = 0$ in Ω, $\partial w/\partial \boldsymbol{n} = f$ on $\partial\Omega$, for some function f, satisfying $\int_{\partial\Omega} f \,\mathrm{d}\sigma = 0$. Let w be normalized by $\int_{\partial\Omega} w(\sigma)\,\mathrm{d}\sigma = 0$. For every $\delta \in (0, \pi)$, there exists a constant C_δ such that*

$$\int_{\partial\Omega} \exp\biggl[\frac{(\pi-\delta)|w(\sigma)|}{\|f\|_{L^1(\partial\Omega)}}\biggr] \mathrm{d}\sigma \leqslant C_\delta.$$

The constant C_δ depends only on δ and Ω.

Proof. Let $H(x, y)$, $y \in \partial\Omega$, denote the solution to

$$\left.\begin{aligned} \Delta_x H(x,y) &= 0 && \text{in } \Omega, \\ \frac{\partial H(x,y)}{\partial \boldsymbol{n}_x} &= \frac{1}{\pi}\frac{(x-y)\cdot \boldsymbol{n}_x}{|x-y|^2} - \frac{1}{|\partial\Omega|} && \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.6}$$

normalized with $\int_{\partial\Omega} H(\sigma, y)\,\mathrm{d}\sigma = 0$. We note that $H(\cdot, y)$ is in $C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$, and

$$C_0 := \max_{x\in\partial\Omega,\ y\in\partial\Omega} |H(x,y)| < \infty.$$

Define now Green's function for the Neumann boundary problem

$$N(x,y) = -\frac{1}{\pi}\log|x-y| + H(x,y),$$

$x \in \bar{\Omega} \setminus \{y\}$, $y \in \partial\Omega$. A simple calculation gives that the function $w(y)$, $y \in \partial\Omega$, is given by

$$w(y) = \int_{\partial\Omega} N(\sigma, y) f(\sigma)\,\mathrm{d}\sigma.$$

From Jensen's inequality (and the convexity of the exponential function), it follows that

$$\exp\biggl[\int_{\partial\Omega} (\pi-\delta)|N(\sigma,y)|\frac{|f(\sigma)|}{\|f\|_{L^1(\partial\Omega)}}\,\mathrm{d}\sigma\biggr] \leqslant \int_{\partial\Omega} \exp[(\pi-\delta)|N(\sigma,y)|]\frac{|f(\sigma)|}{\|f\|_{L^1(\partial\Omega)}}\,\mathrm{d}\sigma,$$

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and so we may estimate

$$\int_{\partial\Omega} \exp\left[\frac{(\pi-\delta)|w(y)|}{\|f\|_{L^1(\partial\Omega)}}\right] \mathrm{d}y \leqslant \int_{\partial\Omega}\int_{\partial\Omega} \exp[(\pi-\delta)|N(\sigma,y)|] \frac{|f(\sigma)|}{\|f\|_{L^1(\partial\Omega)}} \,\mathrm{d}\sigma \mathrm{d}y$$
$$\leqslant C_1 \mathrm{e}^{(\pi-\delta)C_0} \int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|\sigma-y|^{1-\delta/\pi}} \,\mathrm{d}y \frac{|f(\sigma)|}{\|f\|_{L^1(\partial\Omega)}} \,\mathrm{d}\sigma$$
$$\leqslant C_1 \mathrm{e}^{(\pi-\delta)C_0}\, D_\delta,$$

where

$$C_0 := \max_{x\in\partial\Omega,\, y\in\partial\Omega} |H(x,y)|, \qquad D_\delta := \max_{\sigma\in\partial\Omega} \int_{\partial\Omega} \frac{1}{|\sigma-y|^{1-\delta/\pi}} \,\mathrm{d}y,$$

and C_1 only depends on the diameter of Ω. □

LEMMA 4.3. *Let $\{w_n\}_n$ be a sequence of classical solutions to*

$$\Delta w_n = 0 \quad \text{in } \Omega,$$
$$\frac{\partial w_n}{\partial \boldsymbol{n}} = f_n \quad \text{on } \partial\Omega.$$

Suppose $\|w_n\|_{L^2(\Omega)} \leqslant C$ and suppose there exists a point $x_0 \in \partial\Omega$, a ball $B_{r_0}(x_0)$ of radius $r_0 > 0$ centred at x_0 and an index $s < \frac{1}{2}$ such that

$$\|f_n\|_{H^{-s}(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant C,$$

with C independent of n and H^{-s} denoting the dual of H^s. Then

$$\|w_n\|_{L^\infty(B_{r_0/2}(x_0)\cap\partial\Omega)} \leqslant \|w_n\|_{L^\infty(B_{r_0/2}(x_0)\cap\Omega)} \leqslant C$$

for some (other) C independent of n.

