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Travelling wavefronts for a non-divergent degenerate and singular parabolic equation with changing sign sources

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We discuss travelling wavefronts of a degenerate and singular parabolic equation in non-divergent form with changing sign sources. Necessary and sufficient conditions will be given for the existence of smooth or non-smooth and non-decreasing or non-increasing solutions. We also study the regularity of such solutions.

1. Introduction

We consider mainly sharp conditions for the existence of smooth travelling wavefronts for the following parabolic equation in non-divergence form

$$\frac{\partial u}{\partial t} = u^m \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q f(u), \quad t \geqslant 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $m \in \mathbb{R}$, $p > 1$, $q > 0$ and $f(s)$ is a smooth and sign-changing function with some typical structure conditions.

Equation (1.1) is degenerate at the points where $u = 0$ if $m > 0$, and at the points where $\nabla u = 0$ if $p > 2$, while it is also singular at the points where $u = 0$ if $m < 0$, and at the points where $\nabla u = 0$ if $1 < p < 2$. Another peculiarity is that the equation is known as the non-divergence-form equation, although, in the case in which $m < 1$, on replacing u by $u^{1/(1-m)}$, (1.1) could be transformed into the following well-known polytropic filtration equation with a source:

$$\frac{\partial u}{\partial t} = (1-m)\operatorname{div}(|\nabla u^\lambda|^{p-2}\nabla u^\lambda) + g(u), \quad \lambda = \frac{1}{1-m} > 0.$$

During the last few years, these kinds of equations have received attention from several authors. Some special properties of solutions have been discovered for the case when $m = 1$ and $p = 2$, which appears in a biological model describing the diffusive process for biological species (see, for example, [1, 4, 5]). In [8], Friedman and McLeod studied another typical case, when $m = p = 2$, appearing in plasma physics. In addition, there are also some further works for the case when $m > 1$ and $p = 2$ arising in the theory of damage mechanics (see, for example, [3, 20, 22, 23]). In these works some properties of solutions, such as the existence and the blow-up properties, were investigated. For the case in which $0 < m < 1$, many works have

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been devoted to properties of solutions (see, for example, [18, 24] and the references therein).

We are mainly interested in travelling wave solutions to equation (1.1). By a travelling-wave solution, we mean a solution $u(x,t)$ of (1.1) in $Q = \{(x,t);\ x \in \mathbb{R}^N,\ t > 0\}$ of the form $u(x,t) = \varphi(\gamma \cdot x + t)$ with γ being a given vector, and φ being a one-dimensional function. It was Luther [12] who first made mention of travelling waves as solutions for a certain reaction diffusion equation in a report drawing an analogy between the conduction of a nerve pulse and a crystallization process. A modern version of his paper can be seen in [2]. The importance of special solutions in travelling-wave form to equations like (1.1) lies in the fact that they give insight into the behaviour of some classes of solutions of the same equation with arbitrary initial conditions. In 1937, an important contribution was made in two separate works [7, 11], both of which are related to the description of the space–time distribution of an advantageous gene in a population which lives in a one-dimensional domain. In [11], Kolmogorov *et al.* introduced a formal way in which one can analyse the existence and the stability of the travelling wave for the case when it is a solution of a type of parabolic equation. They stated their results on existence for the one-dimensional heat equation with a source like $u(1-u)$ and with the Heaviside function as an initial condition and also showed the stability of travelling wave solutions. Since then, much research has been carried out in an attempt to extend the original results to more complicated equations which arise in a variety of fields. Many kinds of linear or semilinear equations have been investigated (see, for example, [6, 10, 21] and the references therein). It should be noted that only a few papers are concerned with travelling-wave solutions of the degenerate or singular diffusion equations [9, 13–16, 19]. In particular, a typical case when $p = 2$ for the non-divergence-form equation has also been investigated in [19].

The aim of this paper is to discuss the existence and regularity of smooth travelling wavefronts of (1.1), namely, special solutions of the form $u(x,t) = \varphi(\gamma \cdot x + t)$ with γ being a given vector. If we simply replace $\gamma \cdot x + t$ by t, then (1.1) is transformed into

$$-|\gamma|^p \varphi^m (|\varphi'|^{p-2}\varphi')'(t) + \varphi'(t) = \varphi^q(t) f(\varphi),$$

which, for convenience of argument, will also be transformed into the following system:

$$\left.\begin{aligned} \psi'(t) &= c\varphi^{-m}(|\psi|^{(2-p)/(p-1)}\psi - \varphi^q f(\varphi)), \\ \varphi' &= |\psi|^{(2-p)/(p-1)}\psi, \end{aligned}\right\} \tag{1.2}$$

where $\psi(t) = |\varphi'|^{p-2}\varphi'$ and $c = 1/|\gamma|^p$ is the wave speed.

Now we introduce the definition of travelling-wave solutions.

DEFINITION 1.1. A function $u(x,t) = \varphi(\gamma \cdot x + t)$ with a given $0 \neq \gamma \in \mathbb{R}^N$ is called a travelling wavefront solution of (1.1) if $\varphi(t) : C(\mathbb{R}) \to [0,1]$ and there exist t_1, t_2 with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$ such that

(i) $\varphi(t) \in C^2(t_1, t_2)$ and satisfies

$$-|\gamma|^p \varphi^m (|\varphi'|^{p-2}\varphi')'(t) + \varphi'(t) = \varphi^q(t) f(\varphi) \quad \text{for any } t \in (t_1, t_2), \tag{1.3}$$

(ii) $\varphi(t_1) = \theta_1$ and $\varphi(t_2) = \theta_2$, where θ_1 and θ_2 are equilibria of (1.3),

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(iii) $\varphi(t)$ is strictly monotone in the interval (t_1, t_2), $\varphi(t) = \theta_1$ for $t \in (-\infty, t_1)$ and $\varphi(t) = \theta_2$ for $t \in (t_2, +\infty)$,

(iv) if $\varphi(t_1) < \varphi(t_2)$, then $\varphi'(t_2) = 0$, while if $\varphi(t_1) > \varphi(t_2)$, then $\varphi'(t_1) = 0$.

Furthermore, if $\varphi'_+(t_1) = \varphi'_-(t_2) = 0$, we call $\varphi(t)$ a smooth travelling-wavefront solution, where φ'_+ and φ'_- denote the right and the left derivatives of φ.

Since it has been adopted by many authors, we restrict ourselves mainly to the typical Huxley source, that is, the bistable case, under which we discuss the existence of smooth travelling wavefronts with at most one wave speed. Throughout this paper, we assume that

(H) $f(1) = f(a) = 0$, $f'(1) < 0$, $f(s) < 0$ for $s \in [0, a)$ and $f(s) > 0$ for $s \in (a, 1)$, where $a \in (0, 1)$ is a given constant.

It is worth noting that the solution may not belong to $C^1(\mathbb{R})$, though we call it a smooth solution. We aim to find smooth travelling wavefronts connecting the two equilibria 0 and 1, which is defined in definition 1.1. In fact, only the so-called smooth travelling wavefront can be classical, namely only this kind of solution may be extended into the whole domain (see § 4).

We focus our attention initially on the existence of smooth travelling wavefronts, including both non-decreasing travelling wavefronts and non-increasing travelling wavefronts. Sufficient and necessary conditions will be given for the existence of non-decreasing solutions, non-increasing solutions and non-existence of solutions. More precisely, (1.1) admits a smooth and non-increasing travelling wavefront if and only if

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0 \quad \text{or} \quad = -\infty,$$

and in this case, the solution is unique, corresponding to a unique wave speed c^*. While (1.1) admits a smooth non-decreasing travelling wavefront if and only if $m < 1$ and

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s > 0,$$

and in this case the solution is still unique, corresponding to a unique wave speed, c^*. In addition to all the above-mentioned situations, (1.1) always admits non-smooth travelling wavefronts. After this, we will turn our attention to the regularity of smooth travelling wavefronts, specifically, the finiteness of t_1 for non-decreasing travelling wavefronts and the finiteness of t_2 for non-increasing travelling wavefronts.

2. The case $m \geqslant 1$

In this section we consider the case $m \geqslant 1$ and focus our attention on non-increasing travelling wavefronts. We find that there exists at most one wave speed with which the corresponding travelling wavefront is smooth. We attempt to determine necessary and sufficient conditions under which the corresponding travelling wavefront is smooth. Clearly, we have $\theta_1 = 0$, $\theta_2 = 1$. We see that there is no smooth and non-decreasing travelling wavefront for the case when $m \geqslant 1$, and only the smooth and non-increasing travelling wavefront possibly exists.

Before going further, we first show that $\varphi(t)$ is a smooth and non-decreasing travelling wavefront of (1.1) with the following asymptotic boundary conditions:

$$\varphi(t_1^+) = 0, \qquad \varphi(+\infty) = 1, \tag{2.1}$$

if and only if $\varphi(t)$ satisfies that

$$\left.\begin{aligned} &\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{\psi^{1/(p-1)}}, \quad \varphi \in (0,1), \\ &\psi(0) = \psi(1) = 0, \\ &\psi(\varphi) > 0, \quad \varphi \in (0,1). \end{aligned}\right\} \tag{2.2}$$

Remark 2.1. By a smooth travelling wavefront we mean that $\psi(0) = \psi(1) = 0$, while a non-smooth travelling wavefront means that $\psi(0) \neq 0$, $\psi(1) = 0$.

Lemma 2.2. *$\varphi(\xi)$ is a smooth and monotone non-decreasing travelling-wave solution of the problem (1.1) with asymptotic boundary conditions (2.1) for some fixed speed $c > 0$ if and only if $\psi(\varphi)$ is a solution of the problem (2.2).*

Proof. The necessary condition is now clear from the discussion above. So it suffices to consider the sufficient one. Let $\psi(\varphi)$ be a solution of the problem (2.2), and let $\varphi(\xi)$ be a solution of the equation

$$\varphi'(\xi) = \psi^{1/(p-1)}(\xi).$$

Without loss of generality, we may assume that $\varphi(0) = \frac{1}{2}$ and that $]\alpha, \beta[\subset \mathbb{R}$ is the maximal existence interval of φ. Firstly, we have

$$\begin{aligned} \varphi^m(|\varphi'|^{p-2}\varphi')'(\xi) &= \varphi^m \psi'(\varphi)\varphi'(\xi) \\ &= \left(c - \frac{c\varphi^q f(\varphi)}{\psi^{1/(p-1)}}\right)\varphi' \\ &= c\varphi' - c\varphi^q f(\varphi). \end{aligned}$$

That is, $\varphi(\xi)$ satisfies (1.3). Moreover, $\varphi(\alpha^+) = 0$ and $\varphi(\beta^-) = 1$. Therefore, when both $\alpha = -\infty$ and $\beta = +\infty$, φ is a smooth travelling wavefront, while if $\alpha > -\infty$, then

$$\lim_{\xi \to \alpha^+} \varphi'(\xi) = \lim_{\varphi \to 0^+} \psi^{1/(p-1)}(\varphi) = 0.$$

Similarly, if $\beta < +\infty$, we have

$$\lim_{\xi \to \beta^-} \varphi'(\xi) = \lim_{\varphi \to 1^-} \psi^{1/(p-1)}(\varphi) = 0.$$

Furthermore, by virtue of (1.3) itself, we also have

$$\lim_{\xi \to \beta^-} (|\varphi'|^{p-2}\varphi')'(\xi) = 0.$$

Then we can extend the solution to $[\beta, +\infty)$ by taking $\varphi(\xi) = 1$ on $[\beta, +\infty)$. Thus, $\varphi(\xi)$ is a smooth travelling wavefront. The proof is complete. □

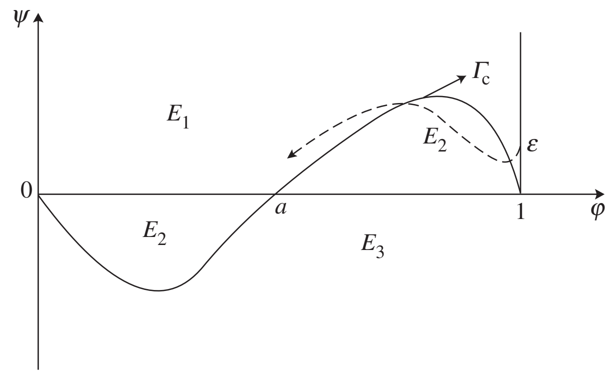

Figure 1. The path curve $\Gamma(\varphi, \psi)$ of the approaching problem.

