LYAPUNOV EXPONENTS FOR EXPANSIVE HOMEOMORPHISMS

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Abstract   We address the problem of defining Lyapunov exponents for an expansive homeomorphism f on a compact metric space (X, dist) using similar techniques as those developed in Barreira and Silva [Lyapunov exponents for continuous transformations and dimension theory, *Discrete Contin. Dynam. Sys.* **13** (2005), 469–490]; Kifer [Characteristic exponents of dynamical systems in metric spaces, *Ergod. Th. Dynam. Sys.* **3** (1983), 119–127]. Under certain conditions on the topology of the space X where f acts we obtain that there is a metric D defining the topology of X such that the Lyapunov exponents of f are different from zero with respect to D for every point $x \in X$. We give an example showing that this may not be true with respect to the original metric dist. But expansiveness of f ensures that Lyapunov exponents do not vanish on a G_δ subset of X with respect to any metric defining the topology of X. We define Lyapunov exponents on compact invariant sets of Peano spaces and prove that if the maximal exponent on the compact set is negative then the compact is an attractor.

Keywords: expansive homeomorphisms; Lyapunov exponents; Peano space

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1. Introduction

In the study of dynamical systems an indication of chaos is given by the so called Lyapunov exponents or characteristic exponents. This is well established in the case of differentiable dynamics, and their use in physics was initially based on the following consideration which in fact goes in the opposite direction: trying to ensure stability of motions. Let the differential equation $\dot{x} = F(x)$ define an autonomous dynamical system where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and Ω is open. For $x_0 \in \Omega$, consider the solution $\varphi(t, x_0)$ of the initial value problem

$$\begin{cases} \dot{x} = F(x), \\ x(0) = x_0. \end{cases} \tag{1}$$

Assume that all solutions of (1) with initial condition x_1 in a neighbourhood of x_0 do exist for $t \in [0, +\infty)$. An experimenter will probably have an error in the measurements for initial data slightly altered and the initial data will be $x_1 = x_0 + y$ instead of x_0,

where y is the error in the measurement which is assumed to be small. The dynamical behaviour of the nearby solution can be described approximately by the linearization of $\dot{x} = F(x)$, that is, by the linear system of differential equations $\dot{y} = DF_x(\varphi(t, x_0))y$ where $\varphi(t, x_0)$ is supposed to be the 'correct' solution. If, for all small y, the solution $\widetilde{\varphi}(t, y)$ of the system $\dot{y} = DF_x(\varphi(t, x_0))y$ tends to zero when $t \to +\infty$ then this is seen as an indication of (asymptotic) stability of the motion. A way to capture this is given by the limit $\chi_{x_0}(y) = \lim_{t \to +\infty}(1/t) \log(\|\widetilde{\varphi}(t, y)\|)$ whenever this limit exists. In this case, this limit gives information about exponential convergence (if $\chi_{x_0}(y) < 0$ for all y small) or divergence (total instability if $\chi_{x_0}(y) > 0$ for all y small) of trajectories with respect to the initial data problem. If the limit does not exist we can instead consider the lim sup if we want to capture this by means of any kind of exponential divergence.

In the discrete case (i.e., $t = n \in \mathbb{Z}$), when a C^1-dynamical system is given by a differentiable map $f : M \to M$, where M is a compact smooth manifold, the Lyapunov exponent is given for $x \in M$ and $v \in T_x M$ by $\chi(x, v) = \limsup_{n \to \infty}(1/n) \log(\|Df_x^n(v)\|)$. Here v takes the place of the 'error' y via the inverse of the exponential map $\exp_x : T_x M \to M$.

One problem with this approach is that in various situations we cannot assume that the system given by f is differentiable and therefore the computations roughly described above make no sense. Moreover, in several cases an experimenter has a collection of data indicating that the map f is continuous and even differentiable, but has not enough data to obtain an approximation of the differential map Df. So it seems of interest to introduce some kind of Lyapunov exponents for the case of a continuous dynamical system. This has been done by Barreira and Silva [1] for continuous maps $f : \mathbb{R}^n \to \mathbb{R}^n$; see also [5] for the case $f : X \to X$ where X is a compact metric space.

Indeed, in physics an approach is usually implemented which avoids the calculus of derivatives for the estimate of the largest Lyapunov exponent, see [10].

We will address the problem of defining Lyapunov exponents for an expansive homeomorphism f on a compact metric space (X, dist) using similar techniques to those developed in [1,5]. Under certain conditions on the topology of the space X where f acts, we obtain that the Lyapunov exponents are different from zero, indicating that f presents a chaotic dynamics. We define Lyapunov exponents $\Lambda(f, \mu)_{\max}$ and $\lambda(f, \mu)_{\min}$ for an f-invariant measure μ. The case where $\Lambda(f, \mu)_{\max} > 0$ and $\lambda(f, \mu)_{\min} < 0$ can be interpreted as a weak form of hyperbolicity for f. We prove that if M is a Peano space then there is $\gamma > 0$ such that $\Lambda(f, \mu)_{\max} > \gamma$ and $\lambda(f, \mu)_{\min} < -\gamma$. We also show that the hypothesis that M is a Peano space is necessary to obtain positive maximal and negative minimal Lyapunov exponent. Moreover, we define Lyapunov exponents for K, a compact f-invariant subset of M and prove that if the maximal Lyapunov exponent of K is negative then K is an attractor. When f is a diffeomorphism on a compact manifold, these Lyapunov exponents coincide with the usual ones.

We point out that another definition of Lyapunov exponents, $\chi_f(x)$, was recently introduced in [9], based on concepts developed by Durand-Cartagena and Jaramillo in [3] using the pointwise Lipschitz constant, similar to that given in [2, Section 2.1], which is obtained by interchanging the limits taken in equations [5, (5), (6)]. The authors compare their definition with others in the literature; among the results in [9] it is proved that for Lyapunov stable points the value of $\chi_f(x)$ is less than or equal to the top Lyapunov exponent $\Lambda^+(x)$ defined in [5], and in [9, Theorem 3.1] it is proved that these exponents coincide with those in [2]. Also in [9] it is proved that the new exponents coincide with

the classical ones for a large class of repellers and for hyperbolic sets of differentiable maps. The authors also discuss the relation of the new exponents with the dimension theory of dynamical systems for invariant sets of continuous transformations.