Proof. From interior elliptic estimates, it follows that

$$\|w_n\|_{H^{3/2-s}(B_{r_0/2}(x_0)\cap\Omega)} \leqslant C(\|f_n\|_{H^{-s}(B_{r_0}(x_0)\cap\partial\Omega)} + \|w_n\|_{L^2(B_{r_0}(x_0)\cap\Omega)}) \leqslant C$$

for any $0 < s < \frac{1}{2}$. Using Sobolev's imbedding theorem, more precisely, the fact that, for the two-dimensional domain Ω and $s < \frac{1}{2}$, one has

$$H^{3/2-s}(B_r(x_0)\cap\Omega) \subset C^0(\overline{B_r(x_0)\cap\Omega}),$$

we now get

$$\|w_n\|_{L^\infty(B_{r_0/2}(x_0)\cap\partial\Omega)} \leqslant \|w_n\|_{L^\infty(B_{r_0/2}(x_0)\cap\Omega)} \leqslant C\|w_n\|_{H^{3/2-s}(B_{r_0/2}(x_0)\cap\Omega)} \leqslant C.$$

□

Let $v^+_{\lambda_n}$ and $v^-_{\lambda_n}$ denote the positive part and the negative part of v_{λ_n}, respectively. Since $\{\lambda_n \sinh(v_{\lambda_n})\}$ is bounded in $L^1(\partial\Omega)$ (cf. (4.4)), it follows immediately that

$$\{\lambda_n \sinh(v^+_{\lambda_n})\} = \{\lambda_n(\sinh(v_{\lambda_n}))^+\} \quad \text{and} \quad \{\lambda_n \sinh(v^-_{\lambda_n})\} = \{\lambda_n(\sinh(v_{\lambda_n}))^-\}$$

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are bounded in $L^1(\partial\Omega)$. We may therefore extract a subsequence (in the following, also denoted v_{λ_n}), so that

$$\lambda_n \sinh(v^+_{\lambda_n}) \to \mu^+ \tag{4.7}$$

and

$$\lambda_n \sinh(v^-_{\lambda_n}) \to \mu^-, \tag{4.8}$$

where μ^+ and μ^- are non-negative regular finite Borel measures and the convergence is in the aforementioned sense of measures. Along this subsequence, we also have

$$\lambda_n \sinh(v_{\lambda_n}) = \lambda_n \sinh(v^+_{\lambda_n}) - \lambda_n \sinh(v^-_{\lambda_n}) \to \mu^+ - \mu^-,$$
$$\lambda_n |\sinh(v_{\lambda_n})| = \lambda_n \sinh(v^+_{\lambda_n}) + \lambda_n \sinh(v^-_{\lambda_n}) \to \mu^+ + \mu^-.$$

We use the notation $|\mu| = \mu^+ + \mu^-$ (this is frequently called the total variation measure associated with μ).

We are now ready to proceed with the proof of theorem 4.1. We formulate the ingredients of this proof in the form of three separate lemmas. The proof of each of the lemmas follows along the lines of the proof of theorem 3 in the aforementioned paper by Brezis and Merle. We first introduce the so-called regular points of $\partial\Omega$.

Definition 4.4. A point $x_0 \in \partial\Omega$ is called *regular* if there exists a continuous function $0 \leqslant \psi \leqslant 1$, with $\psi \equiv 1$ in a neighbourhood of x_0 such that $\int_{\partial\Omega} \psi \, \mathrm{d}|\mu| < \frac{1}{2}\pi$.