By using a simple method similar to [19], it is easy to obtain the following result.

THEOREM 2.3. *Assume that $m \geqslant 1$, and that (H) is satisfied. Then problem (2.2) admits no solution for any $c > 0$, which means that there is no increasing and smooth travelling wavefront for problem (1.1) with $\theta_1 = 0$, $\theta_2 = 1$.*

We are now in a position to establish the existence results. In preparation, we give the following basic existence result for non-decreasing travelling wavefronts, with the smoothness determined later.

THEOREM 2.4. *Assume that $m \geqslant 1$, and that (H) is satisfied. Then problem (2.2) admits a unique non-decreasing travelling wavefront with $\theta_1 = a$, $\theta_2 = 1$ and $\varphi'_+(t_1) \geqslant 0$, $\varphi'_-(t_2) = 0$ for any $c > 0$, namely, the following problem admits a unique solution:*

$$\left.\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{\psi^{1/(p-1)}}, \quad \varphi \in (a, 1), \\ \psi(1) = 0, \\ \psi(\varphi) > 0, \quad \varphi \in (a, 1). \end{aligned}\right\} \tag{2.3}$$

Proof. The proof will be completed by using a phase-plane analysis approach and then using an approximation to obtain the existence of solutions in the interval $\varphi \in [a, 1]$.

Consider the following approaching problem:

$$\left.\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{\psi^{1/(p-1)}}, \quad \varphi \in (a, 1), \\ \psi(1) = \varepsilon, \\ \psi(\varphi) > 0, \quad \varphi \in (a, 1). \end{aligned}\right\} \tag{2.4}$$

We are now able to apply the phase-plane arguments to this problem to show the existence of solutions. In what follows, we denote the solution of (2.4) by ψ_ε, and denote the curve $\psi(\varphi) = \varphi^{q(p-1)} f^{p-1}(\varphi)$ by Γ_c. From figure 1, we see that curve Γ_c divides the plane into three parts, E_1, E_2 and E_3. From the first equation of problem (2.4), we see that $\psi'(\varphi) > 0$ if the point $(\varphi, \psi) \in E_1$, while $\psi'(\varphi) < 0$ if $(\varphi, \psi) \in E_2$. Therefore, starting from point $(1, \varepsilon)$, trajectory $\Gamma(\varphi, \psi)$ of (2.4) must be increasing before intersecting with Γ_c. It is also easy to see that $\Gamma(\varphi, \psi)$ does not intersect with the φ-axis when $\varphi \in (a, 1)$, since $\psi'(\varphi) < 0$ when $(\varphi, \psi) \in E_2$.

Next we show that ψ_ε is increasing in ε. Otherwise, there exist ε_1 and ε_2 with $\varepsilon_1 > \varepsilon_2 > 0$ and a point $0 < x_0 < 1$ such that $\psi_{\varepsilon_1}(x_0) < \psi_{\varepsilon_2}(x_0)$. Let

$$x^* = \inf\{x;\ x > x_0,\ \psi_{\varepsilon_1}(x) \geqslant \psi_{\varepsilon_2}(x)\}.$$

Then, for any $\varphi \in (x_0, x^*)$, we have

$$\begin{aligned}\psi'_{\varepsilon_1} - \psi'_{\varepsilon_2} &= c\varphi^{q-m} f(\varphi) \frac{\psi_{\varepsilon_1}^{1/(p-1)} - \psi_{\varepsilon_2}^{1/(p-1)}}{(\psi_{\varepsilon_1}\psi_{\varepsilon_2})^{1/(p-1)}} \cdot \frac{1}{\psi_{\varepsilon_1} - \psi_{\varepsilon_2}} (\psi_{\varepsilon_1} - \psi_{\varepsilon_2}) \\ &= G(\varphi)(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}).\end{aligned}$$

Obviously, we have $G(\varphi) > 0$ for $\varphi \in (x_0, x^*)$. Integrating from x_0 to φ yields

$$(\psi_{\varepsilon_1} - \psi_{\varepsilon_2})(\varphi) = (\psi_{\varepsilon_1} - \psi_{\varepsilon_2})(x_0) \exp\biggl\{ \int_{x_0}^{\varphi} G(s)\,\mathrm{d}s \biggr\}.$$

Letting $\varphi \to x^*$, we clearly have $(\psi_{\varepsilon_1} - \psi_{\varepsilon_2})(x^*) < 0$, which contradicts the definition of x^*. Therefore, ψ_ε is non-decreasing on ε. In addition, we also note that $\psi_\varepsilon \leqslant \max_{\varphi\in(a,1)}\{\varepsilon + \varphi^{q(p-1)} f^{p-1}(\varphi)\}$, which implies that ψ_ε is uniformly bounded. Also, since ψ_ε is a solution of problem (2.4), we also have

$$\psi_\varepsilon^{p/(p-1)}(\varphi) = \varepsilon^{p/(p-1)} - \frac{pc}{p-1} \int_{\varphi}^{1} s^{-m}(\psi_\varepsilon^{1/(p-1)}(s) - s^q f(s))\,\mathrm{d}s.$$

Since ψ_ε is bounded uniformly and increasing on ε, there must exist a function $\psi \geqslant 0$ such that $\psi_\varepsilon \to \psi$ almost everywhere in $(a, 1)$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0^+$ in the above equality and recalling the Lebesgue dominated convergence theorem yields

$$\psi^{p/(p-1)}(\varphi) = -\frac{pc}{p-1} \int_{\varphi}^{1} s^{-m}(\psi^{1/(p-1)}(s) - s^q f(s))\,\mathrm{d}s.$$

A phase-plane analysis approach yields that $\psi(\varphi) > 0$ for any $\varphi \in (a, 1)$, which implies that ψ is a solution of problem (2.3).

In what follows, we show the uniqueness.

Suppose to the contrary that there exists a $c > 0$ such that problem (2.3) admits at least two solutions ψ_1, ψ_2. Without loss of generality, we assume that there exists a $\varphi_0 \in (a, 1)$ such that $\psi_1(\varphi_0) < \psi_2(\varphi_0)$. Then we have

$$(\psi_2 - \psi_1)'(\varphi_0) = c\varphi^{q-m} f(\varphi_0) \biggl(\frac{1}{\psi_1^{1/(p-1)}(\varphi_0)} - \frac{1}{\psi_2^{1/(p-1)}(\varphi_0)} \biggr) > 0,$$

which means that $(\psi_2 - \psi_1)(\varphi) > (\psi_2 - \psi_1)(\varphi_0)$ for any $\varphi > \varphi_0$, which contradicts $\psi_2(1^-) = \psi_1(1^-) = 0$. Summing up, the proof is complete. □

Remark 2.5. From the proof of theorem 2.4 we see that theorem 2.4 is valid for $m \in \mathbb{R}$.

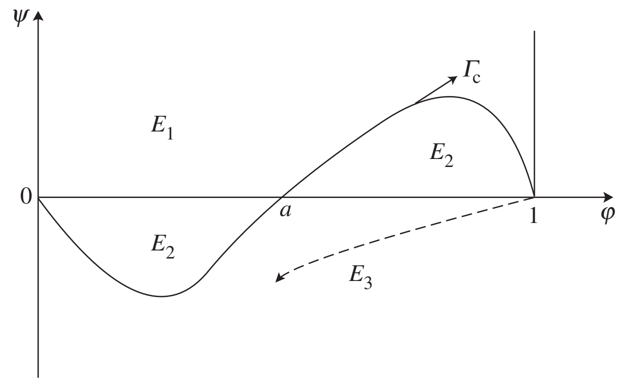

Figure 2. The path curve of $\Gamma(\varphi, \psi)$.

In what follows, we investigate the decreasing travelling wavefronts of problem (1.1), and we will see that $\varphi(t)$ is a decreasing and smooth travelling wavefront with $\theta_1 = 1$, $\theta_2 = 0$ if and only if $\psi(\varphi)$ satisfies

$$\left.\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}, \quad \varphi \in (0,1), \\ \psi(0) = \psi(1) = 0, \\ \psi(\varphi) < 0, \quad \varphi \in (0,1). \end{aligned}\right\} \tag{2.5}$$

Consider the following problem:

$$\left.\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}, \quad \varphi \in (\theta,1), \\ \psi(1) = 0, \\ \psi(\varphi) < 0, \quad \varphi \in (\theta,1), \end{aligned}\right\} \tag{2.6}$$

where $\theta \geqslant 0$, and $(\theta, 1)$ is the maximal existence interval of solutions. Using an approaching method with $\psi(1) = -\varepsilon < 0$ for problem (2.6), similarly to the above arguments, we see that there exists a path curve $\Gamma(\varphi, \psi)$ of problem (2.6) entering from $(1, 0)$ into the region $E = \{(\varphi, \psi);\ 0 < \varphi < 1, \psi < 0\}$.

We need the following lemmas.

LEMMA 2.6. *Assume that $m \geqslant 1$. Let $\Gamma(\varphi, \psi)$ be the path curve of the first equation of (2.5) entering into E from $(1, 0)$, and let $\psi(\varphi)$ be the corresponding solution. Then for sufficiently large $c > 0$, $\Gamma(\varphi, \psi)$ does not intersect with the φ-axis when $\varphi \in [0, 1)$ and $\psi(\varphi) \to -\infty$ as $\varphi \to 0^+$.*

Proof. From figure 2, we see that curve $\Gamma_c : \psi(\varphi) = \varphi^{q(p-1)}|f|^{p-2}f$ divides the plane $(0, 1) \times \mathbb{R}$ into three parts, E_1, E_2 and E_3: $\psi' > 0$ in E_1 and E_3 and $\psi' < 0$ in E_2. Therefore, we see that $\psi(\varphi)$ is increasing for $\varphi \in [a, 1]$ along the orbit $\Gamma(\varphi, \psi)$, which implies that $\psi(\varphi) < 0$ for $\varphi \in [a, 1)$. Next, we only need to find an appropriately large c such that ψ does not intersect with Γ_c (for details, see [19]). □

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Before going further, we give a comparison lemma as follows.