Let us observe that to us, from the mathematical point of view, there is no reason to choose one or other definition of Lyapunov exponents to work with, and we adopted that of [5], which seems more appropriate to apply in experimental sciences, where the knowledge of derivatives is often substituted by numerical approximations (see [10]).

2. Lyapunov exponents for expansive homeomorphisms

Let $f : M \to M$ be a homeomorphism defined on a compact metric space (M, dist). Following [5], we define the maximal and minimal Lyapunov exponents with respect to the distance $\mathrm{dist} : M \times M \to \mathbb{R}$ for a homeomorphism f. Assume M has no isolated points.

Given $N \in \mathbb{N}$ and $x \in M$, we define the N-dynamical ball at x with radius δ by

$$B_x^*(\delta, N) = \{y \in M \setminus \{x\} : \mathrm{dist}(f^j(x), f^j(y)) < \delta, \ \forall \ j = 0, 1, \ldots, N\}.$$

If $N < 0$, define

$$B_x^*(\delta, N) = \{y \in M \setminus \{x\} : \mathrm{dist}(f^j(x), f^j(y)) < \delta, \ \forall \ j = N, N+1, \ldots, -1, 0\}.$$

For $n \in \mathbb{Z}$, $\delta > 0$ and $x \in M$, define

$$A_\delta(x, n) = \sup_{y \in B_x^*(\delta, n)} \left\{ \frac{\mathrm{dist}(f^n(x), f^n(y))}{\mathrm{dist}(x, y)} \right\}, a_\delta(x, n) = \inf_{y \in B_x^*(\delta, n)} \left\{ \frac{\mathrm{dist}(f^n(x), f^n(y))}{\mathrm{dist}(x, y)} \right\}.$$

Remark 2.1. Note that $A_\delta(x, n)$ and $a_\delta(x, n)$ can be interpreted respectively as the maximal and minimal distortion of f on $B_x^*(\delta, N)$.

Let μ be a Borel f-invariant probability measure and assume that there is $\varepsilon_0 > 0$ such that, for all $0 < \delta < \varepsilon_0$,

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(A_\delta(x, n))| \mu(\mathrm{d}x) < \infty \tag{2}$$

$$\left| \inf_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(a_\delta(x, n))| \mu(\mathrm{d}x) \right| < \infty.$$

In this case we define

$$\Lambda_\delta^+(x) = \limsup_{n \to +\infty} \frac{1}{n} \log(A_\delta(x, n)) \quad \text{and} \quad \lambda_\delta^+(x) = \limsup_{n \to +\infty} \frac{1}{n} \log(a_\delta(x, n))$$

and, for $n < 0$,

$$\Lambda_\delta^-(x) = -\limsup_{n \to -\infty} \frac{1}{n} \log(A_\delta(x, n)) \quad \text{and} \quad \lambda_\delta^-(x) = -\limsup_{n \to -\infty} \frac{1}{n} \log(a_\delta(x, n)).$$

The following result is proved in [5, Theorem 1].

Theorem 2.2. *For μ-almost every $x \in M$, the limits*

$$\Lambda_\delta^+(x) = \lim_{n \to +\infty} \frac{1}{n} \log(A_\delta(x,n)), \quad \lambda_\delta^+(x) = \lim_{n \to +\infty} \frac{1}{n} \log(a_\delta(x,n)),$$

$$\Lambda_\delta^-(x) = -\lim_{n \to -\infty} \frac{1}{n} \log(A_\delta(x,n)), \quad \lambda_\delta^-(x) = -\lim_{n \to -\infty} \frac{1}{n} \log(a_\delta(x,n))$$

exist. Moreover, $\Lambda_\delta^+(x) = -\lambda_\delta^-(x)$, $\lambda_\delta^+(x) = -\Lambda_\delta^-(x)$, $\Lambda_\delta^+(x)$ and $\lambda_\delta^+(x)$ are f-invariant μ-almost everywhere (a.e.). Similarly for $\Lambda_\delta^-(x)$ and $\lambda_\delta^-(x)$.

Since we are assuming that (2) is valid and $A_\delta(x,n)$ decreases when δ decreases to zero, the limit $\Lambda^+(x) = \lim_{\delta \to 0} \Lambda_\delta^+(x)$ exists. Analogously, since $a_\delta(x,n)$ increases when δ decreases, the limit $\lambda^+(x) = \lim_{\delta \to 0} \lambda_\delta^+(x)$ exists. Similarly, $\Lambda^-(x)$ and $\lambda^-(x)$ exist μ-a.e. Thus we introduce the following definition.

Definition 2.1. The Lyapunov exponents for f at $x \in M$ are defined by

$$\Lambda^+(x) = \lim_{\delta \to 0} \Lambda_\delta^+(x), \quad \lambda^+(x) = \lim_{\delta \to 0} \lambda_\delta^+(x)$$

and similarly for $\Lambda^-(x)$ and $\lambda^-(x)$. As proved above, these quantities exist μ-a.e. and are f-invariant.

3. Lyapunov exponents with respect to Fathi's metric

Next we compute these Lyapunov exponents for an expansive homeomorphism. To do so, let us recall that a homeomorphism $f : X \to X$, X a compact metric space, is expansive if there exists $\alpha > 0$ such that, for all $x, y \in X$, if $x \neq y$ then there is $n \in \mathbb{Z}$ such that $\operatorname{dist}(f^n(x), f^n(y)) > \alpha$. We will obtain these Lyapunov exponents with respect to a hyperbolic metric adapted to the expansive homeomorphism, given by [**4**, Theorem 5.1].