Lemma 4.5. *Let v_{λ_n} denote the subsequence extracted above. Given any regular point $x_0 \in \partial\Omega$, there exists a positive number r_0 and a constant C, independent of n, such that*

$$\|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant \|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\Omega)} \leqslant C. \tag{4.9}$$

Proof. We decompose $v^0_{\lambda_n}$ into two parts,

$$v^0_{\lambda_n} = v^{(1)}_{\lambda_n} + v^{(2)}_{\lambda_n},$$

where $v^{(1)}_{\lambda_n}$ solves

$$\begin{aligned} \Delta v^{(1)}_{\lambda_n} &= 0 && \text{in } \Omega, \\ \frac{\partial v^{(1)}_{\lambda_n}}{\partial \boldsymbol{n}} &= \psi\lambda_n \sinh(v_{\lambda_n}) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \psi\lambda_n \sinh(v_{\lambda_n})\,\mathrm{d}\sigma && \text{on } \partial\Omega \end{aligned}$$

and $v^{(2)}_{\lambda_n}$ solves

$$\begin{aligned} \Delta v^{(2)}_{\lambda_n} &= 0 && \text{in } \Omega, \\ \frac{\partial v^{(2)}_{\lambda_n}}{\partial \boldsymbol{n}} &= \lambda_n \sinh(v_{\lambda_n}) - \frac{\partial v^{(1)}_{\lambda_n}}{\partial \boldsymbol{n}} && \text{on } \partial\Omega. \end{aligned}$$

Here, $0 \leqslant \psi \leqslant 1$ is a $C^{1,\alpha}$ function such that $\psi \equiv 1$ near x_0 and

$$\int_{\partial\Omega} \psi \, \mathrm{d}|\mu| < \tfrac{1}{2}\pi.$$

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The existence of such a smooth ψ also immediately follows from definition 4.4 (and the assumption that x_0 is a regular point). Both functions $v^{(i)}_{\lambda_n}$ are normalized by $\int_{\partial\Omega} v^{(i)}_{\lambda_n}\,\mathrm{d}\sigma = 0$. Elliptic regularity theory gives that $v_{\lambda_n} \in C^{\infty}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$, and the same is true for $v^{(1)}_{\lambda_n}$ (and $v^{(2)}_{\lambda_n}$). Duality and elliptic estimates give that

$$\|v^0_{\lambda_n}\|_{H^{s+2}(\Omega)} \leqslant C_s \|\lambda_n \sinh(v_{\lambda_n})\|_{H^{s+1/2}(\partial\Omega)}$$

for any $-2 \leqslant s \leqslant -1$, in particular, for $s = -\frac{3}{2}$, i.e.

$$\|v^0_{\lambda_n}\|_{H^{1/2}(\Omega)} \leqslant C \|\lambda_n \sinh(v_{\lambda_n})\|_{H^{-1}(\partial\Omega)}. \tag{4.10}$$

We have

$$\begin{aligned}
\|\lambda_n \sinh(v_{\lambda_n})\|_{H^{-1}(\partial\Omega)} &= \sup_{\|w\|_{H^1(\partial\Omega) \leqslant 1}} \left| \int_{\partial\Omega} \lambda_n \sinh(v_{\lambda_n}(\sigma))\; w(\sigma)\,\mathrm{d}\sigma \right| \\
&\leqslant \sup_{\|w\|_{H^1(\partial\Omega) \leqslant 1}} \|\lambda_n \sinh(v_{\lambda_n})\|_{L^1(\partial\Omega)} \|w\|_{L^\infty(\partial\Omega)} \\
&\leqslant C.
\end{aligned}$$

For the last inequality, we used (4.4) and the fact that $H^1(\partial\Omega) \subset L^\infty(\partial\Omega)$. By a combination of this estimate and (4.10),

$$\|v^0_{\lambda_n}\|_{H^{1/2}(\Omega)} \leqslant C. \tag{4.11}$$

The same argument works for $v^{(1)}_{\lambda_n}$, so that

$$\|v^{(1)}_{\lambda_n}\|_{H^{1/2}(\Omega)} \leqslant C. \tag{4.12}$$

Since $v^0_{\lambda_n} = v^{(1)}_{\lambda_n} + v^{(2)}_{\lambda_n}$, inequalities (4.11) and (4.12) immediately imply

$$\|v^{(2)}_{\lambda_n}\|_{H^{1/2}(\Omega)} \leqslant C. \tag{4.13}$$

The component $v^{(2)}_{\lambda_n}$ has

$$\frac{\partial v^{(2)}_{\lambda_n}}{\partial \boldsymbol{n}} = \text{const.} \quad \left(= \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \psi \lambda_n \sinh(v_{\lambda_n})\,\mathrm{d}\sigma \right)$$

near x_0, and local elliptic estimates thus show that there exists $r_0 > 0$ such that

$$\begin{aligned}
\|v^{(2)}_{\lambda_n}\|_{L^\infty(B_{2r_0}(x_0) \cap \partial\Omega)} &\leqslant \|v^{(2)}_{\lambda_n}\|_{L^\infty(B_{2r_0}(x_0) \cap \Omega)} \\
&\leqslant C\left(\|v^{(2)}_{\lambda_n}\|_{H^{1/2}(\Omega)} + \frac{1}{|\partial\Omega|} \left| \int_{\partial\Omega} \psi \lambda_n \sinh(v_{\lambda_n})\,\mathrm{d}\sigma \right| \right) \\
&\leqslant C.
\end{aligned} \tag{4.14}$$