LEMMA 2.7. *Let ψ_i, $i = 1, 2$, be the solutions of the following problems with $0 \leqslant \alpha < \beta$, $c_i > 0$:*

$$\left.\begin{aligned} \frac{\mathrm{d}\psi_i}{\mathrm{d}\varphi} &= c_i\varphi^{-m} - \frac{c_i\varphi^{q-m}f(\varphi)}{|\psi_i|^{1/(p-1)-1}\psi_i}, \quad \varphi \in (\alpha, \beta) \\ \psi_i(\beta) &= \beta_i. \end{aligned}\right\} \tag{2.7}$$

If $\psi_1(\varphi)\psi_2(\varphi) > 0$ in (α, β) and there exists $\gamma \in [\alpha, \beta)$ such that $f(\varphi) \geqslant 0$ for $\varphi \in (\gamma, \beta)$, then if $c_1 > c_2$ and $c_1^{(p-1)/p}\beta_2 \geqslant c_2^{(p-1)/p}\beta_1$, we have $c_1^{(p-1)/p}\psi_2 > c_2^{(p-1)/p}\psi_1$ for any $\varphi \in (\alpha, \beta)$.

Proof. From (2.7) we see that

$$\begin{aligned} &\frac{\mathrm{d}(c_2^{(p-1)/p}\psi_1 - c_1^{(p-1)/p}\psi_2)}{\mathrm{d}\varphi} - \frac{c_1^{(p-1)/p}c_2^{(p-1)/p}\varphi^{q-m}f(\varphi)}{|\psi_1\psi_2|^{1/(p-1)}} \\ &\qquad \times (|c_2^{(p-1)/p}\psi_1|^{1/(p-1)-1}c_2^{(p-1)/p}\psi_1 - |c_1^{(p-1)/p}\psi_2|^{1/(p-1)-1}c_1^{(p-1)/p}\psi_2) \\ &\qquad = c_1^{(p-1)/p}c_2^{(p-1)/p}\varphi^{-m}(c_1^{1/p} - c_2^{1/p}). \end{aligned}$$

For simplicity, we define $\omega_1 = c_2^{(p-1)/p}\psi_1$, $\omega_2 = c_1^{(p-1)/p}\psi_2$ and

$$F(\varphi) = \begin{cases} \dfrac{|\omega_1|^{1/(p-1)-1}\omega_1(\varphi) - |\omega_2|^{1/(p-1)-1}\omega_2(\varphi)}{\omega_1(\varphi) - \omega_2(\varphi)} & \text{if } \omega_1(\varphi) \neq \omega_2(\varphi), \\ \dfrac{1}{p-1}|\omega_1|^{1/(p-1)-1}(\varphi) & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} &\frac{\mathrm{d}(\omega_1 - \omega_2)}{\mathrm{d}\varphi} - \frac{c_1^{(p-1)/p}c_2^{(p-1)/p}\varphi^{q-m}f(\varphi)}{|\psi_1\psi_2|^{1/(p-1)}}F(\varphi)(\omega_1 - \omega_2) \\ &\qquad = c_1^{(p-1)/p}c_2^{(p-1)/p}\varphi^{-m}(c_1^{1/p} - c_2^{1/p}). \end{aligned}$$

Let

$$G(\varphi) = (\omega_1 - \omega_2)(\varphi)\exp\biggl\{-c_1^{(p-1)/p}c_2^{(p-1)/p}\int_\gamma^\varphi \frac{s^{q-m}f(s)F(s)}{|\psi_1\psi_2|^{1/(p-1)}}\,\mathrm{d}s\biggr\}, \quad \varphi \in (\alpha, \beta).$$

Then we have

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}\varphi} &= c_1^{(p-1)/p}c_2^{(p-1)/p}\varphi^{-m}(c_1^{1/p} - c_2^{1/p}) \\ &\qquad \times \exp\biggl\{-c_1^{(p-1)/p}c_2^{(p-1)/p}\int_\gamma^\varphi \frac{s^{q-m}f(s)F(s)}{|\psi_1\psi_2|^{1/(p-1)}}\,\mathrm{d}s\biggr\}. \end{aligned}$$

Recalling that $c_1 > c_2$, $c_1^{(p-1)/p}\beta_2 \geqslant c_2^{(p-1)/p}\beta_1$, we obtain

$$\frac{\mathrm{d}G}{\mathrm{d}\varphi} > 0, \quad \lim_{\varphi \to \beta^-} G(\varphi) \leqslant 0, \quad \varphi \in (\alpha, \beta).$$

Therefore, we have

$$G < 0 \quad \text{for } \varphi \in (\alpha, \beta),$$

which implies $c_1^{(p-1)/p}\psi_2 > c_2^{(p-1)/p}\psi_1$ for $\varphi \in (\alpha, \beta)$. The proof is complete. □

LEMMA 2.8. *Assume that $m \geqslant 1$. Let $\Gamma(\varphi, \psi)$ be the path curve of the first equation of (2.5) entering into E from $(1, 0)$, and let $\psi(\varphi)$ be the corresponding solution.*

(i) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0,$$

then, for any $c > 0$, $\Gamma(\varphi, \psi)$ does not intersect the φ-axis when $\varphi \in [0, 1)$ and $\psi(\varphi) \to -\infty$ as $\varphi \to 0^+$.

(ii) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0,$$

then, for sufficiently small $c > 0$, $\Gamma(\varphi, \psi)$ intersects the φ-axis when $\varphi \in [0, 1)$.

Proof. The proof is simple and similar to [19], so only an outline is given.

(i) Firstly, by throwing out the first term $c\varphi^{-m}$, and with the help of the fact that

$$\int_\varphi^1 s^{q-m} f(s)\,\mathrm{d}s > 0$$

for any $1 > \varphi > 0$, it is easy to obtain that $\Gamma(\varphi, \psi)$ does not intersect with the φ-axis when $\varphi \in [0, 1)$. Furthermore, by using this result, we consider this problem in the domain $(0, \delta)$ for an appropriately small $\delta > 0$ and obtain that $\psi(\varphi) \to -\infty$ as $\varphi \to 0^+$.

(ii) This result can be obtained simply by taking a large wave speed c_0 such that $\Gamma(\varphi, \psi_0)$ does not intersect with the φ-axis when $\varphi \in [0, 1)$ (see lemma 2.6) and combining with comparison lemma 2.7 (see [19] for more details). □

Now we give a necessary and sufficient condition for the existence of smooth and non-increasing travelling wavefronts.

PROPOSITION 2.9. *Assume that $m \geqslant 1$ and that (H) holds.*

(i) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0,$$

then, for any $c > 0$, (2.5) admits no solution. That is, there is no decreasing and smooth travelling wavefront of (1.1) with $\theta_1 = 1$, $\theta_2 = 0$.

(ii) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0,$$

then there exists a unique wave speed $c^* > 0$ such that (2.5) admits a unique solution. That is, (1.1) admits a unique decreasing and smooth travelling wavefront with $\theta_1 = 1$, $\theta_2 = 0$ if and only if $c = c^*$.

Proof. Part (i) is a direct consequence of lemma 2.8. So it suffices to show (ii). Let

$$\Lambda = \{c > 0;\ \Gamma(\varphi, \psi) \text{ intersects the } \varphi\text{-axis when } \varphi \in [0, 1)\},$$

where $\Gamma(\varphi, \psi)$ is the orbit of (1.2) entering from $(1, 0)$ into $E = \{(\varphi, \psi);\ 0 < \varphi < 1, \psi < 0\}$. Lemmas 2.6 and 2.8(ii) imply that Λ is bounded and non-empty. Let $c^* = \sup \Lambda$. Then $c^* > 0$. Let Γ^* be the corresponding orbit, and let $\psi^*(\varphi)$ be the corresponding solution of (1.2). We shall show that c^* is the wave speed required.

We first show that $c^* \in \Lambda$. Suppose the contrary. Similarly to the proof of the last part of lemma 2.8(i), it is easy to show that $\lim_{\varphi \to 0^+} \psi^*(\varphi) = -\infty$. Take $\{c_i\}_i^\infty \subset \Lambda$ with $c_i \nearrow c^*$ and $\psi_i(\varphi)$, Γ_i being the corresponding solutions and orbit. Let φ_i be the first point of intersection with the φ-axis on $[0, 1)$ along the orbit Γ_i starting from $(1, 0)$. By lemma 2.7, we see that $\varphi_i \geqslant 0$ is non-increasing. Let $\varphi_0 = \lim_{i\to\infty} \varphi_i \geqslant 0$. If $\varphi_0 > 0$, then $\psi^*(\varphi_0) = 0$ from $\psi_i(\varphi_i) = 0$, which contradicts the statement that Γ^* does not intersect with the φ-axis when $\varphi \in [0, 1)$. Thus, $\varphi_0 = 0$; namely, $\lim_{i\to+\infty} \varphi_i = 0$. Furthermore, similar to the proof of proposition 3.2, we obtain

$$|\psi^*(\varphi)|^{p/(p-1)} = \frac{pc^*}{p-1} \int_0^\varphi s^{-m} |\psi^*(s)|^{1/(p-1)-1} \psi^*(s)\, \mathrm{d}s - \frac{pc^*}{p-1} \int_0^\varphi s^{q-m} f(s)\, \mathrm{d}s$$

by letting $i \to \infty$.