Theorem 3.1. *Let $f : M \to M$ be an expansive homeomorphism of the compact metric space (M, dist). Then there exist a metric $d : M \times M \to \mathbb{R}$ on M, defining the same topology as dist, and numbers $k > 1$, $\varepsilon_0 > 0$ such that,*

$$\forall x, y \in M, \quad \max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{kd(x,y), \varepsilon_0\}.$$

Moreover, both f and f^{-1} are Lipschitz for d.

Remark 3.2. The existence of an expansive homeomorphism on M implies that the topological dimension of M is finite; see [**7**].

To define $\Lambda^\pm(x)$ and $\lambda^\pm(x)$ for $x \in M$, we need to show that condition (2) is fulfilled. To this end, we first verify the following lemma.

Lemma 3.3. *Let μ be a Borel probability measure invariant by $f : M \to M$. If f is expansive and d is the distance defined by Theorem 3.1 then*

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(A_\delta(x, n))| \mu(\mathrm{d}x) < \infty$$

and

$$\left| \inf_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(a_\delta(x, n))| \mu(\mathrm{d}x) \right| < \infty.$$

Proof. By Theorem 3.1, f and f^{-1} are Lipschitz with respect to the metric d, that is, there is a constant $K > 1$ such that

$$\forall x, y \in M : x \neq y, \quad \frac{d(f(x), f(y))}{d(x, y)} \leq K \quad \text{and} \quad \frac{d(f^{-1}(x), f^{-1}(y))}{d(x, y)} \leq K.$$

From the last inequality it follows that, for all $x, y \in M$, $x \neq y$, $d(f(x), f(y))/d(x, y) \geq 1/K$. Thus, $\sup_{x, y \in M, x \neq y} (d(f^n(x), f^n(y))/d(x, y)) \leq K^{|n|}$ for all $n \in \mathbb{Z}$, which implies that $\log(|A_\delta(x, n)|) \leq |n| \log(K)$ for all $\delta > 0$, $x \in M$ and $n \in \mathbb{Z}$. Therefore

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(A_\delta(x, n))| \mu(\mathrm{d}x) < \infty \quad \text{and condition (2) holds.}$$

Moreover, since

$$a_\delta(x, n) = \inf_{y \in B_x^*(\delta, n)} \{d(f^n(x), f^n(y))/d(x, y)\} = \left(\sup_{y \in B_x^*(\delta, n)} \{d(x, y)/d(f^n(x), f^n(y))\} \right)^{-1}$$

$$= \frac{1}{A_\delta(f^n(x), -n)}$$

and μ is f-invariant, we also have that

$$\left| \inf_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} \int_M |\log(a_\delta(x, n))| \mu(\mathrm{d}x) \right| < \infty.$$

The proof is complete. $\qquad\qquad\square$

Note that Lemma 3.3 and Theorem 2.2 imply that, for any f-invariant measure μ, the numbers $\Lambda^+(x)$, $\lambda^+(x)$, $\Lambda^-(x)$, $\lambda^-(x)$ do exist μ-a.e. and are f invariant.

4. Lyapunov exponents for expansive homeomorphisms defined on Peano spaces

Recall that M is a Peano space if it is a connected, locally connected compact metric space. Next we give a positive lower bound of $\Lambda^+(x)$ and a negative lower bound of $\lambda^+(x)$ for an expansive homeomorphism $f : M \to M$ defined on a Peano space. As remarked above, this can be interpreted as a weak kind of hyperbolicity condition.

Theorem 4.1. *Let (M, d) be a Peano space, $f : M \to M$ an expansive homeomorphism and $\gamma = \log(k)$, $k > 1$, the constant given by Theorem 3.1. Then, for all $x \in M$,*

$$\Lambda^+(x) \geq \gamma \quad \text{and} \quad \lambda^+(x) \leq -\gamma.$$

Proof. Given a point $x \in M$, there is $y \in M \backslash \{x\}$ close to x such that $d(f(x), f(y)) \geq k\, d(x, y)$, where $d(\cdot, \cdot)$ is the distance given by Theorem 3.1. Otherwise, by the theorem mentioned, for some $\delta > 0$ and every point $y \in B(x, \delta)$ we have $d(f(x), f(y)) < k\, d(x, y)$, and therefore for all $y \in B(x, \delta)$ we have $d(f^{-1}(x), f^{-1}(y)) \geq k\, d(x, y)$. Thus $B(f^{-1}(x), \delta) \subset f^{-1}(B(x, \delta))$. Moreover, we also have for all $y \in B(f^{-1}(x), \delta)$ that $d(f^{-2}(x), f^{-1}(y)) \geq k\, d(f^{-1}(x), y)$. For we already know that, for every point $z \in B(f^{-1}(x), \delta)$, the inequality $d(f(z), f(f^{-1}(x))) \leq (1/k)\, d(f^{-1}(x), z)$ holds. By induction we obtain a sequence of balls $B(f^{-n}(x), \delta)$ such that, for all $y \in B(f^{-n}(x), \delta)$, we have $d(f^{-n-1}(x), f^{-1}(y)) \geq k\, d(f^{-n}(x), y)$. Let z be an α-limit point of the sequence $\{f^{-n}(x)\}$. Then z is a Lyapunov stable point of f, contradicting that there are no such points if $f : M \to M$ is expansive and M is compact connected and locally connected; see [**6**, Proposition 2.7]. Hence, for every $\delta > 0$; there is $y \in B(x, \delta) \backslash \{x\}$ such that $d(f(x), f(y)) \geq k\, d(x, y)$. Thus, given $n > 0$, taking $\varepsilon_0 > 0$ as in Theorem 3.1, and $\delta > 0$ sufficiently small in $B_x^*(\delta, n) = \{y \in M \setminus \{x\} : d(f^j(x), f^j(y)) \leq \delta \ \forall \ j = 0, 1, \ldots, n\}$, we obtain $d(f^j(x) f^j(y)) \leq \varepsilon_0$ for all $j = 1, 2 \ldots, n$. Therefore

$$A_\delta(x, n) = \sup_{y \in B_x^*(\delta, n)} \{d(f^n(x), f^n(y))/d(x, y)\} \geq k^n,$$

implying that $\Lambda_\delta^+(x) = \lim_{n \to +\infty} (1/n) \log(A_\delta(x, n)) \geq \log(k) = \gamma > 0$. Similarly, $\lambda_\delta^-(x) \leq -\log(k) = -\gamma$. Since this is valid for any small $\delta > 0$, letting $\delta \to 0$, we obtain that $\Lambda^+(x) \geq \gamma$ and $\lambda^+(x) \leq -\gamma$, finishing the proof. $\square$