For the last estimate, we used (4.13) and the fact that the sequence $\{\lambda_n \sinh(v_{\lambda_n})\}$ is bounded in $L^1(\partial\Omega)$. Concerning the component $v^{(1)}_{\lambda_n}$, one has

$$\begin{aligned}
\left\| \frac{\partial v^{(1)}_{\lambda_n}}{\partial \boldsymbol{n}} \right\|_{L^1(\partial\Omega)} &\leqslant \|\psi \lambda_n \sinh(v_{\lambda_n})\|_{L^1(\partial\Omega)} + \int_{\partial\Omega} |\psi \lambda_n \sinh(v_{\lambda_n})|\,\mathrm{d}\sigma \\
&= 2\|\psi \lambda_n \sinh(v_{\lambda_n})\|_{L^1(\partial\Omega)}.
\end{aligned}$$

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Since

$$\int_{\partial\Omega} \psi\lambda_n|\sinh(v_{\lambda_n})|\,\mathrm{d}\sigma \to \int_{\partial\Omega} \psi\,\mathrm{d}|\mu|,$$

and since $\int_{\partial\Omega}\psi\,\mathrm{d}|\mu| < \frac{1}{2}(\pi-\delta_0)$, for some $\delta_0 > 0$ sufficiently small, we conclude that

$$\|\psi\lambda_n\sinh(v_{\lambda_n})\|_{L^1(\partial\Omega)} < \tfrac{1}{2}(\pi-\delta_0)$$

for n sufficiently large. As a consequence,

$$\left\|\frac{\partial v^{(1)}_{\lambda_n}}{\partial \boldsymbol{n}}\right\|_{L^1(\partial\Omega)} \leqslant \pi-\delta_0$$

for n sufficiently large. Due to lemma 4.2, there exists $p > 1$ such that

$$\int_{\partial\Omega}(\mathrm{e}^{|v^{(1)}_{\lambda_n}|})^p\,\mathrm{d}\sigma \leqslant C. \tag{4.15}$$

One may, for instance, take

$$p = \frac{\pi-\frac{1}{2}\delta_0}{\pi-\delta_0}.$$

We also have

$$\lambda_n|\sinh(v_{\lambda_n})| \leqslant \lambda_n\exp(|v_{\lambda_n}|) \leqslant \lambda_n\exp(|v^{(1)}_{\lambda_n}| + |v^{(2)}_{\lambda_n}| + |s_{\lambda_n}|),$$

with

$$s_{\lambda_n} = s(v^0_{\lambda_n}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v_{\lambda_n}\,\mathrm{d}\sigma.$$

The component $v^{(2)}_{\lambda_n}$ is uniformly bounded sufficiently close to x_0 (cf. (4.14)). Due to the estimate (4.5), it follows that

$$|\lambda_n\sinh(v_{\lambda_n})|^p \leqslant C(\mathrm{e}^{|v^{(1)}_{\lambda_n}|})^p \tag{4.16}$$

in some neighbourhood $B_{2r_0}(x_0)\cap\partial\Omega$. By a combination of the estimates (4.15) and (4.16),

$$\left\|\frac{\partial v^{(0)}_{\lambda_n}}{\partial \boldsymbol{n}}\right\|_{L^p(B_{2r_0}(x_0)\cap\partial\Omega)} \leqslant C$$

for some $p > 1$, uniformly in n. Sobolev's imbedding theorem therefore yields

$$\left\|\frac{\partial v^{(0)}_{\lambda_n}}{\partial \boldsymbol{n}}\right\|_{H^{-s}(B_{2r_0}(x_0)\cap\partial\Omega)} \leqslant C$$

uniformly in n, for some $s < \frac{1}{2}$. Using the interior elliptic estimates of lemma 4.3, we arrive at

$$\|v^{(0)}_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant \|v^{(0)}_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\Omega)} \leqslant C$$

uniformly in n. This is the desired estimate. □

Definition 4.6. Let S denote the set of singular points of $\partial\Omega$, that is, the points which are not regular in the sense of definition 4.4: $S = \partial\Omega\setminus\{\text{regular points}\}$.