This is clearly a contradiction, since $\psi^*(\varphi) \to -\infty$ as $\varphi \to 0^+$. Therefore, we have $c^* \in \Lambda$, which means that Γ^* intersects the φ-axis when $\varphi \in [0, 1)$. Denote the point of intersection by $(\varphi^*, \psi^*(\varphi^*))$. In what follows, we show that $\varphi^* = 0$. Suppose the contrary; specifically, assume that $\varphi^* > 0$. Note that $\psi^*(\varphi)$ is increasing in $(a, 1)$ (see figure 2), which implies that $\varphi^* < a$. Recall that $f(s) < 0$ for $s \in [0, a)$ and

$$\frac{\mathrm{d}\psi_c}{\mathrm{d}\varphi} = c\varphi^{-m}\biggl(1 - \frac{\varphi^q f(\varphi)}{|\psi_c|^{1/(p-1)-1}\psi_c}\biggr).$$

By the continuity of the curve with respect to c, indeed, by replacing ψ with $-\psi$, we see that the above equality equals

$$\psi^{p/(p-1)}(\varphi) = \frac{pc}{p-1} \int_\varphi^1 (s^{-m}\psi^{1/(p-1)}(s) + s^{q-m} f(s))\, \mathrm{d}s.$$

It is clear that ψ is continuous on c for $\psi > \varepsilon$ for any fixed $\varepsilon > 0$, while, for the case in which ψ is near 0, by the fact that the term $\psi^{1/(p-1)}$ is small, we still have continuous independence on c. Then we have

$$\lim_{c \searrow c^*, \varphi \to \varphi^*} \frac{\mathrm{d}\psi_c}{\mathrm{d}\varphi} = -\infty.$$

Therefore $\Gamma(\varphi, \psi_c)$ must intersect with Γ_c. By using a phase-plane analysis, it is not difficult to see that $\psi_c(\varphi)$ must admit two extreme points φ_1, φ_2 with $0 < \varphi_1 < \varphi^* < \varphi_2 < a$ as $c > c^*$ sufficiently approaches c^*, since $\Gamma(\varphi, \psi_c)$ does not intersect with the φ-axis according to the definition of c^* and Λ. Thus, we have

$$\frac{\mathrm{d}\psi_c(\varphi_1)}{\mathrm{d}\varphi} = \frac{\mathrm{d}\psi_c(\varphi_2)}{\mathrm{d}\varphi} = 0,$$

which implies that

$$\psi_c(\varphi_i) = \varphi^{q(p-1)} |f(\varphi_i)|^{p-2} f(\varphi_i), \quad i = 1, 2.$$

Letting $c \searrow c^*$ yields

$$\psi^*(\varphi^*) = \varphi^{*q(p-1)} |f(\varphi^*)|^{p-2} f(\varphi^*),$$

which implies that $f(\varphi^*) = 0$ since $\psi^*(\varphi^*) = 0$, that is, $\varphi^* = a$, which is a contradiction. In addition, the uniqueness of c^* can be obtained from the proof of lemma 2.7.

Finally, we discuss the uniqueness of solutions when the speed $c = c^*$. Suppose the contrary, that is, that problem (2.5) admits at least two solutions ψ_1 and ψ_2. Without loss of generality, we assume that there exists a $\varphi_0 \in (0, 1)$ such that $\psi_1(\varphi_0) < \psi_2(\varphi_0)$. Note that

$$(\psi_2 - \psi_1)'(\varphi_0) = c\varphi^{q-m} f(\varphi_0) g(\varphi_0)(\psi_2 - \psi_1)(\varphi_0),$$

where

$$g(\varphi) = \begin{cases} \dfrac{|\psi_2|^{1/(p-1)-1}\psi_2 - |\psi_1|^{1/(p-1)-1}\psi_1}{(\psi_2\psi_1)^{1/(p-1)}(\psi_2 - \psi_1)} & \text{if } \psi_2 \neq \psi_1, \\ \dfrac{1}{p-1} |\psi_1|^{-1/(p-1)-1} & \text{if } \psi_2 = \psi_1. \end{cases}$$

Thus, if $\varphi_0 \in [a, 1)$, then we have $(\psi_2 - \psi_1)(\varphi) > (\psi_2 - \psi_1)(\varphi_0)$ for any $\varphi > \varphi_0$, which contradicts $\psi_2(1^-) = \psi_1(1^-) = 0$. If $\varphi_0 \in (0, a)$, then $(\psi_2 - \psi_1)(\varphi_0) < (\psi_2 - \psi_1)(\varphi)$ for any $\varphi < \varphi_0$, which contradicts $\psi_2(0^+) = \psi_1(0^+) = 0$. Summing up, we complete the proof. □

Combining lemmas 2.6 and 2.8 with proposition 2.9, we obtain the following detailed conclusions for the existence of smooth or non-smooth solutions.

THEOREM 2.10. *Assume that $m \geqslant 1$ and that (H) holds, and let c^* be given as in proposition 2.9.*

(i) *If*

$$\int_0^1 s^{q-m} f(s)\, \mathrm{d}s \geqslant 0,$$

then, for any $c > 0$, (1.1) admits a unique decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 1$, $\theta_2 = 0$ and $\varphi'_+(t_1) = 0$, $\varphi'_-(t_2) = -\infty$.

(ii) *If*

$$\int_0^1 s^{q-m} f(s)\, \mathrm{d}s < 0,$$

then, for $c > c^*$, (1.1) admits a unique decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 1$, $\theta_2 = 0$ and $\varphi'_+(t_1) = 0$, $\varphi'_-(t_2) = -\infty$. For $0 < c < c^*$, (1.1) admits a unique decreasing travelling wavefront $\psi(\varphi)$ with $\theta_1 = 1$, $\theta_2 = a$ and $\varphi'_+(t_1) = 0$, $-\infty < \varphi'_-(t_2) < 0$. If $c = c^*$, (1.1) admits a unique decreasing and smooth travelling wavefront with $\theta_1 = 1$, $\theta_2 = 0$.

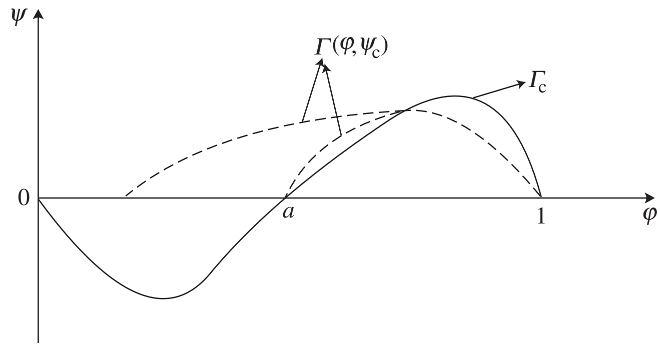

Figure 3. The case in which travelling wavefronts are increasing for any $c > 0$.

We give a group of figures depicting trajectories for the case in which $m \geqslant 1$.

The cases in which travelling wavefronts are decreasing are shown in figures 4 and 5.

3. The case $m < 1$

In this section we consider the case in which $m < 1$ with $f(s)$ satisfying assumption (H). We first study non-decreasing travelling wavefronts.

Consider the following variational equation:

$$\left.\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m}f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}, \quad \varphi \in (\theta, 1),\\ \psi(1) = 0,\\ \psi(\varphi) > 0, \quad \varphi \in (\theta, 1),\end{aligned}\right\} \tag{3.1}$$

where $\theta \geqslant 0$ and $(\theta, 1)$ is the maximal existence interval of φ.

The local existence of the solution $\psi(\varphi)$ is known from remark 2.5. As a preparation for establishing necessary and sufficient conditions for the existence of smooth non-decreasing travelling wavefronts, we first present the following non-existence result.

Proposition 3.1. *Assume that $m < 1$. If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \leqslant 0,$$

then, for any speed $c > 0$, there is no non-decreasing smooth travelling wavefront connecting 0 and 1. However, problem (1.3) admits a unique non-decreasing travelling wavefront with $\theta_1 = a$, $\theta_2 = 1$ and $\varphi'_+(t_1) \geqslant 0$, $\varphi'_-(t_2) = 0$ for any $c > 0$. That is, problem (3.1) with $\theta = a$ admits a unique solution.

Proof. Suppose the contrary. Recall that

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m}\left(1 - \frac{\varphi^q f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}\right)$$

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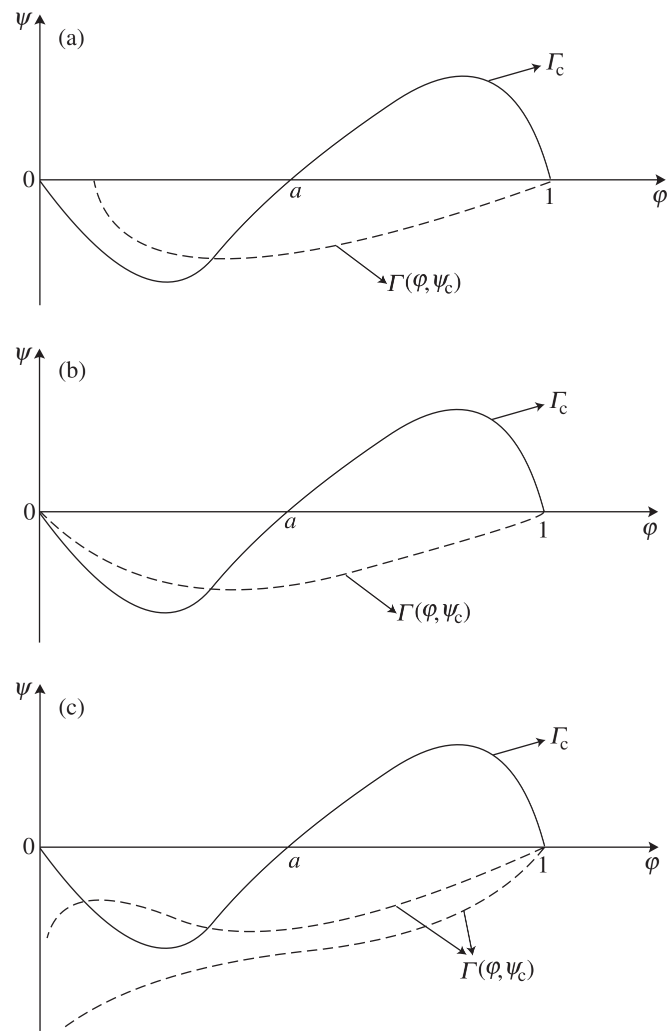

Figure 4. The case in which $\int_0^1 s^{q-m} f(s)\,ds < 0$: (a) $0 < c < c^*$, (b) $c = c^*$, (c) $c > c^*$.

implies that

$$\frac{p-1}{p}\frac{d|\psi|^{(p-1)/p}}{d\varphi} = c\varphi^{-m}|\psi|^{1/(p-1)-1}\psi - c\varphi^{q-m}f(\varphi) \geqslant -c\varphi^{q-m}f(\varphi),$$

since $\psi \geqslant 0$. Integrating from φ to 1 yields

$$|\psi|^{(p-1)/p}(\varphi) < \int_{\varphi}^{1} cs^{q-m}f(s)\,ds.$$

We therefore have

$$|\psi|^{(p-1)/p}(\varphi) < 0$$

as $\varphi \to 0^+$. This is a contradiction.

The second conclusion is a direct result of theorem 2.4 and remark 2.5. □

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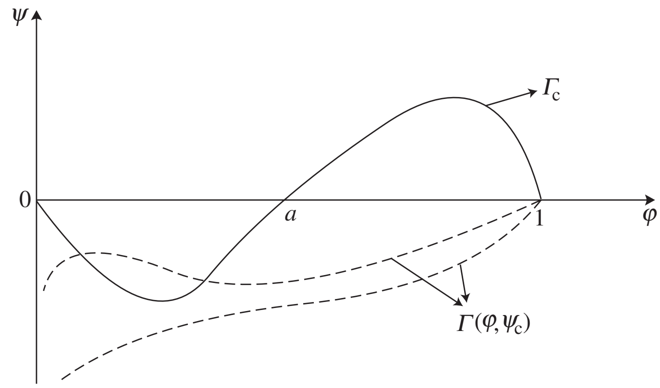

Figure 5. The case in which $\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0$ for any wave speed $c > 0$.