Remark 4.2. As proved in [**5**] (see also [**1**]), when $f : M \to M$ is a diffeomorphism on a compact manifold these Lyapunov exponents coincide with the usual ones. The relations between classic Lyapunov exponents (under smoothness assumptions) and exponents with topological flavour are treated by Kifer in [**5**, Remark 1], Barreira and Silva in [**1**, Theorem 5] and Bessa and Silva in [**2**, Theorem A]. In [**1**] some assumptions are considered beyond the differentiability. They considered a $C^{1+\alpha}$ map $f : \mathbb{R}^m \to \mathbb{R}^m$ with an f-invariant compact repeller on which f has α-bunched derivative and prove that in this case the new Lyapunov exponents coincide with the classical ones. Let us recall that, for $\alpha \in (0, 1]$, we say that a differentiable map $f : \mathbb{R}^m \to \mathbb{R}^m$ has α-bunched derivative on the set J if Df_x is invertible and $\|(Df_x)^{-1}\|^{1+\alpha} \|Df_x\| < 1$ for every $x \in J$. But the definitions of Lyapunov exponents given in [**2**, Subsection 2.1] enable the authors to obtain, in the differentiable case, the equality of the classical Lyapunov exponents with those that they defined without additional assumptions about the derivatives; see [**2**, Theorem 2.3].

Next we construct an example of an expansive homeomorphism defined on a compact connected metric space exhibiting Lyapunov stable points, showing that the hypothesis of locally connectedness cannot be neglected in Theorem 4.1.

Theorem 4.3. *The hypotheses of locally connectedness cannot be neglected in Theorem 4.1.*

Proof. Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the Anosov map in the 2-torus $\mathbb{T}^2$ induced by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{3}$$

Let p the fixed point of f_A corresponding to the origin and $v_p, \|v_p\| = 1$, be an eigenvector for the larger eigenvalue $\lambda < 1$ of A. Then the natural projection of $\{tv_p, t \in \mathbb{R}\}$ into $\mathbb{T}^2$ is dense in $\mathbb{T}^2$ and corresponds to the stable manifold $W^s(p)$ of the fixed point p.

Now consider the interval $\mathbb{I} = [p, q] \subset \mathbb{R}^3$, perpendicular to the tangent space $T_{\mathbb{T}^2}(p) \subset \mathbb{R}^3$, and in $\mathbb{R} \times \mathbb{R} \times \mathbb{I}$ let $\mathcal{H} = \{(p, q, q + tv_p); t \geq 0\}$ and let $A(p, q, q + tv) = (p, q, q + \lambda tv)$. Thus A induces a homeomorphism $A : \mathbb{R}^2 \times \mathcal{H} \to \mathbb{R}^2 \times \mathcal{H}$. Note that $\mathcal{H}$ is a copy of the stable manifold $W^s(p)$. Factoring out the integer lattice $\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{R}^2$, we obtain a homeomorphism $f : \mathbb{T}^2 \cup H \to \mathbb{T}^2 \cup \mathcal{H}$. As $\mathcal{H}$ is a copy of $W^s(p)$, it is a curve asymptotic to $\mathbb{T}^2$ and we obtain that $X = \mathbb{T}^2 \cup \mathcal{H}$ is compact and connected. Furthermore, it is also closed for $W^s(p)$ is asymptotic to $\mathbb{T}^2$. We finally define a dynamics in X: in $\mathbb{T}^2$ the dynamics is induced by A and in $\mathcal{H}$ is the dynamics of $W^s(p)$. It turns out that this dynamics in X is expansive. But the points in $W^s(p) = \mathcal{H}$ are stable. In particular, so is the point q, implying that the q has a unique Lyapunov exponent, which is strictly less than zero, finishing the proof. $\qquad \square$

5. Lyapunov exponents with respect to the original metric

A question that arises is what occurs when we compute Lyapunov exponents with respect to the original metric. Observe that the bound $\gamma > 0$ is uniform with respect to every point $x \in M$. But this may not be true with respect to another metric defining the topology of M. Indeed, consider in $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ the diffeomorphism given by

$$F(x, y) = \left(2x + y - \frac{1}{2\pi} \sin(2\pi x) \bmod 1, \ x + y - \frac{1}{2\pi} \sin(2\pi x) \bmod 1 \right).$$

The fixed point $(x, y) = (0, 0)$ is non-hyperbolic (the matrix $DF_{(0,0)} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$) and hence the classical Lyapunov exponents of F in $(0, 0)$ vanish. But F is an expansive homeomorphism of T^2 and hence conjugated to an Anosov diffeomorphism (in fact the linear one given by $A(x, y) = (2x + y, x + y)$), [**6**].

Observe that, with respect to the maximal Lyapunov exponent as defined in Definition 2.1, there is a positive continuous function $\gamma(x, y)$ such that $\gamma(x, y) \to 0$ when $(x, y) \to (0, 0)^*$ and $\|DF_{(x,y)}^k\| \leq (1 + \gamma(x, y)^k)\|DF_{(0,0)}^k\|$ for all $(x, y) \in B_{(0,0)}^*(\epsilon, n)$ and $k = 0, \ldots, n$.