Lemma 4.7. *The set S consists of finitely many points, and it is non-empty.*

Proof. From the definition of a regular point in definition 4.4, it follows that if $x_0 \in S$, then $\int_{\partial\Omega} \psi \, \mathrm{d}|\mu| \geqslant \frac{1}{2}\pi$ for any continuous function $0 \leqslant \psi \leqslant 1$, with $\psi \equiv 1$ in a neighbourhood of x_0. Since the measure μ is regular, we conclude that $|\mu(\{x_0\})| \geqslant \frac{1}{2}\pi$ for any $x_0 \in S$. Due to the finiteness of the measure $|\mu|$, it now follows that S consists of a finite number of points, with

$$\#S \leqslant \frac{\int_{\partial\Omega} \mathrm{d}|\mu|}{\inf_{x_0 \in S} |\mu(\{x_0\})|} \leqslant 2\frac{\int_{\partial\Omega} \mathrm{d}|\mu|}{\pi}.$$

If S were empty, then lemma 4.5, together with the compactness of $\partial\Omega$, would imply that

$$\|v^0_{\lambda_n}\|_{L^\infty(\partial\Omega)} \leqslant C.$$

Since $|s_{\lambda_n}| \leqslant \log(1/\lambda_n) + D$ (see (4.5)), it would follow that

$$\|\lambda_n \sinh(v_{\lambda_n})\|_{L^\infty(\partial\Omega)} \leqslant C,$$

and therefore

$$\|\nabla v^0_{\lambda_n}\|_{L^2(\Omega)} \leqslant C\|\lambda_n \sinh(v_{\lambda_n})\|_{L^2(\partial\Omega)} \leqslant C.$$

From lemma 3.1, we could now conclude that

$$|s_{\lambda_n}| \leqslant C_1 + C_2\|\nabla v^0_{\lambda_n}\|^2_{L^2(\Omega)} \leqslant C,$$

or

$$\|v_{\lambda_n}\|_{L^\infty(\partial\Omega)} \leqslant \|v^0_{\lambda_n}\|_{L^\infty(\partial\Omega)} + |s_{\lambda_n}| \leqslant C.$$

We would therefore have the estimate

$$\begin{aligned} E(v_{\lambda_n}) &= \frac{1}{2}\int_\Omega |\nabla v_{\lambda_n}|^2 \, \mathrm{d}x - \lambda_n \int_{\partial\Omega} (\cosh(v_{\lambda_n}) - 1) \, \mathrm{d}\sigma \\ &= \lambda_n \int_{\partial\Omega} (\tfrac{1}{2}\sinh(v_{\lambda_n})v_{\lambda_n} - \cosh(v_{\lambda_n}) + 1) \, \mathrm{d}\sigma \\ &\leqslant C\lambda_n \to 0 \end{aligned}$$

as $n \to \infty$, which clearly contradicts (4.2). As a consequence, S must be non-empty. □

We proceed to show the following result.

Lemma 4.8. *Let v_{λ_n} be the subsequence extracted in connection with (4.7) and (4.8). The set S may alternatively be characterized as*

$$\begin{aligned} S &= \{x \in \partial\Omega : \exists x_n \to x, \ x_n \in \bar{\Omega}, \ \textit{with } |v^0_{\lambda_n}(x_n)| \to \infty\} \\ &= \{x \in \bar{\Omega} : \exists x_n \to x, \ x_n \in \bar{\Omega}, \ \textit{with } |v^0_{\lambda_n}(x_n)| \to \infty\} \end{aligned}$$

and

$$S = \{x \in \partial\Omega : \mu(\{x\}) \neq 0\}.$$

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Proof. The fact that the set

$$\{x \in \partial\Omega : \exists x_n \to x,\ x_n \in \bar{\Omega},\ \text{with } |v^0_{\lambda_n}(x_n)| \to \infty\}$$

is a subset of S is already established. Indeed, if this inclusion did not hold, then there would exist a regular point x_0 and a sequence $x_n \to x_0$, $x_n \in \bar{\Omega}$, with $|v^0_{\lambda_n}(x_n)| \to \infty$. However, this would contradict the already verified fact that

$$\|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant \|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\Omega)} \leqslant C$$

for some $r_0 > 0$ (lemma 4.5).