As a supplementation of the previous proposition, we consider the case in which

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s > 0.$$

Proposition 3.2. *Assume that $m < 1$ and that*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s > 0.$$

Let $\Gamma(\varphi, \psi_c)$ be the path curve of problem (3.1) starting from $(1, 0)$, and let $\psi_c(\varphi)$ be the corresponding solution. There is $c^* > 0$ such that

(i) *if $c = c^*$, $\Gamma(\varphi, \psi_c)$ connects 0 and 1,*

(ii) *if $c < c^*$, $\Gamma(\varphi, \psi_c)$ does not intersect with the φ-axis when $\varphi \in [0, 1)$,*

(iii) *if $c > c^*$, $\Gamma(\varphi, \psi_c)$ intersects the φ-axis for $\varphi \in (0, a]$.*

Proof. For a fixed $0 < \varphi_0 < a$, let

$$M = \sup_{\varphi \in (a,1)} \frac{\varphi^q f(\varphi)}{(\varphi - \varphi_0)^{1/(p-1)}}, \qquad \alpha = (2M)^{p-1} \quad \text{and} \quad \bar{\psi} = \alpha(\varphi - \varphi_0).$$

Then we have

$$\frac{\mathrm{d}\bar{\psi}_c}{\mathrm{d}\varphi} = \alpha.$$

A direct calculation yields

$$\frac{\mathrm{d}\bar{\psi}_c}{\mathrm{d}\varphi} \leqslant c\varphi^{-m}\biggl(1 - \frac{\varphi^q f(\varphi)}{\bar{\psi}_c^{1/(p-1)}}\biggr)$$

for any $c > 2\alpha$, $\varphi \geqslant \varphi_0$. In addition, we also note that $\bar{\psi}_c(1^-) > \psi_c(1^-)$. Then we have $\bar{\psi}_c(\varphi) > \psi_c(\varphi)$, which implies that $\psi_c(\varphi)$ intersects the φ-axis at some point $\varphi \geqslant \varphi_0$. In addition, by a phase-plane argument, we see that $\Gamma(\varphi, \psi_c)$ does not intersect with the φ-axis for $\varphi > a$. Therefore, for sufficiently large $c > 0$, ψ_c must intersect with the φ-axis at $(0, a]$.

In what follows, we show that when c is sufficiently small $\Gamma(\varphi, \psi_c)$ does not intersect with the φ-axis at $[0,1)$. Let $\tilde{\psi}_c = c^{-(p-1)/p}\psi_c$ for any $c > 0$. Then consider the following problem:

$$\left.\begin{aligned} \frac{\mathrm{d}\tilde{\psi}_c}{\mathrm{d}\varphi} = c^{1/p}\varphi^{-m} - \frac{\varphi^{q-m}f(\varphi)}{\tilde{\psi}_c^{1/(p-1)}}, \quad \varphi \in (\theta, 1), \\ \tilde{\psi}(1) = 0, \\ \tilde{\psi}(\varphi) > 0, \quad \varphi \in (\theta, 1). \end{aligned}\right\} \tag{3.2}$$

where $(\theta, 1)$ is the maximal existence interval of $\tilde{\psi}_c$. Firstly, we have

$$\frac{\mathrm{d}\tilde{\psi}_c}{\mathrm{d}\varphi} \geqslant -\frac{\varphi^{q-m}f(\varphi)}{\tilde{\psi}_c^{1/(p-1)}} \geqslant -M\frac{\varphi^{q-m}}{\tilde{\psi}_c^{1/(p-1)}},$$

where $M = \max_{\varphi\in[a,1]} f(\varphi)$. Integrating from φ to 1 yields

$$\tilde{\psi}_c^{p/(p-1)}(\varphi) \leqslant \frac{Mp}{(p-1)(q+1-m)}(1 - \varphi^{q+1-m}). \tag{3.3}$$

In addition, by (3.2), we further obtain that

$$\begin{aligned} \tilde{\psi}_c^{p/(p-1)}(\varphi) &= -\frac{c^{1/p}p}{p-1}\int_\varphi^1 s^{-m}\tilde{\psi}_c^{1/(p-1)}(s)\,\mathrm{d}s + \frac{p}{p-1}\int_\varphi^1 s^{q-m}f(s)\,\mathrm{d}s \\ &\geqslant -c^{1/p}\left(\frac{p}{p-1}\right)^{(p+1)/p}\left(\frac{M}{q+1-m}\right)^{1/p}\int_\varphi^1 s^{-m}(1 - s^{q+1-m})^{1/p}\,\mathrm{d}s \\ &\quad + \frac{p}{p-1}\int_\varphi^1 s^{q-m}f(s)\,\mathrm{d}s. \end{aligned}$$

Clearly, when c is appropriately small, we have $\tilde{\psi}_c(\varphi) > 0$ for any $\varphi \in [0,1)$, since

$$\int_\varphi^1 s^{q-m}f(s)\,\mathrm{d}s > \int_0^1 s^{q-m}f(s)\,\mathrm{d}s > 0.$$

In what follows, we show that there exists $c^* > 0$ such that $\Gamma(\varphi, \tilde{\psi}_{c^*})$ intersects the φ-axis at $(0,0)$. Let $c^* = \inf\{c; c \in \tilde{\Lambda}\}$, where

$$\tilde{\Lambda} = \{c > 0; \Gamma(\varphi, \tilde{\psi}_c) \text{ intersects the } \varphi\text{-axis when } \varphi \in [0,1)\}.$$

From the above discussion, we see that c^* is well defined. We now show that $\tilde{\psi}_{c^*}$ connects 0 and 1. We first show that $c^* \in \tilde{\Lambda}$. From the definition of c^*, we see that there exists a sequence $c_i \in \tilde{\Lambda}$ with $c_i \searrow c^*$. Correspondingly, let φ_i be the first point of intersection with the φ-axis on $[0,a]$ along the orbit $\Gamma(\varphi, \tilde{\psi}_{c_i})$ starting from $(1,0)$. By lemma 2.7, we see that $\tilde{\psi}_{c_i}$ is increasing on i, which implies that φ_i is decreasing, namely $\varphi_i \geqslant \varphi_{i+1}$ for any $i \in \mathbb{N}$. Let

$$\varphi^* = \lim_{i\to+\infty}\varphi_i, \qquad \tilde{\psi}_{c^*}(\varphi) = \lim_{i\to+\infty}\tilde{\psi}_{c_i}(\varphi).$$

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Noting that $\psi_i(\varphi_i) = 0$ and that

$$\frac{\mathrm{d}(|\tilde{\psi}_i|^{p/(p-1)})}{\mathrm{d}\varphi} = \frac{pc^{1/p}}{p-1}\varphi^{-m}|\tilde{\psi}_i|^{1/(p-1)-1}\tilde{\psi}_i - \frac{p}{p-1}\varphi^{q-m}f(\varphi),$$

by integrating from φ_i to φ we see that

$$\tilde{\psi}_i(\varphi)^{p/(p-1)} = \frac{pc^{1/p}}{p-1}\int_{\varphi_i}^{\varphi} s^{-m}\tilde{\psi}_i(s)^{1/(p-1)}\,\mathrm{d}s - \frac{p}{p-1}\int_{\varphi_i}^{\varphi} s^{q-m}f(s)\,\mathrm{d}s.$$

From Levi's theorem [17], and noting that $\lim_{i\to+\infty}\varphi_i = \varphi^*$, we obtain

$$\lim_{i\to+\infty}\int_{\varphi_i}^{\varphi} s^{-m}\tilde{\psi}_i(s)^{1/(p-1)}\,\mathrm{d}s = \int_{\varphi^*}^{\varphi} s^{-m}\tilde{\psi}^*(s)^{1/(p-1)}\,\mathrm{d}s \quad \text{for any } \varphi \in (\varphi^*, 1).$$

Thus, letting $i \to \infty$, we obtain

$$\tilde{\psi}^*(\varphi)^{p/(p-1)} = \frac{pc^{1/p}}{p-1}\int_{\varphi^*}^{\varphi} s^{-m}\tilde{\psi}_i(s)^{1/(p-1)}\,\mathrm{d}s - \frac{p}{p-1}\int_{\varphi^*}^{\varphi} s^{q-m}f(s)\,\mathrm{d}s$$

and

$$\tilde{\psi}^*(\varphi^*) = 0.$$

In what follows, we show that $\varphi^* = 0$; otherwise $0 < \varphi^* < a$. We have

$$\lim_{c\nearrow c^*,\varphi\to\varphi^*}\frac{\mathrm{d}\tilde{\psi}_c}{\mathrm{d}\varphi} = +\infty,$$

since

$$\frac{\mathrm{d}\tilde{\psi}_c}{\mathrm{d}\varphi} = c^{1/p}\varphi^{-m} - \frac{\varphi^{q-m}f(\varphi)}{\tilde{\psi}_c^{1/(p-1)}}$$

from the continuity of the curve with respect to c. Therefore, we also have that

$$\lim_{c\nearrow c^*}\frac{\mathrm{d}\tilde{\psi}_c}{\mathrm{d}\varphi} = +\infty$$

for $\varphi \in [\varphi^*/2, \varphi^*]$, since $\tilde{\psi}_c$ is increasing for $\varphi \in (0, a)$. In addition, recalling (3.3), we also see that $\tilde{\psi}_c$ is bounded uniformly. Thus, $\Gamma(\varphi, \tilde{\psi}_c)$ must intersect with the φ-axis as $c < c^*$ sufficiently approaches c^*, which contradicts the definition of c^*. Moreover, by the definition of c^* we see that, for any $c < c^*$, $\Gamma(\varphi, \tilde{\psi}_c)$ does not intersect with the φ-axis for $\varphi \in [0, 1)$. In addition, by the monotonicity of $\tilde{\psi}_c$ on the c, and combining with the proof of lemma 2.7, it is also easy to see that, for any $c > c^*$, $\Gamma(\varphi, \tilde{\psi}_c)$ must intersect with the φ-axis at $(0, a)$. Summing up, we complete the proof. □

Combining propositions 3.1 and 3.2, we present the following existence results for non-decreasing travelling wavefronts. In particular, a necessary and sufficient condition for the existence of smooth solutions is given in the theorem.

THEOREM 3.3. *Assume that $m < 1$ and that (H) holds, and let c^* be given by proposition 3.2.*

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(i) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \leqslant 0,$$

then, for any $c > 0$, (1.1) admits no non-decreasing and smooth travelling wavefront. However, it does admit a unique non-decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = a$, $\theta_2 = 1$ and $0 \leqslant \varphi'_+(t_1) < +\infty$, $\varphi'_-(t_2) = 0$.

(ii) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s > 0,$$

then, for $c \neq c^*$, (1.1) admits no non-decreasing and smooth travelling wavefront with $\theta_1 = 0$, $\theta_2 = 1$. However, for $c > c^*$, (1.1) admits a unique non-decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = a$, $\theta_2 = 1$ and $0 \leqslant \varphi'_+(t_1) < +\infty$, $\varphi'_-(t_2) = 0$. For $0 < c < c^*$, (1.1) admits at least one non-decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 0$, $\theta_2 = 1$ and $\varphi'_-(t_2) = 0$, $0 < \varphi'_+(t_1) < +\infty$. If $c = c^*$, (1.1) admits a unique non-decreasing and smooth travelling wavefront with $\theta_1 = 0$, $\theta_2 = 1$.