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* Indeed $\gamma(x, y) = 4(1 - \cos(2\pi \sqrt{x^2 + y^2}))$ is a bound, as can be easily checked.

Let $v \in \mathbb{R}^2$ such that $(x, y) = \exp_{(0,0)} v$, where $\|v\| = r$. Then if $n + 2$ is a linear bound to $\|DF^n_{(0,0)}\|$ we get

$$\text{dist}(F^n(x, y), F^n(0, 0)) \leq \int_0^1 \|DF^n(sv)\| \mathrm{d}s \leq (1 + \gamma(x, y))^n \, (n + 2).$$

It follows that $A_\delta(0, 0) \leq (1 + \gamma(x, y))^n \, (n + 2)$ and so $\Lambda^+_\delta((0, 0) \leq \log(1 + \gamma(\delta))$, where $\|(x, y)\| \leq \delta$, implying that $\Lambda^+(0, 0)$ vanishes when $\delta \to 0$. Thus we cannot expect to have positive Lyapunov exponents at every point of M.

But, although we do not in general have a positive bound for $\Lambda^+(x)$ in the original metric, Theorem 5.5 below proves that Λ^+ is positive for a Baire set of the second category (i.e., a G_δ subset of M), and that $\lambda^+ < 0$ also for a G_δ subset. Since the intersection of G_δ subsets is again a G_δ subset it follows in particular that $\Lambda^+ > 0$ and $\lambda^+ < 0$ simultaneously in a residual (hence dense) subset of M.

Remark 5.1. By the definition of $A^+_\epsilon(x, n)$ it is easy to prove that $A^+_\epsilon(x, n + m) \leq A^+_\epsilon(x, n) \cdot A^+_\epsilon(f^n(x), m)$, that is, $\log(A^+_\epsilon(x, n))$ is subadditive.

Let us recall that in a compact Peano space M, given a pair of points $x, y \in M$, a geodesic (hence rectifiable) arc joining them is defined; see [8]. This arc, say ξ, can be parameterized by a function $\varphi : [0, 1] \to M$, $\xi = \varphi([0, 1])$, $\varphi(0) = x$ and $\varphi(1) = y$. Hence we may speak of a point z between x and y inside the arc ξ; meaning that there is $t \in (0, 1)$ such that $\varphi(t) = z$.

We will denote a subarc $\varphi([t_0, t_1])$ where $\varphi(t_0) = z$ and $\varphi(t_1) = w$ by $[zw]$.

Theorem 5.2. *Let M be a compact Peano space equipped with a metric* $\text{dist} : M \times M \to \mathbb{R}^+ \cup \{0\}$ *and* $f : M \to M$ *an expansive homeomorphism such that there is $K > 0$ such that* $\text{dist}(f(x), (f(y))/\text{dist}(x, y) \leq K$ *for every pair $x, y \in M$, $x \neq y$. Then $\Lambda^+(x) < +\infty$ and there are points $x \in M$ such that $\Lambda^+(x) > 0$.*

Proof. The existence of K ensures that $\Lambda^+(x) < +\infty$. Let $0 < \epsilon < \alpha/2$, where α is an expansivity constant. Let $x, y \in M$, $\epsilon/2 > \text{dist}(x, y) = \delta > 0$. Join them with a geodesic arc ξ of length, say, ℓ. There is $j_0 \in \mathbb{Z}$ such that $\text{dist}(f^{j_0}(x), f^{j_0}(y)) \geq \epsilon$. We may assume without loss of generality that $j_0 > 0$.

We pick $y' \in \xi$ such that, for every point between x and y', no point z has the property that $\text{dist}(f^{j_0}(x), f^o j(z)) \geq \epsilon$, which implies that $\text{dist}(f^{j_0}(x), f^{j_0}(y')) = \epsilon$ Moreover, we choose $0 < j_0$ minimum with the property that $\text{dist}(f^{j_0}(x), f^{j_0}(y')) = \epsilon$. Clearly length$([xy']) \leq \ell$. Renaming the points if necessary, let us assume that $y = y'$. Let z be the middle point of $[xy]$ in the arc ξ. By the triangle inequality either $\text{dist}(f^{j_0}(x), f^{j_0}(z)) \geq \epsilon/2$ or $\text{dist}(f^{j_0}(y), f^{j_0}(z)) \geq \epsilon/2$. Again, replacing z by another point z' if necessary, we may assume that equality holds and the point z' is at a distance less than or equal to z from the respective point x or y and, for every point w between $[xz]$ or between $[zy]$, we have that $\text{dist}(f^{j_0}(x), f^{j_0}(w)) < \epsilon/2$. To fix ideas, suppose that $\text{dist}(f^{j_0}(x), f^{j_0}(z)) = \epsilon/2$. By compactness of M there is $N_0 > 0$, depending only on $\epsilon/2$ and f, such that $\text{dist}(f^{j_0+j_1}(x), f^{j_0+j_1}(z) \geq \epsilon$ for some $0 < j_1 < N_0$. Again we may assume that equality holds, reducing the distance between x and z if necessary, and that, for all points w of $[xz]$ different from z, we have $\text{dist}(f^{j_0+j_1}(x), f^{j_0+j_1}(w)) < \epsilon$. Moreover, we choose j_1 such that,