To establish the other inclusion, we simply take $x_0 \in S$ and note that, for such x_0 and for any $r > 0$, one necessarily has

$$\|v^0_{\lambda_n}\|_{L^\infty(B_r(x_0)\cap\partial\Omega)} \to \infty \quad \text{as } n \to \infty. \tag{4.17}$$

If this were not the case, then there would exist $r_1 > 0$ and a subsequence $v_k = v_{\lambda_{n_k}}$ such that

$$\|v_k\|_{L^\infty(B_{r_1}(x_0)\cap\partial\Omega)} \leqslant \|v^0_{\lambda_{n_k}}\|_{L^\infty(B_{r_1}(x_0)\cap\partial\Omega)} + |s_{\lambda_{n_k}}| \leqslant \log\biggl(\frac{1}{\lambda_{n_k}}\biggr) + D.$$

Here we have relied on (4.5) for the last estimate. We would thus have

$$\int_{B_r(x_0)\cap\partial\Omega} \lambda_{n_k}|\sinh(v_k)|\,\mathrm{d}\sigma \leqslant C|B_r(x_0)\cap\partial\Omega| \leqslant Cr$$

for any $0 < r < r_1$ and any k. Consequently, there would exist a continuous function $0 \leqslant \psi \leqslant 1$, with $\psi \equiv 1$ in a neighbourhood of x_0, so that

$$\int_{\partial\Omega} \psi\lambda_{n_k}|\sinh(v_k)|\,\mathrm{d}\sigma < \tfrac{1}{4}\pi \quad \text{for all } k.$$

However, this would imply that

$$\int_{\partial\Omega} \psi\,\mathrm{d}|\mu| = \lim_{k\to\infty}\int_{\partial\Omega} \psi\lambda_{n_k}|\sinh(v_k)|\,\mathrm{d}\sigma < \tfrac{1}{2}\pi,$$

and so x_0 would be a regular point—a contradiction. Thus we conclude that (4.17) must hold. We now pick $N_1 < N_2 < \cdots < N_k < N_{k+1} < \cdots$, so that

$$n \geqslant N_k \quad \Rightarrow \quad \|v^0_{\lambda_n}\|_{L^\infty(B_{1/k}(x_0)\cap\partial\Omega)} \geqslant k,$$

and then we select points x_n, $N_k \leqslant n \leqslant N_{k+1} - 1$, so that

$$x_n \in B_{1/k}(x_0) \cap \partial\Omega, \text{ and } \quad |v^0_{\lambda_n}(x_n)| \geqslant k - 1.$$

With this selection of points $x_n \in \partial\Omega$, it is clear that $x_n \to x_0$ and $|v^0_{\lambda_n}(x_n)| \to \infty$ as $n \to \infty$. This verifies the first alternative characterization of S. The fact that there are no blow-up points inside Ω is a direct consequence of interior elliptic estimates, the bound (4.11) and the identity $\Delta v^0_{\lambda_n} = 0$ in Ω. This verifies the second alternative characterization of S.

When it comes to the third characterization, we have already seen in the proof of the last lemma that $|\mu(\{x\})| \geqslant \frac{1}{2}\pi$ for any $x \in S$ (this was used to prove that S

consists of a finite number of points). It remains to be seen that $\mu(\{x_0\}) = 0$ for any regular point x_0. From lemma 4.5, we know that there exists $B_{r_0}(x_0)$ such that

$$\|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant C.$$

As above, this implies that

$$\|v_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} \leqslant \|v^0_{\lambda_n}\|_{L^\infty(B_{r_0}(x_0)\cap\partial\Omega)} + |s_{\lambda_n}| \leqslant \log\left(\frac{1}{\lambda_n}\right) + D,$$

and therefore

$$\int_{B_r(x_0)\cap\partial\Omega} \lambda_n |\sinh(v_{\lambda_n})|\,\mathrm{d}\sigma \leqslant C|B_r(x_0)\cap\partial\Omega| \leqslant Cr$$

for any $0 < r < r_0$ and any n. We conclude that $|\mu(\{x_0\})| \leqslant Cr$ for any $0 < r < r_0$, or $|\mu(\{x_0\})| = 0$. □

A combination of lemma 4.7 and 4.8 now immediately establishes theorem 4.1.

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