Next we consider the non-increasing travelling fronts.

Consider the following variational problem:

$$\left.\begin{gathered}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}, \quad \varphi \in (\theta, 1),\\ \psi(1) = 0,\\ \psi(\varphi) < 0, \quad \varphi \in (\theta, 1),\end{gathered}\right\} \tag{3.4}$$

where $(\theta, 1)$ is the maximal existence interval of φ.

As for the non-decreasing solutions, we first present a non-existence result for non-increasing solutions.

PROPOSITION 3.4. *Assume that $m < 1$. If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0,$$

then, for any speed $c > 0$, the trajectory of problem (3.4) satisfies $\psi(\varphi) < 0$ for any $\varphi \in [0, 1)$.

Proof. Firstly, we see that

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} &= c\varphi^{-m} - \frac{c\varphi^{q-m} f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}\\ &> -\frac{c\varphi^{q-m} f(\varphi)}{|\psi|^{1/(p-1)-1}\psi}.\end{aligned}$$

Since $\psi_c < 0$, then we also have

$$\frac{p-1}{p}|\psi|^{p/(p-1)}(\varphi) > c\int_\varphi^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0,$$

which means that $\psi < 0$ for any $\varphi \in [0, 1)$; that is, ψ_c does not intersect with the φ-axis. The proof is complete. □

The following proposition supplements the results of the previous proposition by considering the case in which

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0.$$

PROPOSITION 3.5. *Assume that $m < 1$ and that*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0.$$

Let $\Gamma(\varphi, \psi_c)$ be the path curve of the first equation of problem (3.4), and let $\psi_c(\varphi) \leqslant 0$ be the corresponding solution. Then there is $c^* > 0$ such that

(i) *if $c = c^*$, $\Gamma(\varphi, \psi_c)$ connects 0 and 1,*

(ii) *if $c > c^*$, $\Gamma(\varphi, \psi_c)$ does not intersect with the φ-axis when $\varphi \in [0, 1)$,*

(iii) *if $0 < c < c^*$, $\Gamma(\varphi, \psi_c)$ intersects the φ-axis for $\varphi \in (0, a)$.*

Proof. We first show that, for sufficiently large $c > 0$, ψ_c does not intersect with the φ-axis for $\varphi \in [0, 1)$. Let

$$M = \sup_{\varphi \in (0,a)} \varphi^{q(p-1)} |f(\varphi)|^{p-1}.$$

Then we have $\psi_c'(\varphi) \geqslant c\varphi^{-m}$ for $\varphi \in [a, 1)$. Integrating from a to 1 yields

$$\psi_c(\varphi) \leqslant -\frac{c}{1-m}(1 - a^{1-m}).$$

Then for any $c \geqslant M(1-m)/(1-a^{1-m})$, ψ_c does not intersect with Γ_c, which also means that ψ_c does not intersect with the φ-axis. Now we show that, for sufficiently small speed c, ψ_c intersects the φ-axis at some point $\varphi \in (0, a)$. By lemma 2.7 we see that, for any fixed c_0 where ψ_{c_0} does not intersect with the φ-axis, we have

$$\begin{aligned}
|\psi_c|^{p/(p-1)} &= \frac{pc}{p-1}\int_\varphi^1 s^{-m}|\psi_c(s)|^{1/(p-1)}\,\mathrm{d}s + \frac{pc}{p-1}\int_\varphi^1 s^{q-m} f(s)\,\mathrm{d}s \\
&\leqslant c\biggl(\frac{p}{p-1}\biggl(\frac{c}{c_0}\biggr)^{1/p}\int_\varphi^1 s^{-m}|\psi_{c_0}(s)|^{1/(p-1)}\,\mathrm{d}s + \frac{p}{p-1}\int_\varphi^1 s^{q-m} f(s)\,\mathrm{d}s\biggr).
\end{aligned}$$

Therefore, for sufficiently small $c > 0$, there must exist a $\varphi_0 \in [0, a)$ such that $\psi_c(\varphi_0) = 0$, since

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s = 0.$$

Similarly to the proof of proposition 3.4, we define $c^* = \sup\{c; c \in \hat{\Lambda}\}$ with

$$\hat{\Lambda} = \{c > 0;\ \Gamma(\varphi, \psi_c) \text{ intersects the } \varphi\text{-axis when } \varphi \in (0, 1]\}.$$

From the above discussion, we see that c^* is well defined. The following arguments are similar to those of proposition 3.4, and so we omit the proof. □

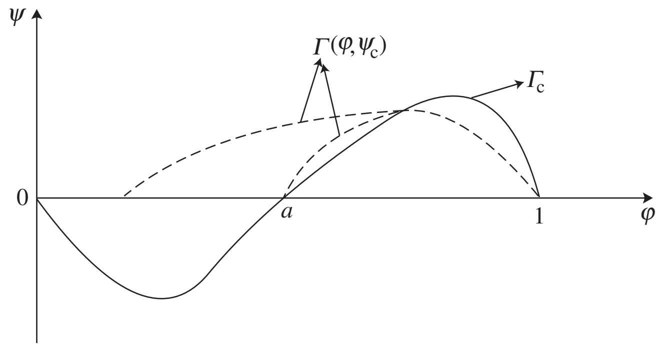

Figure 6. The case in which $\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \leqslant 0$ for any $c > 0$.

REMARK 3.6. Similarly to the proof of theorem 2.4 and proposition 2.9, it is not difficult to obtain the uniqueness of trajectories connecting the two points $(1, 0)$ and $(0, 0)$. It is also easy to obtain the uniqueness of trajectories starting from $(1, 0)$ and going to (a, ψ).

Combining propositions 3.4, 3.5 and remark 3.6, we present the following existence results for non-increasing travelling wavefronts. In particular, a necessary and sufficient condition for the existence of smooth solutions is given in the theorem.

THEOREM 3.7. *Assume that $m < 1$ and that (H) holds, with c^* given by proposition 3.5.*

(i) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0,$$

then, for any $c > 0$, (1.1) admits no decreasing and smooth travelling wavefront. However, it does admit a unique decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 1$, $\theta_2 = 0$ and $\varphi'_+(t_1) = 0$, $-\infty < \varphi'_-(t_2) < 0$.

(ii) *If*

$$\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0,$$

then, for $c \neq c^*$, (1.1) admits no decreasing and smooth travelling wavefront with $\theta_1 = 1$, $\theta_2 = 0$. However, for $c > c^*$, (1.1) admits a unique decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 1$, $\theta_2 = 0$ and $\varphi'_+(t_1) = 0$, $-\infty < \varphi'_-(t_2) < 0$. For $0 < c < c^*$, (1.1) admits a unique decreasing and non-smooth travelling wavefront $\varphi(t)$ with $\theta_1 = 1$, $\theta_2 = a$ and $\varphi'_+(t_1) = 0$, $-\infty < \varphi'_-(t_2) < 0$. If $c = c^*$, (1.1) admits a unique decreasing and smooth travelling wavefront with $\theta_1 = 1$, $\theta_2 = 0$.

Figures 6–9 depict the trajectories for the case $m < 1$. Figures 6 and 7 show the case in which travelling waves are increasing, while figures 8 and 9 show the case in which travelling waves are decreasing.

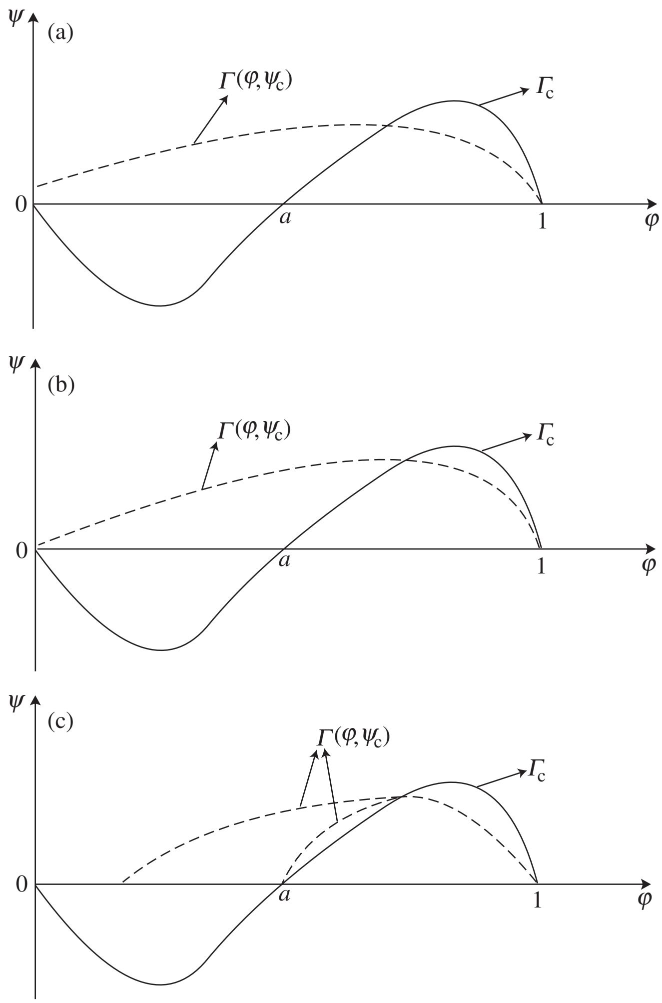

Figure 7. The case in which $\int_0^1 s^{q-m} f(s)\,\mathrm{d}s > 0$: (a) $0 < c < c^*$; (b) $c = c^*$; (c) $c > c^*$.

4. Regularity of smooth wavefronts

Now we turn our attention to the regularity of travelling wavefronts. More precisely, for non-decreasing travelling wavefronts, we investigate the finiteness of t_1, and for decreasing travelling wavefronts, we investigate the finiteness of t_2. It should be noted that if $t_1 = -\infty$ or $t_2 = +\infty$, then the corresponding solutions are classical in the whole domain $\mathbb{R} \times (0, +\infty)$. From §§ 2 and 3 we see that the equation may have smooth non-increasing and non-decreasing travelling wavefronts. The following theorem shows the regularity for non-increasing solutions, namely, the finiteness of t_2.

THEOREM 4.1. *Let $\varphi(t)$ be the non-increasing and smooth travelling wavefront of (1.1) with f satisfying (H) corresponding to c^* as determined in proposition 2.9.*

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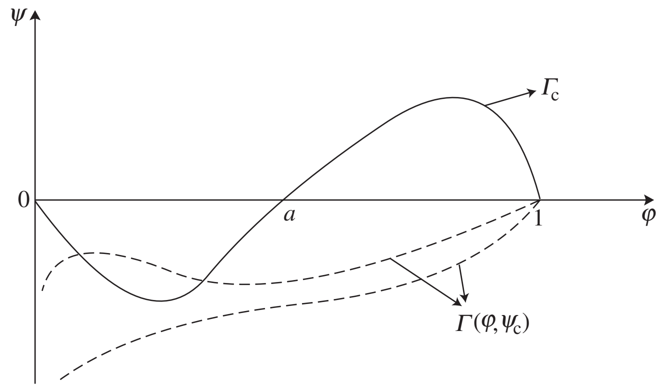

Figure 8. The case in which $\int_0^1 s^{q-m} f(s)\,\mathrm{d}s \geqslant 0$ for any wave speed $c > 0$.