for every $0 < i < j_1$ and every point $w \in [xz]$, the inequality $\text{dist}(f^{j_0+i}(x), f^{j_0+i}(w)) < \epsilon$ holds. Let us rename $x_1 = x$ and $y_1 = z$. On the other hand, if $\text{dist}(f^{j_0}(y), f^{j_0}(z)) = \epsilon/2$ then there would exist j_1, $0 < j_1 < N_0$, such that $\text{dist}(f^{j_0+j_1}(y), f^{j_0+j_1}(z) \geq \epsilon$, and assuming, as we can, that equality holds and that all conditions similar to those described above hold, we rename $x_1 = y$ and $y_1 = z$. We then repeat this procedure, finding a point z_1 in the arc ξ such that $[x_1 z_1]$ and $[z_1 y_1]$ have the same length and such that either $\text{dist}(f^{j_0+j_1}(x_1), f^{j_0+j_1}(z_1)) \geq \epsilon/2$ or $\text{dist}(f^{j_0+j_1}(y_1), f^{j_0+j_1}(z_1)) \geq \epsilon/2$. As we noticed previously, we may assume that equality holds and all the conditions previously prescribed are fulfilled. In the case where $\text{dist}(f^{j_0+j_1}(x_1), f^{j_0+j_1}(z_1) = \epsilon/2$ we rename $x_2 = x_1$ and $y_2 = z_1$, otherwise $x_2 = y_1$ and $y_2 = z_1$. Again there exist $j_2 > 0$, $j_2 < N_0$ such that $\text{dist}(f^{j_0+j_1+j_2}(x_2), f^{j_0+j_1+j_2}(y_2)) \geq \epsilon$ and, repeating the previous procedure infinitely many times, we find a pair of sequences $\{x_n\}$ and $\{y_n\}$ in ξ such that the arcs $[x_n, y_n] \subset \xi$ have the property that $[x_n, y_n] \subset [x_{n-1}, y_{n-1}]$ for every $n \geq 1$, $\text{dist}(x_n, y_n) \to 0$ and the length of $[x_n, y_n]$ is less than or equal to $\ell/2^n$. Moreover, we have that

$$\text{dist}(f^{j_0+\sum_{k=1}^n j_k}(x_n), f^{j_0+\sum_{k=1}^n j_k}(y_n)) = \epsilon \quad \text{for all } 0 < j_k < N_0,$$

$$\text{and} \quad k = 1, 2, \ldots, n,$$

$$\text{dist}(f^{j_0+\sum_{k=1}^{n-1} j_k + i}(x_n), f^{j_0+\sum_{k=1}^{n-1} j_k + i}(y_n)) < \epsilon \quad \text{for } 0 \leq i < j_n.$$

The previous conditions imply that in the arc ξ there is a unique β such that $x_n \to \beta$ and $y_n \to \beta$ when $n \to +\infty$. Now, it is easy to see that for every $n \geq 1$ we have

$$\text{dist}(f^{j_0+\sum_{k=1}^n j_k}(x_n), f^{j_0+\sum_{k=1}^n j_k}(\beta)) \geq \epsilon/2$$

or

$$\text{dist}(f^{j_0+\sum_{k=1}^n j_k}(y_n), f^{j_0+\sum_{k=1}^n j_k}(\beta)) \geq \epsilon/2.$$

This implies that $A_\epsilon(\beta, j_0 + \sum_{k=1}^n j_k) \geq \epsilon/2/\ell/2^n = 2^{n-1}\epsilon/\ell$ and so

$$\frac{1}{j_0 + \sum_{k=1}^n j_k} \log \left(A_\epsilon \left(\beta, j_0 + \sum_{k=1}^n j_k \right) \right)$$

$$\geq \frac{1}{j_0 + nN_0} \log(2^{n-1}\epsilon/\ell) \to \frac{1}{N_0} \log(2) \quad \text{when } n \to \infty.$$

Thus,

$$\Lambda_\epsilon^+(\beta) = \frac{1}{N_0} \log(2) > 0.$$

If we take $\epsilon/2 < \epsilon' < \epsilon$ the same bound N_0 can be used, because if

$$\text{dist}(f^{j+\sum_{k=1}^n j_k}(x_n), f^{j+\sum_{k=1}^n j_k}(y_n)) = \epsilon$$

then *a fortiori*

$$\text{dist}(f^{j+\sum_{k=1}^n j_k}(x_n), f^{j+\sum_{k=1}^n j_k}(y_n)) \geq \epsilon',$$

and we may take points x_n' between x_n and β, and y_n' between β and y_n, such that equality above holds. Therefore $\Lambda_{\epsilon'}^+(\beta) \geq (1/N_0) \log(2)$ too.

Finally, if $0 < \epsilon' \leq \epsilon$ we may choose x_n and y_n so close to each other such that $\mathrm{dist}(x_n, y_n) < \epsilon'$ and since

$$\mathrm{dist}(f^{j_0 + \sum_{k=1}^n j_k}(x_n), f^{j_0 + \sum_{k=1}^n j_k}(y_n)) = \epsilon, \quad \text{for } 0 < j_k < N_0 \text{ and for all } k = 1, 2, \ldots, n,$$

there are $1 \leq h < n$ and $0 < i < N_0$ such that

$$\mathrm{dist}(f^{j_0 + \sum_{k=1}^h j_k + i}(x_n), f^{j_0 + \sum_{k=1}^h j_k + i}(y_n)) \geq \epsilon',$$
$$\text{with } 0 < j_k < N_0 \text{ for every } k = 1, 2, \ldots, h,$$

and also with the property that $\mathrm{dist}(f^s(x_n), f^s(y_n)) < \epsilon'$ for every $0 \leq s < j + \sum_{k=1}^h j_k + i$. Moreover, $h \to +\infty$ when $n \to +\infty$ since $\mathrm{dist}(x_n, y_n) \to 0$ when $n \to +\infty$. Therefore, $\limsup_{n \to +\infty} \Lambda_{\epsilon'}^+(\beta) \geq (1/N_0) \log(2)$ for every $\epsilon' > 0$. Indeed, we have

$$A_{\epsilon'}\left(\beta, j_0 + \sum_{k=1}^h j_k + i\right) \geq \frac{\epsilon'/2}{\ell/2^n} = 2^{n-1}\epsilon'/\ell$$

and hence

$$\frac{1}{j_0 + \sum_{k=1}^h j_k + i} \log\left(A_{\epsilon'}\left(\beta, j_0 + \sum_{k=1}^h j_k + i\right)\right) \geq \frac{1}{j_0 + (h+1)N_0} \log(2^{n-1}\epsilon'/\ell)$$
$$\geq \frac{1}{j_0 + nN_0} \log(2^{n-1}\epsilon'/\ell) \to \frac{1}{N_0} \log(2) \quad \text{when } n \to +\infty.$$

Letting $\epsilon \searrow 0$, it follows that $\Lambda^+(\beta) > 0$ and, moreover, its value is bounded from below, finishing the proof. $\square$

Remark 5.3. Since the initial points x, y are chosen arbitrarily we conclude that the set of points where $\Lambda^+ > 0$ constitute a dense subset D of M. Moreover, $\Lambda^+ > 0$ is bounded away from zero on D. Similarly, we can prove that the set of points where $\lambda^+ < 0$ is dense in M too and that λ^+ is bounded from above.