(i) *If* $m \geqslant 1$ *with* $0 < q < 1$, *or* $m < 1$ *with* $0 < q < 1$ *and* $m + p > 1 + q$, *then* $t_2 < +\infty$.

(ii) *If* $q \geqslant 1$, *or* $0 < q < 1$ *with* $m < 1$ *and* $m + p \leqslant 1 + q$, *then* $t_2 = +\infty$.

Proof. (i) When $m \geqslant 1$, let

$$A = \min_{0 \leqslant \varphi \leqslant a/2} \{|f(\varphi)|\}, \qquad \sigma = \min\left\{\frac{a}{2}, \left(\frac{2^{p-1}c^*}{q(p-1)A^{p-1}}\right)^{1/(m+q(p-1)-1)}\right\}$$

and

$$\psi^*(\varphi) = -\left(\frac{A}{2}\right)^{p-1} \varphi^{q(p-1)}.$$

We assert that $\psi(\varphi) < \psi^*(\varphi)$ for all $\varphi \in (0, \sigma)$. Indeed, if the assertion were not true, then there would exist $\varphi_0 \in (0, \sigma)$ such that $\psi(\varphi_0) \geqslant \psi^*(\varphi_0)$. Thus, we have

$$\begin{aligned}\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} &= c^*\varphi_0^{-m}\left(1 - \frac{\varphi_0^q|f(\varphi_0)|}{|\psi(\varphi_0)|^{1/(p-1)}}\right)\\ &\leqslant c^*\varphi_0^{-m}\left(1 - \frac{\varphi_0^q|f(\varphi_0)|}{|\psi^*(\varphi_0)|^{1/(p-1)}}\right)\\ &\leqslant -c^*\varphi_0^{-m}.\end{aligned}$$

In addition, we also note that

$$\begin{aligned}\left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} &= -q(p-1)\left(\frac{A}{2}\right)^{p-1}\varphi_0^{q(p-1)-1}\\ &> -c^*\varphi_0^{-m}\\ &\geqslant \left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0},\end{aligned}$$

since

$$\varphi_0 < \sigma \leqslant \left(\frac{2^{p-1}c^*}{q(p-1)A^{p-1}}\right)^{1/(m+q(p-1)-1)}.$$

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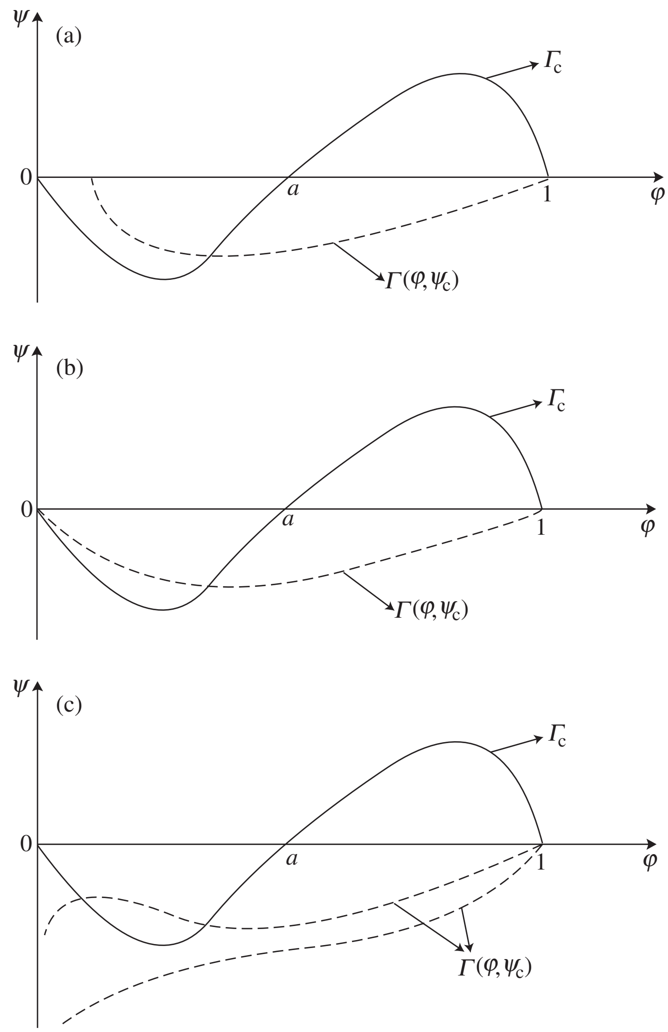

Figure 9. The case in which $\int_0^1 s^{q-m} f(s)\,\mathrm{d}s < 0$: (a) $0 < c < c^*$; (b) $c = c^*$; (c) $c > c^*$.

Let

$$G = \{\varphi \in [0, \varphi_0);\ \psi(\varphi) < \psi^*(\varphi)\}.$$

We will show that $G = \emptyset$. Otherwise, let

$$\varphi_1 = \sup\{\varphi \in [0, \varphi_0);\ \psi(\varphi) < \psi^*(\varphi)\}.$$

We clearly have $\varphi_1 > 0$, since $\psi(0) = \psi^*(0) = 0$, and we also have $\psi(\varphi_1) = \psi^*(\varphi_1)$ from the definition of φ_1. According to the above arguments, we obtain

$$\left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} > \left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1}.$$

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From the definition of φ_1, we also have

$$\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\bigg|_{\varphi=\varphi_1} \leqslant \frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\bigg|_{\varphi=\varphi_1}.$$

This leads to a contradiction. Thus, we have $\psi(\varphi) \geqslant \psi^*(\varphi)$ for all $\varphi \in [0, \varphi_0)$. We further obtain

$$\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi} > \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} \quad \text{for any } \varphi \in (0, \varphi_0).$$

Integrating from 0 to φ yields

$$\psi(0) > \psi(\varphi) - \psi^*(\varphi) + \psi^*(0) \geqslant 0,$$

which contradicts the fact that $\psi(0) = \psi^*(0) = 0$. Hence, we have, for all $\varphi \in (0, \sigma)$,

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= |\psi(\varphi)|^{1/(p-1)-1}\psi(\varphi)\\ &< |\psi^*(\varphi)|^{1/(p-1)-1}\psi^*(\varphi)\\ &= -\tfrac{1}{2}A\varphi^q.\end{aligned}$$

Take $\tau_1 \in (t_1, t_2)$ such that $\varphi(\tau_1) = \sigma/2$. Then $\varphi(t) \in (0, \sigma)$ for all $t \in (\tau_1, t_2)$. Integrating from τ_1 to τ_2 with $t_1 < \tau_1 < \tau_2 < t_2$, we obtain

$$\tau_2 - \tau_1 \leqslant -\frac{2}{A}\int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \frac{1}{s^q}\,\mathrm{d}s \leqslant \frac{2}{A}\int_0^1 \frac{1}{s^q}\,\mathrm{d}s = \frac{2}{A(1-q)},$$

since $0 < q < 1$. According to the arbitrariness of τ_2 with $\tau_2 < t_2$, we conclude that $t_2 < +\infty$.

When $m < 1$, let $A = \min_{0 \leqslant \varphi \leqslant a/2}\{|f(\varphi)|\}$,

$$\psi^* = -\alpha\varphi^\beta,$$

where $\beta = \max\{(p-1)(1-m+q)/p, q(p-1)\}$ and α is a sufficiently small constant that is to be determined. In what follows, we show that $\psi(\varphi) < \psi^*(\varphi)$ for $\varphi \in (0, a/2)$. Suppose, to the contrary, that there exists a $\varphi_0 \in (0, a/2)$ such that $\psi(\varphi_0) \geqslant \psi^*(\varphi_0)$. Then we have

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\bigg|_{\varphi=\varphi_0} &= c^*\varphi_0^{-m}\biggl(1 - \frac{\varphi_0^q|f(\varphi_0)|}{|\psi(\varphi_0)|^{1/(p-1)}}\biggr)\\ &\leqslant c^*\varphi_0^{-m}\biggl(1 - \frac{A\varphi_0^q}{\alpha^{1/(p-1)}\varphi_0^{\beta/(p-1)}}\biggr).\end{aligned}$$

By a direct calculation, we see that

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\bigg|_{\varphi=\varphi_0} < \frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\bigg|_{\varphi=\varphi_0}$$

is ensured by

$$c^*\varphi_0^{1-m-\beta} + \alpha\beta < \frac{c^*A}{\alpha^{1/(p-1)}}\varphi_0^{1-m-\beta+q-\beta/(p-1)}.$$

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Clearly, if we choose sufficiently small α then the above inequality holds, since $\beta \geqslant q(p-1)$, $1-m-\beta+q-\beta/(p-1) \leqslant 0$. We claim that $\psi(\varphi) \geqslant \psi^*(\varphi)$ for $\varphi \in (0,\varphi_0)$. Indeed, if this is not true, we define

$$\Gamma = \{\varphi \in (0,\varphi_0);\ \psi(\varphi) < \psi^*(\varphi)\}.$$

Then $\Gamma \neq \emptyset$. Let

$$\varphi_1 = \sup_{\varphi \in \Gamma} \varphi.$$

Clearly, we have $0 < \varphi_1 < \varphi_0$ and $\psi(\varphi_1) = \psi^*(\varphi_1)$, since $\psi(0) = \psi^*(0)$ and $\psi(\varphi_0) > \psi^*(\varphi_0)$. Then we have

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} < \left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1}.$$

On the other hand, by the definition of φ_1, we also have

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} \geqslant \left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1},$$

which is a contradiction. That is, $\psi(\varphi) \geqslant \psi^*(\varphi)$ for $\varphi \in (0,\varphi_0)$. As above, we further have

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} < \frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}.$$

Integrating from 0 to φ for $\varphi \in (0,\varphi_0)$ yields

$$\psi(\varphi) < \psi^*(\varphi),$$

which is also a contradiction. Accordingly, we have $\psi(\varphi) < \psi^*(\varphi)$ for $\varphi \in (0,a/2)$. Set τ_1 with $\varphi(\tau_1) \in (0,a/2)$. Then $\varphi(t) \in (0,a/2)$ for all $t \in (\tau_1,t_2)$. Integrating from τ_1 to τ_2 with $t_1 < \tau_1 < \tau_2 < t_2$, we obtain

$$\begin{aligned}
\tau_2 - \tau_1 &= \int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \frac{1}{|\psi|^{1/(p-1)-1}\psi(s)}\,\mathrm{d}s \\
&= \int_{\varphi(\tau_2)}^{\varphi(\tau_1)} \frac{1}{|\psi|^{1/(p-1)}}\,\mathrm{d}s \\
&\leqslant \int_0^1 \alpha^{-1/(p-1)} s^{-\beta/(p-1)}\,\mathrm{d}s \\
&\leqslant \frac{p-1}{p-1-\beta}\alpha^{-1/(p-1)},
\end{aligned}$$

since $\beta < p-1$. Furthermore, by the arbitrariness of τ_2 with $\tau_2 < t_2$, we conclude that $t_2 < +\infty$.