Proposition 5.4. *Let M and f as in Theorem 5.2. Let $A_\epsilon^+(x_0, n) = \gamma$ and $\delta > 0$. There is $r > 0$ such that, for every $x \in B(x_0, r)$, the inequality $A_\epsilon^+(x, n) > \gamma - \delta$ holds, that is, $A_\epsilon^+(x, n)$ is bounded from below by a positive number. That is, $A_\epsilon^+(\cdot, n)$ is lower semi-continuous.*

Proof. $A_\epsilon(x_0, n)$ being a supremum, for any δ, $0 < \delta < \epsilon$, there is $y_n \in B_{x_0}^*(\epsilon, n)$ such that $\gamma \geq \mathrm{dist}(f^n(x_0), f^n(y_n))/\mathrm{dist}(x_0, y_n) > \gamma - \delta/2$. Since f^n is continuous there is $r > 0$ such that if $x \in B(x_0, r) \backslash \{x_0\}$ then $x \in B_{x_0}^*(\epsilon, n)$, $\mathrm{dist}(f^j(x_0), f^j(x)) < \eta$, for every $j = 0, 1, \ldots, n$, and $\mathrm{dist}(f^n(x), f^n(y_n))/\mathrm{dist}(x, y_n) > \gamma - \delta$. Choosing $0 < \eta < \min\{\epsilon - \mathrm{dist}(f^j(x_0), f^j(y_n)) \, j = 0, 1, \ldots, n\}$ and the corresponding $r > 0$, we have that $A_\epsilon(x, n) > \gamma - \delta$. Thus, the thesis follows. $\square$

Theorem 5.5. *There is a residual subset $\mathcal{R}^+$ of M such that, for every $x \in \mathcal{R}^+$, we have $\Lambda^+(x) > 0$.*

Proof. Since $\log(A_\epsilon^+(x))$ is subadditive, there exists a limit $\lim_{n\to+\infty}(\log(A_\epsilon(x,n))/n) = \Lambda_\epsilon^+(x)$. Since $A_\epsilon(x,n)$ is lower semi-continuous on x, it is positive and bounded from below in a dense open subset by Proposition 5.4 and Remark 5.3. Thus $\Lambda_\epsilon^+(x)$ is positive and bounded from below in that open dense subset. Taking the countable intersection on $1/k$, $k \in \mathbb{Z}^+$, of the open dense subsets where $\Lambda_{1/k}^+(x) > 0$, we find a residual subset where $\Lambda^+ > 0$. $\qquad\square$

Analogously we may find $\mathcal{R}^-$ residual where $\lambda^- < 0$. Since the intersection of residual subsets is again a residual subset we may conclude that, for $f : M \to M$ an expansive homeomorphism defined on a compact Peano space, there is a residual (in particular dense) subset of M where both Lyapunov exponents defined do not vanish.

6. Negative exponents on compact sets and attractors

In this section we extend the definition of the Lyapunov exponent for compact invariant sets $K \subset M$. The main result in this section establishes that negative (respectively, positive) Lyapunov exponents for compact invariant sets on Peano spaces imply that the compact set is an attractor (respectively, repeller), improving [**9**, Theorem 5.1].

Let M be a (non-trivial) compact Peano space and $f : M \to M$ a homeomorphism. For $A \subset M$, $A \neq \emptyset$ and $x \in M$, we define $\operatorname{dist}(x, A) = \inf\{\operatorname{dist}(x, y) : y \in A\}$.

Let $K \subset M$ be a compact f-invariant subset of M (i.e., $f(K) = K$). For $N \in \mathbb{N}$, define

$$B_K^*(\delta, N) = \{y \in M \setminus K : \operatorname{dist}(f^j(y), K) < \delta, \ \forall \ j = 0, 1, \ldots, N\}.$$

If $N < 0$, define

$$B_K^*(\delta, N) = \{y \in M \setminus K : \operatorname{dist}(f^j(y), K) < \delta, \ \forall \ j = N, N+1, \ldots, -1, 0\}.$$

For $n \in \mathbb{Z}$ and $\delta > 0$, let us define

$$A_\delta(K, n) = \sup_{y \in B_K^*(\delta, n)} \left\{ \frac{\operatorname{dist}(K, f^n(y))}{\operatorname{dist}(K, y)} \right\},$$

$$a_\delta(K, n) = \inf_{y \in B_K^*(\delta, n)} \left\{ \frac{\operatorname{dist}(K, f^n(y))}{\operatorname{dist}(K, y)} \right\}.$$

Let us also define

$$\Lambda_\delta^+(K) = \limsup_{n\to+\infty} \frac{1}{n} \log(A_\delta(K, n)) \quad \text{and} \quad \lambda_\delta^+(K) = \limsup_{n\to+\infty} \frac{1}{n} \log(a_\delta(K, n)),$$

$$\Lambda_\delta^-(K) = -\limsup_{n\to-\infty} \frac{1}{n} \log(A_\delta(K, n)) \quad \text{and} \quad \lambda_\delta^-(K) = -\limsup_{n\to-\infty} \frac{1}{n} \log(a_\delta(K, n)).$$