(ii) First we consider the case in which $q \geqslant 1$. Let $B = \max_{0 \leqslant \varphi \leqslant a} |f(\varphi)|^{p-1} > 0$. For any fixed $\varphi_0 \in (0,a)$, if

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} \leqslant 0,$$

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then

$$\psi(\varphi_0) \geqslant \varphi_0^{q(p-1)} |f(\varphi_0)|^{p-2} f(\varphi_0) \geqslant -B\varphi_0^{q(p-1)},$$

while if

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} > 0,$$

let

$$\varphi_1 = \inf\left\{\varphi \in (0, \varphi_0);\ \left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=s} > 0 \text{ for all } s \in (\varphi, \varphi_0)\right\}.$$

Then

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} = 0 \quad \text{and} \quad \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} > 0 \quad \text{for all } \varphi \in (\varphi_1, \varphi_0).$$

It is obvious that $0 < \varphi_1 < \varphi_0$, since $\psi(0) = 0$ and $\psi(\varphi) < 0$ for $\varphi \in (0, 1)$. Therefore, we also have

$$\begin{aligned}\psi(\varphi_0) &> \psi(\varphi_1)\\ &= \varphi_1^{q(p-1)} |f(\varphi_1)|^{p-2} f(\varphi_1)\\ &\geqslant -B\varphi_1^{q(p-1)}\\ &> -B\varphi_0^{q(p-1)}.\end{aligned}$$

Thus, we obtain that, for all $\varphi \in (0, a)$,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = |\psi(\varphi)|^{1/(p-1)-1}\psi(\varphi) \geqslant -B^{1/(p-1)}\varphi^q.$$

Integrating the above inequality from τ_1 to τ_2 with $t_1 < \tau_1 < \tau_2 < t_2$, we obtain

$$\begin{aligned}B^{1/(p-1)}(\tau_2 - \tau_1) &\geqslant -\int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \frac{1}{s^q}\,\mathrm{d}s\\ &= \int_{\varphi(\tau_2)}^{\varphi(\tau_1)} \frac{1}{s^q}\,\mathrm{d}s.\end{aligned}$$

Letting $\tau_2 \to t_2$, and noting that $\varphi(t_2) = 0$ and $q \geqslant 1$, we obtain $t_2 = +\infty$.

Furthermore, if $q < 1$, $m < 1$ and $m + p \leqslant 1 + q$, by noting that

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} &= c^*\varphi^{-m} - \frac{c^*\varphi^{q-m}f(\varphi)}{|\psi(\varphi)|^{1/(p-1)-1}\psi(\varphi)}\\ &\geqslant -\frac{c^*\varphi^{q-m}f(\varphi)}{|\psi(\varphi)|^{1/(p-1)-1}\psi(\varphi)},\end{aligned}$$

we may infer that

$$\frac{\mathrm{d}|\psi|^{p/(p-1)}}{\mathrm{d}\varphi} \leqslant -\frac{pc^*}{p-1}\varphi^{q-m}f(\varphi) \leqslant \frac{Apc^*}{p-1}\varphi^{q-m}.$$

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Integrating from 0 to φ yields

$$|\psi(\varphi)| \leqslant \left(\frac{Apc^*}{(p-1)(q-m+1)}\varphi^{q-m+1}\right)^{(p-1)/p},$$

which implies that

$$\psi(\varphi) \geqslant -\left(\frac{Apc^*}{(p-1)(q-m+1)}\varphi^{q-m+1}\right)^{(p-1)/p}.$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= |\psi(\varphi)|^{1/(p-1)-1}\psi(\varphi)\\ &\geqslant -\left(\frac{Apc^*}{(p-1)(q-m+1)}\varphi^{q-m+1}\right)^{1/p}.\end{aligned}$$

Integrating the above inequality from τ_1 to τ_2 with $t_1 < \tau_1 < \tau_2 < t_2$, we obtain

$$\left(\frac{Apc^*}{(p-1)(q-m+1)}\right)^{1/p}(\tau_2-\tau_1) \geqslant -\int_{\varphi(\tau_1)}^{\varphi(\tau_2)} s^{-(q-m+1)/p}\,\mathrm{d}s.$$

Letting $\tau_2 \to t_2$, and noting that $\varphi(t_2) = 0$ and $(q-m+1)/p \geqslant 1$, we obtain $t_2 = +\infty$.

The proof of the theorem is now complete. □

We now turn to a discussion of the regularity of smooth non-decreasing travelling wavefronts, and more specifically, the finiteness of t_1.

THEOREM 4.2. *Let $\varphi(t)$ be the non-decreasing and smooth travelling wavefront of (1.1) with f satisfying (H) for $c = c^*$, which is determined by proposition 3.2. Then*

(i) *if $m + p > \min\{2, 1 + q\}$, then $t_1 > -\infty$,*

(ii) *if $m + p \leqslant \min\{2, 1 + q\}$, then $t_1 = -\infty$.*

Proof. (i) Let $A = \min_{0\leqslant\varphi\leqslant a/2}\{|f(\varphi)|\}$. Then by the equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = c^*\varphi^{-m} - \frac{c^*\varphi^{q-m}f(\varphi)}{\psi^{1/(p-1)}}$$

it is easy to see that

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} \geqslant c^*\varphi^{-m} \quad \text{and} \quad \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} \geqslant -\frac{c^*\varphi^{q-m}f(\varphi)}{\psi^{1/(p-1)}}$$

for $0 < \varphi \leqslant a/2$. Integrating from 0 to φ for $\varphi < a/2$ yields

$$\psi(\varphi) \geqslant \frac{c^*}{1-m}\varphi^{1-m}$$

and

$$\psi(\varphi) \geqslant \left(\frac{Apc^*}{(p-1)(q+1-m)}\varphi^{q+1-m}\right)^{(p-1)/p}.$$

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Setting τ_2 with $\varphi(\tau_2) \in (0, a/2)$, for any τ_1 with $t_1 < \tau_1 < \tau_2 < t_2$, we see that

$$\tau_2 - \tau_1 = \int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \frac{1}{\psi^{1/(p-1)}(s)}\,\mathrm{d}s \\ \leqslant \int_{\varphi(\tau_1)}^{\varphi(\tau_2)} Bs^{-q^*}\,\mathrm{d}s,$$

where

$$B = \max\left\{\left(\frac{1-m}{c^*}\right)^{1/(p-1)}, \left(\frac{(p-1)(q+1-m)}{Apc^*}\right)^{1/p}\right\}$$

and

$$q^* = \min\left\{\frac{1-m}{p-1}, \frac{q+1-m}{p}\right\}.$$

By a simple calculation we see that $q^* < 1$, since $m + p > \min\{2, 1+q\}$. Letting $\tau_1 \to t_1$ and noting that $\varphi(t_1) = 0$, we obtain $t_1 > -\infty$.

(ii) Let $A = \max_{0 \leqslant \varphi \leqslant a}\{|f(\varphi)|\}$ and $\psi^* = \alpha\varphi^\beta$, where

$$\beta = \min\left\{1-m, \frac{(p-1)(q+1-m)}{p}\right\}$$

and α is a sufficiently large constant to be determined. We claim that $\psi(\varphi) \leqslant \psi^*(\varphi)$ for $\varphi \in (0, a/2)$. Otherwise there exists a $\varphi_0 \in (0, \sigma)$ such that $\psi^*(\varphi_0) < \psi(\varphi_0)$. Then

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} = c^*\varphi_0^{-m}\left(1 - \frac{\varphi_0^q f(\varphi_0)}{\psi^{1/(p-1)}(\varphi_0)}\right) \\ \leqslant c^*\varphi_0^{-m}\left(1 - \frac{\varphi_0^q f(\varphi_0)}{\alpha^{1/(p-1)}\varphi_0^{\beta/(p-1)}}\right).$$

A direct calculation yields

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0} < \left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_0}, \tag{4.1}$$

which is ensured by

$$c^*\varphi_0^{1-m-\beta} + \frac{c^*A\varphi_0^{1-m+q-(p\beta)/(p-1)}}{\alpha^{1/(p-1)}} < \alpha\beta.$$

Observe that $1-m-\beta \geqslant 0$, $1-m+q-(p\beta)/(p-1) \geqslant 0$ since

$$\beta = \min\left\{1-m, \frac{(p-1)(q+1-m)}{p}\right\}.$$

Therefore, for sufficiently large α, (4.1) holds. Thus, we have

$$\psi(\varphi) \geqslant \psi^*(\varphi) \quad \text{for } \varphi \in (0, \varphi_0).$$

Indeed, if this were not true, we let

$$\Gamma = \{\varphi \in (0, \varphi_0);\ \psi(\varphi) < \psi^*(\varphi)\}.$$

Then $\Gamma \neq \emptyset$. Let

$$\varphi_1 = \sup_{\varphi \in \Gamma} \varphi.$$

Clearly, we have $0 < \varphi_1 < \varphi_0$ and $\psi(\varphi_1) = \psi^*(\varphi_1)$, since $\psi(0) = \psi^*(0)$ and $\psi(\varphi_0) > \psi^*(\varphi_0)$. Then, similarly to (4.1), we have

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} < \left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1}.$$

Furthermore, by the definition of φ_1, we also have

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1} \geqslant \left.\frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi}\right|_{\varphi=\varphi_1},$$

which is a contradiction. Hence $\psi(\varphi) \geqslant \psi^*(\varphi)$ for $\varphi \in (0, \varphi_0)$. Similarly to (4.1), we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} < \frac{\mathrm{d}\psi^*}{\mathrm{d}\varphi} \quad \text{for } \varphi \in (0, \varphi_0).$$

Integrating from 0 to φ for $\varphi \in (0, \varphi_0)$, we obtain

$$\psi(\varphi) < \psi^*(\varphi),$$

which is also a contradiction. Therefore,

$$\psi(\varphi) \leqslant \psi^*(\varphi) \quad \text{for } \varphi \in (0, a/2).$$

Hence, fixing τ_2 with $\varphi(\tau_2) \in (0, a/2)$, and for any τ_1 with $t_1 < \tau_1 < \tau_2 < t_2$, we obtain

$$\begin{aligned}\tau_2 - \tau_1 &= \int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \frac{1}{\psi^{1/(p-1)}(s)}\,\mathrm{d}s \\ &\geqslant \int_{\varphi(\tau_1)}^{\varphi(\tau_2)} \alpha^{-1/(p-1)}\varphi^{-\beta/(p-1)}\,\mathrm{d}s.\end{aligned}$$

Note that $\beta/(p-1) \geqslant 1$, since $m + p \leqslant \min\{2, 1+q\}$. Letting $\tau_1 \to t_1$, and noting that $\varphi(t_1) = 0^+$, we have $t_1 = -\infty$. The proof is complete. □

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