Since $f^n(K) = K$ and $B_K^*(\delta, n+k) \subset B_K^*(\delta, n)$, if $k \geq 0$, we have

$$A_\delta(K, n+k) = \sup_{y \in B_K^*(\delta, n+k)} \{\operatorname{dist}(K, f^{n+k}(y))/d(K, y)\}$$

$$= \sup_{y \in B_K^*(\delta, n+k)} \left\{ \left(\frac{\operatorname{dist}(K, f^n(y))}{d(K, y)} \right) \cdot \left(\frac{\operatorname{dist}(K, f^{n+k}(y))}{d(K, f^n(y))} \right) \right\}$$

$$\leq \sup_{y \in B_K^*(\delta,n)} \{ \mathrm{dist}(K, f^n(y))/d(K,y) \} \cdot \sup_{z \in B_K^*(\delta,k)} \{ \mathrm{dist}(K, f^k(z))/d(K,z) \}$$

$$= A_\delta(K,n) \cdot A_\delta(K,k).$$

Therefore, letting $Y(\delta, K, n) = \log(A_\delta(K,n))$, we obtain a subadditive function and there is the limit $\Lambda^+(K, \delta)$ of $(1/n) \log(A_\delta(K,n))$ for $n \to +\infty$. Since $\log(A_\delta(K,n))$ is monotone in δ, there exists $\Lambda^+(K) = \lim_{\delta \to 0} \Lambda^+(K, \delta)$.

As for the case of a point $x \in M$, the following theorem can be proved.

Theorem 6.1. $\Lambda_\delta^+(K) = -\lambda_\delta^-(K)$ and $\lambda_\delta^+(K) = -\Lambda_\delta^-(K)$.

Proof. We have that

$$a_\delta(K,n) = \inf_{y \in B_K^*(\delta,n)} \left\{ \frac{\mathrm{dist}(K, f^n(y))}{\mathrm{dist}(K,y)} \right\} = \left(\sup_{y \in B_K^*(\delta,n)} \left\{ \frac{\mathrm{dist}(K,y)}{\mathrm{dist}(K, f^n(y))} \right\} \right)^{-1}$$

$$= \left(\sup_{y \in B_K^*(\delta,-n)} \left\{ \frac{\mathrm{dist}(K, f^{-n}(y))}{\mathrm{dist}(K,y)} \right\} \right)^{-1} = A_\delta^{-1}(K, -n).$$

Therefore $\lambda_\delta^+(K) = \lim_{n \to +\infty}(1/n)\log(a_\delta(K,n)) = -\lim_{n \to -\infty}(1/|n|)\log(A_\delta(K,-n)) = -\Lambda_\delta^-(K)$. Similarly, it can be proved that $\Lambda_\delta^+(K) = -\lambda_\delta^-(K)$. □

Again, since $\Lambda_\delta^+(K)$ is monotone with δ, the limit $\Lambda^+(K) = \lim_{\delta \to 0^+} \Lambda_\delta^+(K)$ exists. Similarly, there exists $\lambda^+(K) = \lim_{\delta \to 0^+} \lambda_\delta^+(K)$.

Given a compact invariant set $K \subset M$, we say that K is an *attractor* if there is a neighbourhood U of K such that if $y \in U$ then $\lim_{n \to +\infty} \mathrm{dist}(f^n(y), K) = 0$. Analogously, K is a *repeller* if there is a neighbourhood U of K such that if $y \in U$ then $\lim_{n \to -\infty} \mathrm{dist}(f^n(y), K) = 0$.

Theorem 6.2. *Let M be a compact Peano space and $K \subset M$ be an invariant compact set. If $\Lambda^+(K) < 0$ then K is an attractor. Analogously, if $\lambda^-(K) > 0$ then K is a repeller.*

Proof. Since $\Lambda^+(K) = \lim_{\delta \to 0} \Lambda^+(K, \delta) < 0$ there is $\delta_0 > 0$ such that, for all $0 < \delta \leq \delta_0$, we have $\Lambda^+(K, \delta) < \frac{2}{3}\Lambda^+(K) < 0$. Since $\lim_{n \to +\infty}(1/n)\log(A_\delta(K,n)) = \Lambda^+(K, \delta)$ there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0 = n_0(\delta_0)$, we have $(1/n)\log(A_{\delta_0}(K,n)) < \frac{1}{2}\Lambda^+(K) < 0$. A fortiori, for all $n \geq n_0$ and $0 < \delta \leq \delta_0$, we have $(1/n)\log(A_\delta(K,n)) < \frac{1}{2}\Lambda^+(K)$ too. Let us write $-\gamma = \frac{1}{2}\Lambda^+(K)$. Choose $\delta_0 > \delta_1 > 0$ such that if $\mathrm{dist}(y, K) < \delta_1$ then $\mathrm{dist}(f^j(y), K) < \delta_0$, for all $j = 0, 1, 2, \ldots, n_0$. Finally, let $U = \{ y \in M : \mathrm{dist}(y, K) < \delta_1 \}$. If $y \in U$, since

$$\frac{1}{n}\log(A_\delta(K,n)) = \frac{1}{n}\log\left(\sup_{y \in B_K^*(\delta,n)} \left\{ \frac{\mathrm{dist}(K, f^n(y))}{\mathrm{dist}(K,y)} \right\} \right) < -\gamma,$$

we have that $\text{dist}(K, f^n(y))/\text{dist}(K, y) < e^{-\gamma n}$. But $\text{dist}(y, K) < \delta_1 < \delta_0$ and so $\text{dist}(K, f^n(y)) < e^{-\gamma n} \delta_0 < \delta_0$ for all $n \geq n_0$ and we can apply induction. Thus $\text{dist}(K, f^n(y))$ tends to zero when $n \to +\infty$ and K is an attractor.

The proof that $\lambda^-(K) > 0$ implies that K is a repeller is similar. $\qquad\square$

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